Objetos inyectivos en estructuras residuadas. Forma algebraica del teorema de Cantor - Bernstein - Schröder

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Objetos inyectivos en estructuras residuadas.

Forma algebraica del teorema de Cantor – Bernstein - Schröder

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Objetos inyectivos en estructuras residuadas. Forma algebraica del teorema de Cantor - Bernstein - Schröder

La presente tesis es un estudio de objetos inyectivos en clases de estructuras residuadas asociadas con la lógica y del teorema de Cantor - Bernstein - Schröder. En la primera parte se investigan inyectivos y retractos absolutos en clases de retículos residuados y pocrims. Algunas de las clases consideradas son las MTL-álgebras, IMTL-álgebras, BL-álgebras, NM-álgebras y los hoops acotados. En la segunda parte es desarrollado un marco algebraico para la validez del teorema de Cantor-Bernstein-Schröder aplicable a álgebras con una estructura subyacente de retículo tal que los elementos centrales de este retículo determinan una descomposición directa del álgebra. Se dan condiciones necesarias y suficientes para la validez del teorema de Cantor-Bernstein-Schröder en estas álgebras. Estos resultados son aplicados para obtener versiones del teorema en retículos ortomodulares, álgebras de Stone, BL-álgebras, MV-álgebras, pseudo MV-álgebras álgebras de Lukasiewicz y álgebras de Post of order n.

Palabras claves: Objetos injectives, Retractos absolutos, Retículos residuados, Bl-álgebras, Elementos Centrales, Variedades.

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Injectives in residuated structures . An algebraic version of the Cantor - Bernstein - Schröder theorem

The present thesis is a study of injectives in several classes of residuated structures associated with logic and the Cantor - Bernstein - Schröder theorem. In the first part we investigate injectives and absolute retracts in classes of residuated lattices and pocrims. Among the classes considered are MTL-algebras, IMTL-algebras, BL-algebras, NM-algebras and bounded hoops. In the second part is developed an algebraic frame for the validity of the Cantor-Bernstein-Schröder theorem, applicable to algebras with an underlying lattice structure and such that the central elements of this lattice determine a direct decomposition of the algebra. Necessary and sufficient conditions for the validity of the Cantor-Bernstein-Schröder theorem for these algebras are given. These results are applied to obtain versions of the Cantor-Bernstein-Schröder theorem for orthomodular lattices, Stone algebras, BL-algebras, MV-algebras, pseudo MV-algebras, Lukasiewicz and Post algebras of order n.

Keywords: Injective objects, Absolute retracts, Residuated lattices, Bl-algebras, Central elements, Varieties.

Prefacio

Las estructuras residuadas, originadas en los trabajos de Dedekind sobre de la teoría de ideales en anillos, aparecen en muchos campos de la matemática, y son particularmente comunes en álgebras asociadas con sistemas lógicos.

Dichas álgebras son estructuras $\langle A, \odot, \rightarrow, \leq \rangle$ donde A es un conjunto no vacío, $\leq$ es un orden parcial en A y $\odot$, $\rightarrow$ son operaciones binarias satisfaciendo la siguiente relación para cada a, b, c in A:

$$a \odot b \leq c \quad \textit{si y solo si} \quad a \leq b \rightarrow c.$$

Importantes ejemplos de estructuras residuadas relacionadas con la lógica son las álgebras de Boole (correspondientes a la lógica clásica), las álgebras de Heyting (correspondientes al intuicionismo), los retículos residuados (correspondientes a la lógica sin regla de contracción [35]), BL-álgebras (correspondientes a la lógica difusa básica de Hájek [26]), MV-álgebras (correspondientes a la lógica multivaluada de Lukasiewicz [10]).

Estos ejemplos, con la excepción de los retículos residuados son *hoops* [5], es decir, satisfacen la ecuación $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$. Todas las estructuras mencionadas son casos particulares de *monoides conmutativos integrales residuados parcialmente ordenados* , o *pocrims* por simplicidad [5].

En los primeros cuatro capítulos de esta tesis se estudian objetos inyectivos y retractos absolutos en clases de retículos residuados y pocrims. En el capítulo 2 se dan también algunos resultados sobre injectivos en variedades más generales.

El conocido teorema de Cantor-Bernstein-Schröder (teorema CBS, por simplicidad) dice que si un conjunto X puede sumergirse en otro Y y viceversa, entonces existe una función biyectiva entre ambos. A finales de los cuarenta, Sikorski [39] (ver también Tarski [40]) mostró que el teorema CBS es un caso particular de un resultado para álgebras de Boole σ-completas. Recientemente muchos autores extendieron el resultado de Sikorski a clases de álgebras más generales que las álgebras de Boole como, por ejemplo, retículos ortomodulares, [16], MV-algebras [15], pseudo MV-algebras [30]. En el último capítulo de la tesis se da un marco algebraico general para la validez del teorema CBS, que permitie derivar todas las versiones mencionadas. Se establecen también, bajo el mismo marco, versiones del teorema CBS para retículos residuados, en particular para BL-algebras, álgebras de Stone [2], álgebras de Lukasiewicz y álgebras de Post de orden n [2, 6].

En más detalle, el contenido de la tesis es el siguiente: el Capítulo 1 presenta definiciones básicas y propiedades de las estructuras residuadas. El único resultado original de este capítulo es la Proposición 1.2.13. En el Capítulo 2 se muestra que bajo ciertas hipótesis no demasiado restrictivas sobre una variedad de álgebras $\mathcal{V}$, la existencia de objetos inyectivos no triviales en $\mathcal{V}$ es equivalente a la existencia de un álgebra auto-inyectiva simple y máxima. Además, con técnicas de ultraproductos, se obtienen propiedades reticulares de inyectivos en variedades de álgebras ordenadas. Los resultados del Capítulo 2 son aplicados en el Capítulo 3 para el estudio de inyectivos en variedades de retículos residuados. Estos resultados están sumarizados en la tabla 3.1. En el Capítulo 4 se investigan inyectivos en clases de pocrims y hoops, estos resultados están sumarizados en la tabla 4.1.

El marco abstracto para el teorema CBS es dado por las $\mathcal{L}$-variedades de álgebras, introducidas en la primera sección del Capítulo 5. En la Sección 5.2 se muestran varios ejemplos $\mathcal{L}$-variedades. En la Sección 5.3 se dan condiciones necesarias y suficientes para la validez del teorema CBS en álgebras pertenecientes a $\mathcal{L}$-variedades. En la Sección 5.4 se muestran algunas condiciones globales sobre álgebras de una $\mathcal{L}$-variedad que resultan ser suficientes para la validez del teorema CBS. En la Sección 5.5 se muestra que los retractos absolutos en una $\mathcal{L}$-variedad satisfacen el teorema CBS. Finalmente, en la Sección 5.6 se da una versión del teorema CBS para conjuntos parcialmente ordenados.

El contenido de los Capítulos 2 y 3, así como los resultados del Capítulo 4 que siguen a la Definición 4.3.7 están reproducidos en el trabajo [22]. Los resultados del Capítulo 4 anteriores a la Definición 4.3.7 están en el trabajo [23]. Los resultados del Capítulo 5, con excepción de los de la Sección 5.5, están en el trabajo [21].

Preface

Residuated structures, rooted in the work of Dedekind on the ideal theory of rings, arise in many fields of mathematics, and are particularly common among algebras associated with logical systems. They are structures $\langle A, \odot, \rightarrow, \leq \rangle$ such that A is a nonempty set, $\leq$ is a partial order on A and $\odot$ and $\rightarrow$ are binary operations such that the following relation holds for each a, b, c in A:

$$a \odot b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c.$$

Important examples of residuated structures related to logic are Boolean algebras (corresponding to classical logic), Heyting algebras (corresponding to intuitionism), residuated lattices (corresponding to logics without contraction rule [35]), BL-algebras (corresponding to Hájek's basic fuzzy logic [26]), MV-algebras (corresponding to Łukasiewicz many-valued logic [10]). All these examples, with the exception of residuated lattices are *hoops* [5], i. e., they satisfy the equation $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$. All the mentioned examples are particular cases of *partially ordered commutative residuated integral monoids*, or *pocrims* for short [5].

In the first four chapters of this thesis we investigate injectives and absolute retracts in classes of residuated lattices and pocrims. In Chapter 2 we also present some results on injectives in more general varieties.

The famous Cantor-Bernstein-Schröder theorem (CBS theorem, for short) states that, if a set X can be embedded into a set Y and viceversa, then there is a one-to-one function of X onto Y. At the end of the forties, Sikorski [39] (see also Tarski [40]) showed that the CBS theorem is a particular case of a statement on σ-complete boolean algebras. Recently several authors extended Sikorski's result to classes of algebras more general than boolean algebras, like orthomodular lattices [16], MV-algebras [15], pseudo MV-algebras [30]. The aim of the last chapter of this thesis is to give a general algebraic frame for the validity of the CBS theorem, from which all the versions mentioned above can be derived, as well as versions of the CBS theorem for residuated lattices, in particular for BL-algebras, and also for Stone algebras [2], Lukasiewicz and Post algebras of order n [2, 6].

