Variedades de BL-álgebras generadas por BLn-cadenas

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2003

Tesis presentada para obtener el grado de Doctor en Ciencias Matemáticas de la Universidad de Buenos Aires

UNIVERSIDAD DE BUENOS AIRES

Facultad de Ciencias Exactas y Naturales

Departamento de Matemática

Variedades de BL-álgebras generadas por BL_n-cadenas

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Trabajo de tesis para optar por el título de Doctor en Ciencias Matemáticas

18 de noviembre de 2003.

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En la presente tesis se estudian subvariedades de BL-álgebras. En una primera etapa, después de haber dado las nociones básicas acerca de BL-álgebras, se estudian las estructuras fundamentales dentro de las variedades de BL-álgebras: las BL-cadenas. Se prueba la descomposición de las mismas en suma ordinal de hoops de Wajsberg y se da también una descomposición en suma ordinal de su subálgebra de elementos regulares y la BL-álgebra generalizada de sus elementos densos. Una vez obtenidos estos resultados, se define una BL_n-cadena como aquella que es suma ordinal de una MV-cadena finita de longitud n y una BL-cadena generalizada para luego proceder al estudio de subvariedades de BL-álgebras generadas por una de estas cadenas. Se da un método para caracterizar ecuacionalmente estas subvariedades y luego se da una descripción de las BL-álgebras libres en estas variedades sobre un conjunto arbitrario de generadores.

Palabras claves: *Lógicas difusas, Hoops, BL-álgebras, Álgebras libres, MV-álgebras*

Varieties of BL-algebras generated by BL_n-chains

The present thesis is a study of subvarieties of BL-algebras. As a first step, after introducing some basic notions about BL-algebras, the most important structures in the varieties of BL-algebras, BL-chains, are studied. A proof of the decomposition of BL-chains into the ordinal sum of Wajsberg hoops is given, and another decomposition of them as ordinal sum of the MV-algebra of regular elements of the chains and the generalized BL-algebra of their dense elements is also presented. After this, the definition of BL_n-chain as the ordinal sum of a finite MV-chain of length n and a generalized BL-chain is introduced. A method to equationally characterize subvarieties of BL-algebras generated by one BL_n-chain is developed and a description of free algebras over an arbitrary set of generators in these varieties is obtained.

Keywords: *Fuzzy logics, Hoops, BL-algebras, Free algebras, MV-algebras.*

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0.1 Introducción

Las lógicas difusas se originan en 1965 en la publicación [37] de L. A. Zadeh, y desde entonces se han desarrollado y han sido aplicadas exitosamente en muchos problemas, principalmente en el diseño de sistemas expertos que puedan tomar decisiones sobre la base de información incompleta, incierta y/o vaga. La lógica básica (BL) es introducida por Hájek (ver [27] y las referencias allí citadas), con el objetivo de formalizar las lógicas difusas en las que la conjunción se interpreta por una t-norma continua en el segmento real [0,1] y la implicación como su correspondiente adjunta. Hájek también introduce en [27] las BL-álgebras como las contrapartes algebraicas de BL. Estas álgebras forman una variedad (o clase ecuacional) de reticulados residuados [27]. Más precisamente, pueden ser caracterizadas como hoops básicos acotados [1, 7]. Las subvariedades de la variedad de BL-álgebras están en correspondencia con las extensiones axiomáticas de BL. Algunas subvariedades importantes de la variedad de BL-álgebras son las MV-álgebras, correspondientes a la lógica multivaluada de Lukasiewicz (ver [16]), las álgebras de Heyting lineales, correspondientes a la extensión de la lógica superintuicionista caracterizada por el axioma $(P \to Q) \vee (Q \to P)$, (ver [36]), las PL-álgebras, correspondientes a la lógica determinada por la t-norma dada por el producto usual en [0,1], (ver [19]), y también las álgebras de Boole, correspondientes a la lógica clásica.

En la presente tesis se estudian ciertas subvariedades de BL-álgebras. Como toda BL-álgebra es un producto subdirecto de BL-álgebras totalmente ordenadas (ver [27]), un primer paso es investigar la estructura de estas álgebras generadoras, a las que usualmente se conoce como BL-cadenas.

Por su importancia en el desarrollo de la teoría acerca de las BL-álgebras, se han realizado varios estudios sobre la estructura de las BL-cadenas. En [17], se descompone toda BL-cadena que es saturada en una suma ordinal de MV-cadenas, cadenas de Gödel y PL-cadenas, siguiendo la descomposición natural de las t-normas continuas. El propósito principal de esta descomposición es la demostración del teorema de completitud de BL. Por otro lado, considerando el hecho que las BL-álgebras poseen como raíz algebraica la teoría de hoops (ver [1]), se da en [2] un teorema de descomposición para BL-cadenas en una calse especial de hoops, llamados hoops de Wajsberg, que no admiten ulteriores descomposiciones. Si bien esto mejora el resultado dado en [17], porque no necesita que la cadena a descomponer satisfaga la condición de saturación, la demostración de la descomposición se basa fundamentalmente en el axioma de elección (este axioma es requerido tres veces a lo largo de la prueba). Una tercera descomposición de BL-cadenas se da en [33]. La idea principal de esta descomposición es definir en cada BL-cadena una relación de equivalencia de modo tal que las clases de equivalencia son estructuras relacionadas con semigrupos abelianos totalmente ordenados a los que llaman formas básicas. Estas estructuras formarán los bloques de la descomposición. A pesar de que la demostración no requiere del uso del axioma de elección, las formas básicas son estructuras ad hoc.

En la presente tesis ofrezco una prueba simple y autocontenida del teorema de descomposición en hoops de Wajsberg definiendo en cada BL-cadena una relación de equivalencia. Esta demostración no requiere del uso de ninguna versión del axioma de elección. Además se prueba la unicidad de dicha descomposición.

Pero hay otra manera de descomponer las BL-cadenas que será de suma utilidad para el desarrollo de los resultados de la tesis. En [21], se estudian dos clases diferentes de elementos en una BL-álgebra: elementos regulares y elementos densos. Se prueba allí que el conjunto de elementos regulares de una BL-álgebra forma una subálgebra que posee una estructura de MV-álgebra. Por otro lado, el conjunto de elementos densos de una BL-álgebra posee una estructura de BL-álgebra generalizada. Con base en estos hechos, en el Teorema 2.2.1, demuestro que cada BL-cadena puede ser descompuesta en la suma ordinal de la MV-álgebra de sus elementos regulares y la BL-álgebra generalizada de sus elementos densos. Esta descomposición permite clasificar a las BL-cadenas de acuerdo a la MV-álgebra de sus elementos regulares. Se llamarán BL_n-cadenas a las BL-cadenas cuyas MV-álgebras de elementos regulares sean MV-cadenas finitas de $n \geq 2$ elementos.

Como indica el título de la presente tesis, estudiaré subvariedades de BL-álgebras generadas por BL_n-cadenas. Para comenzar con este estudio, y siguiendo ideas utilizadas en [2], doy un método de caracterización ecuacional para las subvariedades de BL-álgebras generadas por una BL_n-cadena. Más precisamente, como las BL_n-cadenas son suma ordinal de una MV-cadena finita $\mathbf{L}_n$, y una BL-cadena generalizada $\mathbf{B}$, demuestro cómo las ecuaciones que caracterizan la subvariedad generada por la BL_n-cadena $\mathbf{L}_n \uplus \mathbf{B}$ dependen de las ecuaciones que caracterizan la variedad de MV-álgebras generada por $\mathbf{L}_n$ y las ecuaciones que caracterizan la variedad de BL-álgebras generalizadas generada por $\mathbf{B}$.

Una vez obtenida esta caracterización, comienzo con el estudio de álgebras libres en subvariedades de BL-álgebras generadas por una BL_n-cadena. La descripción de las álgebras libres da la representación concreta en término de funciones de las proposiciones de BL, puesto que las proposiciones, bajo equivalencia lógica, forman una BL-álgebra libre. Para algunas subvariedades de BL-álgebras dichas álgebras libres ya han sido estudiadas. El ejemplo más conocido es la representación de proposiciones clásicas por funciones booleanas. Otro ejemplo es la descripción de MV-álgebras libres en términos de funciones lineales continuas a trozos dada por Mc Naughton [34] (ver también [16]). Las álgebras libres finitamente generadas en la variedad de álgebras de Heyting lineales fue dada por Horn [30], y una descripción de PL-álgebras libres finitamente generadas se da en [19]. Las álgebras de Heyting lineales y las PL-álgebras son ejemplos de variedades de BL-álgebras que satisfacen la propiedad de la retracción booleana. Las álgebras libres en estas subvariedades de BL-álgebras fueron descriptas en [20].

Para la descripción de las álgebras libres en variedades de BL-álgebras generadas por una BL_n-cadena utilizo la representación de BL-álgebras como producto booleano débil de álgebras directamente indescomponibles dada en [20]. Dicha representación, llamada representación de Pierce, consiste en tomar los cocientes del álgebra libre por los filtros implicativos generados por los ultrafiltros de la subálgebra de elementos booleanos del álgebra libre en cuestión. Utilizando los resultados de [21], pruebo que la subálgebra de elementos booleanos del álgebra libre es la subálgebra de elementos booleanos de un álgebra libre en $\mathcal{MV}_n$, la variedad de MV-álgebras generada por la cadena finita $\mathbf{L}_n$. Finalmente se caracteriza el álgebra de elementos booleanos de esta álgebra libre en $\mathcal{MV}_n$: es el álgebra de Boole libre sobre un conjunto parcialmente ordenado que es suma cardinal de cadenas de longitud $n-1$. En la demostración de este resultado juegan un rol fundamental los reductos de álgebras de Moisil de las álgebras en $\mathcal{MV}_n$. Una vez obtenida una caracterización de los booleanos del álgebra libre realizo un estudio de los cocientes del álgebra libre por los filtros generados por los ultrafiltros booleanos. Concluyo que las álgebras libres en variedades de BL-álgebras generadas por una BL_n-cadena $\mathbf{L}_n \uplus \mathbf{B}$ son productos booleanos débiles de BL-álgebras que son suma ordinal de una subálgebra de $\mathbf{L}_n$ y una BL-álgebra generalizada libre en la variedad de BL-álgebras generalizadas generada por $\mathbf{B}$.

Por último, presento un método alternativo de descripción de estas álgebras libres cuando el conjunto de generadores del álgebra es finto. Basándome en el hecho que la subálgebra de elementos booleanos de estas álgebras libres finitamente generadas es finita, caracterizo los átomos del álgebra de Boole. Los elementos del álgebra libre se pueden vizualizar como funciones finitas y el conocimiento de los átomos de la subálgebra de Boole permite una descripción detallada de cada factor indescomponible. Como el producto booleano débil sobre espacios finitos discretos es un producto directo, obtengo una caracterización de las álgebras libres finitamente generadas como el procuto directo de álgebras indescomponibles. Estos resultados han sido aceptados para ser publicados en Algebra Universalis ([11]). Ambas descripciones coinciden cuando el conjunto de generadores libres es finito, pero esta última descripción permite obtener mayor información acerca de los factores de la descomposición.

La tesis está organizada como sigue: En un primer capítulo se revisan las nociones básicas concernientes a BL-álgebras necesarias para el desarrollo del trabajo. En el segundo capítulo se presentan dos teoremas de descomposición de BL-cadenas: en el primero se presenta la descomposición en hoops de Wajsberg dada en [2], pero se ofrece una demostración más simple del mismo que, a diferencia de la de [2] no requiere del uso del axioma de elección. En el segundo se da una descomposición en la MV-álgebra de elementos regulares y la BL-álgebra generalizada de elementos densos. En el tercer capítulo, una vez introducida la noción de BL_n-cadena, se ofrece un método para la caracterización ecuacional de las subvariedades de BL-álgebras generadas por una de estas cadenas. También se ofrecen ejemplos de caracterizaciones ecuacionales para algunas de estas subvariedades. En el capítulo cuatro, se obtiene una descripción de las álgebras libres en variedades de BL-álgebras generadas por una BL_n-cadena en términos de productos booleanos débiles. Por último, en el capítulo cinco, se presenta una descripción alternativa de dichas álgebras libres cuando el conjunto de generadores es finito, y se comparan los resultados con los del capítulo anterior. La tesis posee un apéndice donde se describen la subálgebras booleanas de elementos idempotentes de las álgebras libres en variedades de MV-álgebras generadas por cadenas finitas.

Denotaré con letras negritas $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ a las estructuras algebraicas y con la misma letra ordinaria $A, B, C, \ldots$ a sus correspondientes universos. Los conceptos de álgebra universal utilizados durante el desarrollo de la tesis se pueden encontrar en [10] y en [22].

0.2 Introduction

Fuzzy logics have their origins in a paper published in 1965 by L. A. Zadeh [37], and since then, have been far developed and applied successfully in many problems, mainly in the design of experts systems that can take decisions based on fuzzy or vague information. Basic Fuzzy Logic (BL for short) was introduced by Hájek (see [27] and the references given there) to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0, 1]$ and the implication by its corresponding adjoint. Hájek also introduced BL-algebras as the algebraic counterparts of BL. These algebras form a variety (or equational class) of residuated lattices [27]. More precisely, they can be characterized as bounded basic hoops [1, 7]. Subvarieties of the variety of BL-algebras are in correspondence with axiomatic extensions of BL. Important examples of subvarieties of BL-algebras are MV-algebras, that correspond to Lukasiewicz many-valued logics (see [16]), linear Heyting algebras, that correspond to the superintuitionistic logic characterized by the axiom $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ (see [36] for a historical account about this logic), PL-algebras, that correspond to the logic determined by the t-norm given by the ordinary product on [0,1], (see [19]), and also boolean algebras that correspond to classical logic.

In the present thesis I study certain subvarieties of BL-algebras. Since each BL-algebra is a subdirect product of totally ordered BL-algebras (see [27, Lemma 2.3.16]), as a first step I investigate the structure of such generating algebras, that are usually called BL-chains.

Since BL-chains are very important in the theory of BL-algebras, they have already been deeply investigated. Following the natural decomposition of continuous t-norms, in [17] each BL-chain which is saturated is decomposed into an ordinal sum of MV-chains, Gödel chains and PL-chains. The main purpose of such decomposition is the proof of completeness of BL. Considering the fact that BL-algebras have as an algebraic root the theory of hoops (see [1]), in [2] a theorem of decomposition for BL-chains (i.e., basic totally ordered bounded hoops) into some special kind of hoops is given. These hoops, named Wajsberg hoops, can not be further decomposed. Although this improves the result given in [17], for it can be applied not only to saturated BL-chains, the given proof strongly relies on the axiom of choice (as a matter of fact, it is invoked three times in the course of the proof). An alternative decomposition of BL-chains is given in [33]. The main idea of such decomposition is to define on each BL-chain an equivalence relation such that the equivalence classes are structures related to ordered abelian semigroups, called basic forms. These structures are the building blocks of the decomposition. Although the decomposition is obtained without appealing to the axiom of choice, basic forms are ad hoc structures.

In the present thesis, I offer a simple and self contained proof of the decomposition given in [2] by means of a suitable equivalence relation on BL-chains, whose equivalence classes are Wajsberg hoops. This proof does not invoke any version of the axiom of choice. I also prove the uniqueness of the decomposition.

But there is another way of decomposing BL-chains that shall be more useful to obtain the results of the thesis. In [21] two different kinds of elements in a BL-algebra are studied: regular elements and dense elements. It is proved that the set of regular elements of a BL-algebra form a subalgebra which is an MV-algebra. On the other hand, the set of dense elements of a BL-algebra form a generalized BL-algebra. Taking these ideas into account, in Theorem 2.2.1, I prove that each BL-chain can be decomposed into the ordinal sum of the MV-algebra of its regular elements and the generalized BL-algebra of its dense elements. This decomposition makes possible the classification of BL-chains according to the MV-algebra of its regular elements. I shall call BL_n-chain each BL-chain whose subalgebra of regular elements form a finite MV-chain of $n \geq 2$ elements.

As it is indicated by the title of the thesis, I shall study subvarieties of BL-algebras generated by BL_n-chains. As a first step, following some ideas of [2], I describe a method to equationally characterize subvarieties of BL-algebras generated by one BL_n-chain. Since these chains are the ordinal sum of a finite MV-chain $\mathbf{L}_n$ and a generalized BL-chain $\mathbf{B}$, I demonstrate how the equations that define the subvariety generated by the BL_n-chain $\mathbf{L}_n \uplus \mathbf{B}$ depends on the equations that define the subvariety of MV-algebras generated by $\mathbf{L}_n$ and the equations that define the subvariety of generalized BL-algebras generated by $\mathbf{B}$.

Once the subvarieties of BL-algebras generated by one BL_n-chain are characterized, I study free algebras in these subvarieties. Since the propositions under BL equivalence form a free BL-algebra, descriptions of free algebras in terms of functions give concrete representations of these propositions. Such descriptions are known for some subvarieties of BL-algebras. The best known example is the representation of classical propositions by boolean functions. Free MV-algebras have been described in terms of continuous piecewise linear functions by McNaughton [34] (see also [16]). Finitely generated free linear Heyting algebras were described by Horn [30], and a description of finitely generated free PL-algebras is given in [19]. Linear Heyting algebras and PL-algebras are examples of varieties of BL-algebras satisfying the boolean retraction property. Free algebras in these varieties were completely described in [20].

To describe free algebras in varieties of BL-algebras generated by one

BL_n-chain, the representation as a weak boolean product of directly indecomposable algebras given in [20] is invoked. The mentioned representation, called the Pierce representation, consists of taking the quotients of the free algebra over the implicative filters generated by the ultrafilters of the subalgebra of boolean elements of the free algebra. Using results of [21], I prove that the subalgebra of boolean elements of the free algebra is the subalgebra of boolean elements of a free algebra in $\mathcal{MV}_n$, the variety of MV-algebras generated by the finite MV-chain $\mathbf{L}_n$. Therefore a characterization of the algebra of boolean elements of this free algebra in $\mathcal{MV}_n$ is obtained: it is the free boolean algebra over a poset which is the cardinal sum of chains of length $n-1$. In the proof of this result a central role is played by the Moisil algebra reducts of algebras in $\mathcal{MV}_n$. I conclude that free algebras in varieties of BL-algebras generated by a single BL_n-chain $\mathbf{L}_n \uplus \mathbf{B}$ are weak boolean products of BL-algebras that are ordinal sums of subalgebras of $\mathbf{L}_n$ and free algebras in the variety of basic hoops generated by $\mathbf{B}$.

I present an alternative description of the free algebras when the set of free generators is finite. Since for the finite case the subalgebra of boolean elements of the free algebra is finite, I characterize the atoms of these boolean algebra. The elements of the free algebra are view then as functions, and the knowledge of the atom that generates each ultrafilter makes possible a complete description of each indecomposable factor. Since weak boolean products over discrete finite spaces are direct products, I give a description of the finitely generated free algebra as a direct product of indecomposable algebras. These results are about to appear in Algebra Universalis (see [11]). Although both descriptions coincide when the set of generators is finite, this last one gives more information about the factors of the decomposition.

The thesis is organized as follows: In the first chapter, all the basic notions concerning BL-algebras needed for the development of the thesis are recalled. In the second one, two different theorems of decomposition of BL-chains are presented: the first one is the decomposition into Wajsberg hoops given in [2], but a much simpler and constructive proof of the Theorem is presented. The second one is the decomposition into regular and dense elements. In chapter three, after introducing the notion of BL_n-chain, a method to equationally characterize the subvarieties of BL-algebras generated by one of these chains is described. I also give examples of the equational characterization for some of such subvarieties. In chapter four, the main one, a description of free algebras in varieties of BL-algebras generated by a BL_n-chain in terms of weak boolean product is given. Lastly, in chapter five, an alternative description of such free algebras is given when the set of free generators is finite, and I compare these results with the ones given in the previous chapter for the general case. An appendix is also add at the end of the thesis. In this appendix a description of the boolean subalgebra of idempotents elements of free algebras in varieties of MV-algebras generated by finite chains is presented.

I denote algebras with bold face letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ and their corresponding universes by the ordinary type of the same letter $A, B, C, \ldots$. The notions of universal algebra used in the development of the thesis can be found in [10] and [22].

Chapter 1

Basic Notions

1.1 Continuous t-norms

Definition 1.1.1 *A* **t-norm** *is s binary operation* $*$ *form* $[0,1]^2$ *into* $[0,1]$ *satisfying the following conditions:*

1. $*$ *is commutative and associative,*

2. $*$ *is non decreasing in both arguments, i.e, for all* $x, y, z \in [0,1]$

$$x \leq y \text{ implies } x * z \leq y * z \text{ and } z * x \leq z * y,$$

3. $1 * x = x$ *and* $0 * x = 0$ *for all* $x \in [0,1]$.

A **continuous t-norm** is a t-norm which is continuous as a map from $[0,1]^2$ into $[0,1]$ in the usual sense. For each continuous t-norm a residuum $\rightarrow$ can be defined (see [27]) satisfying

$$x * z \leq y \text{ iff } x \leq z \rightarrow y.$$

Example 1.1.2 *The following are the most important examples of continuous t-norms and their corresponding residuum:*

1. *Lukasiewicz t-norm:* $x * y = max\ (0, x + y - 1)$,

   Lukasiewicz implication: $x \rightarrow y = min(1, 1 - x + y)$,

2. *Gödel t-norm:* $x * y = min(x, y)$,

   Gödel implication: $x \rightarrow y = \begin{cases} y & \text{if } x > y, \\ 1 & \text{if } x \leq y, \end{cases}$

3. *Product t-norm:* $x * y = x.y$,

Goguen implication: $x \to y = \begin{cases} y/x & \text{if } x > y, \\ 1 & \text{if } x \leq y. \end{cases}$

In [27] for each fixed continuous t-norm $*$ a propositional calculus PC($*$) is presented whose truth values are in the real segment $[0, 1]$, $*$ is taken for the truth function of the (strong) conjunction and the residuum $\to$ of $*$ becomes the truth function of the implication. Hajėk formulated logical axioms for BL and he proved that each provable formula in BL is a tautology in each PC($*$) (soundness of BL). To prove the completeness of the logic he starts an algebraization of BL. Is then when BL-algebras are introduced.

1.2 Hoops, Generalized BL-algebras and BL-algebras

A **hoop** is an algebra $\mathbf{A} = (A, *, \to, \top)$ of type $(2, 2, 0)$, such that $(A, *, \top)$ is a commutative monoid and for all $x, y, z \in A$:

1. $x \to x = \top$,
2. $x * (x \to y) = y * (y \to x)$,
3. $x \to (y \to z) = (x * y) \to z$.

Hoops were introduced in an unpublished manuscript [9] by Büchi and Owens and they were deeply investigated in [1], [6], [7] and [26]. Some basic properties of hoops are enumerated in the next proposition:

Proposition 1.2.1 *Let* $\mathbf{A} = (A, *, \to, \top)$ *be a hoop. Then:*

1. $(A, *, \top)$ *is a naturally ordered residuated commutative monoid, where the order is defined by* $x \leq y$ *iff* $x \to y = \top$ *and the residuation is*

$$x * y \leq z \text{ iff } x \leq y \to z.$$

2. *The partial order on any hoop is a semilattice order, where* $x \wedge y = x * (x \to y)$.

3. *For any* $x, y, z \in A$ *the following hold:*

   (a) $\top \to x = x$,

   (b) $x \to \top = \top$, *i.e.,* $\top$ *is the largest element in the order,*

(c) $x \to y \leq (z \to x) \to (z \to y)$,

(d) $x \leq y \to x$,

(e) $x \leq (x \to y) \to y$,

(f) $x \to (y \to z) = y \to (x \to z)$,

(g) $x \to y \leq (y \to z) \to (x \to z)$,

(h) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,

(i) $x \leq y$ *implies* $x * z \leq y * z$,

(j) $x * y \leq x$.

A **generalized BL-algebra or basic hoop** is a hoop that satisfies the equation:

$$(((x \to y) \to z) * ((y \to x) \to z)) \to z = \top \tag{1.1}$$

In every generalized BL-algebra $\mathbf{A}$ an operation $\vee$ can be defined by

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x),$$

thus $\mathbf{L}(\mathbf{A}) = (A, \wedge, \vee, \top)$ is a lattice with greatest element $\top$. Besides, every generalized BL-algebra $\mathbf{A}$ satisfies the equation:

$$(x \to y) \vee (y \to x) = \top.$$

A **BL-algebra** is a bounded generalized BL-algebra (or bounded basic hoop), that is, it is an algebra $\mathbf{A} = (A, *, \to, \perp, \top)$ of type $(2, 2, 0, 0)$ such that $(A, *, \to, \top)$ is a generalized BL-algebra, and $\perp$ is the lower bound of $\mathbf{L}(\mathbf{A})$. Then the set $B \subseteq A$ is the universe of a subalgebra of a BL-algebra $\mathbf{A}$ iff $\top, \perp \in B$ and B is closed under $*$ and $\to$. Besides, if $C \subseteq A$ is a set closed under $*$ and $\to$ such that $\top \in C$, then $\mathbf{C} = (C, *, \to, \top)$ is a generalized BL-algebra. For any integer k, a **BL-term** in the variables $x_i, x_2, \ldots x_k$ is a string over the set $S_k = \{*, \to, \perp, \top, x_1, x_2, \ldots, x_k, (,)\}$ that arises from a finite number of application of the following rules:

- $\perp, \top, x_1, x_2, \ldots, x_k$ are BL-terms,
- if τ_1 and τ_2 are BL-terms, then $(\tau_1 * \tau_2)$ and $(\tau_1 \to \tau_2)$ are BL-terms.