In more detail, the content of the thesis is as follows: In Chapter 1 we recall some basic definitions and properties of residuated structures. The only original result of this chapter is Proposition 1.2.13. In Chapter 2 we show that under some mild hypothesis on a variety $\mathcal{V}$ of algebras, the existence of nontrivial injectives is equivalent to the existence of a self-injective maximum simple algebra. Moreover, we use ultrapowers to obtain lattice properties of the injectives in varieties of ordered algebras. The results of Chapter 2 are applied in Chapter 3 to the study of injectives in varieties of residuated lattices. The results obtained are summarized in Table 3.1. In Chapter 4 we investigate injectives in classes of pocrims and hoops. The results are summarized in Table 4.1.

The abstract frame for the CBS theorem is given by the $\mathcal{L}$-varieties of algebras, introduced in the first section of Chapter 5. In Section 5.2 we show that there are many examples of $\mathcal{L}$-varieties. Necessary and sufficient conditions for the validity of the CBS theorem in algebras belonging to an $\mathcal{L}$-variety are given in Section 5.3, which is the main section of this paper. In Section 5.4 we look for some simple global conditions on algebras of an $\mathcal{L}$-variety that are sufficient for the validity of the CBS theorem. In Section 5.5 we show that absolute retracts in $\mathcal{L}$-varieties satisfy the CBS theorem. Finally, in Section 5.6 we give a version of the CBS theorem for partially ordered sets.

The content of Chapters 2 and 3, as well as the results following Definition 4.3.7 in Chapter 4, are reproduced in the paper [22]. The results of Chapter 4, until Definition 4.3.7, are in the paper [23]. The results of Chapter 5, with the exception of those in Section 5.5 are in the paper [21].

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Chapter 1

Residuated Structures

1.1 Basic Notions

We recall from from [2] and [7] some basic notions of injectives and universal algebra. Let $\mathcal{A}$ be a class of algebras. For all algebras A, B in $\mathcal{A}$, $[A, B]_{\mathcal{A}}$ will denote the set of all homomorphisms $g : A \rightarrow B$. In this case, classes of algebras are considered as categories. A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ if reflective is there is a functor $\mathcal{R} : \mathcal{A} \rightarrow \mathcal{B}$, called **reflector**, such that for each $A \in \mathcal{A}$ there exists a morphism $\Phi_{\mathcal{R}}(A) \in [A, \mathcal{R}(A)]_{\mathcal{A}}$ with the following properties:

i) If $f \in [A, A']_{\mathcal{A}}$ then the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \Phi_{\mathcal{R}}(A)\downarrow & \equiv & \downarrow \Phi_{\mathcal{R}}(A') \\ \mathcal{R}(A) & \xrightarrow[\mathcal{R}(f)]{} & \mathcal{R}(A') \end{array}$$

ii) If $B \in \mathcal{B}$ and $f \in [A, B]_{\mathcal{A}}$ then there exists a unique morphism $f' \in [\mathcal{R}(A)B]_{\mathcal{B}}$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Phi_{\mathcal{R}}(A)\downarrow & \equiv & \nearrow f' \\ \mathcal{R}(A) & & \end{array}$$

An algebra A in $\mathcal{A}$ is **injective** iff for every monomorphism $f \in [B, A]_{\mathcal{A}}$ and every $g \in [B, C]_{\mathcal{A}}$ there exists $h \in [C, A]_{\mathcal{A}}$ such that the following diagram is commutative

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ g \downarrow & \equiv \;\; \nearrow_{h} & \\ C & & \end{array}$$

A is **self-injective** iff every homomorphism from a subalgebra of A into A, extends to an endomorphism of A.

An algebra B is a **retract** of an algebra A iff there exists $g \in [B, A]_{\mathcal{A}}$ and $f \in [A, B]_{\mathcal{A}}$ such that $fg = 1_B$. Notice that g is necessarily a monomorphism and f is an epimorphism. Also, if the morphisms are functions, then g is injective and f is surjective. An algebra B is called an **absolute retract** in $\mathcal{A}$ iff it is a retract of each of its extensions in $\mathcal{A}$. It is well-known (and easy to verify) that *a retract of an injective object is injective.*

A non-trivial algebra T is said to be **minimal** in $\mathcal{A}$ iff for each non-trivial algebra A in $\mathcal{A}$, there exists a monomorphism $f : T \to A$.

For each algebra A, we denote by $Con(A)$ the congruence lattice of A, the diagonal congruence is denoted by Δ and the largest congruence A^2 is denoted by ∇. A congruence θ_M is said to be maximal iff $\theta_M \neq \nabla$ and there is no congruence θ such that $\theta_M \subset \theta \subset \nabla$. An algebra I is **simple** iff $Con(I) = \{\Delta, \nabla\}$. A simple algebra is **hereditarily simple** iff all its subalgebras are simple. An algebra A is **semisimple** iff it is a subdirect product of simple algebras.

An algebra A has the **congruence extension property** (CEP) iff for each subalgebra B and $\theta \in Con(B)$ there is a $\phi \in Con(A)$ such that $\theta = \phi \cap A^2$. A variety $\mathcal{V}$ satisfies CEP iff every algebra in $\mathcal{V}$ has the CEP. It is clear that if $\mathcal{V}$ satisfies CEP then every simple algebra is hereditarily simple.

An algebra A is **rigid** iff the identity homomorphism is the only automorphism.

Let τ be a type of algebras. A quasi-identity of type τ is either an identity $p = q$ or a formula of the form $(p_1 = q_1) \wedge \cdots \wedge (p_n = q_n) \Rightarrow (p = q)$ where $p_1 \cdots p_n, q_1 \cdots q_n, p, q$ are terms in the language τ. A quasivariety is a class $\mathcal{A}$ of algebras of the same type that can be axiomatized by a set of quasi-identities, called a basis for $\mathcal{A}$. A subquasivariety $\mathcal{B}$ of a quasivariety $\mathcal{A}$ is a **relative subvariety** of $\mathcal{A}$ provided that a basis of $\mathcal{B}$ can be obtained by adding only identities to a basis of $\mathcal{A}$. Let $\mathcal{A}$ be a quasivariety and

$A \in \mathcal{A}$. A congruence θ in A is said **relative to the quasivariety** $\mathcal{A}$ iff $A/\theta \in \mathcal{A}$. We denote by $Con_{\mathcal{A}}(A)$, the relative congruences lattice of A. The set $Con_{\mathcal{A}}(A)$ is closed under arbitrary intersections and hence form a complete lattice. Note that Δ and ∇ are always congruences relatives to $\mathcal{A}$. We say that an algebra $A \in \mathcal{A}$ is **simple relative** to $\mathcal{A}$ provided $Con_{\mathcal{A}}(A) = \{\Delta, \nabla\}$ and we say that A is **semisimple relative** to $\mathcal{A}$ iff it is a subdirect product of algebras simple relative to $\mathcal{A}$.

1.2 Pocrims and Hoops

Definition 1.2.1 *A pocrim [5] is an algebra $\langle A, \odot, \rightarrow, 1\rangle$ of type $\langle 2,2,0\rangle$ satisfying the following axioms:*

1. $\langle A, \odot, 1\rangle$ *is an abelian monoid,*
2. $x \rightarrow 1 = 1$,
3. $1 \rightarrow x = x$,
4. $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$,
5. $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$,
6. *If* $x \rightarrow y = 1$ *and* $y \rightarrow x = 1$ *then* $x = y$.

We denote by $\mathcal{M}$ the class of all pocrims. $\mathcal{M}$ is a quasivariety which is not a variety [29]. If $A \in \mathcal{M}$, we can define an order in A by $x \leq y$ iff $x \rightarrow y = 1$. With this order, the structure $\langle A, \odot, \rightarrow, 1, \leq\rangle$ is a commutative partial ordered monoid in which 1 is the upper bound.

An element $x \in A$ is called **idempotent** iff $x \odot x = x$, and the set of all idempotent elements in A is denoted by $Idp(A)$.

For all $a \in A$, we inductively define $a^1 = a$ and $a^{n+1} = a^n \odot a$.

It is easy to verify the following proposition:

Proposition 1.2.2 *The following assertions hold in every pocrim A, where x, y, z denote arbitrary elements of A:*

(1) $x \rightarrow x = 1$,

(2) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(3) $x \odot y \leq z$ *iff* $x \leq y \rightarrow z$,

(4) $x \leq y$ *iff* $1 = x \rightarrow y$,

(5) $x \odot y \leq y$,

(6) $x \odot (x \rightarrow y) \leq y$,

(7) $a \leq b \Longrightarrow x \rightarrow a \leq x \rightarrow b$,

(8) $a \leq b \Longrightarrow a \rightarrow x \geq b \rightarrow x$,

(9) $a \leq b \Longrightarrow a \odot x \leq b \odot x$,

(10) $a \rightarrow b = \bigvee\{x \in A : a \odot x \leq b\}$. □

We recall now some well-known facts about implicative filters and congruences on pocrims. Let A be a pocrim and $F \subseteq A$. Then F is an **implicative filter** iff it satisfies the following conditions:

1) $1 \in F$,

2) if $x \in F$ and $x \rightarrow y \in F$, then $y \in F$.

It is easy to verify that a non-empty subset F of a pocrim A is an implicative filter iff for all $a, b \in A$:

- If $a \in F$ and $a \leq b$, then $b \in F$,
- if $a, b \in F$, then $a \odot b \in F$.

The intersection of any family of implicative filters is again an implicative filter. We denote by $\langle X \rangle$ the implicative filter generated by $X \subseteq A$, i.e., the intersection of all implicative filters of A containing X. We abbreviate this as $\langle a \rangle$ when $X = \{a\}$ and it is easy to verify that

$$\langle X \rangle = \{x \in A : \exists\ w_1, \cdots, w_n \in X \ \textit{such that}\ x \geq w_1 \odot \cdots \odot w_n\}.$$

The set $Filt(A)$ of all implicative filters of A, ordered by inclusion, is a bounded lattice. For any implicative filter F of A,

$$\theta_F = \{(x, y) \in A^2 : x \rightarrow y, y \rightarrow x \in F\}$$

is a congruence relative to $\mathcal{M}$. Moreover $F = \{x \in A : (x, 1) \in \theta_F\}$. Conversely, if $\theta \in Con_{\mathcal{M}}(A)$ then $F_\theta = \{x \in A : (x, 1) \in \theta\}$ is an implicative filter and $(x, y) \in \theta$ iff $(x \to y, 1) \in \theta$ and $(y \to x, 1) \in \theta$. Thus *the correspondence $F \to \theta_F$ establishes an order isomorphism between $Con_{\mathcal{M}}(A)$ and $Filt(A)$.*

If $F \in Filt(A)$, we shall write A/F instead of A/θ_F, and for each $x \in A$ we shall write $[x]_\theta$ (or simply $[x]$ when θ is understand) for the equivalence class of x.

Definition 1.2.3 A *bounded pocrim* is an algebra $\langle A, \odot, \to, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that:

1. $\langle A, \odot, \to, 1\rangle$ is a pocrim
2. $0 \to x = 1$

The quasivariety of bounded pocrims is denoted by $\mathcal{M}_0$. Observe that since 0 is in the clone of operations, then we require that for each morphism f, $f(0) = 0$. Observe that $\{0, 1\}$ is a subalgebra of each non-trivial $A \in \mathcal{M}_0$, which is a boolean algebra. Hence $\{0, 1\}$, with its natural boolean algebra structure, is the minimal algebra in each subquasivariety of $\mathcal{M}_0$. Thus the variety of boolean algebras $\mathcal{BA}$ is a relative variety of all subquasivarieties of bounded pocrims.

On each bounded pocrim A we can define a unary operation $\neg$ by

$$\neg x = x \to 0.$$

Note that an implicative filter F of a bounded pocrim is proper iff 0 does not belong to F. Hence a standar application of Zorn's Lemma gives that *every implicative filter in a bounded pocrim is contained in a maximal filter.*

Let A be a bounded pocrim. An element a in A is called **nilpotent** iff there exists a natural number n such that $a^n = 0$. The minimum n such that $a^n = 0$ is called the **nilpotence order** of a. An element a in A is called **dense** iff $\neg a = 0$, and it is called a **unity** iff for all natural numbers n, $\neg(a^n)$ is nilpotent. The set of dense elements of A will be denoted by $Ds(A)$. It is easy to verify that $Ds(A)$ is an implicative filter.

A bounded pocrim A is called **dense free** iff $Ds(A) = \{1\}$. If $\mathcal{A}$ is a relative subvariety of $\mathcal{M}_0$, we denote by $\mathcal{DF}(\mathcal{A})$ the full subcategory of $\mathcal{A}$ whose elements are the dense free algebras of $\mathcal{A}$.

Proposition 1.2.4 *Let $\mathcal{A}$ be a relative subvariety of $\mathcal{M}_0$. Then we have:*

1. $\mathcal{DF}(\mathcal{A}) = \{A/Ds(A) : A \in \mathcal{A}\}$

2. *$\mathcal{DF}(\mathcal{A})$ is the subquasivariety of $\mathcal{A}$ characterized by the quasiequation* $\neg\neg x = 1 \Rightarrow x = 1$.

Proof: To prove 1., we need to prove that $Ds(A/Ds(A)) = \{[1]\}$. Let $[x]$ be a dense element in $A/Ds(A)$. Therefore $[\neg x] = [0]$ and then $\neg x \to 0 = \neg\neg x \in Ds(A)$. Thus $\neg x = \neg\neg\neg x = 0$, that is $x \in Ds(A)$. Hence $[x] = [1]$. 2. is immediate. □

Definition 1.2.5 If A is a bounded pocrim then we define:

$$Rad(A) = \bigcap\{F : F \text{ is a maximal implicative filter of } A\}.$$

Proposition 1.2.6 *Let A be a bounded pocrim. Then:*

1. $Rad(A) = \{a \in A : a \text{ is a unity}\}$.
2. $Ds(A) \subseteq Rad(A)$.

Proof: 1) Suppose that $a \notin Rad(A)$. Then there exists a maximal implicative filter F in A such that $a \notin F$. Since F is maximal, there exists $b \in F$ and a natural number n such that $a^n \odot b = 0$. By Proposition 1.2.2 $b \leq \neg a^n$. Hence $\neg a^n \in F$, and $(\neg a^n)^m \neq 0$ for each natural number m. Thus a is not a unity. On the other hand, if a is not a unity then $(\neg a^n)^m \neq 0$ for each natural number m. We consider the implicative filter generated by $\neg a^n$, i.e., $\langle \neg a^n \rangle$. By Zorn's Lemma there exists a maximal implicative filter F containing $\langle \neg a^n \rangle$. Since $\neg a^n \in F$, a is not an element of F. Therefore $a \notin Rad(A)$. 2) Is an obvious consequence of 1). □

Proposition 1.2.7 *Let A be a bounded pocrim. Then A is relative semisimple iff $Rad(A) = \{1\}$*

Proof: Suppose that A is relative semisimple. Let $f : A \to \prod_{i\in I} L_i$, be a subdirect embedding with L_i a relative simple bounded pocrim for each $i \in I$. Then $F_i = Ker(\pi_i f)$ is maximal implicative filter in A. Thus $Rad(A) \subseteq \bigcap_{i\in I} F_i = \{1\}$. Conversely, if $Rad(A) = \{1\}$ then $A = A/Rad(A)$ and A can be subdirectly embedded in $\prod_{i\in I} A/F_i$, with F_i a maximal implicative filter for each $i \in I$. Hence A is relative semisimple. □

If $\mathcal{A}$ is a relative subvariety of $\mathcal{M}_0$, we denote by $\mathcal{S}em(\mathcal{A})$ the full subcategory of $\mathcal{A}$ whose elements are relative semisimple algebras of $\mathcal{A}$.

Proposition 1.2.8 *If $\mathcal{A}$ is a relative subvariety of $\mathcal{M}_0$ then $Sem(\mathcal{A}) = \{A/Rad(A) : A \in \mathcal{A}\}$*

Proof: Since $Filt(A/Rad(A)) = [Rad(A), A]$, then F is a maximal implicative filter in A iff it is maximal in $A/Rad(A)$. Thus $Rad(A/Rad(A)) = \{1\}$ and $A/Rad(A)$ is relative semisimple. □

If $Rad(A)$ has a least element a, i.e., $Rad(A) = [a)$, then a is called the **principal unity** of A. It is clear that the principal unity is the minimum unit. Hence it is an idempotent element, and obviously, it generates the radical.

Lemma 1.2.9 *Let A be a bounded pocrim having principal unity a. If $x \in Rad(A)$ then, $x \rightarrow \neg a = \neg a$.*

Proof: $x \rightarrow \neg a = \neg(x \odot a) = \neg a$ since a is the minimum unity. □

Proposition 1.2.10 *Let A be a linearly ordered bounded pocrim. Then:*

1. *a is a unity in A iff a is not a nilpotent element.*
2. *If a is a unity in A, then $\neg a < a$.*

Proof: 1) If $a < 1$ and there exists a natural number n such that $a^n = 0$, then $\neg(a^n) = 1$ and a is not a unity. Conversely, suppose that a is not a unity. Since A is linearly ordered, we must have $a^n \leq \neg\neg(a^n) < \neg(a^n)$. Hence $a^{2n} = 0$ and a is nilpotent, which is a contradiction. 2) Is an obvious consequence of 1). □

Corollary 1.2.11 *Let A be a bounded pocrim such that there exists an embedding $f : A \rightarrow \prod_{i \in I} L_i$, with L_i a linearly ordered bounded pocrim for each $i \in I$. Then a is a unity in A iff for each $i \in I$, $a_i = \pi_i f(a)$ is a unity in L_i, where π_i is the projection onto L_i .*

Proof: If a is a unity in A then $a_i = \pi_i f(a)$ is a unity in L_i, because homomorphisms preserve unites. Conversely, suppose that a is not a unity. Therefore there is an n such that $\neg(a^n)$ is not nilpotent, and hence $\neg(a^n) \not\leq \neg\neg(a^n)$. Since f is an embedding and since L_i is linearly ordered for each $i \in I$, there exists $j \in I$ such that $\neg\neg(a_j^n) \leq \neg(a_j^n)$, and by Proposition 1.2.10 a_j is not a unity in L_j.