For each continuous t-norm $*$, the structure $([0, 1], *, \to, 0, 1)$ is a BL-algebra, where $\to$ is the residuum of $*$. As a matter of fact, each BL-algebra structure on the segment $[0, 1]$ is given by a continuous t-norm, because the continuity of $*$ is equivalent to the condition $x * (x \to y) = y * (y \to x)$ (see, for intance, [25]).

On each BL-algebra, the unary operation $\neg$ (negation) is defined by the equation:

$$\neg x = x \to \bot.$$

The BL-algebra $\mathbf{A}$ with only one element, that is $\bot = \top$, is called the **trivial BL-algebra**. The varieties of BL-algebras and of generalized BL-algebras will be denoted by $\mathcal{BL}$ and $\mathcal{GBL}$, respectively. These are varieties of residuated lattices, hence they are varieties of BCK-algebras. It is known (see [31]) that both varieties are congruence distributive and congruence permutable.

Let $\mathbf{A}$ be a generalized BL-algebra. As mentioned in Proposition 1.2.1, we denote by $\leq$ the (partial) order defined on A by the lattice $\mathbf{L}(\mathbf{A})$, i.e. for $a, b \in A$, $a \leq b$ iff $a = a \wedge b$ iff $b = a \vee b$. This order is called the **natural order** of $\mathbf{A}$. When this natural order is total (i.e., for each $a, b, \in A$, $a \leq b$ or $b \leq a$), $\mathbf{A}$ is called **generalized BL-chain** (**BL-chain** in case $\mathbf{A}$ is a BL-algebra).

The following theorem makes obvious the importance of BL-chains and can be easily derived from [27, Lemma 2.3.16].

Theorem 1.2.2 *Each BL-algebra is a subdirect product of BL-chains.*

Indeed, since BL-algebras are bounded basic hoops the previous result also follows from [1, Theorem 2.8].

1.3 Important subvarieties of BL-algebras.

Some subvarieties of $\mathcal{BL}$ have been studied for their own importance, since they are the algebraic counterpart of some well known logics. **MV-algebras**, for instance, the algebras of Lukasiewicz infinite-valued logic, form the subvariety of $\mathcal{BL}$ characterized by the equation:

$$\neg\neg x = x$$

(see [27]). For references about these algebras see [16]. The variety of MV-algebras is denoted by $\mathcal{MV}$ and a totally ordered MV-algebra is an **MV-chain**. If $\mathbf{A}$ is a BL-algebra, consider

$$MV(\mathbf{A}) = \{x \in A : \neg\neg x = x\}.$$

Then $\mathbf{MV}(\mathbf{A}) = (MV(\mathbf{A}), *, \to, \bot, \top)$ is an MV-algebra (see [21]) which is a subalgebra of $\mathbf{A}$.

For $n \geq 2$, we define:

$$L_n = \{\frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-1}{n-1}\}.$$

The set L_n equipped with the operations $x * y = \max(0, x+y-1)$, $x \to y = \min(1, 1-x+y)$, and with $\bot = 0$ and $\top = 1$ defines a finite MV-algebra which shall be denoted by $\mathbf{L}_n$.

A **linear Heyting algebra** $\mathbf{H} = (H, \wedge, \to, \bot, \top)$ is a Heyting algebra (or relative pseudocomplemented bounded distributive lattice, see [4]) which satisfies the equation:

$$(x \to y) \vee (y \to x) = \top.$$

These algebras are the algebraic counterpart of the superintuitionistic logic characterized by the axiom $(P \Rightarrow Q) \vee (Q \Rightarrow P)$. Observe that any linear Heyting algebra $\mathbf{H}$ satisfies the equations:

$$x \wedge y = x \wedge (x \to y),$$

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Then $\mathbf{H}$ is a BL-algebra in which $\wedge = *$, i.e., it satisfies the equation $x * y = x \wedge y$.

A **PL-algebra** is a BL-algebra that satisfies the following two equations:

$$(\neg\neg z * ((x * z) \to (y * z))) \to (x \to y) = \top,$$

$$x \wedge \neg x = \bot.$$

PL-algebras correspond to **product fuzzy logic**, see [19] and [27].

It follows from Theorem 1.2.2 that for each BL-algebra $\mathbf{A}$ the lattice $\mathbf{L}(\mathbf{A})$ is distributive. The complemented elements of $L(\mathbf{A})$ form a subalgebra $\mathbf{B}(\mathbf{A})$ of $\mathbf{A}$ which is a boolean algebra. Elements of $B(\mathbf{A})$ are called **boolean elements of A**.

1.4 Implicative filters

Definition 1.4.1 *An* **implicative filter** *of a BL-algebra* $\mathbf{A}$ *is a subset* $F \subseteq A$ *satisfying the following conditions:*

1. $\top \in F$,

2. *If* $x \in F$ *and* $x \to y \in F$, *then* $y \in F$.

An implicative filter is called **proper** provided $F \neq A$. If W is a subset of a BL-algebra $\mathbf{A}$, the implicative filter generated by W will be denoted by $\langle W \rangle$. If U is a filter of the boolean subalgebra $\mathbf{B}(\mathbf{A})$, then the implicative filter $\langle U \rangle$ is called **Stone filter of A**. An implicative filter F of a BL-algebra $\mathbf{A}$ is called **maximal** iff it is proper and no proper implicative filter of $\mathbf{A}$ strictly contains F.

Implicative filters characterize congruences in BL-algebras. Indeed, if F is an implicative filter of a BL-algebra $\mathbf{A}$ it is well known (see [27, Lemma 2.3.14]), that the binary relation $\equiv_F$ on A defined by:

$$x \equiv_F y \quad \text{iff} \quad x \to y \in F \text{ and } y \to x \in F$$

is a congruence of $\mathbf{A}$. Moreover, $F = \{x \in A : x \equiv_F \top\}$. Conversely, if $\equiv$ is a congruence relation on A, then the set $F = \{x \in A : x \equiv \top\}$ is an implicative filter, and $x \equiv y$ iff $x \to y \equiv \top$ and $y \to x \equiv \top$. Therefore, the correspondence

$$F \mapsto \equiv_F$$

is a bijection from the set of implicative filters of $\mathbf{A}$ onto the set of congruences of $\mathbf{A}$.

Given a BL-algebra $\mathbf{A}$ and a filter F of $\mathbf{A}$, we will denote the quotient set $\mathbf{A}/\equiv_F$ by $\mathbf{A}/F$. Since $\equiv_F$ is a congruence, defining on the set $\mathbf{A}/F$ the operations

$$(x/F) * (y/F) = (x * y)/F$$

and

$$(x/F) \to (y/F) = (x \to y)/F,$$

the system $(A/F, *, \to, \bot/F, \top/F)$ becomes a BL-algebra called the **quotient algebra of A by the implicative filter** F. Moreover, the correspondence

$$x \mapsto x/F$$

defines an homomorphism h_F from $\mathbf{A}$ onto the quotient algebra $\mathbf{A}/F$.

Lemma 1.4.2 *(see [20]) Let* $\mathbf{A}$ *be a BL-algebra, and let* U *be a filter of* $\mathbf{B}(\mathbf{A})$. *Then*

$$(\equiv_U) = \{(a, b) \in A \times A : a \wedge c = b \wedge c \text{ for some } c \in U\}$$

is a congruence relation on $\mathbf{A}$ *that coincides with the congruence relation given by the implicative filter* $\langle U \rangle$ *generated by* U.

1.5 Initial segments

An element u of a BL-algebra $\mathbf{A}$ is called **idempotent** provided that $u * u = u$. If $\mathbf{A}$ is a BL-algebra and u, v are idempotents in A such that $u < v$, then the segment $[u, v] = \{x \in A : u \leq x \leq v\}$ is closed by $*$. It is not hard to verify that the boolean elements of a BL-algebra $\mathbf{A}$ are idempotents. Therefore new BL-algebras can be defined from a BL-algebra $\mathbf{A}$ by taking segments between boolean elements. The following two results can be found in [15].

Theorem 1.5.1 *Let* $\mathbf{A} = (A, *, \rightarrow, \bot, \top)$ *be a BL-algebra. For each* $u \neq \bot$, $u \in B(\mathbf{A})$, *the system* $\mathbf{A}_u = ([\bot, u], *, \Rightarrow_u, \bot, u)$ *is a BL-algebra where*

$$x \Rightarrow_u y = (x \rightarrow y) \wedge u.$$

Theorem 1.5.2 *If* $\mathbf{A}$ *is a BL-algebra and* $a \in B(\mathbf{A})$, *then the correspondence* $x \mapsto (x \wedge a, x \wedge \neg a)$ *is an isomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}_a \times \mathbf{A}_{\neg a}$.

A BL-algebra $\mathbf{A}$ is called **directly indecomposable** iff $\mathbf{A}$ is non trivial and when it is decomposed into a direct product of two BL-algebras then one of them must be trivial ([21]). In consequence a BL-algebra $\mathbf{A}$ is directly indecomposable iff it is not trivial and $\mathbf{B}(\mathbf{A}) = \{\bot, \top\}$.

Recall that an **atom** of a boolean algebra $\mathbf{B}$ is an element $x \in B$ such that $x > \bot$ and if $y \in B$ and $y < x$, then $y = \bot$. Our next theorem is the analogous for BL-algebras of Corollary 3.8 in [19].

Theorem 1.5.3 *Let* $\mathbf{A}$ *be a BL-algebra and suppose that* $\mathbf{B}(\mathbf{A})$ *is finite. Let* $At(\mathbf{A}) = \{a_1, a_2, \ldots, a_n\}$ *be the set of atoms of* $\mathbf{B}(\mathbf{A})$. *Then*

$$\mathbf{A} \cong \mathbf{A}_{a_1} \times \mathbf{A}_{a_2} \times \cdots \times \mathbf{A}_{a_n}.$$

Each algebra $\mathbf{A}_{a_i}$ *is directly indecomposable.*

Proof: From the definition of atom we have that:

1. $a_1 \vee a_2 \vee \ldots \vee a_n = \top$,

2. if $i \neq j$, $a_i \wedge a_j = \bot$.

Let $h : \mathbf{A} \rightarrow \mathbf{A}_{a_1} \times \mathbf{A}_{a_2} \times \cdots \times \mathbf{A}_{a_n}$ be given by

$$h(a) = (a \wedge a_1, a \wedge a_2, \ldots, a \wedge a_n).$$

Clearly h is a homomorphism. From (1) we obtain that $\bigcap_{i=1}^{n} [a_i, \top] = \{\top\}$, consequently h is an embedding. Besides, for each element $(x_1, x_2, \ldots, x_n)$ in $\mathbf{A}_{a_1} \times \mathbf{A}_{a_2} \times \cdots \times \mathbf{A}_{a_n}$, we have that $(x_1, x_2, \ldots, x_n) = h(x_1 \vee x_2 \vee \ldots \vee x_n)$, thus h is surjective. Hence we conclude that h is an isomorphism. ∎

1.6 Ordinal sums

From Theorem 1.2.2 we can deduce that BL-chains play a key role in the structure of BL-algebras. One way of characterizing BL-chains consist of studying the number and form of some of their subhoops. To describe how the operations of a BL-chain **A** behave between some of its proper subhoops we use the notion of ordinal sum introduced by Büchi and Owens in their unpublished manuscript [9] and recalled in [26]. It is worth to remark that this notion does not coincide with the notion of ordinal sum given in [17].

Let $\mathbf{R} = (R, *_{\mathbf{R}}, \to_{\mathbf{R}}, \top)$ and $\mathbf{S} = (S, *_{\mathbf{S}}, \to_{\mathbf{S}}, \top)$ be two hoops such that $R \cap S = \{\top\}$. We define the **ordinal sum** $\mathbf{R} \uplus \mathbf{S}$ of these two hoops as the hoop given by $(R \cup S, *, \to, \top)$ where the operations $(*, \to)$ are defined as follows:

$$x * y = \begin{cases} x *_{\mathbf{R}} y & \text{if } x, y \in R, \\ x *_{\mathbf{S}} y & \text{if } x, y \in S, \\ x & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in R \setminus \{\top\},\ y \in S, \\ x \to_{\mathbf{R}} y & \text{if } x, y \in R, \\ x \to_{\mathbf{S}} y & \text{if } x, y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

If $R \cap S \neq \{\top\}$, **R** and **S** can be replaced by isomorphic copies whose intersection is $\{\top\}$, thus their ordinal sum can be defined. Observe that when **R** is a generalized BL-chain and **S** is a generalized BL-algebra, the hoop resulting from their ordinal sum satisfies equation (1.1). Thus $\mathbf{R} \uplus \mathbf{S}$ is a generalized BL-algebra. Moreover, if **R** is a BL-chain, then $\mathbf{R} \uplus \mathbf{S}$ is a BL-algebra, where $\bot = \bot_{\mathbf{R}}$. In this case it is obvious that the chain $\mathbf{R} \uplus \mathbf{S}$ is subdirectly irreducible if and only if **S** is subdirectly irreducible. Notice also that for any generalized BL-algebra **S**, $\mathbf{L_2} \uplus \mathbf{S}$ is the BL-algebra that arises from adjoining a bottom element to **S**.

The definition of ordinal sum can be extended for a family of hoops. Let $(I, \leq)$ be a totally ordered set. For each $i \in I$ let $\mathbf{A_i} = (A_i, *_i, \to_i, \top)$ be a hoop such that for every $i \neq j$, $A_i \cap A_j = \{\top\}$. Then we can define the ordinal sum as the hoop $\uplus_{i \in I} \mathbf{A_i} = (\cup_{i \in I} A_i, *, \to, \top)$ where the operations $*, \to$ are given by:

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in A_i, \\ x & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j \text{ and } i < j, \\ y & \text{if } y \in A_i \setminus \{\top\},\ x \in A_j \text{ and } i < j. \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j \text{ and } i < j, \\ x \to_i y & \text{if } x, y \in A_i, \\ y & \text{if } y \in A_i,\ x \in A_j \text{ and } i < j. \end{cases}$$

Remark 1.6.1 Since generalized BL-algebras with a lower bound are reducts of BL-algebras, with an abuse of notation we shall refer to both algebras by the same symbol, and we will deduce their structure from the context. For example, $\mathbf{L}_n$ will denote the MV-chain $(L_n, *, \to, 0, 1)$ as well as the generalized BL-algebra $(L_n, *, \to, 1)$, and then we shall understand $\mathbf{L}_n \uplus \mathbf{L}_m$ as the ordinal sum of the MV-chain $\mathbf{L}_n$ and the generalized BL-chain $\mathbf{L}_m$. We are also going to refer to the ordinal sum of BL-chains, but, except from the first summand, we are considering generalized BL-chains.

Chapter 2

Decomposition of BL-chains

2.1 Decomposition into irreducible hoops

Hájek conjectured that a propositional formula ϕ is deducible in the logic BL iff ϕ is a tautology for all continuous t-norms. His conjecture was proved in [28] under some supplementary conditions. In order to show that these conditions were redundant, a first decomposition of saturated BL-chains as ordinal sums of MV-chains, Gödel chains and PL-chains was given in [17], generalizing a well known decomposition of continuous t-norms. The notion of ordinal sum used in such decomposition differs from the one presented in the previous chapter, and does not allow to decompose BL-chains that are not saturated. To avoid this restriction, Aglianò and Montagna give in [2] a theorem of decomposition for BL-chains into an ordinal sum of some special kind of hoops, named Wajsberg hoops, which can not be further decomposed. Although this improves the result given in [17], the given proof is non constructive, because the axiom of choice is invoked three times in the course of the proof.

In the present section a rather simple and self contained proof of the Aglianò - Montagna decomposition is going to be offered without appealing to any version of the axiom of choice.

A **trivial hoop** is a hoop whose only element is $\top$. When the order of a hoop $\mathbf{A}$ is total, we say that $\mathbf{A}$ is an **o-hoop**. A **Wajsberg hoop** is a hoop that satisfies the equation:

$$(x \to y) \to y = (y \to x) \to x.$$

A **bounded hoop** is an algebra $\mathbf{A} = (A, \to, *, \bot, \top)$ such that $(A, \to, *, \top)$ is a hoop and $\bot \leq a$ for each $a \in A$. A **Wajsberg algebra** is a bounded

Wajsberg hoop, and Wajsberg algebras are equivalent to MV-algebras (see [16]). Recall also that a BL-algebra is a bounded basic hoop and a BL-chain is a totally ordered BL-algebra, that is, a bounded basic o-hoop.

Definition 2.1.1 *An o-hoop is* **irreducible** *if it can not be written as the ordinal sum of two non trivial o-hoops.*

The following results can be found in [2].

Lemma 2.1.2 *Let* $\mathbf{A}$ *be any basic o-hoop and let* $a \neq \top$ *be an element in* A. *Let* $F_a = \{x \in A \setminus \{\top\} : a \to x = x\}$. *Then* F_a *is downwards closed, and* $F_a \cup \{\top\}$ *is the domain of a subhoop* $\mathbf{F}_a$ *of* A.

Theorem 2.1.3 *For an o-hoop (BL-chain)* $\mathbf{A}$ *the following are equivalent:*

1. $\mathbf{A}$ *is irreducible;*
2. *For all* $a, b \in A$, $b \to a = a$ *implies* $b = \top$ *or* $a = \top$;
3. $\mathbf{A}$ *is a Wajsberg o-hoop (Wajsberg chain).*

Definition 2.1.4 *A* **tower of irreducible o-hoops** *is a family* $\tau = (\mathbf{C_i} : i \in I)$ *index by a totally ordered set* $(I, \leq)$ *with first element* 0 *such that:*

- $\mathbf{C_i} = (C_i, *_i, \to_i, \top)$ *is an irreducible o-hoop,*
- $C_i \cap C_j = \{\top\}$ *for each* $i \neq j$,
- $\mathbf{C_0}$ *is a bounded o-hoop.*

It is easy to see that for each tower $\tau = (\mathbf{C_i} : i \in I)$ of irreducible o-hoops, $\mathbf{A}_\tau = \biguplus_{i \in I} \mathbf{C_i}$ is a BL-chain. We shall demonstrate the following theorem that gives the unique decomposition of each BL-chain into an ordinal sum of irreducible hoops:

Theorem 2.1.5 *Each BL-chain* $\mathbf{A}$ *is isomorphic to an algebra of the form* $\mathbf{A}_\tau$ *for some tower* τ *of irreducible o-hoops.*

Proof: We have already noticed that if an algebra is of the form $\mathbf{A}_\tau$ for some tower of irreducible o-hoops τ, then the algebra is a BL-chain.

To prove that each BL-chain $\mathbf{A}$ has this form, as in Lemma 2.1.2, for each $a \in A$, $a \neq \top$ let $F_a = \{x \in A \setminus \{\top\} : a \to x = x\}$ and let $F_\top = \{\top\}$. We give an equivalence relation $\sim$ on $\mathbf{A}$ by:

$$a \sim b \text{ iff } \forall x \in A, a \to x = x \Leftrightarrow b \to x = x \text{ iff } F_a = F_b.$$

Clearly $\sim$ is an equivalence relation. We will see that for each equivalence class C, the structure $\mathbf{C}' = (C \cup \{\top\}, *, \to, \top)$ is a Wajsberg o-hoop.

1. **C′ is totally ordered**, because the order of **A** is inherited.

2. **C′ is closed by $*$**. Indeed, if $a, b \in C$ (that is, $a \sim b$) we can check that $a * b \sim a$ in the following way:

   Let $x \in F_{a*b}$. Then $a * b \rightarrow x = x$. Since $a * b \leq a$ we have that

   $$x \leq a \rightarrow x \leq a * b \rightarrow x = x,$$

   and $x \in F_a$. Now let $x \in F_a = F_b$. Then

   $$a * b \rightarrow x = b \rightarrow (a \rightarrow x) = b \rightarrow x = x,$$

   and $x \in F_{a*b}$.

   Besides, since $x * \top = x$ for every $x \in C$ and $\top * \top = \top$, we obtain that **C′** is closed by $*$.

3. **C′ is closed by $\rightarrow$** We need the following results to obtain this conclusion:

   (a) *If $a \leq b$, then $F_a \subseteq F_b$.*
   If $x \in F_a$, then $a \rightarrow x = x$ and so $x \leq b \rightarrow x \leq a \rightarrow x = x$ and $x \in F_b$.

   (b) *If $a < b$ and $F_a \neq F_b$, then $a \in F_b$.*
   Let an element $y \in F_b \setminus F_a$, that means $b \rightarrow y = y$ and $y < a \rightarrow y$. Suppose $a \notin F_b$. Then $a \leq (a \rightarrow y) \rightarrow y \notin F_b$, since F_b is downwards closed. But

   $$b \rightarrow ((a \rightarrow y) \rightarrow y) = (a \rightarrow y) \rightarrow (b \rightarrow y) = (a \rightarrow y) \rightarrow y$$

   thus $(a \rightarrow y) \rightarrow y \in F_b$ which is a contradiction that arises from the hypothesis that $a \notin F_b$.

   (c) *If a is not equivalent to b, then $a * b = min(a, b)$.*
   Suppose $a < b$. Since a is not equivalent to b, we have that $F_a \neq F_b$, and by (3b) $a \in F_b$. Thus $b * a = b * (b \rightarrow a) = b \wedge a$, and since **A** is a BL-chain, $a \wedge b = \min(a, b)$. This happens analogously if $b < a$.

   Now let $a, b \in C$. We intend to see that $b \rightarrow a \in C'$.

   - If $b \leq a$, then $b \rightarrow a = \top \in C'$.

- If $a < b$, since $a \to a = \top \neq a$, then $a \notin F_a = F_b$. Therefore $a < b \to a$ and $b * (b \to a) = b \wedge a = a \neq b$. Then we have that $b * (b \to a) \neq \min(b, b \to a)$, and by (3c) $b \to a$ and b are equivalent, and that means that $b \to a \in C'$.

On the other hand, if $a \in C$, then $\top \to a = a$, $a \to \top = \top$. Since $\top \to \top = \top$, we conclude that $\mathbf{C}'$ is closed under $\to$.

Up to here, we deduce from (1), (2) and (3) that for each equivalence class C, $\mathbf{C}'$ is an o-hoop.

4. **$\mathbf{C}'$ is irreducible.** Suppose conversely that $\mathbf{C}' \cong \mathbf{A_1} \uplus \mathbf{A_2}$, for some non trivial hoops $\mathbf{A_1}$ and $\mathbf{A_2}$. Let x be in $\mathbf{A_1} \setminus \{\top\}$ and y be in $\mathbf{A_2} \setminus \{\top\}$. Then $x, y \in C$, and this implies that $F_x = F_y$. From the definition of ordinal sum we have that $y \to x = x < \top$, and this means that $x \in F_y$, which implies that $x \in F_x$. But this can not happen because $x \to x = \top \neq x$. Then $\mathbf{C}'$ is irreducible.

5. **C is a convex set.** Assume that $a, b \in C$ and $a < u < b$. From (3a) we have that $F_a \subseteq F_u \subseteq F_b$. But since $a \sim b$, we have that $F_a = F_u = F_b$.

Let I be the set of equivalence classes C. Since each equivalence class is a convex set, the order of $\mathbf{A}$ induces an order on I, defined for $C, D \in I$, by $C \preceq D$ iff either $C = D$ or for all $x \in C$ and for all $y \in D$ one has that $x \leq y$. Thus I is a totally ordered set. We shall denote by C_0 the equivalence class that contains the bottom element of $\mathbf{A}$, and by C_1 the class that contains the element $\top$. Therefore we have that:

- For each $C \in I$, $\mathbf{C}'$ is an irreducible o-hoop.
- For each $C \neq D \in I$, $C' \cap D' = \{\top\}$, for equivalence classes are pairwise disjoint.
- $\mathbf{C}'_0$ has a least element, because $\mathbf{A}$ is a BL-chain.

Therefore we have that $\tau = (\mathbf{C}', C \in I)$ is a tower of irreducible o-hoops. Notice that if $a \in C$ and $b \in D$ with $C \preceq D$ and $C \neq D$, by (3c) we have that $b \to a = a$ and by (3b) $a * b = b$. Then it is easy to see that $\mathbf{A}_\tau = \oplus_{C \in I} \mathbf{C}' \cong \mathbf{A}$. ∎

Remark 2.1.6 It is clear that $C'_1 = \{\top\}$. Notice that, with the exception of $\mathbf{C}'_1$, the previous Theorem offers a constructive method for decomposing BL-chains into non trivial irreducible o-hoops.

Theorem 2.1.7 *Each non trivial BL-chain admits a unique decomposition into non trivial irreducible hoops.*

Proof: Suppose that $\mathbf{A} = \biguplus_{i\in I}\mathbf{C_i} = \biguplus_{j\in J}\mathbf{D_j}$, where $\mathbf{C_i}$ and $\mathbf{D_j}$ are non trivial irreducible o-hoops for each $i \in I$ and $j \in J$, I and J are totally ordered sets. For each $i \in I$ and $j \in J$, the possibilities are:

1. $C_i \cap D_j = \{\top\}$ or,
2. there exists $a \in C_i \cap D_j$ such that $a < \top$.

We only need to see that if the second case happens, then $\mathbf{C_i} = \mathbf{D_j}$. Suppose that $\mathbf{C_i} \neq \mathbf{D_j}$, and let $a < \top$ be in $C_i \cap D_j$. Without loss of generality we can think that there exists $b \in C_i \setminus D_j$ and $a < b$. Since $b \notin D_j$, necessarily $b < \top$. Since $b \in A$ there exists $k \in J$ such that $b \in D_k$ and clearly $j < k$. Therefore, from the definition of ordinal sum we obtain that $b \to a = a$. But, since $a, b \in C_i$ and $\mathbf{C_i}$ is irreducible, Theorem 2.1.3 asserts that $b = \top$ or $a = \top$. The contradiction arises from the hypothesis that $\mathbf{C_i} \neq \mathbf{D_j}$. ■

Since irreducible o-hoops coincide with Wajsberg o-hoops, we shall investigate the structure of Wajsberg o-hoops. We have already noticed that Wajsberg bounded o-hoops (irreducible bounded o-hoops) coincide with MV-chains. A characterization of bounded and unbounded o-hoops is given in [7, Section 1]. A hoop is cancellative if its basic monoid is cancellative. Cancellative hoops form a variety characterized by the equation

$$y = x \to (y * x).$$

Cancellative o-hoops coincide with Wajsberg unbounded o-hoops (see [7]).

If $\mathbf{G} = (G, +, 0)$ is an abelian o-group (totally ordered group), and we define $G^- = \{x \in G : x \leq 0\}$, then $\mathbf{P(G)} = (G^-, *, \to, 0)$ is a Wajsberg o-hoop where the operations $*$ and $\to$ are given by:

$$x * y = x + y, \quad \text{and} \quad x \to y = 0 \wedge (y - x).$$

In fact, if $\mathbf{G}$ is an abelian ℓ-group (lattice ordered group), then $\mathbf{P(G)} = (G^-, *, \to, 0)$ with $*, \to$ as defined above is a generalized BL-algebra. The following result can be deduced from [3] (see also [6] and [19]).

Theorem 2.1.8 *The following conditions are equivalent for a generalized BL-algebra* $\mathbf{A}$:

1. $\mathbf{A}$ *is a cancellative hoop,*

2. *there is an ℓ-group* $\mathbf{G}$ *such that* $\mathbf{A} \cong \mathbf{P}(\mathbf{G})$,

3. $\mathbf{A}$ *is in the variety of generalized BL-algebras generated by* $\mathbf{P}(\mathbf{Z})$, *where* $\mathbf{Z}$ *denotes the additive group of integers with the usual order.*

Therefore we conclude:

Theorem 2.1.9 *If* $\mathbf{A}$ *is an irreducible o-hoop then either* $\mathbf{A}$ *is an MV-chain (in case* $\mathbf{A}$ *is bounded), or* $\mathbf{A}$ *is isomorphic to* $\mathbf{P}(\mathbf{G})$ *for some totally ordered abelian group* $\mathbf{G}$ *(if* $\mathbf{A}$ *is unbounded).*

Corollary 2.1.10 *Each BL-chain is an ordinal sum of a family of MV-chains and hoops of the form* $\mathbf{P}(\mathbf{G})$ *for a totally ordered abelian group* $\mathbf{G}$.

Let consider a BL-algebra $\mathbf{A}$. We proved that there exists a unique tower of non trivial irreducible o-hoops $\tau = (\mathbf{C}_i, i \in I)$ such that $\mathbf{A} = \uplus_{i \in I} \mathbf{C}_i$. If 0 denotes the first element in I then $\mathbf{C}_0$ is a bounded Wajsberg o-hoop, i.e, $\mathbf{C}_0$ is an MV-chain. Besides, $\mathbf{B} = \uplus_{i \in I \setminus \{0\}} \mathbf{C}_i$ is an implicative filter of $\mathbf{A}$. Recall that a BL-algebra $\mathbf{A}$ is said to be **simple** provided it is non trivial and the only proper implicative filter of $\mathbf{A}$ is the singleton $\{\top\}$. Therefore a BL-chain is simple iff it is a Wajsberg chain.

Since finite BL-chains are bounded, from Theorem 2.1.9 and Corollary 2.1.10 we have that finite BL-chains can be uniquely decomposed into an ordinal sum of finite MV-chains. From [16, Corollary 3.5.4] we have that an MV-chain is simple and finite iff it is isomorphic to an MV-chain of the form $\mathbf{L}_n$ for some integer n. This implies the following theorem:

Theorem 2.1.11 *Each finite BL-chain* $\mathbf{C}$ *is isomorphic to a chain of the form*

$$\uplus_{i=0}^{k} \mathbf{L}_{r_i}$$

for an integer k *and where each* r_i *is an integer for* $i = 0, 1, \ldots, k$.

Remark 2.1.12 It is worth to notice that for each finite BL-chain $\mathbf{C}$, the number of idempotent elements different from $\top$ coincides with the number of irreducible MV-chains that compose $\mathbf{C}$, since they are the bottom element of each of the non trivial irreducible parts.

2.2 Decomposition into regular and dense elements

There is another way of decomposing BL-chains that will be very useful to our purpose. In order to use the results given in [21] about free algebras in the following chapters, we shall decompose each BL-chain into two parts. Given a BL-algebra $\mathbf{A}$ we can consider the set

$$D(\mathbf{A}) = \{x \in A : \neg x = \bot\}.$$

As indicated by [21], $\mathbf{D}(\mathbf{A}) = (D(\mathbf{A}), *, \rightarrow, \top)$ is a generalized BL-algebra. The elements in $D(\mathbf{A})$ will be called **dense elements of A**. Recall that if $MV(\mathbf{A}) = \{x \in A : \neg\neg x = x\}$, then $\mathbf{MV}(\mathbf{A}) = (MV(\mathbf{A}), *, \rightarrow, \bot, \top)$ is subalgebra of $\mathbf{A}$ which is an MV-algebra. The elements of $MV(\mathbf{A})$ will be called **regular elements of A**.

Theorem 2.2.1 *For each BL-chain* $\mathbf{A}$ *we have that*

$$\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}).$$

Proof: From Theorem 2.1.5 we now that there exists a tower $\tau = (\mathbf{C_i} : i \in I)$ of irreducible o-hoops, such that $\mathbf{A} = \uplus_{i\in I}\mathbf{C_i}$. Let 0 be the least element of I and let $\mathbf{B} = \uplus_{i\in I\setminus\{0\}}\mathbf{C_i}$. Clearly $\mathbf{B}$ is a generalized BL-chain and $\mathbf{A} = \mathbf{C_0}\uplus\mathbf{B}$. Therefore it is enough to prove that $\mathbf{MV}(\mathbf{A}) = \mathbf{C_0}$ and that $\mathbf{D}(\mathbf{A}) = \mathbf{B}$.

- $\mathbf{D}(\mathbf{A}) = \mathbf{B}$.

  It is clear that if $x \in B$ from the definition of ordinal sum $\neg x = \bot$, then $x \in D(\mathbf{A})$. Therefore we have that $B \subseteq D(\mathbf{A})$. Let suppose that $x \in D(\mathbf{A}) \setminus B$. Obviously $x \neq \top$. Since $x \in D(\mathbf{A})$, we have that $\neg\neg x = \top$. On the other hand, since $x \in C_0 \setminus \{\top\}$, we obtain that $\neg\neg x = x$ because $\mathbf{C_0}$ is an MV-chain. Hence we arrive to the contradiction $x = \top$, and we conclude that $\mathbf{B} = \mathbf{D}(\mathbf{A})$.

- $\mathbf{MV}(\mathbf{A}) = \mathbf{C_0}$

  $C_0 \subseteq MV(\mathbf{A})$, since $\mathbf{C_0}$ is an MV-chain. Suppose now that there exists $x \in MV(\mathbf{A}) \setminus C_0$. Again we have that $x \neq \top$. Hence $x \in B$, and $\neg\neg x = \top$. But since $x \in MV(\mathbf{A})$, $\neg\neg x = x$. The contradiction $x = \top$ arrives from the hypothesis $MV(\mathbf{A}) \setminus C_0 \neq \emptyset$, hence we may conclude that $\mathbf{MV}(\mathbf{A}) = \mathbf{C_0}$.

■

Indeed, the condition of $\mathbf{A}$ of being a BL-chain can be released in the following way:

Theorem 2.2.2 *Let* $\mathbf{A}$ *be a BL-algebra such that* $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_n$ *for some integer* n. *Then*

$$\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}) \cong \mathbf{L}_n \uplus \mathbf{D}(\mathbf{A}).$$

Proof: In order to prove this result the following notation is introduced: given x and y in a BL-algebra we define $x \oplus y = \neg(\neg x * \neg y)$. For each positive integer k, the operations x^k and $k \cdot x$ are inductively defined as follows:

- $x^1 = x$ and $x^{k+1} = x^k * x$,
- $1 \cdot x = x$ and $(k+1) \cdot x = (k \cdot x) \oplus x$.

Notice that if $x \in L_n \setminus \{\top\}$, then $x^n = \bot$, and if $x \in L_n \setminus \{\bot\}$, then $n.x = \top$. From Theorem 1.2.2, we can think of each non trivial BL-algebra $\mathbf{A}$ as a subdirect product of a family $(\mathbf{A}_i, i \in I)$ of non trivial BL-chains, that is, there exists an embedding

$$e : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i,$$

such that $\pi_i(e(\mathbf{A})) = \mathbf{A}_i$ for each $i \in I$, where π_i denotes each projection. We shall identify $\mathbf{A}$ with $e(\mathbf{A})$. Then each element of A is a tuple $\mathbf{x}$ and coordinate i is $x_i \in A_i$. With this notation we have that for each $\mathbf{x} \in A$, $\pi_i(\mathbf{x}) = x_i$. We will proof the following items:

1. *For each* $i \in I$, $\mathbf{MV}(\mathbf{A}_i)$ *is isomorphic to* $\mathbf{L}_n$.

   Since for each $i \in I$, π_i is a homomorphism and $\pi_i(MV(\mathbf{A})) \subseteq A_i$, we have that $\pi_i(MV(\mathbf{A})) \subseteq MV(\mathbf{A}_i)$. Then $\pi_i(\mathbf{MV}(\mathbf{A}))$ is a subalgebra of $\mathbf{MV}(\mathbf{A}_i)$. On the other hand, given $i \in I$, let $x_i \in MV(\mathbf{A}_i)$. Then $\neg\neg x_i = x_i$ and there exists an element $\mathbf{x} \in A$ such that $\pi_i(\mathbf{x}) = x_i$. Taking $\mathbf{y} = \neg\neg\mathbf{x} \in MV(\mathbf{A})$ we have that $\pi_i(\mathbf{y}) = x_i$ and $x_i \in \pi_i(MV(\mathbf{A}))$. Hence $MV(\mathbf{A}_i) \subseteq \pi_i(MV(\mathbf{A}))$.

   In conclusion $\mathbf{MV}(\mathbf{A}_i) = \pi_i(\mathbf{MV}(\mathbf{A})) = \pi_i(\mathbf{L}_n)$ and since $\mathbf{L}_n$ is a simple algebra and $\mathbf{MV}(\mathbf{A}_i)$ is non trivial we have that $\mathbf{MV}(\mathbf{A}_i) \cong \mathbf{L}_n$.

2. *If* $\mathbf{x} \in A$, *then* $\mathbf{x} \in MV(\mathbf{A}) \cup D(\mathbf{A})$.

   Let $\mathbf{x} \in A$ and let $\mathbf{y} = n.(\neg\mathbf{x})$. If $x_i \in L_n \setminus \{\top\}$, then $\neg x_i \in L_n \setminus \{\bot\}$ and $y_i = n.(\neg x_i) = \top$. On the other hand if $\neg x_i = \bot$, then $y_i = n.(\neg x_i) = \bot$. Now let $\mathbf{z} = (\neg\neg\mathbf{x})^n$. If $x_i \in L_n \setminus \{\top\}$, then $z_i = \bot$, but if $\neg\neg x_i = \top$, then $z_i = \top$.

   Suppose there exists $\mathbf{x} \in A$ such that $\mathbf{x} \notin MV(\mathbf{A})$ and $\mathbf{x} \notin D(\mathbf{A})$. It follows from Theorem 2.2.1 that for each $i \in I$, $\mathbf{A}_i = \mathbf{MV}(\mathbf{A}_i) \uplus \mathbf{D}(\mathbf{A}_i)$, then there exist $i, j \in I$, such that $x_i \in MV(\mathbf{A}_i) \setminus \{\top\} = L_n \setminus \{\top\}$ and $x_j \in D(\mathbf{A}_j) \setminus \{\top\}$.

Let $\mathbf{y} = n.(\neg\mathbf{x})$. Then $y_i = \top$, $y_j = \bot$ and $y_k \in \{\bot, \top\}$ for each $k \in I \setminus \{i, j\}$. Now let $\mathbf{z} = (\neg\neg\mathbf{x})^n$. We have that $z_j = \top$, $z_i = \bot$ and $z_k \in \{\bot, \top\}$ for each $k \in I \setminus \{i, j\}$. It follows that $\mathbf{y}$ and $\mathbf{z}$ are elements in the chain $MV(\mathbf{A}) = L_n$ which are not comparable, a contradiction.

3. *If* $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ *and* $\mathbf{y} \in D(\mathbf{A})$, *then* $\mathbf{x} < \mathbf{y}$.

   The statement is clear if $x_i \in MV(\mathbf{A}_i) \setminus \{\top\}$ for every $i \in I$ or if $y_i = \top$ for each $i \in I$. Otherwise, let $S = \{i \in I : x_i = \top\} \neq \emptyset$. Since $\mathbf{x} \neq \top$ we have that S is a proper subset of I. If $y_i = \top$ for each $i \in S$, then $\mathbf{x} < \mathbf{y}$. If not, let $j \in S$ be such that $y_j \neq \top$. Let $\mathbf{z} = \mathbf{x} \wedge \mathbf{y}$. Since operations are coordinatewise, $z_i \in MV(\mathbf{A}) \setminus \{\top\}$ for each $i \in I \setminus S$ and $z_j \in D(\mathbf{A}) \setminus \{\top\}$. Hence $\mathbf{z} \notin MV(\mathbf{A})$ and $\mathbf{z} \notin D(\mathbf{A})$ contradicting the previous item.

4. *If* $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ *and* $\mathbf{y} \in D(\mathbf{A})$, *then* $\mathbf{y} \to \mathbf{x} = \mathbf{x}$ *and* $\mathbf{y} * \mathbf{x} = \mathbf{x}$.

   Since $\neg\mathbf{y} = \bot$ we have that

$$\mathbf{y} \to \mathbf{x} = \mathbf{y} \to \neg\neg\mathbf{x} = \mathbf{y} \to (\neg\mathbf{x} \to \bot) = \neg\mathbf{x} \to (\mathbf{y} \to \bot) =$$

$$= \neg\mathbf{x} \to \bot = \neg\neg\mathbf{x} = \mathbf{x},$$

   and

$$\mathbf{x} = \mathbf{y} \wedge \mathbf{x} = \mathbf{y} * (\mathbf{y} \to \mathbf{x}) = \mathbf{y} * \mathbf{x}.$$

From the previous items it follows that

$$\mathbf{A} \cong \mathbf{MV(A)} \uplus \mathbf{D(A)} = \mathbf{L}_n \uplus \mathbf{D(A)}.$$

■

Remark 2.2.3 Notice that if $x \in D(\mathbf{A})$ then $\neg\neg x = \top$, thus $\neg\neg x \to x = x$, and if $x \in MV(\mathbf{A})$ then $\neg\neg x = x$ and $\neg\neg x \to x = \top$.

Chapter 3

Characterization of varieties of BL-algebras generated by BL$_n$-chains.

3.1 Equational characterization of the subvarieties of BL-algebras generated by a BL$_n$-chain.

The purpose of the present section is to find an equational characterization of certain subvarieties of $\mathcal{BL}$ generated by a single chain. Following the decomposition given in Theorem 2.2.1, the idea is to see how the equations that characterize the subvariety of $\mathcal{BL}$ generated by a chain $\mathbf{A} = \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A})$, depend on the ones that characterize the subvariety of $\mathcal{MV}$ generated by $\mathbf{MV}(\mathbf{A})$ and the ones that characterize the subvariety of $\mathcal{GBL}$ generated by $\mathbf{D}(\mathbf{A})$. In order to do so we follow the ideas given by Aglianò and Montagna in [2]. It is worth to note that certain subvarieties of BL-algebras are characterized in [29]. The main difference with the present work is that I do not introduce new constants to the original algebraic system.

We shall denote by $\mathcal{MV}_n$ the subvariety of $\mathcal{MV}$ generated by $\mathbf{L}_n$. The elements of $\mathcal{MV}_n$ are called **MV$_n$-algebras**. The following result can be found in [16].

Theorem 3.1.1 *A finite MV-chain* $\mathbf{L}_m$ *belongs to the variety generated by* $\mathbf{L}_n$ *iff* $m-1$ *is a divisor of* $n-1$.

From Theorem 1.2.2 and the previous theorem we conclude that every

MV_n-algebra is a subdirect product of a family of algebras $(\mathbf{L}_{m_i}, i \in I)$ where $m_i - 1$ divides $n-1$ for each $i \in I$.

We define a **$\mathbf{BL}_n$-chain** as a BL-chain which is an ordinal sum of the MV_n-chain $\mathbf{L}_n$ and a generalized BL-chain $\mathbf{B}$. To continue with our work we set a fixed BL_n-chain

$$\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}.$$

We shall denote by $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{T}_n$ and by $\mathcal{W}$ the subvariety of $\mathcal{GBL}$ generated by $\mathbf{B}$.

Let $\mathcal{MV}_n^t$ and $\mathcal{W}^t$ denote the classes of totally ordered members of $\mathcal{MV}_n$ and $\mathcal{W}$ respectively. Following [2], we denote

$$\mathcal{MV}_n \uplus^t \mathcal{W}$$

the variety generated by $\{\mathbf{A}_1 \uplus \mathbf{A}_2 : \mathbf{A}_1 \in \mathcal{MV}_n^t, \mathbf{A}_2 \in \mathcal{W}^t\}$. From Theorem 3.1.1 we know that $\mathcal{MV}_n^t = \{\mathbf{L}_m : m-1 \text{ divides } n-1\}$. We shall characterize equationally the variety

$$\mathcal{MV}_n \uplus^t \mathcal{W}.$$

Let $\{e_i, i \in I\}$ be the set of equations that define $\mathcal{MV}_n$ as a subvariety of $\mathcal{BL}$, and $\{d_j, j \in J\}$ be the set of equations that define $\mathcal{W}$ as a subvariety of $\mathcal{GBL}$, i.e., an MV-algebra $\mathbf{A}_1$ belongs to $\mathcal{MV}_n$ iff the elements of A_1 satisfy e_i for each $i \in I$, and a generalized BL-algebra $\mathbf{A}_2$ belongs to $\mathcal{W}$ iff the elements of A_2 satisfy equations d_j for each $j \in J$. For each $i \in I$, let e'_i be the equation that results from substituting $\neg\neg x$ for each variable x in e_i, and for each $j \in J$, let d'_j the equation that results from substituting $\neg\neg x \to x$ for each variable x in the equation d_j. Let $\mathcal{V}'$ the variety of BL-algebras characterized by the equations of BL-algebras plus the equations $\{e'_i, i \in I\} \cup \{d'_j, j \in J\}$. From Remark 2.2.3, a BL-algebra $\mathbf{A}$ is in $\mathcal{V}'$ iff its regular elements satisfy equations e_i for each $i \in I$ and its dense elements satisfy equations d_j for each $j \in J$.

Lemma 3.1.2 $\mathcal{V}' = \mathcal{MV}_n \uplus^t \mathcal{W}$.

Proof: Let $\mathbf{A} = \mathbf{A}_1 \uplus \mathbf{A}_2$, with $\mathbf{A}_1 \in \mathcal{MV}_n^t$ and $\mathbf{A}_2 \in \mathcal{W}^t$. For each $x \in A$, we have that $\neg\neg x \in A_1$ and $\neg\neg x \to x \in A_2$. Therefore $\mathbf{A}$ satisfies equations e'_i for each $i \in I$ and d'_j for each $j \in J$, and $\mathcal{MV}_n \uplus^t \mathcal{W} \subseteq \mathcal{V}'$. Now let $\mathbf{A}$ be a BL-chain in $\mathcal{V}'$, that is, a BL-chain that satisfies equations $e'_i, i \in I$ and equations $d'_j, j \in J$. From Theorem 2.2.1 we know that

$$\mathbf{A} = \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}).$$

Since for each $x \in MV(\mathbf{A})$ we have that $\neg\neg x = x$, and $\mathbf{MV}(\mathbf{A})$ is in $\mathcal{V}'$, we obtain that for each $i \in I$, $\mathbf{MV}(\mathbf{A})$ satisfies the equation e_i. Then

$\mathbf{MV}(\mathbf{A})$ is a chain in $\mathcal{MV}_n$. On the other hand, since for each $x \in D(\mathbf{A})$ we have that $\neg\neg x \to x = x$, $\mathbf{D}(\mathbf{A})$ satisfies equation d_j for each $j \in J$. Then $\mathbf{D}(\mathbf{A})$ is a generalized BL-chain in $\mathcal{W}$. Then $\mathbf{A} \in \mathcal{MV}_n \uplus^t \mathcal{W}$ and by Theorem 1.2.2 we conclude that $\mathcal{V}' = \mathcal{MV}_n \uplus^t \mathcal{W}$. ∎

Following the arguments in the proofs of [2, Lemma 7.1 and Theorem 7.4], we shall see that

$$\mathcal{V} = \mathcal{MV}_n \uplus^t \mathcal{W}.$$

To accomplish such aim, for each class of algebras $\mathcal{K}$, let $\mathbf{H}(\mathcal{K}), \mathbf{I}(\mathcal{K}), \mathbf{S}(\mathcal{K})$, $\mathbf{P}(\mathcal{K})$ and $\mathbf{P}_u(\mathcal{K})$ denote the classes of homomorphic images, of isomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from $\mathcal{K}$ respectively. If $\mathbf{O_1}$ and $\mathbf{O_2}$ denote two operators we write $\mathbf{O_1O_2}$ for their composition and $\mathbf{O_1}(\mathcal{K}_1) \uplus \mathbf{O_2}(\mathcal{K}_2)$ will denote the algebras in the class $\{\mathbf{B}_1 \uplus \mathbf{B}_2 : \mathbf{B}_i \in \mathbf{O}_i(\mathcal{K}_i)\}$. From [2, Proposition 3.1, Proposition 3.2 and Proposition 3.4] we obtain the following three results:

Lemma 3.1.3 *Given two hoops $\mathbf{A}_1$ and $\mathbf{A}_2$, the subalgebras of $\mathbf{A}_1 \uplus \mathbf{A}_2$ are of the form $\mathbf{C}_1 \uplus \mathbf{C}_2$, where $\mathbf{C}_1$ is a subalgebra (possibly trivial) of $\mathbf{A}_1$ and $\mathbf{C}_2$ is a subalgebra (possibly trivial) of $\mathbf{A}_2$.*

Lemma 3.1.4 *Let $\mathbf{A_1}$ and $\mathbf{A_2}$ be hoops. Then the set of homomorphic images of $\mathbf{A_1} \uplus \mathbf{A_2}$ is $\mathbf{H}(\mathbf{A}_1) \cup \{\mathbf{A_1} \uplus \mathbf{C} : \mathbf{C} \in \mathbf{H}(\mathbf{A_2})\}$.*

Lemma 3.1.5 *The ultraproducts $\mathbf{P}_u(\mathbf{L}_n \uplus \mathbf{B})$ consist of algebras of the form $\mathbf{A}_1 \uplus \mathbf{A}_2$, where $\mathbf{A}_1 \in \mathbf{IP}_u(\mathbf{L}_n)$ and $\mathbf{A}_2 \in \mathbf{IP}_u(\mathbf{B})$.*

Lemma 3.1.6 $\mathbf{ISP}_u(\mathbf{L}_n \uplus \mathbf{B}) = \mathbf{I}(\mathbf{SP}_u(\mathbf{L}_n) \uplus \mathbf{SP}_u(\mathbf{B}))$.

Proof: From the previous lemmas we have that

$$\mathbf{ISP}_u(\mathbf{L}_n \uplus \mathbf{B}) \subseteq \mathbf{I}(\mathbf{SP}_u(\mathbf{L}_n) \uplus \mathbf{SP}_u(\mathbf{B})).$$

Let $\mathbf{A} \in \mathbf{SP}_u(\mathbf{L}_n)$ and let $\mathbf{C} \in \mathbf{SP}_u(\mathbf{B})$. Then there exists an embedding of $\mathbf{A}$ into a power $\mathbf{L}_n^I/U$, and then $\mathbf{A} \uplus \mathbf{C}$ embeds into $(\mathbf{L}_n \uplus \mathbf{C})^I/U$. Now let $\mathbf{B}^J/V$ be the ultrapower of $\mathbf{B}$ in which $\mathbf{C}$ embeds. Then $\mathbf{L}_n \uplus \mathbf{C}$ embeds into $(\mathbf{L}_n \uplus \mathbf{B})^J/V$. Therefore we obtain that

$$\mathbf{A} \uplus \mathbf{C} \in \mathbf{ISP}_u(\mathbf{SP}_u(\mathbf{L}_n \uplus \mathbf{B})) \subseteq \mathbf{ISP}_u(\mathbf{L}_n \uplus \mathbf{B}).$$

∎

We recall that Jónsson's Lemma (see [10]) asserts that, since $\mathcal{V}$ is a congruence distributive variety, if $\mathbf{C}$ is a subdirectly irreducible algebra in $\mathcal{V}$, then $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{L}_n \uplus \mathbf{B})$ and $\mathbf{HSP}_u(\mathbf{L}_n \uplus \mathbf{B}) \subseteq \mathcal{V}$.

Theorem 3.1.7 $\mathcal{V} = \mathcal{MV}_n \uplus^t \mathcal{W}$.