□

Remark 1.2.12 If a bounded pocrim A is subdirect product of linearly ordered bounded pocrims, then the radical of A is characterized by equations. More precisely:

$$Rad(A) = \{x \in A : \forall n \in N, (\neg(x^n))^2 = 0\}$$

Proposition 1.2.13 *Let $\mathcal{A}$ be a relative subvariety of $\mathcal{M}_0$. Then $\mathcal{DF}(\mathcal{A})$ and $\mathcal{S}em(\mathcal{A})$ are reflective subcategories of $\mathcal{A}$, and the respective reflectors preserve monomorphisms.*

Proof: If $A \in \mathcal{A}$, for each $x \in A$, $[x]$ will denote the $Rad(A)$-congruence class of x. We define $\mathcal{S}(A) = A/Rad(A)$, and for each $f \in [A, A']_{\mathcal{A}}$, we let $\mathcal{S}(f)$ be defined by $\mathcal{S}(f)([x]) = [f(x)]$ for each $x \in A$. Since homomorphisms preserve unities, we obtain a well defined function $\mathcal{S}(f) : A/Rad(A) \to A'/Rad(A')$. It is easy to check that $\mathcal{S}$ is a functor from $\mathcal{A}$ to $\mathcal{SEM}(\mathcal{A})$. To show that $\mathcal{S}$ is a reflector, note first that if $p_A : A \to A/Rad(A)$ is the canonical projection, then the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ p_A \downarrow & \equiv & \downarrow p_{A'} \\ A/Rad(A) & \xrightarrow[\mathcal{S}(f)]{} & A/Rad(A') \end{array}$$

Suppose that $B \in \mathcal{S}(\mathcal{A})$ and $f \in [A, B]_{\mathcal{A}}$. Since $Rad(B) = \{1\}$, the mapping $[x] \mapsto f(x)$ defines a homomorphism $g : A/Rad(A) \to B$ that makes the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p_A \downarrow & \equiv & \nearrow g \\ A/Rad(A) & & \end{array}$$

and it is obvious that g is the only homomorphism in $[A/Rad(A), B]_{\mathcal{S}em(\mathcal{A})}$ making the triangle commutative. Therefore we have proved that $\mathcal{S}$ is a reflector. We proceed to prove that $\mathcal{S}$ preserves monomorphisms. Let $f \in [A, B]_{\mathcal{A}}$ be a monomorphism and suppose that $(\mathcal{S}(f))(x) = (\mathcal{S}(f))(y)$, i.e., $[f(x)] = [f(y)]$. Then for each number n there exists a number m such that $0 = (\neg((f(x) \to f(y))^n))^m = f((\neg((x \to y)^n))^m)$. Since f is a monomorphism then $(\neg((x \rightarrow y)^n))^m = 0$ and $x \rightarrow y \in Rad(A)$. Interchanging x and y, we obtain $[x] = [y]$ and $\mathcal{S}(f)$ is a monomorphism. The statements about $\mathcal{DF}(\mathcal{A})$ can be proved with similar arguments. □

Corollary 1.2.14 *Let $\mathcal{A}$ be a relative subvariety of $\mathcal{M}_0$. If A is injective either in $\mathcal{DF}(\mathcal{A})$ or in $Sem(\mathcal{A})$, then A is injective in $\mathcal{A}$.*

Proof: It is well-known that if $\mathcal{D}$ is a reflective subcategory of $\mathcal{A}$ such that the reflector preserves monomorphisms then an injective object in $\mathcal{D}$ is also injective in $\mathcal{A}$ [2, I.18]. Then the theorem follows from Propositions 1.2.13. □

Definition 1.2.15 A *Hoop* [5] is a pocrim satisfying the following condition: $x \leq y$ iff $x = x \odot (x \rightarrow y)$.

Every hoop is a meet semilattice, where the meet operation is given by $x \wedge y = x \odot (x \rightarrow y)$.

Observe that in a hoop A, $x \leq y$ iff there is $z \in A$ such that $x = z \odot y$.

Theorem 1.2.16 *An algebra $\langle A, \odot, \rightarrow, 1\rangle$ is a hoop iff*

1. $\langle A, \odot, 1\rangle$ *is an abelian monoid,*
2. $x \rightarrow x = 1$,
3. $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y$,
4. $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.

Proof: See ([5, Theorem 1.2]) □

Hence, the class of all hoops form a variety. This variety is noted by $\mathcal{HO}$. In hoops, all congruences are identified to implicative filters.

Proposition 1.2.17 *If $\mathcal{A}$ is a variety of hoops then $\mathcal{A}$ satisfies CEP.*

Proof: Let A be a hoop and let B be a subhoop of A. For each implicative filter F of B, let $\langle F\rangle_A$ be the implicative filter of A generated by F. Clearly $F \subseteq \langle F\rangle_A$. To see the converse, let $b \in B \cap \langle F\rangle_A$. Then there exists

$a_1, \cdots, a_n \in F$ such that $a_1 \odot a_2 \odot \cdots \odot a_n \leq b$. Since $b \in B$ and F is an implicative filter of B, hence upward closed, it follows that $b \in F$. □

Let k be a natural number. A **k-potent hoop** [5] is a hoop satisfying $x^k = x^{k+1}$. We denote the class of k-potent hoop by $\mathcal{HO}(k)$. It is clear that $\mathcal{HO}(2)$ is the variety of brouwerian semilattices [33].

A **basic hoop** [1] is an algebra $\langle A, \wedge, \vee, \odot, \rightarrow, 1\rangle$ of type $\langle 2,2,2,2,0\rangle$ such that:

1. $\langle A, \odot, \rightarrow, 1\rangle$ is a hoop,
2. $\langle A, \wedge, \vee, 1\rangle$ is lattice with greatest element 1,
3. $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Basic hoops are also known as **generalized BL-algebras** [13]. We denote by $\mathcal{BH}$ the variety whose element are basic hoops.

1.3 Residuated Lattices

Definition 1.3.1 A *residuated lattice* [35] or *commutative integral residuated* $0,1$*-lattice* [31], is an algebra $\langle A, \wedge, \vee, \odot, \rightarrow, 0, 1\rangle$ of type $\langle 2,2,2,2,0,0\rangle$ such that:

1. $\langle A, \odot, \rightarrow, 1, 0\rangle$ is a bounded pocrim
2. $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice,
3. $(x \wedge y) \rightarrow y = 1$.

Residuated lattices form a variety $\mathcal{RL}$ defined by the following equations:

1. $\langle A, \odot, 1\rangle$ is an abelian monoid,
2. $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice,
3. $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
4. $((x \rightarrow y) \odot x) \wedge y = (x \rightarrow y) \odot x$,

5. $(x \wedge y) \rightarrow y = 1$.

A is called an *involutive residuated lattice* or a *Girard monoid* [27] if it also satisfies the equation:

6. $\neg\neg x = x$.

A is called *distributive* if satisfies 1.–5. as well as:

7. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

The subvariety of Girard monoids is noted by $\mathcal{GM}$. Following the notation used in [31], the variety of residuated lattices that satisfy the distributive law is denoted by $\mathcal{DRL}$, and $\mathcal{DGM}$ will denote the variety of distributive Girard monoids. In residuated lattices, congruences are in correspondence with implicative filters. In the next proposition we collect some easy consequences of the definition of residuated lattices.

Proposition 1.3.2 *Let A be a residuated lattice and $Z \subseteq A$. Then:*

1. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
2. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
3. $(x \vee y) \rightarrow z) = (x \rightarrow z) \wedge (x \rightarrow y)$,
4. $x \odot y \leq x \wedge y$,
5. $x \leq y \Longrightarrow \neg x \leq \neg y$,
6. *if* $\bigvee Z$ *exists, then* $a \odot \bigvee_{z \in Z} z = \bigvee_{z \in Z} a \odot z$,
7. *if* $\bigvee Z$ *exists, then* $\bigvee_{z \in Z} z \rightarrow a = \bigwedge_{z \in Z} z \rightarrow a$,
8. *if* $\bigwedge Z$ *exists, then* $a \rightarrow \bigwedge_{z \in Z} z = \bigwedge_{z \in Z} a \rightarrow z$

□

Proposition 1.3.3 *Let A be a residuated lattice. Then the following conditions are equivalent:*

1. $(a \rightarrow b) \vee (b \rightarrow a) = 1$ (prelinearity),

2. $a \to (b \vee c) = (a \to b) \vee (a \to c)$,

3. $(a \wedge b) \to c = (a \to c) \vee (b \to c)$.