Proof: Clearly $\mathcal{V} \subseteq \mathcal{MV}_n \uplus^t \mathcal{W}$. Let **A** be a subdirectly irreducible BL-algebra in $\mathcal{MV}_n \uplus^t \mathcal{W}$. From Theorem 1.2.2, **A** is a BL-chain and from the proof of Lemma 3.1.2, $\mathbf{A} = \mathbf{L}_s \uplus \mathbf{C}$ with $s-1$ dividing $n-1$ and some chain $\mathbf{C} \in \mathcal{W}$. Clearly **C** is subdirectly irreducible. Since $\mathcal{GBL}$ is a congruence distributive variety by Jónsson's Lemma, $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{B})$. Then, from Lemma 3.1.4 and Lemma 3.1.6,

$$\mathbf{A} \in \mathbf{ISP}_u(\mathbf{L}_n) \uplus \mathbf{HSP}_u(\mathbf{B}) \subseteq \mathbf{HSP}_u(\mathbf{L}_n \uplus \mathbf{B}) \subseteq \mathcal{V}.$$

■

Corollary 3.1.8 $\mathcal{V} = \mathcal{V}'$.

In conclusion we have that the variety $\mathcal{V}$ of BL-algebras generated by the BL_n-chain $\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}$ is equationally characterized by the equations of BL-algebras plus the equations that result from substituting $\neg\neg x$ for each variable x in the equations that characterize $\mathcal{MV}_n$ and the equations that result from substituting $\neg\neg x \to x$ for each variable x in the equations that characterize $\mathcal{W}$ as subvariety of $\mathcal{GBL}$.

3.2 Equational characterization of subalgebras of regular elements

Following the notation established in the previous chapter, for x and y in a BL-algebra we define $x \oplus y = \neg(\neg x * \neg y)$, and for each positive integer k, the operations x^k and $k \cdot x$ are inductively defined as follows:

- $x^1 = x$ and $x^{k+1} = x^k * x$,
- $1 \cdot x = x$ and $(k+1) \cdot x = (k \cdot x) \oplus x$.

The following three results can be found in [16, Chapters 3 and 8].

Theorem 3.2.1 *Let* **A** *be an MV-algebra and* $n \geq 2$ *an integer. Then* **A** *satisfies the equation*

$$x^{n-1} = x^n, \tag{3.1}$$

if and only if **A** *is a subdirect product of algebras* $\mathbf{L}_k$*, with* $2 \leq k \leq n$.

Theorem 3.2.2 *An MV-algebra* **A** *belongs to* $\mathcal{MV}_2$ *iff* **A** *satisfies the equation:*

$$x^2 = x \tag{3.2}$$

An MV-algebra belongs to $\mathcal{MV}_3$ *iff it satisfies the equation:*

$$x^3 = x^2 \tag{3.3}$$

Theorem 3.2.3 *For every integer* $n \geq 4$ *and every MV-algebra* **A**, *the following conditions are equivalent:*

1. **A** *satisfies the equations:*
$$x^{n-1} = x^n, \tag{3.4}$$
and
$$(p \cdot x^{p-1})^n = n \cdot x^p, \tag{3.5}$$
for every integer $p = 2, \ldots, n-2$ *that does not divide* $n-1$*;*

2. $\mathbf{A} \in \mathcal{MV}_n$

The following lemmas follow from the previous theorems and Corollary 3.1.8

Lemma 3.2.4 *Let* $\mathcal{V}$ *be a variety generated by a* BL_n*-chain. If* $\mathbf{C} \in \mathcal{V}$, *every* $x \in C$ *satisfies the following equations:*

$$(\neg\neg x)^{(n-1)} = (\neg\neg x)^n, \tag{3.6}$$

and if $n \geq 4$, *for every integer* $p = 2, \ldots, n-2$ *that does not divide* $n-1$*:*

$$(p \cdot (\neg\neg x)^{p-1})^n = n \cdot (\neg\neg x)^p. \tag{3.7}$$

Lemma 3.2.5 *Let* $n \geq 2$ *be an integer and let* **C** *be a BL-chain. Then* **MV(C)** *belongs to the variety generated by* $\mathbf{L}_n$ *iff* **C** *satisfies the following equations:*

$$(\neg\neg x)^{(n-1)} = (\neg\neg x)^n, \tag{3.8}$$

and if $n \geq 4$, *for every integer* $p = 2, \ldots, n-2$ *that does not divide* $n-1$*:*

$$(p \cdot (\neg\neg x)^{p-1})^n = n \cdot (\neg\neg x)^p. \tag{3.9}$$

Proof: Suppose the chain **C** satisfies equations (3.8) and (3.9). From the fact that if $x \in MV(\mathbf{C})$, then $\neg\neg x = x$, we deduce that the elements of the MV-algebra $MV(\mathbf{C})$ satisfy equations in Theorem 3.2.3 in case $n \geq 4$ or, otherwise, the corresponding equation in Theorem 3.2.2.

Now let **C** be a BL-chain such that $MV(\mathbf{C})$ is an MV_n-algebra. From Remark 2.2.3 we know that $\neg\neg x \in MV(\mathbf{C})$, thus equations (3.8) and (3.9) are satisfied. ■

3.3 Counting dense elements

In the following lemma we characterize with an equation BL-chains that have at most m dense elements.

Lemma 3.3.1 *A BL-chain* $\mathbf{C}$ *satisfies the equation*

$$(\neg\neg x_m \to x_m) \vee ((\neg\neg x_m \to x_m) \to (\neg\neg x_{m-1} \to x_{m-1})) \vee \cdots \tag{3.10}$$

$$\ldots \vee ((\neg\neg x_2 \to x_2) \to (\neg\neg x_1 \to x_1)) = \top,$$

iff $D(\mathbf{C})$ *has at most* m *elements.*

Proof: Let $\mathbf{C}$ be a BL-chain such that $D(\mathbf{C})$ has $r \leq m$ elements. Let $x_1, \ldots, x_m$ be in $\mathbf{C}$. If $x_i \in MV(\mathbf{C})$, for some $i = 1, 2, \ldots, m$, (3.10) is satisfied because $\neg\neg x_i \to x_i = \top$. Otherwise, necessarily $x_i \in D(\mathbf{C}) \setminus \{\top\}$ for every $i = 1, \ldots, m$. This implies $\neg\neg x_i \to x_i = x_i$. There are $r-1$ different elements in $D(\mathbf{C}) \setminus \{\top\}$, and since $r - 1 < m$, there exist i, j such that $j \leq i$, $x_i \leq x_j$ and $(\neg\neg x_i \to x_i) \to (\neg\neg x_j \to x_j) = x_i \to x_j = \top$, thus (3.10) is satisfied.

Suppose conversely that $\mathbf{C}$ is a chain such that $D(\mathbf{C})$ has $r > m$ elements. Let them be $a_1 < \ldots < a_m < a_{m+1} \leq \ldots \leq a_{r-1} \leq a_r = \top$. For $i \leq m$, we know that $\neg\neg a_i \to a_i = a_i < \top$, then if $j < i \leq m$ we have that

$$(\neg\neg a_i \to a_i) \to (\neg\neg a_j \to a_j) = a_i \to a_j < \top.$$

Taking $x_i = a_i$ in equation (3.10) we obtain that

$$(a_m \vee (a_m \to a_{m-1}) \vee \ldots \vee (a_2 \to a_1)) < \top,$$

because $D(\mathbf{C})$ is totally ordered. Hence equation (3.10) is not satisfied by all the elements of $\mathbf{C}$. ∎

The next lemma is given in [2, Lemma 4.2] to characterize BL-chains that have at most $(k + 1)$ non trivial irreducible parts.

Lemma 3.3.2 *Let* $\mathbf{A} = \biguplus_{i=0}^{n} \mathbf{A}_i$ *be a BL-chain, where every* $\mathbf{A}_i$ *is a totally ordered non trivial Wajsberg hoop, and consider for any* k *the equation:*

$$\bigwedge_{i=0}^{k}((x_{i+1} \to x_i) \to x_i) \to \bigvee_{i=0}^{k+1} x_i = \top. \tag{3.11}$$

Then the equation is satisfied in $\mathbf{A}$ *if and only if* $n \leq k$.

3.4 Examples

As it has been proved, the variety generated by one BL_n-chain can be equationally characterized in terms of the equations that define $\mathcal{MV}_n$ and the equations that define $\mathcal{W}$ as a subvariety of $\mathcal{GBL}$. Notice that the equations that characterize $\mathcal{MV}_n$ as a subvariety of $\mathcal{MV}$ are explicitly shown in theorems 3.2.2 and 3.2.3. But in order to apply the method presented in the first section for a given BL_n-chain $\mathbf{L}_n \uplus \mathbf{B}$, we should also know the equations that characterize the subvariety of generalized BL-algebras generated by $\mathbf{B}$. These equations are often unknown. For finite BL-chains, we know that $\mathbf{B}$ is a finite ordinal sum of finite MV-chains (see Theorem 2.1.11). But when a finite MV-chain $\mathbf{L}_m$, with $m \geq 4$ is considered as a generalized BL-algebra, $\perp$ is no longer a constant, and equation (3.5) in Theorem 3.2.3 can not be expressed in the language of basic hoops. Then, even in this simple case the equations that characterize the variety of generalized BL-algebras generated by the dense part of the BL_n-chain are unknown.

Indeed the given method of characterizing subvarieties of $\mathcal{BL}$ was introduced because of its theoretical interest in the description of free algebras in the following chapter. For some cases we can explicitly show the equations that characterize the subvariety of $\mathcal{BL}$ generated by one finite BL_n-chain in an alternative way.

To achieve such aim, we will firstly describe varieties of BL-algebras generated by finite BL_n-chains that satisfy some special conditions. Let n, m, r and k be integers such that $n \geq 2$, $m \geq 2$, $k+1 \leq r$ and $k.(m-1)+1 \geq r$. Let $\mathcal{V}_{(n,m,r,k)}$ be the variety of BL-algebras generated by the BL-chains that are of the form

$$\uplus_{i=0}^{k} \mathbf{L}_{r_i},$$

where $r_0 = n$, $(\sum_{i=1}^{k} r_i) - (k-1) \leq r$ and $2 \leq r_i \leq m$ for each $i = 1, \ldots k$. Then the generating chains of $\mathcal{V}_{(n,m,r,k)}$ can be decomposed into $(k+1)$ non trivial irreducible parts, they have at most r dense elements and at least $(k+1)$ dense elements, and each of the irreducible hoops that compose their generalized BL-algebra of dense elements has at most m elements.

Proposition 3.4.1 *A BL-algebra* $\mathbf{C}$ *is in* $\mathcal{V}_{(n,m,r,k)}$ *if and only if the following identities hold for every* $x, x_1, \ldots, x_r$ *in* $\mathbf{C}$*:*

$$(\neg\neg x)^{n-1} = (\neg\neg x)^n, \tag{3.12}$$

If $n \geq 4$*, for every integer* $p = 2, \ldots, n-2$ *that does not divide* $n-1$*:*

$$(p \cdot (\neg\neg x)^{p-1})^n = n \cdot (\neg\neg x)^p, \tag{3.13}$$

$$\bigwedge_{i=0}^{k}((x_{i+1} \to x_i) \to x_i) \to \bigvee_{i=0}^{k+1} x_i = \top \tag{3.14}$$

$$(\neg\neg x \to x)^{m-1} = (\neg\neg x \to x)^m, \tag{3.15}$$

$$\begin{aligned}(\neg\neg x_r \to x_r) \vee ((\neg\neg x_r \to x_r) \to (\neg\neg x_{r-1} \to x_{r-1})) \vee \cdots \\ \ldots \vee ((\neg\neg x_2 \to x_2) \to (\neg\neg x_1 \to x_1)) = \top,\end{aligned} \tag{3.16}$$

Proof: Firstly, we see that each of the chains that generates the variety satisfies equations (3.12), (3.13), (3.14), (3.15) and (3.16). Notice that if **A** is one of such chains, then **A** is a BL_n-chain and from Lemma 3.2.4, **A** satisfies equations (3.12) and (3.13).

Let consider $D(\mathbf{A})$. This generalized BL-chain is an ordinal sum of k chains each one of the form $\mathbf{L}_{r_i}$, with $2 \leq r_i \leq m$. Thus, from Theorem 3.2.1, for each $x \in D(\mathbf{A})$, we have that $(\neg\neg x \to x)^m = x^m = x^{m-1} = (\neg\neg x \to x)^{m-1}$, and equation (3.15) is satisfied by all the elements in $D(\mathbf{A})$. Of course equation (3.15) is also satisfied by elements in $MV(\mathbf{A})$, hence , from Theorem 2.2.1, the equation is satisfied by all the elements in **A**. Since $(\sum_{i=1}^{k} r_i) - (k-1) \leq r$, Lemma 3.3.1 asserts that **A** satisfies equation (3.16). Finally, by Lemma 3.3.2 equation (3.14) is satisfied.

Let **A** be a BL-chain satisfying (3.12), (3.13), (3.14), (3.15) and (3.16). By Lemma 3.2.5 and Theorem 3.1.1 we know that equations (3.12) and (3.13) imply that $MV(\mathbf{A}) \cong \mathbf{L}_{d+1}$ for d dividing $n-1$. From equation (3.16) we obtain that $D(\mathbf{A})$ has at most r elements; from Theorem 2.2.1, the previous results indicate that the chain **A** is finite. Equation (3.14) asserts that **A** can be decomposed in at most $k+1$ non trivial irreducible hoops. Since they are finite, they must be all reducts of MV-chains, and from Theorem 3.2.1 and equation (3.15) we know that $D(\mathbf{A}) \cong \uplus_{i=1}^{p} \mathbf{L}_{r_i}$ where $r_i \leq m$ for every $i = 1, \ldots, p$, for some $p \leq k$

Using once more Theorem 2.2.1, we have

$$\mathbf{A} \cong \mathbf{MV(A)} \uplus \mathbf{D(A)} \cong \uplus_{i=0}^{p} \mathbf{L}_{r_i},$$

where $r_0 - 1$ divides $n-1$, $r_i \leq m$ for each $i = 1, \ldots, p$, $p \leq k$ and $(\sum_{i=1}^{p} r_i) - (p-1) \leq r$. It is easy to corroborate that **A** is a subalgebra of one of the chains that generate the variety $\mathcal{V}_{(n,m,r,k)}$. From Theorem 1.2.2, each BL-algebra $\mathbf{C}$ satisfying (3.12), (3.13), (3.14), (3.15) and (3.16) is in $\mathcal{V}_{(n,m,r,k)}$. ■

The variety $\mathcal{V}_{(5,3,6,2)}$ is the variety generated by $\mathbf{L}_5 \uplus \mathbf{L}_3 \uplus \mathbf{L}_3$, $\mathbf{L}_5 \uplus \mathbf{L}_2 \uplus \mathbf{L}_3$, $\mathbf{L}_5 \uplus \mathbf{L}_3 \uplus \mathbf{L}_2$ and $\mathbf{L}_5 \uplus \mathbf{L}_2 \uplus \mathbf{L}_2$. The variety $\mathcal{V}_{(5,3,5,2)}$ is generated by the chains $\mathbf{L}_5 \uplus \mathbf{L}_2 \uplus \mathbf{L}_3$, $\mathbf{L}_5 \uplus \mathbf{L}_3 \uplus \mathbf{L}_2$ and $\mathbf{L}_5 \uplus \mathbf{L}_2 \uplus \mathbf{L}_2$. Since $\mathbf{L}_5 \uplus \mathbf{L}_2 \uplus \mathbf{L}_2$ is a subalgebra of $\mathbf{L}_5 \uplus \mathbf{L}_2 \uplus \mathbf{L}_3$ and of $\mathbf{L}_5 \uplus \mathbf{L}_3 \uplus \mathbf{L}_2$, we can say that $\mathcal{V}_{(5,3,5,2)}$ is the variety generated by these two algebras.

Now we shall characterize equationally varieties of BL-algebras generated by one specific BL_n-chain.

3.4.1 The ordinal sum of two finite MV-chains

We define the BL_n-algebra $\mathbf{L}_n^m$ as the ordinal sum of $\mathbf{L}_n$ and $\mathbf{L}_m$, that is, $\mathbf{L}_n^m = \mathbf{L}_n \uplus \mathbf{L}_m$ for $m \geq 2$ and $n \geq 2$. Notice that the elements in $\mathbf{L}_m$ are dense in $\mathbf{L}_n^m$. To describe equationally the variety $\mathcal{V}$ generated by the chain $\mathbf{L}_n^m$, we define the following operations:

$$\diamond x = x \to x^{m-1},$$

and inductively,

$$1 \circ x = x \text{ and } (k+1) \circ x = ((k \circ x) \to x^{m-1} * \diamond x) \to x^{m-1}.$$

Proposition 3.4.2 *Let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{L}_n^m$. A BL-algebra $\mathbf{C}$ is in $\mathcal{V}$ if and only if the following identities hold for every $x, x_1, \ldots, x_m$ in $\mathbf{C}$:*

$$(\neg\neg x)^{n-1} = (\neg\neg x)^n, \tag{3.17}$$

if $n \geq 4$, for every integer $p = 2, \ldots, n-2$ that does not divide $n-1$:

$$(p \cdot (\neg\neg x)^{p-1})^n = n \cdot (\neg\neg x)^p, \tag{3.18}$$

$$\bigwedge_{i=0}^{1}((x_{i+1} \to x_i) \to x_i) \to \bigvee_{i=0}^{2} x_i = \top. \tag{3.19}$$

$$(\neg\neg x \to x)^{m-1} = (\neg\neg x \to x)^m, \tag{3.20}$$

if $n \geq 4$, for every integer $p = 2, \ldots, m-2$ that does not divide $m-1$:

$$(p \circ (\neg\neg x \to x)^{p-1})^m = m \circ (\neg\neg x \to x)^p. \tag{3.21}$$

$$(\neg\neg x_m \to x_m) \vee ((\neg\neg x_m \to x_m) \to (\neg\neg x_{m-1} \to x_{m-1})) \vee \cdots \tag{3.22}$$
$$\ldots \vee ((\neg\neg x_2 \to x_2) \to (\neg\neg x_1 \to x_1)) = \top,$$

Proof: Firstly we see that $\mathbf{L}_n^m$ satisfies equations (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22). Since $\mathbf{L}_n^m$ is a generator of the variety $\mathcal{V}_{(n,m,m,1)}$, we know that equations (3.17), (3.18), (3.19), (3.20) and (3.22) are satisfied. If $x \in MV(\mathbf{L}_n^m) = L_n$, then $\neg\neg x \to x = \top$ and equation (3.21) is satisfied. Otherwise, $x \in D(\mathbf{L}_n^m) \setminus \{\top\} = L_m \setminus \{\top\}$ and $\neg\neg x \to x = x$. This being the case, $(\neg\neg x \to x)^{m-1}$ is the lower bound of L_m, and equation (3.21) is equivalent to equation (3.5) in Theorem 3.2.3, therefore, it is also satisfied by the elements of $D(\mathbf{L}_n^m)$.

From equations (3.17), (3.18), (3.19), (3.20) and (3.22), we know that $\mathcal{V}$ is a subvariety of $\mathcal{V}_{(n,m,m,1)}$. Remind from Proposition 3.4.1 that a chain $\mathbf{A}$ is in this last variety, then it is of the form

$$\mathbf{L}_{d+1} \uplus \mathbf{L}_s$$

where d divides $n-1$ and $s \le m$, or $\mathbf{A}$ is a subalgebra of $\mathbf{L}_n$. Now let $\mathbf{A}$ be one of these BL-chains that also satisfies equation (3.21). If $\mathbf{A}$ is a subalgebra of $\mathbf{L}_n$ then $\mathbf{A}$ is a subalgebra of $\mathbf{L}_n^m$. Otherwise, $D(\mathbf{A}) \setminus \{\top\} \neq \emptyset$. Notice that for every $x \in D(\mathbf{A}) \setminus \{\top\}$, $x^{m-1} = (\neg\neg x \to x)^{m-1} = (\neg\neg x \to x)^m = x^m$ is an idempotent element different from $\top$ and different from $\bot$. Since there is at most one non-extreme idempotent element, then for every $x, y \in D(\mathbf{A}) \setminus \{\top\}$ we obtain that $x^{m-1} = y^{m-1}$. We shall denote by c to this element. From its definition we have that $c \le x$ for every $x \in D(\mathbf{A})$. Consider the structure $\mathbf{D}'(\mathbf{A}) = (D(\mathbf{A}), *, \to, c, \top)$. If we consider c as a constant, $\mathbf{D}'(\mathbf{A})$ is a finite MV-chain. From Theorem 3.2.3 we have that the MV-chain $\mathbf{D}'(\mathbf{A})$ is in the variety generated by $\mathbf{L}_m$ iff every $x \in D'(\mathbf{A})$ satisfies the following equations:

$$x^{(m-1)} = x^m. \tag{3.23}$$

For $m \ge 4$ and every integer $p = 2, \ldots, m-2$ that does not divide $m-1$:

$$(p \odot x^{p-1})^m = m \odot x^p, \tag{3.24}$$

where the operation $\odot$ is inductively defined by

$$1 \odot x = x \text{ and } (k+1) \odot x = \neg'(\neg'(k \odot x) * \neg' x),$$

and $\neg'$ denotes the negation in $\mathbf{D}'(\mathbf{A})$. From the fact that for $x \in D(\mathbf{A})$, $\neg\neg x \to x = x$, equation (3.20) asserts that equation (3.23) is satisfied in $\mathbf{D}(\mathbf{A})$. Now we will check equation (3.24). It is easy to see that $k \odot \top = \top$ for every integer k and that $\top$ satisfies equation (3.24). If $x \in D'(\mathbf{A}) \setminus \{\top\}$, since $\neg' x = x \to c = x \to x^{m-1} = \diamond x$, from equation (3.21) we have

$$(p \odot x^{p-1})^m = (p \circ x^{p-1})^m = m \circ x^p = m \odot x^p.$$

Therefore, $\mathbf{D}'(\mathbf{A})$ is an MV-chain in the variety generated by $\mathbf{L}_m$ and $\mathbf{D}(\mathbf{A}) \cong \mathbf{L}_{s+1}$ as generalized BL-chain for some s that divides $m-1$. In conclusion, the BL-chains in $\mathcal{V}$ are of the form

$$\mathbf{L}_{d+1} \uplus \mathbf{L}_{s+1}$$

and where d divides $n-1$ and s must divide $m-1$. It is easy to corroborate that these are all subalgebras of $\mathbf{L}_n^m$, thus $\mathcal{V}$ is the variety of BL-algebras generated by $\mathbf{L}_n^m$. ■

3.4.2 The ordinal sum of a finite number of finite MV-chains of the same length

In a similar manner as in Proposition 3.4.2 we can characterize equationally varieties generated by chains whose dense part is an ordinal sum of k chains of the form $\mathbf{L}_m$.

Let

$$\mathbf{L}_n^{(m,k)} = \uplus_{i=0}^k \mathbf{L}_{r_i}$$

where $r_0 = n$ and $r_i = m$ for each $i = 1, \ldots, k$, and let $r = k(m-1)+1$. Once more we set

$$\diamond x = x \to x^{m-1},$$

and inductively,

$$1 \circ x = x \text{ and } (k+1) \circ x = ((k \circ x) \to x^{m-1} * \diamond x) \to x^{m-1}.$$

Proposition 3.4.3 *Let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{L}_n^{(m,k)}$. A BL-algebra $\mathbf{C}$ is in $\mathcal{V}$ if and only if the following identities hold for every $x, x_1, \ldots, x_s$ in $\mathbf{C}$:*

$$(\neg\neg x)^{n-1} = (\neg\neg x)^n, \tag{3.25}$$

if $n \geq 4$, for every integer $p = 2, \ldots, n-2$ that does not divide $n-1$:

$$(p \cdot (\neg\neg x)^{p-1})^n = n \cdot (\neg\neg x)^p, \tag{3.26}$$

$$\bigwedge_{i=0}^{k}((x_{i+1} \to x_i) \to x_i) \to \bigvee_{i=0}^{k+1} x_i = \top. \tag{3.27}$$

$$(\neg\neg x \to x)^{m-1} = (\neg\neg x \to x)^m, \tag{3.28}$$

if $n \geq 4$, for every integer $p = 2, \ldots, m-2$ that does not divide $m-1$:

$$(p \circ (\neg\neg x \to x)^{p-1})^m = m \circ (\neg\neg x \to x)^p. \tag{3.29}$$

$$(\neg\neg x_r \to x_r) \vee ((\neg\neg x_r \to x_r) \to (\neg\neg x_{r-1} \to x_{r-1})) \vee \cdots \tag{3.30}$$

$$\ldots \vee ((\neg\neg x_2 \to x_2) \to (\neg\neg x_1 \to x_1)) = \top,$$

Proof: Verifying that $\mathbf{L}_n^{(m,k)}$ satisfies equations (3.25), (3.26), (3.27), (3.28), (3.29) and (3.30) is analogous to the proof of Proposition 3.4.2.

Again equations (3.25), (3.26), (3.27), (3.28) and (3.30), asserts that $\mathcal{V}$ is a subvariety of $\mathcal{V}_{(n,m,r,k)}$. So let $\mathbf{A}$ be one of the chains in $\mathcal{V}_{(n,m,r,k)}$, that also satisfies equation (3.29). We know that $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_{d+1}$ for some d that divides $n-1$ and that $\mathbf{D}(\mathbf{A}) = \uplus_{i=1}^{p} \mathbf{L}_{r_i}$ where $p \leq k$ and $r_i \leq m$. We also know that $D(\mathbf{A})$ has at most r elements and that the number of dense idempotent elements is $p+1$, p of which correspond to the bottom element of each chain L_{r_i}. We shall call these dense idempotents 0_i for $i = 1, \ldots, p$. Then, for each $x \in D(\mathbf{A})$ there exists $i = 1, \ldots, p$ such that $(\neg\neg x \to x)^{m-1} = x^{m-1} = 0_i$. Now if $\neg_i$ represents negation in $\mathbf{L}_{r_i}$ then for each $x \in L_{r_i} \setminus \{\top\}$ we have that

$$\neg_i x = x \to 0_i = x \to x^{m-1} = \diamond x.$$

Reminding once more that, if $x \in D(\mathbf{A})$, then $\neg\neg x \to x = x$, from Theorem 3.2.3, equations (3.28) and (3.29) asserts that $\mathbf{L}_{r_i}$ is in the variety generated by $\mathbf{L}_m$, and Theorem 3.1.1 asserts that $r_i - 1$ is a divisor of $m-1$. Then the chains that in $\mathcal{V}$ are of the form

$$\uplus_{i=0}^{p} \mathbf{L}_{r_i}$$

where $r_0 = d+1$ for some d that divides $n-1$, $p \leq k$ and $r_i - 1$ divides $m-1$ for each $i = 1, \ldots, p$, that means they are subalgebras of $\mathbf{L}_n^{(m,k)}$ and the theorem follows. ∎

3.4.3 The ordinal sum of a finite MV-chain and a finite Heyting chain

Recall that a **linear Heyting algebra** $\mathbf{H} = (H, *, \rightarrow, \bot, \top)$ is a BL-algebra in which $\wedge = *$, i.e., it satisfies the equation $x * y = x \wedge y$. Clearly, each element x of $\mathbf{H}_m$ is an idempotent element, since

$$x^2 = x * x = x \wedge x = x.$$

This chain can be decomposed to obtain : $\mathbf{H}_{m+1} \cong \uplus_{i=1}^{m} \mathbf{L}_2$. Hence $H_{m+1} \setminus \{\bot\}$ is the universe of a generalized BL-chain isomorphic to $\mathbf{D}(\mathbf{H}_{m+1})$. Notice from Remark 2.1.12 that if $\mathbf{A}$ is a finite BL-chain such that for each $x \in A$, x is idempotent, then $\mathbf{A} \cong \uplus_{i=1}^{m} \mathbf{L}_2$ for some integer m. Hence $\mathbf{A}$ is a finite Heyting chain.

Given $m \geq 2$ and $n \geq 2$, we define the BL-algebra $\mathbf{T}_n^m$ as the ordinal sum of the MV-chain $\mathbf{L}_n$ and the Heyting chain $\mathbf{H}_m$, that is , $\mathbf{T}_n^m = \mathbf{L}_n \uplus \mathbf{H}_m$. The subalgebras of $\mathbf{T}_n^m$ are of the form $\mathbf{T}_{d+1}^j$ if d divides $n-1$ and $j \leq m$.

Proposition 3.4.4 *Let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{T}_n^m$. A BL-algebra $\mathbf{C}$ is in $\mathcal{V}$ if and only if it satisfies the following identities for every $x, x_1, x_2, \ldots, x_m$ in $\mathbf{C}$:*

$$(\neg\neg x)^{(n-1)} = (\neg\neg x)^n. \tag{3.31}$$

If $n \geq 4$, for every integer $p = 2, \ldots, n-2$ that does not divide $n-1$:

$$(p \cdot (\neg\neg x)^{p-1})^n = n \cdot (\neg\neg x)^p, \tag{3.32}$$

$$(\neg\neg x_m \rightarrow x_m) \vee ((\neg\neg x_m \rightarrow x_m) \rightarrow (\neg\neg x_{m-1} \rightarrow x_{m-1})) \vee \cdots \tag{3.33}$$

$$\ldots \vee ((\neg\neg x_2 \rightarrow x_2) \rightarrow (\neg\neg x_1 \rightarrow x_1)) = \top,$$

$$(\neg\neg x \rightarrow x)^2 = \neg\neg x \rightarrow x. \tag{3.34}$$

Proof: As in the proof of Proposition 3.4.1, it is easy to verify that $\mathbf{T}_n^m$ satisfy these equations. Let now $\mathbf{A}$ be a chain that satisfy equations (3.31), (3.32), (3.33) and (3.34). Lemma 3.2.5 asserts that $MV(\mathbf{A}) = \mathbf{L}_{d+1}$ for some d that divides $n-1$ and Lemma 3.3.1 indicates that $D(\mathbf{A})$ has at most m elements. By equation (3.34) dense elements are all idempotents, thus $D(\mathbf{A}) \cong \mathbf{H}_j$ for some $j \leq m$. Clearly, $\mathbf{A}$ is a subalgebra of the generator $\mathbf{T}_n^m$ and the result follows from Theorem 1.2.2. ■

Remark 3.4.5 In [2] the equations that characterize certain subvarieties of $\mathcal{BL}$ generated by a BL-chain which is a finite ordinal sum of Wajsberg hoops are given. But such equations also depend on the equations that characterize certain subvarieties of Wajsberg hoops that aren't explicitly shown.

Chapter 4

Free algebras in varieties of BL-algebras generated by a BL_n-chain.

4.1 Introduction

Since the propositions under logical equivalence form a free BL-algebra, descriptions of free algebras are important from the point of view of algebra as well as from the point of view of logic. In [21], it is shown that free algebras in varieties of BL-algebras can be described in terms of free MV-algebras and free algebras in certain varieties of hoops. The aim of this chapter is to apply the methods of [21] to obtain a description of the free algebras in varieties of BL-algebras generated by one BL_n-chain $\mathbf{L}_n \uplus \mathbf{B}$. These free algebras are going to be described in terms of weak boolean products of BL-algebras that are ordinal sums of subalgebras of $\mathbf{L}_n$ and free algebras in the variety of basic hoops generated by $\mathbf{B}$. The boolean products are taken over the Stone spaces of the boolean algebras of idempotent elements of free algebras in $\mathcal{MV}_n$, which are described in Appendix A. An important role is played by the axiomatization of the variety generated by $\mathbf{L}_n \uplus \mathbf{B}$ in terms of the equations defining the variety generated by $\mathbf{L}_n$ and the variety generated by $\mathbf{B}$ (Corollary 3.1.8).

4.2 Characterization of the free algebra as a weak boolean product

Recall that an algebra $\mathbf{A}$ in a variety $\mathcal{K}$ is said to be **free over a set** Y if and only if for every algebra $\mathbf{C}$ in $\mathcal{K}$ and every function $f : Y \to \mathbf{C}$, f can be uniquely extended to a homomorphism of $\mathbf{A}$ into $\mathbf{C}$. Given a variety $\mathcal{K}$ of algebras, we denote by $\mathbf{Free}_{\mathcal{K}}(X)$ the free algebra in $\mathcal{K}$ over X.

Let $\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}$ be a BL_n-chain and let again $\mathcal{V}$ be the variety generated by $\mathbf{T}_n$. We shall describe $\mathbf{Free}_{\mathcal{V}}(X)$, the free BL-algebra in $\mathcal{V}$ over a set X of generators.

Recall that a **weak boolean product** of a family $(A_y, y \in Y)$ of algebras over a boolean space Y is a subdirect product $\mathbf{A}$ of the given family such that the following conditions hold:

- if $a, b \in A$, then $[a = b] = \{y \in Y : a_y = b_y\}$ is open,
- if $a, b \in A$ and Z is a clopen in X, then $a|_Z \cup b|_{X \setminus Z} \in A$.

An algebra $\mathbf{A}$ is **representable as a weak boolean product** when it is isomorphic to a weak boolean product. Since the variety $\mathcal{BL}$ is congruence distributive, it has the boolean Factor Congruence property. Therefore each non trivial BL-algebra can be represented as a weak boolean product of directly indecomposable BL-algebras (see [5] and [24]). The explicit representation of each BL-algebra as a weak boolean product of directly indecomposable algebras is given in [20] by the following lemma:

Lemma 4.2.1 *Let $\mathbf{A}$ be a BL-algebra and let $Sp\ \mathbf{B}(\mathbf{A})$ be the boolean space of ultrafilters of the boolean algebra $\mathbf{B}(\mathbf{A})$. The correspondence:*

$$a \mapsto (a/\langle U \rangle)_{U \in Sp\ \mathbf{B}(\mathbf{A})}$$

gives a representation of $\mathbf{A}$ as a weak boolean product of the family

$$(\mathbf{A}/\langle U \rangle) : U \in Sp\ \mathbf{B}(\mathbf{A})$$

over the boolean space $Sp\ \mathbf{B}(\mathbf{A})$. This representation is called **the Pierce representation**. *Any other representation of $\mathbf{A}$ as a weak boolean product of a family of directly indecomposable algebras is equivalent to the Pierce representation.*

Therefore, to describe $\mathbf{Free}_{\mathcal{V}}(X)$ we need to describe $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ and the quotients $\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle$ for each $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. In the next section we will obtain a characterization of the boolean algebra $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. Once this aim is achieved, we shall consider the quotients $\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle$.

4.3 The boolean subalgebra of the free BL-algebra

The next two results can be found in [21].

Theorem 4.3.1 *For each BL-algebra* $\mathbf{A}$, $\mathbf{B}(\mathbf{A}) \cong \mathbf{B}(\mathbf{MV}(\mathbf{A}))$.

Theorem 4.3.2 *For each variety* $\mathcal{K}$ *of BL-algebras and each set* X *one has that:*

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{K}}(X)) \cong \mathbf{Free}_{\mathcal{MV}\cap\mathcal{K}}(\neg\neg X).$$

Theorem 4.3.3 $\mathcal{V} \cap \mathcal{MV}$ *is the variety* $\mathcal{MV}_n$.

Proof: Since $\mathbf{MV}(\mathbf{T}_n) = \mathbf{L}_n$ is in $\mathcal{V} \cap \mathcal{MV}$, we have that $\mathcal{MV}_n \subseteq \mathcal{V} \cap \mathcal{MV}$. On the other hand, let $\mathbf{A}$ be an MV-algebra in $\mathcal{V} \cap \mathcal{MV}$. Suppose $\mathbf{A}$ is not in $\mathcal{MV}_n$. Then there exists an equation $e(x_1, \dots, x_p) = \top$ that is satisfied by $\mathbf{L}_n$ and it is not satisfied by $\mathbf{A}$, that is, there exist elements $a_1, \dots, a_p$ in A such that $e(a_1, \dots, a_p) \neq \top$. Since $(\neg\neg b_1, \dots, \neg\neg b_p)$ is in $(L_n)^p$, for each tuple $(b_1, \dots, b_p)$ in $(T_n)^p$, the equation $e'(x_1, \dots, x_p) = e(\neg\neg x_1, \dots, \neg\neg x_p) = \top$ is satisfied in $\mathcal{V}$. Since $\mathbf{A} \in \mathcal{V} \cap \mathcal{MV}$, it follows that

$$\top = e'(a_1, \dots, a_p) = e(\neg\neg a_1, \dots, \neg\neg a_p) = e(a_1, \dots, a_p) \neq \top,$$

a contradiction. Hence $\mathcal{MV}_n = \mathcal{V} \cap \mathcal{MV}$. ∎

From these results we obtain:

Theorem 4.3.4

$$\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X)) \cong \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)).$$

Boolean elements of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ depend on some operators

$$\sigma_i^n : \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X) \to \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)),\ i = 1, \dots, n-1,$$

that can be defined on each MV_n-algebra. Such operators provide each MV_n-algebra with an n-valued Moisil algebra structure. Notions concerning these algebras are study in Appendix A. In Theorem A.0.11, it is proved that for a set Z of generators, $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ is the free boolean algebra generated by the poset $Y = \{\sigma_i^n(z) : z \in Z, i = 1, \dots, n-1\}$. From Theorem 4.3.4 we obtain:

Corollary 4.3.5 $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *is the free boolean algebra generated by the poset* $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$.

Remark 4.3.6 Notice that if $n = 2$, i.e, the variety considered $\mathcal{V}$ is generated by a BL_2-chain, then $\sigma_1^2(x) = x$ for each $x \in X$. Therefore, in this case, $Y = \{\neg\neg x : x \in X\}$, and the cardinality of Y equals the cardinality of X. It follows that $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free boolean algebra over the set Y.

Our next aim is to describe $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ for each ultrafilter U in the free boolean algebra generated by $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$, where $\langle U\rangle$ is the implicative filter generated by the boolean filter U. The plan is to prove that $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$ is a subalgebra of $\mathbf{L}_n$ and then, using Theorem 2.2.2, decompose each quotient $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ into an ordinal sum.

4.4 Regular elements of the indecomposable factors

Theorem 4.4.1 *Let* $\mathbf{A}$ *be a BL-algebra and* $U \in Sp\,\mathbf{B}(\mathbf{A})$. *Then*

$$\mathbf{MV}(\mathbf{A}/\langle U\rangle) \cong \mathbf{MV}(\mathbf{A})/(\langle U\rangle \cap \mathbf{MV}(\mathbf{A})).$$

Proof: Let $V = \langle U\rangle \cap \mathbf{MV}(\mathbf{A})$ and let $f : \mathbf{MV}(\mathbf{A})/V \to \mathbf{MV}(\mathbf{A}/\langle U\rangle)$ be given by

$$f(a/V) = a/\langle U\rangle,$$

for each $a \in MV(\mathbf{A})$. It is easy to see that f is a homomorphism into $\mathbf{MV}(\mathbf{A}/\langle U\rangle)$. Besides, we have that:

1. *f is injective*

   Let $a/\langle U\rangle = b/\langle U\rangle$, with $a, b \in MV(\mathbf{A})$. From Lemma 1.4.2 we know that there exists $u \in U$ such that $a \wedge u = b \wedge u$. Since $U \subseteq MV(\mathbf{A})$, then $u \in V$. From the fact that u is boolean (see [20, Lemma 2.2]), we have that: $a * u = a \wedge u = b \wedge u \leq b$, thus $u \leq a \to b$ and similarly $u \leq b \to a$. Then $a \to b$ and $b \to a$ are in V and this means that $a/V = b/V$.

2. *f is surjective*

   Let $a/\langle U\rangle \in MV(\mathbf{A}/\langle U\rangle)$. Then

   $$a/\langle U\rangle = \neg\neg(a/\langle U\rangle) = \neg\neg a/\langle U\rangle,$$

   and since $\neg\neg a \in MV(\mathbf{A})$ we obtain that $f(\neg\neg a/V) = a/\langle U\rangle$. ■

By Theorem 4.3.4, if $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, then U is an ultrafilter in $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X))$. Moreover,

$$\langle U \rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)) = \langle U \rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$$

is the Stone ultrafilter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ generated by U. From [16, Chapter 6.3], we have that $\langle U \rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ is a maximal filter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$. Since the only simple algebras in $\mathcal{MV}_n$ are the subalgebras of the chain $\mathbf{L}_n$ (see [16, Corollary 3.5.4]), from Theorem 4.4.1 it follows that:

Theorem 4.4.2

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle) \cong \mathbf{L}_s$$

with $s - 1$ *dividing* $n - 1$.

From Theorems 2.2.2 and 4.4.2 we obtain:

Theorem 4.4.3 *For each* $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *we have that*

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle \cong \mathbf{L}_s \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle)$$

for some $s - 1$ *dividing* $n - 1$.

4.5 Dense elements of the indecomposable factors

In order to obtain a complete description of $\mathbf{Free}_{\mathcal{V}}(X)$ there is only left to find a description of $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle)$ for each $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. This last description will depend on the characterization of the variety $\mathcal{W}$ of generalized BL-algebras generated by the generalized BL-chain $\mathbf{B}$. We recall from Corollary 3.1.8 that $\mathcal{V}$ can be characterized in terms of the equations that define $\mathcal{MV}_n$ and $\mathcal{W}$.

Theorem 4.5.1 *The variety* $\mathcal{W}$ *of generalized BL-algebras generated by* $\mathbf{B}$ *consist of the generalized BL-algebras* $\mathbf{C}$ *such that* $\mathbf{L}_n \uplus \mathbf{C}$ *belongs to* $\mathcal{V}$.

Proof: Given $\mathbf{C} \in \mathcal{W}$, for each $x \in L_n \uplus C$, we have that $\neg\neg x$ satisfies the equations that defines $\mathcal{MV}_n$ as a subvariety of $\mathcal{MV}$ and $\neg\neg x \to x$ satisfies the equations that define $\mathcal{W}$ as a subvariety of $\mathcal{GBL}$. From Corollary 3.1.8, we deduce that $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}$. On the other hand, if $\mathbf{C}$ is a generalized BL-algebra such that $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}$, since $\neg\neg x \to x = x$ for each $x \in C$, again Corollary 3.1.8 asserts that the elements of C satisfy the equations that define $\mathcal{W}$. Hence $\mathbf{C}$ is in $\mathcal{W}$. ■

Every upwards closed subset of the poset

$$Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$$

is in correspondence with an increasing function from Y onto $\mathbf{2}$, the two elements boolean algebra. From the definition of free algebra over a poset we know that every increasing function from Y can be extended to a homomorphism from $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ into $\mathbf{2}$. We know that the homomorphisms from a boolean algebra into the two elements boolean algebra are in bijective correspondence with the ultrafilters of the boolean algebra. Then we can conclude that the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ are in bijective correspondence with the upwards closed subsets of Y. This is summarized in the following lemma:

Lemma 4.5.2 *Consider the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$. The correspondence that assigns to each upwards closed subset $S \subseteq Y$ the boolean filter U_S generated by the set*

$$\{\sigma_i^n(\neg\neg x) : \sigma_i^n(\neg\neg x) \in S\} \cup \{\neg\sigma_i^n(\neg\neg x) : \sigma_i^n(\neg\neg x) \notin S\},$$

defines a bijection from the set of upwards closed subsets of Y onto the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$.

Taking this fact into account, we shall refer to each member of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ by U_S making explicit reference to the upwards closed subset S that correspond to it.

Lemma 4.5.3 *Let $U_S \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ and let $\mathbf{F}_S$ be the subalgebra of the generalized BL-algebra $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ generated by the set $X_S = \{x/\langle U_S\rangle : x \in X, \neg\neg x \in \langle U_S\rangle\}$. Then*

$$\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

Proof: $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ is the BL-algebra generated by the set $Z_S = \{x/\langle U_S\rangle : x \in X\}$. From Theorem 4.4.3 there exists an integer m such that

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle = \mathbf{L}_m \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

Hence each element of Z_S is either in $L_m \setminus \{\top\}$ or it is in $D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$.

If $X_S = \emptyset$, then $F_S = D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) = \{\top\}$. So let suppose $X_S \neq \emptyset$. Let $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$. Recalling that $\mathbf{F}_S$ is the generalized BL-algebra generated by X_S, we will check that y is in F_S. Since $y \in Free_{\mathcal{V}}(X)/\langle U_S\rangle$, y is given by a term on the elements $x/\langle U_S\rangle \in Z_S$. Making induction on the complexity of y we have:

- If y is a generator, i.e, $y = x/\langle U_S\rangle$ for some $x/\langle U_S\rangle \in Z_S$, since $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, we have that $\top = \neg\neg y = \neg\neg(x/\langle U_S\rangle) = (\neg\neg x)/\langle U_S\rangle$. This happens only if $\neg\neg x \in X_S$.

- Suppose that for each element $z \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ of complexity less than k, z can be written as a term in the variables $x/\langle U_S\rangle$ in X_S. Let $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ be an element of complexity k. The possible cases are the following:

  1. $y = a \to b$ for some elements a, b of complexity $< k$. In this case the possibilities are:
     (a) $a \leq b$ in which case $a \to b = \top$ and y can be written as $x/\langle U_S\rangle \to x/\langle U_S\rangle$ for any $x/\langle U_S\rangle \in X_S$, thus $y \in F_S$,
     (b) $a > b$. Since $y = a \to b$ is in $D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, the only possibility is that $a, b \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ and by inductive hypothesis y is in F_S.
  2. $y = a * b$ for some elements a, b of complexity $< k$. In this case necessarily $a, b \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ and by inductive hypothesis y is in F_S.

Then for each $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, y can be written as a term on the elements of X_S, therefore $y \in F_S$ and we conclude that

$$\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

■

With the notation of the previous lemma, we have:

Theorem 4.5.4 *For each U_S in Sp $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$,*

$$\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S).$$

Proof: As a consequence of Theorem A.0.4 and Lemma 4.5.2 we have that $\neg\neg x \in \langle U_S\rangle$ iff $\sigma_1^n(\neg\neg x) \in S$ iff $\sigma_i^n(\neg\neg x) \in S$ for $i = 1, \ldots, n-1$. Hence if $\neg\neg x \notin \langle U_S\rangle$ there is a j such that $\sigma_j^n(\neg\neg x) \notin S$. We define for each $x \in X$,

$$j_x = \begin{cases} \bot & \text{if } \neg\neg x \in \langle U_S\rangle, \\ \max\{i \in \{1, \ldots, n-1\} : \sigma_i^n(\neg\neg x) \notin S\} & \text{otherwise.} \end{cases}$$

Let $\mathbf{C} \in \mathcal{W}$ and let $\mathbf{C}' = \mathbf{L}_n \uplus \mathbf{C}$. From Theorem 4.5.1, $\mathbf{C}'$ is in $\mathcal{V}$. Given a function $f : X_S \to \mathbf{C}$, define $\hat{f} : X \to \mathbf{C}'$ by the prescriptions:

$$\hat{f}(x) = \begin{cases} f(x/\langle U_S\rangle) & \text{if } \neg\neg x \in \langle U_S\rangle, \\ \frac{n-j_x-1}{n-1} & \text{otherwise.} \end{cases}$$

There is a unique homomorphism

$$\hat{h} : \mathbf{Free}_{\mathcal{V}}(X) \to \mathbf{C}'$$

such that $\hat{h}(x) = \hat{f}(x)$ for each $x \in X$. We have that $U_S \subseteq \hat{h}^{-1}(\{\top\})$. Indeed, if $\neg\neg x \in \langle U_S \rangle$, then $\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n(\neg\neg(\hat{h}(x)) = \sigma_i^n(\neg\neg f(x/\langle U_S \rangle)) = \sigma_i^n(\top) = \top$. If $\neg\neg x \notin \langle U_S \rangle$, then

$$\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n(\neg\neg\frac{n - j_x - 1}{n-1}) = \sigma_i^n(\frac{n - j_x - 1}{n-1}) = \begin{cases} \bot & \text{if } i \leq j_x, \\ \top & \text{otherwise.} \end{cases}$$

Hence there is a unique homomorphism

$$h_1 : \mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle \to \mathbf{C}'$$

such that $h_1(a/\langle U_S \rangle) = \hat{h}(a)$ for all $a \in \mathbf{Free}_{\mathcal{V}}(X)$. By Lemma 4.5.3, $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle)$ is the algebra generated by X_S. Then the restriction h of h_1 to $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle)$ is a homomorphism

$$h : \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) \to \mathbf{C},$$

and for each x such that $\neg\neg x \in \langle U_S \rangle$, we have that

$$h(x/\langle U_S \rangle) = h_1(x/\langle U_S \rangle) = \hat{h}(x) = \hat{f}(x) = f(x/\langle U_S \rangle).$$

Therefore we conclude that $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S)$. ■

As a conclusion of all the results developed in the present chapter we have:

Theorem 4.5.5 *The free BL-algebra* $\mathbf{Free}_{\mathcal{V}}(X)$ *can be represented as a weak boolean product of the family*

$$(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) : U_S \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$$

where $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *is the free boolean algebra over the poset* $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$. *Moreover, for each* $U_S \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *there exists* $m \geq 2$ *such that* $m-1$ *divides* $n-1$ *and*

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle = \mathbf{L}_m \uplus \mathbf{Free}_{\mathcal{W}}(X_S)$$

where $X_S = \{x/\langle U_S \rangle : \neg\neg x \in \langle U_S \rangle\}$ *and* $\mathcal{W}$ *is the variety of generalized BL-algebras generated by* $\mathbf{B}$.

4.6 Free PL-algebras

We have already noticed in chapter 2, that if $\mathbf{G}$ is a lattice-ordered abelian group (ℓ-group) and $G^- = \{x \in G : x \leq 0\}$ is its negative cone, then $\mathbf{P(G)} = (G^-, *, \rightarrow, 0)$ is a generalized BL-algebra, where

$$x * y = x + y \quad \text{and} \quad x \rightarrow y = 0 \wedge (y - x).$$

From Theorem 2.1.8 we know that, if $\mathbf{Z}$ denotes the group of integers, then $\mathbf{P(Z)}$ generates variety of cancellative hoops. So let consider $\mathcal{W}$, the variety of generalized BL-algebras generated by $\mathbf{P(Z)}$. In [18] a description of $\mathbf{Free}_{\mathcal{W}}(X)$ is given for any set X of free generators. Therefore we can have a complete description of free algebras in varieties of BL-algebras generated by the ordinal sum

$$\mathbf{PL}_n = \mathbf{L}_n \uplus \mathbf{P(Z)}.$$

Indeed, if we denote by $\mathcal{PL}_n$ the variety of BL-algebras generated by $\mathbf{PL}_n$, from Theorem 4.5.5 we obtain that $\mathbf{Free}_{\mathcal{PL}_n}(X)$ is a weak boolean product of algebras of the form

$$\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$$

with $s-1$ dividing $n-1$ and some set X' of cardinality less or equal than X. Therefore, in the present case, the BL-algebra $\mathbf{Free}_{\mathcal{PL}_n}(X)$ can be completely described as a weak boolean product of ordinal sums of two known algebras.