Proof: See ([27, Theorem 2.3]) □

Lemma 1.3.4 *Let A be a residuated lattice satisfying the prelinearity equation. Then following conditions are valid:*

1. $a^2 \wedge b^2 \leq a \odot b \leq a^2 \vee b^2$,

2. $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$,

3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

Proof: See ([27, Lemma 2.4]) □

Proposition 1.3.5 *Let A be a Girard monoid. Then following conditions are valid:*

1. $(a \to b) = \neg(a \odot \neg b)$,

2. $\neg(a \wedge b) = \neg a \vee \neg b$.

Proof: See ([27, Proposition 2.8]) □

Proposition 1.3.6 *Let A be a Girard monoid. Then following conditions are equivalent:*

1. *A satisfies the prelinearity equation,*

2. $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$.

Proof: See ([27, Proposition 2.9]) □

Lemma 1.3.7 *Let A be a Girard monoid satisfying the prelinearity equation. Then the negation has at most one fixed point.*

Proof: See ([27, Lemma 2.10]) □

Proposition 1.3.8 *Let A be a residuated lattice. Then following conditions are equivalent:*

1. $a \leq b \Longrightarrow \exists x \in A$ *s.t.* $b = a \odot x$, (divisibility)

2. $a \wedge b = a \odot (a \rightarrow b)$,

3. $a \rightarrow (b \wedge c) = (a \wedge b) \odot ((a \wedge b) \rightarrow c)$.

Proof: See ([27, Lemma 2.5]) □

Proposition 1.3.9 *Let A be a residuated lattice with divisibility. Then following conditions are valid:*

1. *If a is idempotent then* $a \wedge b = a \odot b$ *for each* $b \in A$,

2. $a \odot b \leq a^2 \vee b^2$,

3. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

Proof: See ([27, Proposition 2.6]) □

Proposition 1.3.10 *If* $\mathcal{A}$ *is a subvariety of* $\mathcal{RL}$, *then* $\mathcal{A}$ *satisfies CEP.*

Proof: This follows from the same argument used in Proposition 1.2.17 . □

It is easy to verify the following proposition:

Proposition 1.3.11 *Let A be a residuated lattice. Then A is simple iff for each* $a < 1$, *a is nilpotent.* □

Chapter 2

Injectives, simple algebras and ultrapowers

2.1 Injectives and simple algebras

Definition 2.1.1 Let $\mathcal{V}$ be a variety. Two constant terms $0, 1$ of the language of $\mathcal{V}$ are called *distinguished constants* iff $A \models 0 \neq 1$ for each non-trivial algebra A in $\mathcal{V}$.

Lemma 2.1.2 *Let $\mathcal{A}$ be variety with distinguished constants $0, 1$ and let A be a non-trivial algebra in $\mathcal{A}$. Then A has maximal congruences, and for each simple algebra $I \in \mathcal{A}$, all homomorphisms $f : I \to A$ are monomorphisms.*

Proof: Since for each homomorphism $f : A \to B$ such that B is a non-trivial algebra, $f(0) \neq f(1)$ then for each $\theta \in Con(A)\backslash\{A^2\}$, $(1, 0) \notin \theta$. Thus a standard application of Zorn lemma shows that $Con(A)\backslash\{A^2\}$ has maximal elements. The second claim follows from the simplicity of I and $f(0) \neq f(1)$. □

Definition 2.1.3 A simple algebra I_M is said to be *maximum simple* [22] iff for each simple algebra I, I can be embedded in I_M.

Theorem 2.1.4 *Let $\mathcal{A}$ be a variety with distinguished constants $0, 1$ having a minimal algebra. If $\mathcal{A}$ has non-trivial injectives, then there exists a maximum simple algebra I.*

Proof: Let A be a non-trivial injective in $\mathcal{A}$. By Lemma 2.1.2 there is a maximal congruence θ of A. Let $I = A/\theta$ and $p : A \to I$ be the canonical projection. Since $\mathcal{A}$ has a minimal algebra it is clear that for each simple algebra J, there exists a monomorphism $h : J \rightarrow A$. Then the composition ph is a monomorphism from J into I. Thus I is a maximum simple algebra. □

We want to establish a kind of the converse of the above theorem.

Theorem 2.1.5 *Let $\mathcal{A}$ be a variety satisfying CEP, with distinguished constants* $0, 1$. *If I is a self-injective maximum simple algebra in $\mathcal{A}$ then I is injective.*

Proof: For each monomorphism $g : A \rightarrow B$ we consider the following diagram in $\mathcal{A}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & I \\ g\downarrow & & \\ B & & \end{array}$$

By CEP, I is hereditarily simple. Hence $f(A)$ is simple and $Ker(f)$ is a maximal congruence of A such that $(0,1) \notin Ker(f)$. Further $Ker(f)$ can be extended to a maximal congruence θ in B. It is clear that $(0,1) \notin \theta$ and $\theta \cap A^2 = Ker(f)$. Thus if we consider the canonical projection $p : B \rightarrow B/\theta$, then there exists a monomorphism $g' : f(A) \rightarrow B/\theta$ such that

$$\begin{array}{ccccc} A & \xrightarrow{f} & f(A) & \xrightarrow{1_{f(A)}} & I \\ g\downarrow & \equiv & g'\downarrow & & \\ B & \xrightarrow[p]{} & B/\theta & & \end{array}$$

Since I is maximum simple, B/θ is isomorphic to a subalgebra of I. Therefore, since that I is self-injective, there exists a monomorphism $\varphi : B/\theta \rightarrow I$ such that $\varphi g' = 1_f(A)$. Thus $(\varphi p)g = f$ and I is injective. □

Lemma 2.1.6 *If A is a rigid simple injective algebra in a variety, then all the subalgebras of A are rigid.* □

2.2 Injectives, ultrapowers and lattice properties

We recall from [4] some basic notions on ordered sets that will play an important role in what follows. An ordered set L is called **bounded** provided it has a smallest element 0 and a greatest element 1. The **decreasing segment** $(a]$ of L is defined as the set $\{ x \in L : x \leq a\}$. The increasing segment $[a)$ is defined dualy. A subset X of L is called **down-directed (upper-directed)** iff for all $a, b \in X$, there exists $x \in X$ such that $x \leq a$ and $x \leq b$ ($a \leq x$ and $b \leq x$).

Lemma 2.2.1 *Let L be a lattice and X be a down (upper) directed subset of L such that X does not have a minimum (maximum) element. If $\mathcal{F}$ is the implicative filter in $\mathcal{P}(X)$ generated by the decreasing (increasing) segments of X, then there exists a non-principal ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subseteq \mathcal{U}$.*

Proof: Let $(a]$, $(b]$ be decreasing segments of X. Since X is a down-directed subset, there exists $x \in X$ such that $x \leq a$ and $x \leq b$, whence $x \in (a] \cap (b]$ and $\mathcal{F}$ is a proper implicative filter of $\mathcal{P}(X)$. By the ultrafilter theorem there exists an ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subseteq \mathcal{U}$. Suppose that $\mathcal{U}$ is the principal filter generated by $(c]$. Since X does not have a minimum element, there exists $x \in X$ such that $x < c$. Thus $(x] \in \mathcal{U}$ and it is a proper subset of $(c]$, a contradiction. Hence $\mathcal{U}$ is not a principal filter. By duality, we can establish the same result when X is an upper-directed set. □

Definition 2.2.2 A variety $\mathcal{V}$ of algebras has *lattice-terms* iff there are terms of the language of $\mathcal{V}$ defining on each $A \in \mathcal{V}$ operations $\vee$, $\wedge$, such that $\langle A, \vee, \wedge\rangle$ is a lattice. $\mathcal{V}$ has *bounded lattice-terms* if, moreover, there are two constant terms 0,1 of the language of $\mathcal{V}$ defining on each $A \in \mathcal{V}$ a bounded lattice $\langle A, \vee, \wedge, 0, 1\rangle$. The order in A, denoted by $L(A)$, is called the *natural order* of A.

Observe that each subvariety of a variety with (bounded) lattice-terms is also a variety with (bounded) lattice-terms.

Remark 2.2.3 Let $\mathcal{V}$ be a variety with lattice-terms and $A \in \mathcal{V}$. $A^X/\mathcal{U}$ will always denote the ultrapower corresponding to a down (upper) directed set X of A with respect to the natural order, without smallest (greatest) element and a non-principal ultrafilter $\mathcal{U}$ of $\mathcal{P}(X)$, containing the filter generated by the decreasing (increasing) segments of X. For each $f \in A^X$, $[f]$ will denote the $\mathcal{U}$-equivalence class of f. Thus $[1_X]$ is the $\mathcal{U}$-equivalence class of the canonical injection $X \hookrightarrow A$ and for each $a \in A$, $[a]$ is the $\mathcal{U}$-equivalence class of the constant function a in A^X. It is well-known that $i_A(a) = [a]$ defines a monomorphism $A \rightarrow A^X/\mathcal{U}$ (see [8, Corollary 4.1.13]).