From [19, Theorem 2.8] $\mathcal{PL}_2$ is the variety of PL-algebras $\mathcal{PL}$ and by Remark 4.3.6, $Sp\,\mathbf{B}(\mathbf{Free}_{\mathcal{PL}}(X))$ is the Cantor space $\mathbf{2}^{|X|}$. Therefore Theorem 4.5.5 asserts that the free PL-algebra over a set X can be described as a weak boolean product over the Cantor space $\mathbf{2}^{|X|}$ of algebras of the form

$$\mathbf{L_2} \uplus \mathbf{Free}_{\mathcal{W}}(X')$$

for some set X' of cardinality less or equal than X.

Given a BL-algebra $\mathbf{A}$, the radical $R(\mathbf{A})$ of $\mathbf{A}$ is the intersection of all maximal implicative filters of $\mathbf{A}$. We have that $\mathbf{r(A)} = (R(\mathbf{A}), *, \rightarrow, \top)$ is a generalized BL-algebra. Let

$$\mathcal{PL}^r = \{\mathbf{R} : \mathbf{R} = \mathbf{r(A)} \text{ for some } \mathbf{A} \in \mathcal{PL}\}.$$

$\mathcal{PL}^r$ is a variety of generalized BL-algebras. In [20] a description of $\mathbf{Free}_{\mathcal{PL}}(X)$ is given. From Example 4.7 and Theorem 5.7 in the mentioned paper we obtained that $\mathbf{Free}_{\mathcal{PL}}(X)$ is the weak boolean product of the family

$$(\mathbf{L}_2 \uplus \mathbf{Free}_{\mathcal{PL}^r}(S) : S \subseteq \mathbf{2}^{|X|})$$

over the Cantor space $\mathbf{2}^{|X|}$. In order to check that our description and the one given in [20] coincide there is only left to check that $\mathcal{PL}^r = \mathcal{W}$. Recalling that $\mathcal{PL} = \mathcal{MV}_2 \uplus^t \mathcal{W}$, from Theorem 4.5.1 we have that $\mathcal{W}$ consist of the generalized BL-algebras $\mathbf{C}$ such that $\mathbf{L}_2 \uplus \mathbf{C} \in \mathcal{PL}$.

Theorem 4.6.1 $\mathcal{PL}^r = \mathcal{W}$.

Proof: Let $\mathbf{C} \in \mathcal{PL}^r$. Then there exists a BL-algebra $\mathbf{A} \in \mathcal{PL}$ such that $\mathbf{r}(\mathbf{A}) = \mathbf{C}$. It is not hard to check that $\mathbf{L}_2 \uplus \mathbf{C}$ is a subalgebra of $\mathbf{A} \uplus \mathbf{C}$, thus $\mathbf{L}_2 \uplus \mathbf{C}$ is in $\mathcal{PL}$. It follows that $\mathbf{C} \in \mathcal{W}$. On the other hand, let $\mathbf{C} \in \mathcal{W}$. Then $\mathbf{L}_2 \uplus \mathbf{C}$ is in $\mathcal{PL}$, and $\mathbf{C} \in \mathcal{PL}^r$. ■

4.7 Free MV$_n$-algebras

We know that $\mathcal{MV}_n$, the variety of MV$_n$-algebras is the subvariety of BL-algebras generated by the finite MV-chain $\mathbf{L}_n$. Of course $\mathbf{L}_n$ is a BL$_n$-chain, because $\mathbf{L}_n = \mathbf{L}_n \uplus \top$, where $\top$ denotes the trivial generalized BL-algebra of one element. Then Theorem 4.5.5 asserts that $\mathbf{Free}_{\mathcal{MV}_n}(X)$ is a weak boolean product of subalgebras of $\mathbf{L}_n$ over the boolean space of the free boolean algebra generated by the poset $Y = \{\sigma_i^n(x) : x \in X, i = 1, \ldots, n-1\}$.

Chapter 5

Finitely generated free algebras in varieties of BL-algebras generated by a BL_n-chain.

5.1 Comparison with the general case

In the previous chapter we described free algebras in varieties of BL-algebras generated by a BL_n-chain. These algebras were taken over an arbitrary set of generators X. Notice that when the set X of generators is finite, of cardinality k, then $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$ is the cardinal sum of k chains of length $n-1$. Therefore the number of upwards closed subsets of Y is n^k. Since weak boolean products over discrete finite spaces coincide with direct products, Theorem 4.5.5 asserts that $\mathbf{Free}_{\mathcal{V}}(X)$ is a direct product of n^k BL-algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$, with $s-1$ dividing $n-1$ and X' a set of cardinality less or equal than k.

When the set X is finite, the boolean elements of $\mathbf{Free}_{\mathcal{V}}(X)$ form a finite boolean algebra. Then it is possible to apply Theorem 1.5.3 to describe the free algebras in terms of directly indecomposable algebras. Indeed, this Theorem also asserts that the free algebra over a set of cardinality k it is a direct product of n^k algebras obtained by taking the quotients by the implicative filters generated by the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. In this chapter, we shall characterize the atoms of the finitely generated free algebra in terms of functions and then, based on the knowledge of the atoms, we shall describe the initial segments of the decomposition. Of course the description we shall present coincide with the one given in the previous chapter, but it will provide more information about the indecomposable factors $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$.

5.2 Alternative description of the finitely generated free algebras

Let $\mathbf{T}_n$ and $\mathcal{V}$ be as in the previous chapter. For every integer k, we will denote by $\mathbf{Free}_{\mathcal{V}}(k)$ the free algebra with k generators in $\mathcal{V}$.

Theorem 5.2.1 *([22, Chapter IV, Theorem 3.13]) If a variety $\mathcal{K}$ of algebras is generated by an algebra $\mathbf{A}$, then $\mathbf{Free}_{\mathcal{K}}(k)$ is isomorphic to the subalgebra of $\mathbf{A}^{\mathbf{A}^k}$ generated by the projection functions $\pi_\alpha : \mathbf{A}^k \to \mathbf{A}$, $\alpha \in k$.*

Then $\mathbf{Free}_{\mathcal{V}}(k)$ is the subalgebra of ${\mathbf{T}_n}^{\mathbf{T}_n^k}$ generated by the projection functions. We shall refer to each element of $\mathbf{Free}_{\mathcal{V}}(k)$ with a function

$$f : (\mathbf{T}_n)^k \to \mathbf{T}_n,$$

and we will denote by $\mathbf{x} = (x_1, \ldots, x_k)$ each k-tuple in the domain of f.

Recall from Theorem 4.3.4 that $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k)) = \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(k))$. From Theorem 5.2.1 we know that $\mathbf{Free}_{\mathcal{MV}_n}(k)$ is a subalgebra of ${\mathbf{L}_n}^{\mathbf{L}_n^k}$, thus $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(k))$ is a finite algebra. Then Theorem 1.5.3 can be applied to describe $\mathbf{Free}_{\mathcal{V}}(k)$ as a direct product of indecomposable algebras. To obtain a complete description of each indecomposable factor we will study the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$. For the sake of readability we shall denote $\top$ by 1 and $\bot$ by 0.

Remark 5.2.2 As it is well known, $f \in Free_{\mathcal{V}}(k)$ iff f is the interpretation of a BL-term. Then, if $\mathbf{D}$ is a subalgebra of $\mathbf{T}_n$ and $f \in Free_{\mathcal{V}}(k)$, we have that $f(\mathbf{x}) \in D$ for every $\mathbf{x} \in D^k$.

Lemma 5.2.3 *Let $f \in \mathbf{Free}_{\mathcal{V}}(k)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ in $(T_n)^k$. Suppose that there exists i such that $x_i = q \in D(\mathbf{T}_n)$. Let $p \in D(\mathbf{T}_n)$ and let $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_k)$ be a k-tuple such that*

$$x'_j = \begin{cases} x_j & \text{if } j \neq i, \\ p & \text{if } j = i. \end{cases}$$

For each $y \in L_n \setminus \{1\}$ one has that: $f(\mathbf{x}) = y$ iff $f(\mathbf{x}') = y$.

Proof: According to Theorem 5.2.1 it is enough to prove that the set

$$P = \{f \in T_n^{T_n^k} : \forall y \in L_n \setminus \{1\}, f(\mathbf{x}) = y \text{ iff } f(\mathbf{x}') = y\}$$

is a subalgebra of $\mathbf{T}_n^{\mathbf{T}_n^k}$ that contains the projection functions. Indeed, we have:

1. Projections functions are trivially in P.

2. If $f, g \in P$, then $f * g \in P$.

   Suppose that $(f*g)(\mathbf{x}) = y$, for some $y \in L_n \setminus \{1\}$. This is only possible if one of the following cases happens:

   (a) $f(\mathbf{x}) = y$ and $g(\mathbf{x}) \in D(\mathbf{T}_n)$. In view of the fact that f and g are in P, $f(\mathbf{x}') = y$ and $g(\mathbf{x}') \in D(\mathbf{T}_n)$. Consequently $(f * g)(\mathbf{x}') = y$ as we wanted.

   (b) $g(\mathbf{x}) = y$ and $f(\mathbf{x}) \in D(\mathbf{T}_n)$, is analogous to the case before.

   (c) $f(\mathbf{x}) = z$ and $g(\mathbf{x}) = z'$ for $z, z' \in L_n \setminus \{1\}$ such that $z * z' = y$. Then $f(\mathbf{x}') = z$, $g(\mathbf{x}') = z'$ and $(f * g)(\mathbf{x}') = y$.

   Therefore $(f * g)(\mathbf{x}) = y$ implies $(f * g)(\mathbf{x}') = y$. By symmetry, $(f * g)(\mathbf{x}') = y$ implies $(f * g)(\mathbf{x}) = y$, so we conclude that

   $$(f * g)(\mathbf{x}) = y \text{ iff } (f * g)(\mathbf{x}') = y.$$

3. If $f, g \in P$, then $f \to g \in P$.

   Suppose that $(f \to g)(\mathbf{x}) = y$, for $y \in L_n \setminus \{1\}$. This is only possible if $f(\mathbf{x}) > g(\mathbf{x})$ and if one of the following conditions is satisfied:

   (a) $f(\mathbf{x}) \in D(\mathbf{T}_n)$ and $g(\mathbf{x}) = y$. Then $g(\mathbf{x}') = y$, $f(\mathbf{x}') \in D(\mathbf{T}_n)$ and $(f \to g)(\mathbf{x}') = y$.

   (b) $f(\mathbf{x})$ and $g(\mathbf{x}) \in L_n \setminus \{1\}$ and $y = 1 - f(\mathbf{x}) + g(\mathbf{x})$. If this happens we have $f(\mathbf{x}') = f(\mathbf{x})$, $g(\mathbf{x}') = g(\mathbf{x})$ and $y = 1 - f(\mathbf{x}') + g(\mathbf{x}')$. It follows that $(f \to g)(\mathbf{x}') = y$.

   Interchanging $\mathbf{x}$ by $\mathbf{x}'$ in the above proof, the other implication is obtained. Therefore

   $$(f \to g)(\mathbf{x}) = y \text{ iff } (f \to g)(\mathbf{x}') = y$$

   which concludes the proof of the theorem.

■

Remark 5.2.4 Since $\mathbf{Free}_{\mathcal{V}}(k)$ is a subalgebra of $\mathbf{T}_n^{\mathbf{T}_n^k}$, an element f in $Free_{\mathcal{V}}(k)$ is boolean iff $f(\mathbf{z}) \in \{0, 1\}$ for each $\mathbf{z} \in (T_n)^k$.

Corollary 5.2.5 *Let $f \in B(\mathbf{Free}_{\mathcal{V}}(k))$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ in $(T_n)^k$. Suppose that there exists i such that $x_i = q \in D(\mathbf{T}_n)$. Let $p \in D(\mathbf{T}_n)$ and let $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_k)$ be a k-tuple such that*

$$x'_j = \begin{cases} x_j & \text{if } j \neq i, \\ p & \text{if } j = i. \end{cases}$$

Then $f(\mathbf{x}) = 1$ iff $f(\mathbf{x}') = 1$.

Proof: Since $f \in B(\mathbf{Free}_{\mathcal{V}}(k))$, from Remark 5.2.4 we know that for every $\mathbf{z} \in (T_n)^k$ either $f(\mathbf{z}) = 0$ or $f(\mathbf{z}) = 1$. It is straightforward from Lemma 5.2.3 that $f(\mathbf{x}) = 0$ iff $f(\mathbf{x}') = 0$. Then $f(\mathbf{x}) = 1$ iff $f(\mathbf{x}') = 1$. ∎

Our next aim is to characterize the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$. For that purpose, for each $y \in L_n$ we will define a function

$$h_y : \mathbf{T}_n \to \mathbf{T}_n$$

and we shall show that every atom in $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ is given by

$$g(\mathbf{x}) = \prod_{i=1}^{k} h_{y_i}(x_i)$$

where $y_i \in L_n$ for each $i = 1, 2, \ldots, k$ and $\prod_{i=1}^{k} h_{y_i}(x_i)$ is inductively defined as follows:

$$\prod_{i=1}^{1} h_{y_i}(x_i) = h_{y_1}(x_1) \quad \text{and} \quad \prod_{i=1}^{k+1} h_{y_i}(x_i) = (\prod_{i=1}^{k} h_{y_i}(x_i)) * h_{y_{k+1}}(x_{k+1}).$$

For each $i = 1, 2, \ldots, n-2$ and $j = 0, 1, \ldots n-1$, let $J_i : \mathbf{L}_n \to \{0, 1\}$ be as in Definition A.0.7, that is,

$$J_i(\frac{j}{n-1}) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

From A.0.7 we know that J_i is the interpretation of an MV-term. Since MV-terms are BL-terms, J_i can be extended to $\mathbf{T}_n$, and its extension $J'_i : \mathbf{T}_n \to \mathbf{T}_n$ is the interpretation of a BL-term on one variable.

We define h_y for each y in L_n in the following way:

$$h_y(x) = \begin{cases} (\neg x)^{n-1} & \text{if } y = 0, \\ \neg((x \vee \neg x)^{n-1}) * J'_i(x) & \text{if } y = \frac{i}{n-1} \text{ for } 0 < i < n-1, \\ \neg\neg(x^{n-1}) & \text{if } y = 1. \end{cases}$$

Then we have

- for $0 \leq i < n-1$,

$$h_{\frac{i}{n-1}}(x) = \begin{cases} 1 & \text{if } x = \frac{i}{n-1}, \\ 0 & \text{if } x \neq \frac{i}{n-1}. \end{cases}$$

$$h_1(x) = \begin{cases} 1 & \text{if } x \in D(\mathbf{T}_n), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.2.6 *Every atom g in $B(\mathbf{Free}_{\mathcal{V}}(k))$ has the form*

$$g(\mathbf{x}) = \prod_{i=1}^{k} h_{y_i}(x_i),$$

where $y_i \in L_n$ for each $i = 1, 2, \ldots, k$.

Proof: Let $g(\mathbf{x}) = \prod_{i=1}^{k} h_{y_i}(x_i)$ with $y_i \in L_n$. We will verify that g is an atom of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$. Observe that each h_y is an interpretation of a BL-term on $\mathbf{T}_n$, thus $g \in Free_{\mathcal{V}}(k)$ and, by Remark 5.2.4, g is a boolean element. Clearly $g > 0$. Let $f \in B(\mathbf{Free}_{\mathcal{V}}(k))$ and $f < g$. It remains to see that $f \equiv 0$ (i.e., f is the zero function). Considering the form of g, the possibilities are:

1. $y_i \neq 1$ for all $i = 1, 2, \ldots, k$. Letting $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ we have that

   $$g(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y}, \\ 0 & \text{otherwise.} \end{cases}$$

   Therefore $f(\mathbf{x}) = 0$ if $\mathbf{x} \neq \mathbf{y}$ and $f(\mathbf{y}) < 1$. By Remark 5.2.4, necessarily $f(\mathbf{y}) = 0$ that implies $f \equiv 0$.

2. $y_i = 1$ for some $i = 1, 2, \ldots, k$. This being the case, let p be the cardinality of the set $P = \{i : y_i = 1\}$ and α the cardinality of $D(\mathbf{T}_n)$. We may infer that there are α^p k-tuples $\mathbf{x}_s$ $(1 \leq s \leq \alpha^p)$ such that

   $$g(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathbf{z} = \mathbf{x}_s \text{ for some } 1 \leq s \leq \alpha^p, \\ 0 & \text{otherwise.} \end{cases}$$

   Indeed, if $y_i = 1$, then $h_{y_i}(a_j) = 1$ for each $a_j \in D(\mathbf{T}_n)$, $1 \leq j \leq \alpha$. Thus for each i such that $y_i = 1$, there are α different values x_i where $h_{y_i}(x_i) = 1$. Since there are p of these cases, the number of k-tuples where the atom g takes the value 1 is α^p. Notice that two k-tuples $\mathbf{x}_t$ and $\mathbf{x}_s$ differ in coordinate i only if $i \in P$.

   The only possibility for $f < g$ is that $f(\mathbf{z}) = 0$ for each $\mathbf{z} \neq \mathbf{x}_s$, $1 \leq s \leq \alpha^p$ and that $f(\mathbf{x}_s) = 0$ for some $1 \leq s \leq \alpha^p$. In this case, by Lemma 5.2.3, $f(\mathbf{x}_s) = 0$ for every $1 \leq s \leq \alpha^p$, thus $f \equiv 0$.

Thus we have proved that for each k-tuple $\mathbf{y} = (y_1, \ldots, y_k) \in L_n$ there exists an atom $g_\mathbf{y}$ such that $g_\mathbf{y} = \prod_{i=1}^k h_{y_i}$. We shall see that these are the only atoms of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(k))$. Suppose $g \in B(\mathbf{Free}_\mathcal{V}(k))$ and g is an atom. From Remark 5.2.4 we know that $g(\mathbf{z}) \in \{0,1\}$ for each $\mathbf{z} \in (T_n)^k$ and that $g > 0$. Hence there must exists $\mathbf{y} = (y_1, \ldots, y_k) \in (T_n)^k$ such that $g(\mathbf{y}) = 1$ and one of the following cases happens:

- Every coordinate of $\mathbf{y}$ is in $L_n \setminus \{1\}$. In this case $g \geq g_\mathbf{y}$. Since $g_\mathbf{y}$ and g are atoms the only possibility is that $g = g_\mathbf{y}$.

- There are $0 < p \leq k$ coordinates $j_1, \ldots j_p$ such that $y_i \notin L_n \setminus \{1\}$ for each $i = j_1, \ldots j_p$. Then $y_i \in D(\mathbf{T}_n)$ for each $i = j_1, \ldots j_p$. From Corollary 5.2.5 we know that $g(\mathbf{x}) = 1$ for each $\mathbf{x} = (x_1, \ldots, x_k)$ such that

  $$x_i = \begin{cases} y_i & \text{if } i \neq j_r \text{ for all } r = 1, \ldots, p, \\ q_r & \text{if } i = j_r, \text{ for some } r = 1, \ldots, p,\ q_r \in D(\mathbf{T}_n) \text{ and } q_r \neq y_{j_r}. \end{cases}$$

  Then $g \geq g_\mathbf{z}$ where $\mathbf{z}$ is the k-tuple given by

  $$z_i = \begin{cases} y_i & \text{if } i \neq j_r \text{ for all } r = 1, \ldots, k, \\ 1 & \text{if } i = j_r, \text{ for some } r = 1, \ldots, k. \end{cases}$$

  Once more we must have that $g = g_\mathbf{z}$.

Therefore we can conclude that all the atoms of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(k))$ are of the form $g_\mathbf{y} = \prod_{i=1}^k h_{y_i}$ for some $\mathbf{y} \in (L_n)^k$. ■

Remark 5.2.7 We recall that Theorem 4.3.4 asserts that $\mathbf{B}(\mathbf{Free}_\mathcal{V}(k))$ is $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(k))$. It is easy to check in [16, Theorem 8.6.1] that the number of atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(k))$ is n^k, hence this could be an alternative way to prove that the functions $g = \prod_{i=1}^k h_{y_i}$ with $y_i \in L_n$ are the only atoms of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(k))$.

Thus we have defined a bijection between $(\mathbf{L}_n)^k$ and the set of atoms of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(k))$ such that

$$\mathbf{y} = (y_1, y_2, \ldots, y_k) \mapsto g_\mathbf{y} = \prod_{i=1}^k h_{y_i}.$$

If $g_1, g_2, \ldots g_{n^k}$ is an enumeration of the atoms of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(k))$, Theorem 1.5.3 asserts that

$$\mathbf{Free}_{\mathcal{V}}(k) = \mathbf{F}_{g_1} \times \mathbf{F}_{g_2} \times \cdots \times \mathbf{F}_{g_{n^k}}$$

where $\mathbf{F}_g = \mathbf{Free}_{\mathcal{V}}(k)_g = ([0,g], *, \Rightarrow_g, 0, g)$. We will apply the previous results to describe $\mathbf{F}_g$ for every atom $g \in Free_{\mathcal{V}}(k)$.

We know from Theorem 4.5.5 that for each atom $g_{\mathbf{y}}$, the initial segment $\mathbf{F}_{g_{\mathbf{y}}}$ is a BL-algebra of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(p)$ for some $s-1$ that divides $n-1$, some $p \leq k$ and where $\mathcal{W}$ denotes the variety of generalized BL-algebras generated by $\mathbf{D}(\mathbf{T}_n) = \mathbf{B}$. In the next Theorem we shall give an alternative proof of this result that will specify for each atom $g_{\mathbf{y}}$ the constants p and s.

To accomplish this, let $g_{\mathbf{y}}$ be a fixed atom and

$$\mathbf{y} = (y_1, y_2 \ldots, y_k) = (\frac{b_1}{n-1}, \frac{b_2}{n-1}, \ldots, \frac{b_k}{n-1})$$

its corresponding associated k-tuple in $(L_n)^k$. We shall denote by $\mathbf{F}_{\mathbf{y}}$ the algebra $\mathbf{F}_{g_{\mathbf{y}}}$ to make explicit reference to the k-tuple $\mathbf{y}$ that characterizes the atom. Associated with the k-tuple $\mathbf{y}$ we define a pair of integers (p, d) where p is the cardinality of the set $P = \{i : y_i = 1\}$, and $d = \frac{n-1}{q}$ with $q = gcd\{b_1, b_2, \ldots b_k, n-1\}$. We shall call this pair of integers **the $g_{\mathbf{y}}$-pair**.

Proposition 5.2.8 *Let $g_{\mathbf{y}}$ be an atom of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ and (p,d) the $g_{\mathbf{y}}$-pair. Then*

$$\mathbf{F}_{\mathbf{y}} = \mathbf{L}_{d+1} \uplus \mathbf{Free}_{\mathcal{W}}(p)$$

where $\mathcal{W}$ is the variety of generalized BL-algebras generated by $\mathbf{D}(\mathbf{T}_n)$ and $\mathbf{Free}_{\mathcal{W}}(0) = \{\top\}$.

Proof: By Theorem 1.5.2 we know that the correspondence

$$f \mapsto f \wedge g_{\mathbf{y}}$$

defines a homomorphism from $\mathbf{Free}_{\mathcal{V}}(k)$ onto $\mathbf{F}_{\mathbf{y}}$. Since Theorem 5.2.1 asserts that $\mathbf{Free}_{\mathcal{V}}(k)$ is the subalgebra of $\mathbf{T}_n^{\mathbf{T}_n^k}$ generated by the projection functions $\pi_1, \pi_2, \ldots, \pi_k$, it follows that $\mathbf{F}_{\mathbf{y}}$ is the BL-algebra generated by the images under this homomorphism of the projections, that is the functions

$$p_i = \pi_i \wedge g_{\mathbf{y}}$$

for $i = 1, 2, \ldots, k$.

Recall that $\mathbf{y} = (y_1, y_2, \ldots y_{n-1}) = (\frac{b_1}{n-1}, \frac{b_2}{n-1} \ldots, \frac{b_k}{n-1})$. Let consider the set $G = \{\mathbf{x} \in (T_n)^k : g_{\mathbf{y}}(\mathbf{x}) = 1\}$. If $\mathbf{x} \in G$, then

$$x_i = \begin{cases} a \text{ for some } a \in D(\mathbf{T_n}) & \text{if } b_i = n-1 \\ \frac{b_i}{n-1} & \text{if } 0 \leq b_i < n-1. \end{cases}$$

Then

$$p_i(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin G \\ x_i & \text{if } \mathbf{x} \in G. \end{cases}$$

Let consider the sets

$$A = \{i : y_i \neq 1\} = \{i : b_i \neq n - 1\}$$

and

$$B = \{i : y_i = 1\} = \{i : b_i = n - 1\}.$$

We are going to consider two algebras:

- Let $\mathbf{A}'$ be the BL-algebra generated by all p_i such that $i \in A$ and $f \equiv 0$ (i.e., the zero function). For each $i \in A$,

  $$p_i(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin G, \\ \frac{b_i}{n-1} \text{ with } b_i < n - 1 & \text{if } \mathbf{x} \in G. \end{cases} \tag{5.1}$$

  Recall that $d = \frac{n-1}{q}$ with $q = gcd\{b_1, b_2, \ldots b_k, n - 1\}$. Clearly $q = gcd\{\{b_i : i \in A\} \cup \{n - 1\}\}$. Notice that $\mathbf{L}_{d+1}$ is the smallest subalgebra of $\mathbf{L}_n$ that contains $\frac{b_i}{n-1}$ for each $i = 1, 2, \ldots, k$. Indeed, there exists an integer c_i such that $b_i = c_i.q$. This implies that $d = \frac{c_i.(n-1)}{b_i}$ and $\frac{b_i}{n-1} = \frac{c_i}{d}$, thus $\frac{b_i}{n-1} \in L_{d+1}$. On the other hand, if there exists $t < d$ such that t divides $n - 1$ and $\frac{b_i}{n-1} \in L_{t+1}$ for each $i = 1, 2, \ldots, k$, then $\frac{b_i}{n-1} = \frac{b_i'}{t}$ for each $i = 1, 2, \ldots, k$. Therefore $q' = \frac{n-1}{t}$ divides b_i for each $i = 1, 2, \ldots, k$ and $q' > q$ contradicting the definition of q.