Theorem 2.2.4 *Let $\mathcal{V}$ be a variety with lattice-terms. If there exists an absolute retract A in $\mathcal{V}$, then each down-directed subset $X \subseteq A$ has an infimum, denoted by $\bigwedge X$. Moreover if $P(x)$ is a first-order positive formula* (see [8]) *of the language of $\mathcal{V}$ such that each $a \in X$ satisfies $P(x)$, then $\bigwedge X$ also satisfies $P(x)$.*

Proof: Let X be a down-directed subset of the absolute retract A. Suppose that X does not admit a minimum element and consider an ultrapower $A^X/\mathcal{U}$. Since A is an absolute retract there exists a homomorphism φ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle i_A}\downarrow & \equiv & \nearrow \\ A^X/\mathcal{U} & & \varphi \end{array}$$

We first prove that $\varphi([1_X])$ is a lower bound of X. Let $a \in X$. Then $[1_X] \leq [a]$ since $\{x \in X : 1_X(x) \leq a(x)\} = \{x \in X : x \leq a\} \in \mathcal{U}$. Thus $\varphi([1_X]) \leq \varphi([a]) = a$ and $\varphi([1_X])$ is a lower bound of X. We proceed now to prove that $\varphi([1_X])$ is the greatest lower bound of X. In fact, if $b \in A$ is a lower bound of X then for each $x \in X$ we have $b \leq x$. Thus $[b] \leq [1_X]$ since $\{x \in X : b(x) \leq 1_X(x)\} = \{x \in X : b \leq x\} = X \in \mathcal{U}$. Now we have $b = \varphi([b]) \leq \varphi([1_X])$. This proves that $\varphi([1_X]) = \bigwedge X$. If each $a \in X$ satisfies the first order formula $P(x)$ then $[1_X]$ satisfies $P(x)$ and, since $P(x)$ is a positive formula, it follows from ([8, Theorem 3.2.4]) that $\varphi([1_X])$ satisfies $P(x)$. □

In the same way, we can establish the dual version of the above theorem. Recalling that a lattice is complete iff there exists the infimum $\bigwedge X$ (supremum $\bigvee X$), for each down-directed (upper-directed) subset X, we have the following corollary:

Corollary 2.2.5 *Let $\mathcal{V}$ be a variety with lattice-terms. If A is an absolute retract in $\mathcal{V}$, then $L(A)$ is a complete lattice.* □

Chapter 3

Injectives in Residuated Lattices

3.1 Radical-dense varieties

Definition 3.1.1 We will say that a variety $\mathcal{A}$ is *radical − dense* [22] provided that $\mathcal{A}$ is a subvariety of $\mathcal{RL}$ and $Rad(A) = Ds(A)$ for each A in $\mathcal{A}$.

An example of radical-dense variety is the variety of Heyting algebras (i.e $\mathcal{RL}$ plus the equation $x \odot y = x \wedge y$). The variety of Heyting algebras is noted by $\mathcal{H}$.

Theorem 3.1.2 *Let $\mathcal{A}$ be a radical-dense variety. If A is a non-semisimple absolute retract in $\mathcal{A}$, then A has a principal unity ϵ and $\{0, \epsilon, 1\}$ is a subalgebra of A isomorphic to the three element Heyting algebra H_3.*

Proof: Let A be a non-semisimple absolute retract. Unities are characterized by the first order positive formula $\neg x = 0$ because $Rad(A) = Ds(A)$. Since $Ds(A)$ is a down-directed set, by Theorem 2.2.4 there exists a minimum dense element ϵ. It is clear that ϵ is the principal unity and since $\epsilon < 1$, $\{0, \epsilon, 1\}$ is a subalgebra of A, which coincides with the three element Heyting algebra H_3. □

Definition 3.1.3 Let $\mathcal{A}$ be a radical-dense variety. An algebra $T \in \mathcal{A}$ is called a *$test_d$-algebra* iff there are $\epsilon, t \in Rad(T)$ such that ϵ is an idempotent element, $t < \epsilon$ and $\epsilon \to t \leq \epsilon$.

An important example of a $test_d$-algebra is the totally ordered four element Heyting algebra $H_4 = \{0 < b < a < 1\}$ whose operations are given as follows:

$$x \odot y = x \wedge y,$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases}$$

Theorem 3.1.4 *Let $\mathcal{A}$ be a radical-dense variety. If $\mathcal{A}$ has a non-trivial injective and contains a $test_d$-algebra T, then all injectives in $\mathcal{A}$ are semisimple.*

Proof: Suppose that there exists a non-semisimple injective A in $\mathcal{A}$. Then by Lemma 3.1.2, there is a monomorphism $\alpha : H_3 \to A$ such that $\alpha(a)$ is the principal unity in A. Let $i : H_3 \to T$ be the monomorphism such that $i(a) = \epsilon$. Since A is injective, there exists a homomorphism $\varphi : T \to A$ such that the following diagram commutes

$$\begin{array}{ccc} H_3 & \xrightarrow{\alpha} & A \\ {\scriptstyle i}\downarrow & \equiv & \nearrow {\scriptstyle \varphi} \\ T & & \end{array}$$

Since $\alpha(a)$ is the principal unity in A and $t \leq \epsilon$, then, by commutativity, $\varphi(\epsilon) = \varphi(t) = \alpha(a)$. Thus $\varphi(\epsilon \to t) = 1$, which is a contradiction since by hypothesis $\varphi(\epsilon \to t) \leq \varphi(\epsilon) = \alpha(a) < 1$. Hence $\mathcal{A}$ has only semisimple injectives. □

3.2 Injectives in $\mathcal{RL}$, $\mathcal{GM}$, $\mathcal{DRL}$ and $\mathcal{DGM}$

Proposition 3.2.1 *Let A be a residuated lattice. Then the set $A^\diamond = \{(a, b) \in A \times A : a \leq b\}$ equipped with the operations*

$$\begin{aligned}
(a_1, b_1) \wedge (a_2, b_2) &:= (a_1 \wedge a_2, b_1 \wedge b_2), \\
(a_1, b_1) \vee (a_2, b_2) &:= (a_1 \vee a_2, b_1 \vee b_2), \\
(a_1, b_1) \odot (a_2, b_2) &:= (a_1 \odot a_2, (a_1 \odot b_2) \vee (a_2 \odot b_1)), \\
(a_1, b_1) \to (a_2, b_2) &:= ((a_1 \to a_2) \wedge (b_1 \to b_2), a_1 \to b_2).
\end{aligned}$$

is a residuated lattice, and the following properties hold:

1. *The map $i : A \to A^\diamond$ defined by $i(a) = (a,a)$ is a monomorphism.*

2. $\neg(a,b) = (\neg b, \neg a)$ *and* $\neg(0,1) = (0,1)$.

3. *A is a Girard monoid iff $A^\diamond$ is a Girard monoid.*

4. *A is distributive iff $A^\diamond$ is distributive.*

Proof: See [27, IV Lemma 3.2.1]. □

Definition 3.2.2 *We say that a subvariety $\mathcal{A}$ of $\mathcal{RL}$ is* **$\diamond$-closed** *iff for all $A \in \mathcal{A}$, $A^\diamond \in \mathcal{A}$.*

Theorem 3.2.3 *If a subvariety $\mathcal{A}$ of $\mathcal{RL}$ is* **$\diamond$-closed,** *then $\mathcal{A}$ has only trivial absolute retracts.*

Proof: Suppose that there exists a non-trivial absolute retract A in $\mathcal{A}$. Then by Proposition 3.2.1 there exists an epimorphism $f : A^\diamond \to A$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle i}\downarrow & \equiv & \nearrow \\ A^\diamond & & f \end{array}$$

Thus there exists $a \in A$ such that $f(0,1) = a = f(a,a)$. Since $(0,1)$ is a fixed point of the negation in $A^\diamond$ it follows that $0 < a < 1$. We have $f(a,1) = 1$. Indeed, $(0,1) \to (a,a) = ((0 \to a) \wedge (1 \to a), 0 \to a) = (a,1)$. Thus $f(a,1) = f((0,1) \to (a,a)) = f(0,1) \to f(a,a) = a \to a = 1$. In view of this we have $1 = f(a,1) \odot f(a,1) = f((a,1) \odot (a,1)) = f(a \odot a, (a \odot 1) \vee (a \odot 1)) = f((a \odot a, a)) \leq f((a,a)) = a$, which is a contradiction since $a < 1$. Hence $\mathcal{A}$ has only trivial absolute retracts. □

Corollary 3.2.4 *$\mathcal{RL}$, $\mathcal{GM}$, $\mathcal{DRL}$ and $\mathcal{DGM}$ have only trivial absolute retracts and injectives.* □

3.3 Injectives in SRL-algebras

Definition 3.3.1 *A SRL-algebra is a residuated lattice satisfying the equation:*

$$(S) \qquad x \wedge \neg x = 0$$

The variety of SRL-algebras is denoted by $\mathcal{SRL}$.