  Consequently, $\frac{b_i}{n-1} \in L_{d+1}$ for each $i = 1, 2, \ldots, k$ and the BL-algebra generated by the coordinates of the k-tuple $\mathbf{y}$ is $\mathbf{L}_{d+1}$. From the definition of d it is easy to see that, unless $\frac{b_i}{n-1} = 1$ for every $i = 1, \ldots, k$, $\mathbf{L}_{d+1}$ is indeed generated by the coordinates $\frac{b_i}{n-1}$ such that $\frac{b_i}{n-1} \neq 1$.

  From this we can conclude that the images of the functions p_i with $i \in A$, generate $\mathbf{L}_{d+1}$. Therefore the functions p_i generate $\mathbf{L}_{d+1}$, that is, for each z in L_{d+1} there is an interpretation of a BL-term q_z such that $z = q_z(\mathbf{y}) = q_z(p_1(\mathbf{y}), \ldots, p_k(\mathbf{y}))$. For each $z \in L_{d+1} \setminus \{1\}$ let $l_z = g_{\mathbf{y}} \wedge q_z$ and let l_1 be the atom $g_{\mathbf{y}}$. Then, for each $z \in L_{d+1}$ the function l_z belongs to $F_{\mathbf{y}}$ and because of Lemma 5.2.3

  $$l_z(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin G, \\ z & \text{if } \mathbf{x} \in G, \end{cases}$$

The correspondence $z \mapsto l_z$ gives an embedding from $\mathbf{L}_{d+1}$ into $\mathbf{A}'$. Now let $f \in A'$. Since $\mathbf{A}'$ is generated by the functions p_i with $i \in A$ and $f \equiv 0$, from equation (5.1) we can deduce that the images of f are in L_{d+1} and if $f \neq g_\mathbf{y}$, then there exists $z \in L_{d+1} \setminus \{1\}$ such that $f(\mathbf{x}) = z$ for some $\mathbf{x} \in G$. Once more Lemma 5.2.3 asserts that $f = l_z$, and we conclude that $\mathbf{A}' \cong \mathbf{L}_{d+1}$. Hence $\mathbf{F}_\mathbf{y}$ has a subalgebra $\mathbf{A}'$ isomorphic to $\mathbf{L}_{d+1}$.

Notice that if $A = \emptyset$, then $\mathbf{A}' = \mathbf{L}_2$, since it is the subalgebra of $\mathbf{F}_\mathbf{y}$ generated by the zero function. On the other hand, if $A \neq \emptyset$, the BL-algebra generated by p_i such that $i \in A$ necessarily contains the zero function. If $B = \emptyset$ it is clear that $\mathbf{F}_\mathbf{y} = \mathbf{A}'$.

- If $B \neq \emptyset$, let $\mathbf{B}'$ be the generalized BL-algebra generated by p_i with $i \in B$. In this case,

$$p_i(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin G, \\ a \in D(\mathbf{T}_n) & \text{if } \mathbf{x} \in G. \end{cases} \tag{5.2}$$

and we deduce that if $f \in B'$, then $f(\mathbf{x}) \in D(\mathbf{T}_n)$ for each $\mathbf{x} \in G$.

We set $i_1 = min\{i : \frac{b_i}{n-1} = 1\}$ and $i_t = min\{i : i \neq i_s \; \forall s < t$ and $\frac{b_i}{n-1} = 1\}$. Then $B = \{i_1, i_2, \ldots, i_p\}$.

By Theorem 5.2.1 we know that $\mathbf{Free}_{\mathcal{W}}(p)$ is the subalgebra of functions from $(\mathbf{D}(\mathbf{T}_n))^p$ in $\mathbf{D}(\mathbf{T}_n)$ generated by the projection functions π'_1, π'_2, $\ldots, \pi'_p$. Letting $t : (D(\mathbf{T}_n))^p \to G$ be given by $\mathbf{x} \mapsto \mathbf{x}'$ where

$$x'_i = \begin{cases} \frac{b_i}{n-1} & \text{if } i \notin B, \\ x_i & \text{if } i = i_s \text{ for some } s = 1, 2, \ldots, p. \end{cases}$$

we obtain a bijection from $(D(\mathbf{T}_n))^p$ onto G and for each $\mathbf{x} \in (D(\mathbf{T}_n))^p$ we have that $\pi'_s(\mathbf{x}) = p_{i_s}(t(\mathbf{x}))$, for $i_s \in B$.

Defining $\varphi : \mathbf{Free}_{\mathcal{W}}(p) \to \mathbf{B}'$ by

$$\varphi(\pi'_s) = p_{i_s},$$

for each $i_s \in B$ we have an isomorphism from $\mathbf{Free}_{\mathcal{W}}(p)$ onto $\mathbf{B}'$. Hence $\mathbf{F}_\mathbf{y}$ has as subalgebra the generalized BL-algebra $\mathbf{B}'$ isomorphic to $\mathbf{Free}_{\mathcal{W}}(p)$.

Clearly $\mathbf{A}'$ is a subalgebra of $\mathbf{MV}(\mathbf{F}_\mathbf{y})$. On the other hand, notice that if $f \neq g_\mathbf{y}$ and $f \in MV(\mathbf{F}_\mathbf{y})$, then $f(\mathbf{x}) = z$ for some $\mathbf{x} \in G$ and $z \in L_{d+1} \setminus \{1\}$.

From Lemma 5.2.3 we know that for any other $\mathbf{x}' \in G$ we have that $f(\mathbf{x}') = z$. Then $f = l_z$ is in A'. From this we obtain that

$$\mathbf{MV}(\mathbf{F_y}) = \mathbf{A}' \cong \mathbf{L}_{d+1}.$$

Then Theorem 2.2.2 asserts that

$$\mathbf{F_y} = \mathbf{L}_{d+1} \uplus \mathbf{D}(\mathbf{F_y}).$$

To obtain the desired result there is only left to prove that $\mathbf{D}(\mathbf{F_y}) = \mathbf{B}'$. In order to achieve such aim, it will be checked that, in fact, if $f \in F_{\mathbf{y}}$, then either $f \in A' \setminus \{g_{\mathbf{y}}\}$ or $f \in B'$. Notice that the only element in $A' \cap B'$ is the atom $g_{\mathbf{y}}$. For the reason that $\mathbf{F_y}$ is the BL-algebra generated by the functions p_i with $i = 1, \dots, k$, the result will be proved by induction on the complexity of f.

- If $f = p_i$ for some $i = 1, \dots, k$, since $i \in A \cup B$ we have that $f \in A' \setminus \{g_{\mathbf{y}}\}$ or $f \in B'$.
- If $f = f_1 * f_2$ we have the following possibilities:
  1. f_1 and $f_2 \in A' \setminus \{g_{\mathbf{y}}\}$ or f_1 and $f_2 \in B'$. Because $\mathbf{A}'$ and $\mathbf{B}'$ are closed by $*$, we have that $f \in A' \setminus \{g_{\mathbf{y}}\}$ or $f \in B'$.
  2. $f_1 \in A' \setminus \{g_{\mathbf{y}}\}$ and $f_2 \in B'$. In this case there exists $z \in L_n \setminus \{1\}$ such that $f_1 = l_z$ and $f_2(\mathbf{x}) \in D(\mathbf{T}_n)$ for each $\mathbf{x} \in G$. Since operations between these two functions are coordinatewise, we have that $f = f_1 \in A' \setminus \{g_{\mathbf{y}}\}$.
  3. $f_2 \in A' \setminus \{g_{\mathbf{y}}\}$ and $f_1 \in B'$. Commutativity of $*$ lead us to the previous case.
- If $f = f_1 \to f_2$ we have the following possibilities:
  1. f_1 and $f_2 \in A' \setminus \{g_{\mathbf{y}}\}$ or f_1 and $f_2 \in B'$. Recalling once more the fact that $\mathbf{A}'$ and $\mathbf{B}'$ are closed by $\to$, we conclude that $f \in A' \setminus \{g_{\mathbf{y}}\}$ or $f \in B'$.
  2. $f_1 \in A' \setminus \{g_{\mathbf{y}}\}$ and $f_2 \in B'$. Again there exists $z \in L_n \setminus \{1\}$ such that $f_1 = l_z$ and $f_2(\mathbf{x}) \in D(\mathbf{T}_n)$ for each $\mathbf{x} \in G$. Thus f is the atom $g_{\mathbf{y}}$, so is in B'.
  3. $f_2 \in A' \setminus \{g_{\mathbf{y}}\}$ and $f_1 \in B'$. In this case we have that $f = f_2$ is in $A' \setminus \{g_{\mathbf{y}}\}$.

Then

$$\mathbf{D}(\mathbf{F_y}) = \mathbf{B}' \cong \mathbf{Free}_{\mathcal{W}}(p),$$

and $\mathbf{F_y}$ is isomorphic to the ordinal sum $\mathbf{L}_{d+1} \uplus \mathbf{Free}_{\mathcal{W}}(p)$, that we denote $\mathbf{F^p_{d+1}}$.

Notice that $B = \emptyset$ iff $p = 0$. Then, if $B = \emptyset$, we obtain

$$\mathbf{F_y} = \mathbf{A}' = \mathbf{A}' \uplus \mathbf{Free}_{\mathcal{W}}(0).$$

When $A = \emptyset$, $\mathbf{F_y}$ is the BL-algebra generated by the functions p_i with $i \in B$. It is not hard to corroborate that this BL-algebra is the BL-algebra that arises from adjoining a bottom element to the generalized BL-algebra $\mathbf{B}' \cong \mathbf{Free}_{\mathcal{W}}(k)$. Thus $\mathbf{F_y} = \mathbf{L}_2 \uplus \mathbf{Free}_{\mathcal{W}}(k)$. Recall that in this case $\mathbf{A}' \cong \mathbf{L}_2$, then $\mathbf{F_y}$ is well defined. ∎

Thus we may conclude:

Theorem 5.2.9 *Let $g_1, \ldots, g_{n^k}$, be the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ and for each $i = 1, \ldots, n^k$ let (d_i, p_i) the g_i-pair. The free algebra $\mathbf{Free}_{\mathcal{V}}(k)$ with k generators in $\mathcal{V}$ is given by*

$$\mathbf{Free}_{\mathcal{V}}(k) = \prod_{i=1}^{n^k} \mathbf{F^{p_i}_{d_i+1}}.$$

where $\mathbf{F^{p_i}_{d_i+1}} = \mathbf{L}_{d_i+1} \uplus \mathbf{Free}_{\mathcal{W}}(p_i)$.

5.3 Remarks on the atoms

As it was mentioned in Lemma 4.5.2, the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ are in bijective correspondence with the upwards closed subsets of the poset $Y = \{\sigma_i^n(\neg\neg x_p) : x_p \in X, i = 1, \ldots, n-1\}$, where σ_i^n denotes the Moisil operators defined on the Appendix A. On the other hand, we have proved that the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ are in bijective correspondence with the elements of $\mathbf{L}_n^k$.

The aim of the present section is to settle a bijective correspondence between atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ and upwards closed subsets of $Y = \{\sigma_i^n(\neg\neg x_p) : x_p \in X, i = 1, \ldots, n-1\}$. To accomplish such aim, we must think of the free generators $x_1, \ldots, x_k$ of $\mathbf{Free}_{\mathcal{V}}(k)$ as a functions

$$x_p : \mathbf{T}_n^k \to \mathbf{T}_n.$$

Therefore we have the functions

$$\neg\neg x_p : \mathbf{T}_n^k \to \mathbf{L}_n,$$

and

$$\sigma_i^n(\neg\neg x_p) : \mathbf{T}_n^k \to \{0,1\}.$$

For each atom $\mathbf{y} \in L_n^k$ and for each $p = 1, \ldots, k$, let $s(p, \mathbf{y})$ be the numerator of $\neg\neg x_p(\mathbf{y})$. For each $i = 1, \ldots, n-1$, we have that

$$\sigma_i^n(\neg\neg x_p)(\mathbf{y}) = \begin{cases} 0 & \text{if } s(p,\mathbf{y}) < n - i; \\ 1 & \text{if } s(p,\mathbf{y}) \geq n - i. \end{cases}$$

For each atom $\mathbf{y} \in B(\mathbf{Free}_{\mathcal{V}}(k))$ and for each $p = 1, \ldots, k$, let $i_p = n - s(p, \mathbf{y})$. Then we have that

$$i_p = \begin{cases} \min \{i : \sigma_i^n(\neg\neg x_p)(\mathbf{y}) = 1\} & \text{if } \{i : \sigma_i^n(\neg\neg x_p)(\mathbf{y}) = 1\} \neq \emptyset; \\ n & \text{otherwise.} \end{cases}$$

Let

$$S_{\mathbf{y}} = \cup_{p=1}^k \cup_{i \geq i_p} \sigma_i^n(\neg\neg x_p).$$

Since for each atom $\mathbf{y} \in B(\mathbf{Free}_{\mathcal{V}}(k))$, the set $S_{\mathbf{y}}$ is an upwards closed subset of $Y = \{\sigma_i^n(\neg\neg x_p) : x_p \in X\}$, from Lemma 4.5.2 $U_{S_{\mathbf{y}}}$ would be the ultrafilter of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$ generated by the sets

$$\{\sigma_i^n(\neg\neg x_p) : \sigma_i^n(\neg\neg x_p) \in S_{\mathbf{y}}\} \text{ and } \{\neg\sigma_i^n(\neg\neg x_p) : \sigma_i^n(\neg\neg x_p) \notin S_{\mathbf{y}}\}.$$

In order to establish a bijection between the upwards closed subset of $Y = \{\sigma_i^n(\neg\neg x_p) : x_p \in X\}$ and the elements of L_n^k it is enough to prove that the atom $g_{\mathbf{y}}$ is in $U_{S_{\mathbf{y}}}$.

Setting $\sigma_0^n(\neg\neg x_p) \equiv 0$ and $\sigma_n^n(\neg\neg x_p) \equiv 1$ for all p, for each $p = 1, \ldots, k$, we define

$$J_{n-i_p}(\neg\neg x_p) = \sigma_{i_p}^n(\neg\neg x_p) \wedge \neg\sigma_{i_p - 1}^n(\neg\neg x_p).$$

Let $g : \mathbf{T}_n^k \to \{0,1\}$ be given by

$$g(\mathbf{x}) = \bigwedge_{p=1}^k J_{n-i_p}(\neg\neg x_p)(\mathbf{x}).$$

Clearly $g \in U_{S_{\mathbf{y}}}$. We are going to prove that this function g is the atom $g_{\mathbf{y}}$. It is clear that $g(\mathbf{y}) = 1$. Besides, if $\mathbf{x} \in L_n^k$ and $\mathbf{x} \neq \mathbf{y}$, then there exists $p \in \{1, \ldots, k\}$ such that $\neg\neg x_p(\mathbf{x}) \neq \neg\neg x_p(\mathbf{y})$. Indeed, since $\mathbf{x} \in L_n^k$, then $\neg\neg x_p(\mathbf{x}) = \neg\neg x_p(\mathbf{y})$ for each $p = 1, \ldots, k$, would imply that $x_p(\mathbf{x}) = x_p(\mathbf{y})$ for each p. Since the atoms are generated by these functions x_p, $p = 1, \ldots, k$, then $g_{\mathbf{x}}(\mathbf{x}) = g_{\mathbf{y}}(\mathbf{x})$, contradicting Theorem 5.2.6. Therefore, for each $\mathbf{x} \in L_n^k$, such that $\mathbf{x} \neq \mathbf{y}$, we have that $g(\mathbf{x}) = 0$. Let consider $\mathbf{x} \in T_n^k$. Let $\mathbf{x}'$ be given by

$$x'_j = \begin{cases} x_j & \text{if } x_j \in L_n \setminus \{1\}; \\ 1 & \text{if } x_j \in D(\mathbf{T}_n). \end{cases}$$

Then $\mathbf{x}' \in L_n^k$ and from Theorem 5.2.5 we know that $g(\mathbf{x}) = g(\mathbf{x}')$. Consequently we have that, if $\mathbf{x}' = \mathbf{y}$, then $g(\mathbf{x}) = 1$, otherwise $g(\mathbf{x}) = 0$. From this we can conclude that $g = g_{\mathbf{y}}$.

5.4 Examples

$\mathbf{L}_2 \uplus \mathbf{B}$

Consider $\mathbf{T} = \mathbf{L_2} \uplus \mathbf{B}$ for any generalized BL-chain $\mathbf{B}$, and let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{T}$. Following Proposition 5.2.8 we obtain that for any integer k

$$\mathbf{Free}_{\mathcal{V}}(k) = \prod_{p=0}^{k} (\mathbf{F_2^p})^{\binom{k}{p}}.$$

$\mathbf{L}_3 \uplus \mathbf{B}$

Now let $\mathbf{T} = \mathbf{L_3} \uplus \mathbf{B}$ for any generalized BL-chain $\mathbf{B}$, and let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{T}$. By Proposition 5.2.8 we have

$$\mathbf{Free}_{\mathcal{V}}(k) = \mathbf{L_2} \times \mathbf{L_3}^{(2^k-1)} \times \prod_{p=1}^{k} (\mathbf{F_2^p})^{\binom{k}{p}} \times \prod_{p=1}^{k-1} (\mathbf{F_3^p})^{\binom{k}{p} \sum_{i=1}^{k-p} \binom{k-p}{i}}$$

$$= \prod_{p=0}^{k} (\mathbf{F_2^p})^{\binom{k}{p}} \times \prod_{p=0}^{k-1} (\mathbf{F_3^p})^{\binom{k}{p} \sum_{i=1}^{k-p} \binom{k-p}{i}}.$$

$\mathbf{L}_5 \uplus \mathbf{B}$

Let

- $\alpha_1 = \binom{k}{i}(2^{k-i} - 1)$,
- $\alpha_2 = \binom{k}{i}(3^{k-i} - 2.2^{k-i} + 1)$,
- $\alpha_3 = \binom{k}{i}(4^{k-i} - 3.3^{k-i} + 3.2^{k-i} - 1)$.

If $\mathbf{T} = \mathbf{L_5} \uplus \mathbf{B}$ for any generalized BL-chain $\mathbf{B}$, and $\mathcal{V}$ is the variety of BL-algebras generated by $\mathbf{T}$, we obtain

$$\mathbf{Free}_{\mathcal{V}}(k) \cong \mathbf{L_2} \times \mathbf{L_3}^{2^k-1} \times \mathbf{L_5}^{4^k-2^k} \times \prod_{i=1}^{k} (\mathbf{F^i_2})^{\binom{k}{i}} \times (\prod_{i=1}^{k-1} (\mathbf{F^i_5})^{\alpha_1})^2 \times$$

$$\times (\prod_{i=1}^{k-2} (\mathbf{F^i_5})^{\alpha_2})^3 \times \prod_{i=1}^{k-1} (\mathbf{F^i_3})^{\alpha_1} \times (\prod_{i=1}^{k-3} (\mathbf{F^i_5})^{\alpha_3}.$$

$\mathbf{L}_n \uplus \mathbf{B}$ (when $n-1 > 2$ is prime).

Let $\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}$ for any $n - 1 \geq 2$ and $\mathcal{V}$ the variety generated by $\mathbf{T}_n$.

For any integer $k > 0$, let denote $t = min(k, n-2)$. We define

$$p_1 = \sum_{i_1=1}^{k} \binom{k}{i_1},$$

for $r = 2, 3, \ldots, t$,

$$p_r = \sum_{i_1=1}^{k_1} \binom{k}{i_1} (\sum_{i_2=1}^{k_2} \binom{k-i_1}{i_2} (\ldots (\sum_{i_r=1}^{k_r} \binom{k-i_1-\ldots-i_{r-1}}{i_r})))$$

where $k_1 = k - (r-1)$ and for $j = 2, 3, \ldots, r$,

$$k_j = k - i_1 - i_2 - \ldots - i_{j-1} - (r-j).$$

For $i = 1, 2, \ldots, k$ and $r_i = 1, 2, \ldots, (t-i)$ we define $t_i = min(k-i, n-2)$ and

$$p'_{r_i} = \sum_{i_1=1}^{k_{i_1}} \binom{k-i}{i_1} (\sum_{i_2=1}^{k_{i_2}} \binom{k-i_1}{i_2} \ldots (\sum_{i_{r_i}=1}^{k_{i_{r_i}}} \binom{k-i_1-\ldots-i_{r_i-1}}{i_{r_i}})))$$

where $k_{i_1} = k - i - (r_i - 1)$ and for $j = 2, 3, \ldots$

$$k_{i_j} = k - i - i_1 - i_2 - \ldots - i_{j-1} - (r_i - j).$$

Then

$$\mathbf{Free}_{\mathcal{V}}(k) \cong \mathbf{L_2} \times \prod_{r=1}^{t} (\mathbf{L}_n^{p_r})^{\binom{n-2}{r}} \times \prod_{r=1}^{k} ((\mathbf{F^r_2})^{\binom{k}{r}}) \times \prod_{i=1}^{k-1} (\prod_{r_i=1}^{t_i} ((\mathbf{F^i_n})^{\binom{k}{i} p'_{r_i}})^{\binom{n-2}{r_i}}).$$

$\mathbf{L}_n \uplus \{\top\}$

In this case we have that $\mathbf{T}_n = \mathbf{L}_n$, thus $\mathcal{V} = \mathcal{MV}_n$. Since the variety $\mathcal{W}$ of generalized BL-algebras generated by $\{\top\}$ has as only element this trivial algebra, $\mathbf{Free}_{\mathcal{W}}(p) = \{\top\}$ for each integer p. Thus $\mathbf{Free}_{\mathcal{MV}_n}(k)$ is a direct product of algebras of the form $\mathbf{L}_s$ with s that divides $n-1$. A description of $\mathbf{Free}_{\mathcal{MV}_n}(k)$ is done in [16, Theorem 8.6.1]. The reader can verify that such description coincides with the one obtained by applying Proposition 5.2.8.

$\mathbf{L}_n \uplus \mathbf{H_m}$

Given $m \geq 2$ and $n \geq 2$, we define the BL-algebra $\mathbf{T_n^m}$ as the ordinal sum of the MV-chain $\mathbf{L}_n$ and the Heyting chain $\mathbf{H_m}$. The free algebra $\mathbf{Free}_{\mathcal{V}}(k)$ for finite k, is described in Proposition 5.2.8. We have that if g_y is an atom in $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$, then $\mathbf{F_y} \cong \mathbf{L}_{d+1} \uplus \mathbf{Free}_{\mathcal{W}}(p) = \mathbf{F^p_{d+1}}$ for d dividing $n-1$, $p \leq k$ and $\mathcal{W}$ the variety of generalized BL-algebras generated by $\mathbf{D}(\mathbf{T_n^m})$. A description of $\mathbf{Free}_{\mathcal{W}}(p)$ is given in [23, Theorem 5.3] in terms of the generalized BL-chains $\mathbf{H_j}$, $j \leq m$. Therefore $\mathbf{Free}_{\mathcal{V}}(k)$ is completely characterized in terms of Lukasiewicz finite chains and the generalized BL-chains of the form $\mathbf{H_j}$.

$\mathbf{L}_n \uplus \mathbf{L}_m$

We define the BL-algebra $\mathbf{L_n^m}$ as the ordinal sum of $\mathbf{L}_n$ and $\mathbf{L}_m$, that is, $\mathbf{L_n^m} = \mathbf{L}_n \uplus \mathbf{L}_m$ for $m \geq 2$ and $n \geq 2$. In this case, since we want to distinguish between the BL-algebra and the generalized BL-algebra, we shall denote $\mathbf{L}_m$ the BL-algebra and $\mathbf{L}'_m$ its corresponding generalized BL-algebra. Therefore $\mathbf{D}(\mathbf{L_n^m}) \cong \mathbf{L}'_m$.

From Proposition 5.2.8, we know that for each atom g_y in $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(k))$, the algebra $\mathbf{F_y}$ is isomorphic to $\mathbf{L}_{d+1} \uplus \mathbf{Free}_{\mathcal{W}}(p)$ for d dividing $n-1$, $p \leq k$ and $\mathcal{W}$ the variety of generalized BL-algebras generated by $\mathbf{L}'_m$.

For each $c \geq 2$ let $\mathbf{Div}(c)$ be the set of divisors of c and $\mathbf{MaxDiv}(c)$ the set of proper and maximal divisors of c, i.e., $d \in \mathbf{Div}(c)$ such that $d \neq c$ and if d divides b and $b \in \mathbf{Div}(c)$, then $b = c$. If $c = 1$ let $\mathbf{MaxDiv}(c) = \emptyset$. For $X \subseteq \mathbf{MaxDiv}(d)$, if $X \neq \emptyset$ we call $gcd(X)$ the greatest common divisor of the elements of X, and we stipulate that $gcd(\emptyset) = d$.