Proposition 3.3.2 *If A is a SRL-algebra, then 0 is the only nilpotent in A.*

Proof: Suppose that there exists a nilpotent element x in A such that $0 < x$, having nilpotence order equal to n. By the residuation property we have $x^{n-1} \leq \neg x$. Thus $x^{n-1} = x \wedge x^{n-1} \leq x \wedge \neg x = 0$, which is a contradiction since x has nilpotence order equal to n. □

Corollary 3.3.3 *Let $\mathcal{A}$ be a subvariety of $\mathcal{SRL}$. Then the two-element boolean algebra is the maximum simple algebra in $\mathcal{A}$ and $Sem(\mathcal{A}) = \mathcal{BA}$.*

Proof: Follows from Propositions 3.3.2 and 1.2.6. □

Corollary 3.3.4 *If $\mathcal{A}$ is a subvariety of $\mathcal{SRL}$ then $\mathcal{A}$ is a radical-dense variety.*

Proof: Let A be an algebra in $\mathcal{A}$ and let a be a unity. Thus $\neg a$ is nilpotent and hence $\neg a = 0$. □

Corollary 3.3.5 *If $\mathcal{A}$ is a subvariety of $\mathcal{SRL}$, then all complete boolean algebras are injectives in $\mathcal{A}$.*

Proof: By Corollary 3.3.3 the two-element boolean algebra is the maximum simple algebra in $\mathcal{A}$. Since it is self-injective, by Theorem 2.1.5 it is injective. Since complete boolean algebras are the retracts of powers of the two-element boolean algebra, the result is proved. □

As an application of this theorem we prove the following results :

Corollary 3.3.6 *In $\mathcal{SRL}$ and $\mathcal{H}$, the only injectives are complete boolean algebras.*

Proof: Follows from Corollary 3.3.5 and Theorem 3.1.4 because the $test_d$-algebra H_4 belongs to both varieties. □

Remark 3.3.7 The fact that injective Heyting algebras are exactly complete boolean algebras was proved in [3] by different arguments.

3.4 MTL-algebras and absolute retracts

Definition 3.4.1 An *MTL-algebra* [19] is a residuated lattice satisfying the pre-linearity equation

$$(Pl) \qquad (x \to y) \vee (y \to x) = 1$$

The variety of MTL-algebras is denoted by $\mathcal{MTL}$.

Proposition 3.4.2 *Let A be a residuated lattice. Then the following conditions are equivalent:*

1. *$A \in \mathcal{MTL}$.*
2. *A is a subdirect product of linearly ordered residuated lattices.*

Proof: [27, Theorem 4.8 p. 76]. □

Corollary 3.4.3 *$\mathcal{MTL}$ is subvariety of $\mathcal{DRL}$.* □

Corollary 3.4.4 *Let A be a MTL-algebra.*

1. *If A is simple, then A is linearly ordered.*
2. *If e is a unity in A, then $\neg e < e$.*

Proof: 1) Is an immediate consequence of Proposition 3.4.2. 2) If we consider that the ith-coordinate $\pi_i f(e)$ of e in the subdirect product $f : A \to \prod_{i\in I} L_i$ is a unity, for each $i \in I$, then by Proposition 1.2.10, $\neg\pi_i f(e) < \pi_i f(e)$. Thus $\neg e < e$. □

To obtain the analog of Theorem 3.1.2 for varieties of MTL-algebras, we cannot use directly Theorem 2.2.4, because the property of being a unity is not a first order property. We need to adapt the proof of Theorem 3.1.2 to this case:

Theorem 3.4.5 *Let $\mathcal{A}$ be a subvariety of $\mathcal{MTL}$. If A is an absolute retract in $\mathcal{A}$ then A has a principal unity e in A.*

Proof: By Proposition 3.4.2 we can consider a subdirect embedding $f : A \to \prod_{i\in I} L_i$ such that L_i is linearly ordered. We define a family $H(L_i)$ in $\mathcal{A}$ as follows: for each $i \in I$

(a) if there exists $e_i = min\{u \in L_i : u \, is \, unity\}$ then $H(L_i) = L_i$,

(b) otherwise, $X = \{u \in L_i : u \, is \, unity\}$ is a down-directed set without least element. Then by Proposition 2.2.4 we can consider an ultraproduct $L_i^X/_{\mathcal{U}}$ of the kind considered after Definition 2.2.2. We define $H(L_i) = L_i^X/_{\mathcal{U}}$. It is clear that $H(L_i)$ is a linearly ordered $\mathcal{A}$-algebra. If we take the class $e_i = [1_X]$ then e_i is a unity in $H(L_i)$ since for every natural number n, $0 < e_i^n$ iff $\{x \in X : 0 < (1_X(x))^n\} \in \mathcal{U}$ and $\{x \in X : 0 < (1_X(x))^n = x^n\} = X \in \mathcal{U}$.

We can take the canonical embedding $j_i : L_i \to H(L_i)$ and then for each $i \in I$ we can consider e_i as a unity lower bound of L_i in $H(L_i)$. By Corollary 1.2.11, $(e_i)_{i\in I}$ is a unity in $\prod_{i\in I} H(L_i)$. Let $j : \prod_{i\in I} L_i \to \prod_{i\in I} H(L_i)$ be the monomorphism defined by $j((x_i)_{i\in I}) = (j_i(x_i))_{i\in I}$. Since A is an absolute retract there exists an epimorphism $\varphi : \prod_{i\in I} H(L_i) \to A$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & \prod_{i\in I} L_i & \xrightarrow{j} & \prod_{i\in I} H(L_i) \\ & & \equiv & & \downarrow \varphi \\ & & \xrightarrow{1_A} & & A \end{array}$$

Let $e = \varphi((e_i)_{i\in I})$. It is clear that e is a unity in A since φ is an homomorphism. If u is a unity in A then $(e_i)_{i\in I} \leq jf(u)$ and by commutativity of the above diagram, $e = \varphi((e_i)_{i\in I}) \leq \varphi jf(u) = u$. Thus $e = min\{u \in A : u \, is \, unity\}$ resulting in $Rad(A) = [e)$.

□

3.5 Injectives in WNM-algebras and $\mathcal{MTL}$

Definition 3.5.1 A *WNM-algebra* (weak nilpotent minimum) [19] is an MTL-algebra satisfying the equation

$$(W) \qquad \neg(x \odot y) \vee ((x \wedge y) \to (x \odot y)) = 1.$$

The variety of WNM-algebras is noted by $\mathcal{WNM}$.

Theorem 3.5.2 *The following conditions are equivalent:*

1. *I is a simple WNM-algebra.*

2. *I has a coatom u and its operations are given by*

$$x \odot y = \begin{cases} 0, & \text{if } x, y < 1 \\ x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \end{cases}$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x = 1 \\ u, & \text{if } y < x < 1. \end{cases}$$

Proof: $\Rightarrow$). For $Card(I) = 2$ this result is trivial. If $Card(I) > 2$ then we only need to prove the following steps:

a) *If $x, y < 1$ in I then $x \odot y = 0$*: Since I is simple, equation (W) implies that $x^2 = 0$ for each $x \in I \setminus \{1\}$. Hence if $x \leq y < 1$, then $x \odot y \leq y \odot y = 0$.

b) *I has a coatom*: Let $0 < x < 1$. We have that $\neg x < 1$ and, since I is simple, we also have $\neg\neg x < 1$. Then by a) it follows that $\neg x \leq \neg\neg x \leq \neg\neg\neg x = \neg x$, i. e., $\neg x = \neg\neg x$. If $0 < x, y < 1$, again by a) we have $\neg x \odot \neg y = 0$. Thus $\neg x \leq \neg\neg y = \neg y$. By interchanging x and y we obtain the equality $\neg x = \neg y$. Now it is clear that if $0 < x < 1$, then $u = \neg x$ is the coatom in I.

c) *If $y < x < 1$ then $x \to y = u$*: Since $x \to y = \bigvee\{t \in I : t \odot x \leq y\}$, this supremum cannot be 1 because $y < x$. Thus, in view of item a), $x \to y$ is the coatom u.

$\Leftarrow$) Immediate.

□

Example 3.5.3 We can build simple WNM-algebras having arbitrary cardinality if we consider an ordinal $\gamma = Suc(Suc(\alpha))$ with the structure given by Proposition 3.5.2, taking $Suc(\alpha)$ as coatom. These algebras will be called *ordinal algebras*.

Proposition 3.5.4 *$\mathcal{WNM}$ and $\mathcal{MTL}$ have only trivial injectives.*

Proof: Follows from Proposition 2.1.4 since these varieties contain all ordinal algebras. □

3.6 Injectives in SMTL-algebras

Definition 3.6.1 An *SMTL-algebra* [20] is a MTL-algebra satisfying equation (S). The variety of SMTL-algebras is denoted by $\mathcal{SMTL}$.