Lemma 5.4.1 *For any two integers $m \geq 2$ and $p \geq 1$, we have that*

$$\mathbf{Free}_{\mathcal{W}}(p) = \prod_{s \in \mathbf{Div}(m-1)} (\mathbf{L}'_{s+1})^{\alpha(m,p,s)}$$

where $\alpha(m,p,s) = \sum_{X \subseteq \mathbf{MaxDiv}(s)} (-1)^{\sharp X} (gcd(X)+1)^p$ *for* $s > 1$ *and* $\alpha(m,p,1) = 2^p - 1$.

Proof: From the development done in [16, Chapter 8.6], we have that

$$\mathbf{Free}_{\mathcal{MV}_m}(p) = \mathbf{S_1} \times \mathbf{S_2} \times \cdots \times \mathbf{S_{m^p}},$$

where each $\mathbf{S_i}$ is the subalgebra of $\mathbf{L}_m$ generated by the elements $e_i(y_j) = \frac{a_{ij}}{(m-1)} \in L_m$ for $j = 1,2,\ldots,p$. Therefore

$$\mathbf{Free}_{\mathcal{W}}(p) = \mathbf{S'_1} \times \mathbf{S'_2} \times \cdots \times \mathbf{S'_{m^p}},$$

and now each $\mathbf{S'_i}$ is the generalized BL-algebra generated by the elements $e_i(y_j) = \frac{a_{ij}}{(m-1)}$ for $j = 1,2,\ldots,p$. Notice that if $a_{ij} \neq 1$ for some $j = 1,2,\ldots,p$, the generalized BL-algebra $\mathbf{S'_i}$ is the BL-algebra $\mathbf{S_i}$ whithout considering 0 as a constant. But if $a_{ij} = 1$ for each $j = 1,2,\ldots,p$, then $\mathbf{S_i} = \mathbf{L_2}$ and $\mathbf{S'_i} = \{1\}$. Thus the theorem follows from [16, Theorem 8.6.1]. ■

Consequently, $\mathbf{Free}_{\mathcal{V}}(k)$ can be completely described in terms of subalgebras of $\mathbf{L}_n$ and $\mathbf{L_m}'$.

Example 5.4.2 Let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{L_3^3}$ and $\mathcal{W}$ the variety of generalized BL-algebras generated by $\mathbf{L_3}'$. Then for any integer $k \geq 1$ we have that

$$\mathbf{Free}_{\mathcal{V}}(k) = \mathbf{L_2} \times \prod_{i=1}^{k} \mathbf{L_3}^{\binom{k}{i}} \times \prod_{i=1}^{k} (\mathbf{L_2} \uplus \mathbf{Free}_{\mathcal{W}}(i))^{\binom{k}{i}} \times \prod_{i=1}^{k-1} (\mathbf{L_3} \uplus \mathbf{Free}_{\mathcal{W}}(i))^{\alpha_i},$$

where $\alpha_i = \binom{k}{i} \sum_{j=1}^{k-i} \binom{k-i}{j}$.Using the previous lemma we have that

$$\mathbf{Free}_{\mathcal{V}}(k) = \mathbf{L_2} \times \prod_{i=1}^{k} \mathbf{L_3}^{\binom{k}{i}} \times \prod_{i=1}^{k} (\mathbf{L_2} \uplus \prod_{s \in \mathbf{Div}(m-1)} (\mathbf{L}'_{s+1})^{\alpha(m,p,s)})^{\binom{k}{i}} \times$$

$$\times \prod_{i=1}^{k-1} (\mathbf{L_3} \uplus \prod_{s \in \mathbf{Div}(m-1)} (\mathbf{L}'_{s+1})^{\alpha(m,p,s)})^{\alpha_i}.$$

PL-algebras

We recall from the last section of the previous chapter that the variety of PL-algebras $\mathcal{PL}$ is generated by $\mathbf{L_2} \uplus \mathbf{P(Z)}$, where $\mathbf{P(Z)}$ is the generalized BL-chain defined in chapter two. In this case, from Proposition 5.2.8, we obtain:

$$\mathbf{Free}_{\mathcal{PL}}(k) = \prod_{p=0}^{k} (\mathbf{L_2} \uplus \mathbf{Free}_{\mathcal{W}}(p))^{\binom{k}{p}} = \mathbf{L_2} \times \prod_{p=1}^{k} (\mathbf{L_2} \uplus \mathbf{Free}_{\mathcal{W}}(p))^{\binom{k}{p}},$$

where $\mathcal{W}$ is the variety of generalized BL-algebras generated by $\mathbf{P(Z)}$ (i.e, the variety of cancellative hoops, see Proposition 2.1.8).

Since a description of $\mathbf{Free}_{\mathcal{PL}}(k)$ is given in [19] we can compare our result with the one in that paper. Let $\mathbf{G}$ be an ℓ-group, and $G^- = \{x \in G : x \leq 0\}$ its negative cone. Let $\perp$ be an element not belonging to G. In [19], the authors define on the set $G^- \cup \{\perp\}$ the binary operations $*$ and $\rightarrow$ as follows:

$$x * y = \begin{cases} x + y & \text{if } x, y \in G^-, \\ \perp & \text{otherwise.} \end{cases}$$

$$x \rightarrow y = \begin{cases} 0 \wedge (y - x) & \text{if } x, y \in G^-, \\ 0 & \text{if } x = \perp, \\ \perp & \text{if } x \in G^- \text{ and } y = \perp. \end{cases}$$

and they obtain a PL-algebra $(G^- \cup \{\perp\}, *, \rightarrow, \perp, 0)$, that they denote by $\mathbf{B(G)}$. Then they describe the free PL-algebra with k generators by

$$\mathbf{Free}_{\mathcal{PL}}(k) = \prod_{p=0}^{k} (\mathbf{B(G_p)})^{\binom{k}{p}},$$

where $G_0 = \{0\}$ and for any $p > 0$, $\mathbf{G_p}$ is an ℓ-group such that $G_p = S_p$ and S_p is the smallest set of functions $f : (\mathbf{P(Z)})^p \rightarrow \mathbf{P(Z)}$ that contains the projections functions $\pi_1, \pi_2, \ldots, \pi_p$ and is closed under addition and truncated subtraction:

$$(f \ominus g)(\mathbf{x}) = min(0, g(\mathbf{x}) - f(\mathbf{x})).$$

With the definitions of chapter two in the present thesis, it is clear that $\mathbf{B(G)} = \mathbf{L_2} \uplus \mathbf{P(G)}$. From the definition of G_0 we conclude that

$$\mathbf{Free}_{\mathcal{PL}}(k) = \mathbf{L_2} \times \prod_{p=1}^{k} (\mathbf{B(G_p)})^{\binom{k}{p}}.$$

By Theorem 5.2.1, $\mathbf{Free}_{\mathcal{W}}(p)$ is the generalized subalgebra of functions from $(\mathbf{P}(\mathbf{Z}))^p$ into $\mathbf{P}(\mathbf{Z})$ generated by the projection functions. It is easy to see that $\mathbf{S_p} = (S_p, +, \ominus, 0)$ is a generalized BL-algebra and that $\mathbf{Free}_{\mathcal{W}}(p) \cong \mathbf{S_p}$. Moreover, $\mathbf{L_2} \uplus \mathbf{Free}_{\mathcal{W}}(p) \cong \mathbf{B}(\mathbf{G_p})$. Therefore both descriptions coincide.

Appendix A

Moisil algebras and boolean elements in free MV_n-algebras

Definition A.0.3 *For each integer $n \geq 2$, an* **n-valued Moisil algebra** *([8] and [12]) or* **n-valued Lukasiewicz algebra** *([4], [13] and [14]) is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$ of type $(2, 2, 1, \ldots, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a distributive lattice with unit 1 and zero 0, and $\neg$, $\sigma_1^n, \ldots, \sigma_{n-1}^n$ are unary operators defined on A that satisfy the following conditions:*

1. $\neg\neg x = x$,
2. $\neg(x \vee y) = \neg x \wedge \neg y$,
3. $\sigma_i^n(x \vee y) = \sigma_i^n x \vee \sigma_i^n y$,
4. $\sigma_i^n x \vee \neg\sigma_i^n x = 1$,
5. $\sigma_i^n \sigma_j^n x = \sigma_j^n x$, *for* $i, j = 1, 2, \ldots n-1$,
6. $\sigma_i^n(\neg x) = \neg(\sigma_{n-i}^n x)$,
7. $\sigma_i^n x \vee \sigma_{i+1}^n x = \sigma_{i+1}^n x$, *for* $i = 1, 2, \ldots, n-2$,
8. $x \vee \sigma_{n-1}^n x = \sigma_{n-1}^n x$,
9. $(x \wedge \neg\sigma_i^n x \wedge \sigma_{i+1}^n y) \vee y = y$, *for* $i = 1, 2, \ldots, n-2$.

Properties and examples of n-valued Moisil algebras can be found in [4] and in [8]. The variety of n-valued Moisil algebras will be denoted $\mathcal{M}_n$. An important property of n-valued Moisil algebras is the following:

Moisil's determination principle: Let $\mathbf{A} \in \mathcal{M}_n$ and let $x, y \in A$. Then $x \leq y$ if and only if $\sigma_i^n x \leq \sigma_i^n y$ for each $i = 1, \ldots, n-1$.

We also have that:

Theorem A.0.4 *(see [12]) Let* $\mathbf{A}$ *be in* $\mathcal{M}_n$. *Then* $x \in B(\mathbf{A})$ *if and only if* $\sigma_{n-1}^n(x) = x$. *Furthermore,*

$$\sigma_{n-1}^n(x) = \min\{b \in B(\mathbf{A}) : x \leq b\} \text{ and } \sigma_1^n(x) = \max\{a \in B(\mathbf{A}) : a \leq x\}.$$

Definition A.0.5 *For each integer* $n \geq 2$, **Post algebra of order** n *is a system*

$$\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, e_1, \ldots, e_{n-1}, 0, 1)$$

such that $(A, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$ *is an n-valued Moisil algebra and* $e_1, \ldots, e_{n-1}$ *are constants that satisfy the following equations:*

$$\sigma_i^n(e_j) = \begin{cases} 0 & \text{if } i + j < n; \\ 1 & \text{if } i + j \geq n. \end{cases}$$

For every $n \geq 2$ we can define one-variable terms $\sigma_1^n(x), \ldots, \sigma_{n-1}^n(x)$ in the language $(\neg, \rightarrow, \top)$ such that evaluated on the algebras $\mathbf{L}_n$ give:

$$\sigma_i^n\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n, \end{cases}$$

for $i = 1, \ldots, n-1$ (see [14] or [35]) . It is easy to check that

$$\mathbf{M}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$$

is a n-valued Moisil algebra. Since these algebras are defined by equations and $\mathbf{L}_n$ generates the variety $\mathcal{MV}_n$, we have that each $\mathbf{A} \in \mathcal{MV}_n$ admits a structure of an n-valued Moisil algebra, denoted by $\mathbf{M}(\mathbf{A})$. The chain $\mathbf{M}(\mathbf{L}_n)$ plays a very important role in the structure of n-valued Moisil algebras, since each n-valued Moisil algebra is a subdirect product of subalgebras of $\mathbf{M}(\mathbf{L}_n)$ (see [4] or [13]). If we add to the structure $\mathbf{M}(\mathbf{L}_n)$ the constants

$$e_i = \frac{i}{n-1},$$

for $i = 1, \ldots, n-1$, then $\mathbf{PT}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, e_1, \ldots, e_{n-1}, 0, 1)$ is a Post algebra.

Not every n-valued Moisil algebra has a structure of MV_n-algebra (see [32]). For example, a subalgebra of $\mathbf{M}(\mathbf{L}_n)$ may not be a subalgebra of $\mathbf{L}_n$ as MV_n-algebra. That is the case of the algebra whose universe is the set

$$C = (\frac{0}{4}, \frac{1}{4}, \frac{3}{4}, \frac{4}{4})$$

which is a subalgebra of $\mathbf{M}(\mathbf{L_5})$, but not a subalgebra of $\mathbf{L_5}$. On the other hand, every Post algebra has a structure of MV_n-algebra (see [35, Theorem 10]).

The next example will play an important role in what follows:

Example A.0.6 Let $\mathbf{C} = (C, \wedge, \vee, \neg, 0, 1)$ be a boolean algebra. We define

$$C^{[n]} = \{\mathbf{z} = (z_1, \ldots, z_{n-1}) \in C^{n-1} : z_1 \leq z_2 \leq \ldots \leq z_{n-1}\}$$

For each $\mathbf{z} = (z_1, \ldots, z_{n-1}) \in C^{[n]}$ we define:

$$\neg_n \mathbf{z} = (\neg z_{n-1}, \ldots \neg z_1),$$

$$\mathbf{0} = (0, \ldots, 0),$$

$$\mathbf{1} = (1, \ldots, 1),$$

$$\sigma_i^n(\mathbf{z}) = (z_i, z_i, \ldots, z_i) \text{ for } i = 1, \ldots, n-1.$$

With $\wedge$ and $\vee$ defined coordinatewise, $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \ldots, \sigma_{n-1}^n, \mathbf{0}, \mathbf{1})$ is an n-valued Moisil algebra (see [8, Chapter 3, Example 1.10]). If we define $\mathbf{e_j} = (e_{j,1}, \ldots, e_{j,n-1})$ by

$$e_{j,i} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i \geq j, \end{cases}$$

then $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \ldots, \sigma_{n-1}^n, \mathbf{e_1}, \ldots, \mathbf{e_{n-1}}, \mathbf{0}, \mathbf{1})$ is a Post algebra. Consequently, $\mathbf{C}^{[n]}$ has a structure of MV_n-algebra.

It is easy to see that for each MV_n-algebra $\mathbf{A}$, the boolean subalgebras $\mathbf{B}(\mathbf{A})$ and $\mathbf{B}(\mathbf{M}(\mathbf{A}))$ are the same. We need to show that the boolean elements of the MV_n-algebra generated by a set G coincide with the boolean elements of the n-valued Moisil algebra generated by the same set. In order to prove this result it is convenient to consider the following operators on each n-valued Moisil algebra $\mathbf{A}$:

Definition A.0.7 *For each* $i = 0, \ldots, n-1$

$$J_i(x) = \sigma_{n-i}^n(x) \wedge \neg\sigma_{n-i-1}^n(x),$$

where $\sigma_0^n(x) = 0$ *and* $\sigma_1^n(x) = 1$.

Notice that in $\mathbf{M}(\mathbf{L}_n)$ we have:

$$J_i(\frac{j}{(n-1)}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Lemma A.0.8 *Let* $\mathbf{A}$ *be an* MV_n*-algebra, and let* $G \subset A$. *If* $< G >_{\mathcal{MV}_n}$ *is the subalgebra of* $\mathbf{A}$ *generated by the set* G *and* $< G >_{\mathcal{M}_n}$ *is the subalgebra of* $\mathbf{M}(\mathbf{A})$ *generated by* G, *then*

$$\mathbf{B}(< G >_{\mathcal{MV}_n}) = \mathbf{B}(< G >_{\mathcal{M}_n}).$$

Proof: Since $< G >_{\mathcal{M}_n}$ is always a subalgebra of $\mathbf{M}(< G >_{\mathcal{MV}_n})$, we have that $\mathbf{B}(< G >_{\mathcal{M}_n})$ is a subalgebra of $\mathbf{B}(< G >_{\mathcal{MV}_n})$.

We will see that $B(< G >_{\mathcal{MV}_n}) \subseteq B(< G >_{\mathcal{M}_n})$. The case $G = \emptyset$ is clear. Suppose that G is a finite set of cardinality $p \geq 1$. Since MV_n-algebras are locally finite (see [10, Chapter II, Theorem 10.16]), we obtain that $< G >_{\mathcal{MV}_n}$ is a finite MV_n-algebra. Since finite MV_n-algebras are direct product of simple algebras, there exists a finite $k \geq 1$ such that

$$< G >_{\mathcal{MV}_n} = \prod_{i=1}^{k} \mathbf{L}_{m_i},$$

where each $m_i - 1$ divides $n - 1$, for each $i = 1, \ldots, k$. If $k = 1$, then $< G >_{\mathcal{M}_n}$ and $< G >_{\mathcal{MV}_n}$ are finite chains whose only boolean elements are their extremes. Otherwise, we can think of the elements of $< G >_{\mathcal{MV}_n}$ as k-tuples, i.e., if $\mathbf{x} \in < G >_{\mathcal{MV}_n}$, then $\mathbf{x} = (x_1, \ldots, x_k)$. We shall denote by $\mathbf{1}^j$ the k-tuple given by

$$(\mathbf{1}^j)_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is clear that for each $j = 1, \ldots, k$, $\mathbf{1}^j$ is in $< G >_{\mathcal{MV}_n}$. From this fact follows that for every pair $i \neq j$, $i, j \in \{1, \ldots, k\}$, there exists an element $\mathbf{x} \in G$ such that $x_j \neq x_i$. Indeed, suppose on the contrary that there exist $i, j \leq k$ such that $x_i = x_j$, for every $\mathbf{x} \in G$. Since these elements generate $< G >_{\mathcal{MV}_n}$, for every $\mathbf{z} \in < G >_{\mathcal{MV}_n}$ we would have $z_i = z_j$ contradicting the fact that $\mathbf{1}^j$ is in $< G >_{\mathcal{MV}_n}$.

It is obvious that in order to see that every boolean element in $< G >_{\mathcal{MV}_n}$ is in $< G >_{\mathcal{M}_n}$ it is enough to prove that $\mathbf{1}^j$ is in $< G >_{\mathcal{M}_n}$ for every $j = 1, \ldots, k$. For a fixed j, for each $i \neq j$, $i = 1, \ldots k$, we choose $\mathbf{x}^i \in G$ such that $x^i_j \neq x^i_i$. Let j_i be the numerator of $x^i_j \in L_n$. It is not hard to verify that

$$\mathbf{1}^j = \bigwedge_{i=1, i \neq j}^{k} J_{j_i}(\mathbf{x}^i).$$

From the definition of the operators J_i, we conclude that $\mathbf{1}^j$ must be in $< G >_{\mathcal{M}_n}$. Hence $B(< G >_{\mathcal{MV}_n}) \subseteq B(< G >_{\mathcal{M}_n})$.

If G is not finite, let $\mathbf{y}$ be a boolean element in $< G >_{\mathcal{MV}_n}$. Hence, there exists a finite subset $G_{\mathbf{y}}$ of G such that $\mathbf{y}$ belongs to the subalgebra of $< G >_{\mathcal{MV}_n}$ generated by $G_{\mathbf{y}}$. Therefore, since $\mathbf{y}$ is boolean, $\mathbf{y}$ belongs to the subalgebra of $< G >_{\mathcal{M}_n}$ generated by $G_{\mathbf{y}}$, and we conclude that

$$B(< G >_{\mathcal{MV}_n}) \subseteq B(< G >_{\mathcal{M}_n})$$

for all sets G. ■

Given an algebra $\mathbf{A}$ in a variety $\mathcal{K}$, a subalgebra $\mathbf{S}$ of $\mathbf{A}$, and an element $x \in A$, we shall denote by $< \mathbf{S}, x >_{\mathcal{K}}$ the subalgebra of $\mathbf{A}$ generated by the set $S \cup \{x\}$ in $\mathcal{K}$.

Lemma A.0.9 *Let* $\mathbf{C}$ *be in* $\mathcal{M}_n$ *and* $x \in C$. *Let* $\mathbf{S}$ *be a subalgebra of* $\mathbf{C}$ *such that* $\sigma_i^n(x)$ *belongs to* $B(\mathbf{S})$ *for each* $i = 1, \ldots, n-1$. *Then*

$$\mathbf{B}(< \mathbf{S}, x >_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S}).$$

Proof: Clearly $\mathbf{B}(\mathbf{S})$ is a subalgebra of $\mathbf{B}(< \mathbf{S}, x >_{\mathcal{M}_n})$, then it is left to check that $B(< \mathbf{S}, x >_{\mathcal{M}_n}) \subseteq B(\mathbf{S})$. To achieve such aim, we shall study the form of the elements in $< \mathbf{S}, x >_{\mathcal{M}_n}$. We define for each $s \in S$

- $\alpha(s) = s \wedge x$,
- $\beta(s) = s \wedge \neg x$,
- $\gamma_i(s) = s \wedge \sigma_i^n(x)$, for $i = 1, \ldots n-1$,
- $\delta_i(s) = s \wedge \neg\sigma_i^n(x)$, for $i = 1, \ldots n-1$.

Notice that for all $s \in S$ we have that $\gamma_i(s)$ and $\delta_i(s)$ are in S for $i = 1, \ldots, n-1$. Let

$$M = \{y = \bigvee_{j=1}^{k_y} \bigwedge_{i=1}^{p_j} f_i(s_i) : f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \ldots \gamma_{n-1}, \delta_{n-1}\} \text{ and } s_i \in S\}.$$

We shall see that $< \mathbf{S}, x >_{\mathcal{M}_n} = \mathbf{M} = (M, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$. Indeed, for all $s \in S$, $s = \gamma_1(s) \vee \delta_1(s)$, then $S \subseteq M$. Besides, $x \in M$ because $x = \alpha(1)$. Lastly, it is easy to see that M is closed under the operations of n-valued Moisil algebra, thus $< \mathbf{S}, x >_{\mathcal{M}_n}$ is a subalgebra of $\mathbf{M}$. From the definition of M, it is obvious that $M \subseteq < \mathbf{S}, x >_{\mathcal{M}_n}$, and the equality follows.

Now let $z \in B(< \mathbf{S}, x >_{\mathcal{M}_n})$. Then

$$z = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)$$

with $f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \ldots \gamma_{n-1}, \delta_{n-1}\}$ and $s_i \in S$. By Theorem A.0.4, we have that $\sigma_{n-1}^n(z) = z$ then

$$z = \sigma_{n-1}^n(z) = \sigma_{n-1}^n(\bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)) = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} \sigma_{n-1}^n(f_i(s_i)),$$

is in $B(\mathbf{S})$ because $\sigma_{n-1}^n(f_i(s_i)) = \gamma_k(\sigma_{n-1}^n(s_i))$ or $\sigma_{n-1}^n(f_i(s_i)) = \delta_k(\sigma_{n-1}^n(s_i))$, for some $k = 1, \ldots, n-1$. ■

Theorem A.0.10 *Let* $\mathbf{C}$ *be an* MV_n*-algebra and* $x \in C$*. Let* $\mathbf{S}$ *be a subalgebra of* $\mathbf{C}$ *such that* $\sigma_i^n(x)$ *belongs to* $B(\mathbf{S})$ *for each* $i = 1, \ldots, n-1$*. Then*

$$\mathbf{B}(< \mathbf{S}, x >_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{S}).$$

Proof: By Lemmas A.0.8 and A.0.9 we obtain:

$$\mathbf{B}(< \mathbf{S}, x >_{\mathcal{MV}_n}) = \mathbf{B}(< \mathbf{S}, x >_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S}).$$

■

Recall that a boolean algebra $\mathbf{B}$ is said to be **free over a poset** Y if for each boolean algebra $\mathbf{C}$ and for each non-decreasing function $f : Y \to \mathbf{C}$, f can be uniquely extended to a homomorphism from $\mathbf{B}$ into $\mathbf{C}$. As before, we shall denote by $\mathbf{Free}_{\mathcal{MV}_n}(Z)$ the free algebra in $\mathcal{MV}_n$ over a set Z.

Theorem A.0.11 $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ *is the free boolean algebra over the poset* $Z' = \{\sigma_i^n(z) : z \in Z, i = 1, \ldots, n-1\}$.

Proof: Let $\mathbf{S}$ be the subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ generated by Z'. Let $\mathbf{C}$ be a boolean algebra and let $f : Z' \to \mathbf{C}$ be a non-decreasing function. The monotonicity of f implies that the prescription

$$f'(z) = (f(\sigma_1^n(z)), \ldots, f(\sigma_{n-1}^n(z)))$$

defines a function $f' : Z \to \mathbf{C}^{[n]}$, where $\mathbf{C}^{[n]}$ is defined as in Example A.0.6. Since $\mathbf{C}^{[n]} \in \mathcal{MV}_n$, there is a unique homomorphism

$$h' : \mathbf{Free}_{\mathcal{MV}_n}(Z) \to \mathbf{C}^{[n]}$$

such that $h'(z) = f'(z)$ for every $z \in Z$. Let $\pi : \mathbf{C}^{[n]} \to \mathbf{C}$ be the projection over the first coordinate. The composition $\pi o h'$ restricted to $\mathbf{S}$ is a homomorphism $h : \mathbf{S} \to \mathbf{C}$, and for $y = \sigma_j^n(z) \in Z'$ we have:

$$h(y) = \pi(h'(\sigma_j^n(z))) = \pi(\sigma_j^n(h'(z))) = \pi(\sigma_j^n(f'(z))) =$$

$$= \pi(\sigma_j^n(f(\sigma_1^n(z)), \ldots, f(\sigma_{n-1}^n(z)))) =$$

$$= \pi(f(\sigma_j^n(z)), \ldots, f(\sigma_j^n(z))) = f(\sigma_j^n(z)) = f(y).$$

Hence $\mathbf{S}$ is the free boolean algebra over the poset Z'. But since $\sigma_j^n(z)$ is in $\mathbf{S}$ for all $z \in Z$ and $j = 1, \ldots n-1$, Theorem A.0.10 asserts that

$$\mathbf{S} = \mathbf{B}(\mathbf{S}) = \mathbf{B}(< \mathbf{S}, z >_{\mathcal{MV}_n})$$

for every $z \in Z$. From the fact that $\mathbf{S}$ is a subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ we obtain:

$$\mathbf{S} = \mathbf{B}(< \mathbf{S}, Z >_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$$

that is, $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ is the free boolean algebra over the poset Z'. ■

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