Proposition 3.6.2 *The only injectives in $\mathcal{SMTL}$ are complete boolean algebras.*

Proof: Follows from Corollary 3.3.5 and Theorem 3.1.4 since the $test_d$-algebra H_4 belongs to $\mathcal{SMTL}$. □

3.7 Injectives in ΠSMTL-algebras

Definition 3.7.1 A *ΠSMTL-algebra* [19] is a SMTL-algebra satisfying the equation:

$$(\Pi) \qquad (\neg\neg z \odot ((x \odot z) \rightarrow (y \odot z))) \rightarrow (x \rightarrow y) = 1.$$

The variety of ΠSMTL-algebras is denoted by $\Pi\mathcal{SMTL}$.

Proposition 3.7.2 *Let A be an ΠSMTL-algebra. Then 1 is the only idempotent dense element in A.*

Proof: By equation Π it is easy to prove that, for each dense element ϵ, if $\epsilon \odot x = \epsilon \odot y$ then $x = y$. Thus if ϵ is an idempotent dense then $\epsilon \odot 1 = \epsilon \odot \epsilon$ and $\epsilon = 1$.

Theorem 3.7.3 *Let $\mathcal{A}$ be a subvariety of $\Pi\mathcal{SMTL}$. Then the injectives in $\mathcal{A}$ are exactly the complete boolean algebras.*

Proof: Follows from Corollary 3.3.5, Theorem 3.1.2 and Proposition 3.7.2.

3.8 Injectives in BL, MV, PL, and in Linear Heyting algebras

Definition 3.8.1 A *BL-algebra* [26] is an MTL-algebra satisfying the equation

$$(B). \qquad x \odot (x \rightarrow y) = x \wedge y$$

We denote by $\mathcal{BL}$ the variety of BL-algebras. Important subvarieties of $\mathcal{BL}$ are the variety $\mathcal{MV}$ of multi-valued logic algebras (MV-algebras for short), characterized by the equation $\neg\neg x = x$ [10, 26], the variety $\mathcal{PL}$ of product logic algebras (PL-algebras for short), characterized by the equations (Π) plus (S) [26, 11], and the variety $\mathcal{HL}$ of linear Heyting algebras, characterized by the equation $x \odot y = x \wedge y$ (also known as Gödel algebras [26]).

Remark 3.8.2 It is well-known that $\mathcal{MV}$ is generated by the MV-algebra $R_{[0,1]} = \langle [0,1], \odot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ such that $[0,1]$ is the real unit segment, $\wedge$, $\vee$ are the natural meet and join on $[0,1]$ and $\odot$ and $\rightarrow$ are defined as follows: $x \odot y := max(0, x + y - 1)$, $x \rightarrow y := min(1, 1 - x + y)$. $R_{[0,1]}$ is the maximum simple algebra in $\mathcal{MV}$ (see [10, Theorem 3.5.1]). Moreover $R_{[0,1]}$ is a rigid algebra (see [10, Corollary 7.2.6]), hence self-injective. Injective MV-algebras were characterized in [25, Corollary 2.11]) as the retracts of powers of $R_{[0,1]}$.

Proposition 3.8.3 *If $\mathcal{A}$ is a subvariety of $\mathcal{PL}$, then the only injectives of $\mathcal{A}$ are the complete boolean algebras.*

Proof: Follows from Theorem 3.7.3 since $\mathcal{PL}$ is a subvariety of $\Pi\mathcal{SMTL}$. □

Proposition 3.8.4 *The only injectives in $\mathcal{HL}$ are the complete boolean algebras.*

Proof: Follows from Corollary 3.3.5 and Theorem 3.1.4 since the algebra $test_d\ H_4$ lies in $\mathcal{SMTL}$. □

Proposition 3.8.5 *$\mathcal{BL}$ is a radical-dense variety.*

Proof: See [12, Theorem 1.7 and Remark 1.9]. □

Proposition 3.8.6 *Injectives in $\mathcal{BL}$ are exactly the retracts of powers of the MV-algebra $R_{[0,1]}$.*

Proof: By Remark 3.8.2 and Propositions 3.8.5 and 2.1.5, retracts of a power of the $R_{[0,1]}$ are injectives in $\mathcal{BL}$. Thus by Theorem 3.1.4, they are the only possible injectives since H_4 lies in $\mathcal{BL}$. □

3.9 Injectives in IMTL-algebras

Definition 3.9.1 An involutive MTL-algebra (or *IMTL-algebra*) [19] is a MTL-algebra satisfying the equation

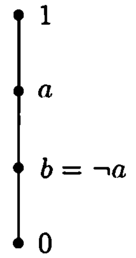
$$(I) \qquad \neg\neg x = x.$$

The variety of IMTL-algebras is noted by $\mathcal{IMTL}$.

An interesting IMTL-algebra, whose role is analogous to H_3 in the radical-dense varieties, is the four element chain I_4 defined as follows:

$\odot$	1	a	b	0
1	1	a	b	0
a	a	a	0	0
b	b	0	0	0
0	0	0	0	0

$\rightarrow$	1	a	b	0
1	1	a	b	0
a	1	1	b	b
b	1	1	1	a
0	1	1	1	1



Theorem 3.9.2 *Let $\mathcal{A}$ be a a subvariety of $\mathcal{IMTL}$. If A is a non-semisimple absolute retract in $\mathcal{A}$, then A has a principal unity ϵ and $\{0, \neg\epsilon, \epsilon, 1\}$ is a subalgebra of A which is isomorphic to I_4.*

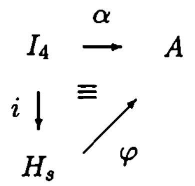
Proof: Follows from Theorem 3.4.5. □

Definition 3.9.3 Let $\mathcal{A}$ be a subvariety of $\mathcal{IMTL}$. An algebra T is called a *test$_I$-algebra* iff, it has a subalgebra $\{0, \neg\epsilon, \epsilon, 1\}$ isomorphic to I_4 and there exists $t \in Rad(T)$ such that $t < \epsilon$.

Theorem 3.9.4 *Let $\mathcal{A}$ be a subvariety of $\mathcal{IMTL}$. If $\mathcal{A}$ has a non-trivial injective and contains a test$_I$-algebra, then injectives are semisimple.*

Proof: Let T be a *test$_I$*-algebra and $t \in Rad(T_i)$ such that $t < \epsilon$. We can consider a subdirect embedding $f : T \rightarrow \prod_{j\in J} H_j$ such that L_j is linearly ordered. Let $x_j = \pi_j f(x)$ for each $x \in T$ and π_j the jth-projection. Since $t < \epsilon$, exists $s \in J$ such that $\neg\epsilon_s < \neg t_s < t_s < \epsilon_s$ and by Corollary 1.2.11 , t_s and ϵ_s are unities in the chain H_s with ϵ_s idempotent. Note that H_s is also a *test$_I$*-algebra. To see that $\epsilon_s \rightarrow t_s \leq \epsilon$, observe first that $0 < \epsilon_s \odot \neg t_s$ since, if $\epsilon_s \odot \neg t_s = 0$ then $\epsilon_s \leq \neg\neg t_s = t_s$ which is a contradiction. Consequently,

$\neg\epsilon_s \leq \epsilon_s \odot \neg t_s$ since, if $\epsilon_s \odot \neg t_s \leq \neg\epsilon_s$ then $\epsilon_s \odot \neg t_s = (\epsilon_s)^2 \odot \neg t_s \leq \neg\epsilon \odot \epsilon = 0$. Thus we can conclude that $\epsilon_s \to t_s = \neg(\epsilon_s \odot \neg t_s) \leq \neg\neg\epsilon_s = \epsilon_s$. Suppose that there exists a non-semisimple injective A in $\mathcal{A}$. Then by Theorem 3.9.2, let $\alpha : I_4 \to A$ be a monomorphism such that $\alpha(a)$ is the principal unity in A. Let $i : I_4 \to H_s$ be the monomorphism such that $i(a) = \epsilon_s$. Since A is injective, there exists a homomorphism $\varphi : H_s \to A$ such that the following diagram commutes:



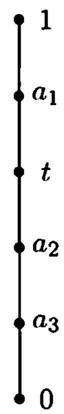
Since $\alpha(a)$ is the principal unity in A and $t_s \leq \epsilon_s$ then, by commutativity, $\varphi(\epsilon_s) = \varphi(t_s) = \alpha(a)$. Thus $\varphi(\epsilon_s \to t_s) = 1$, which is a contradiction since $\varphi(\epsilon_s \to t_s) \leq \varphi(\epsilon_s) = \alpha(a) < 1$. Hence $\mathcal{A}$ has only semisimple injectives. □

Proposition 3.9.5 *$\mathcal{IMTL}$ has only trivial injectives.*

Proof: Suppose that there exists non-trivial injectives in $\mathcal{IMTL}$. By Theorem 2.1.4 there is a simple maximum algebra I in $\mathcal{IMTL}$. We consider the six elements $IMTL$ chain I_6 defined as follows:

$\odot$	1	a_1	t	a_2	a_3	0
1	1	a_1	t	a_2	a_3	0
a_1	a_1	a_2	a_3	a_3	0	0
t	t	a_3	a_3	0	0	0
a_2	a_2	a_3	0	0	0	0
a_3	a_3	0	0	0	0	0
0	0	0	0	0	0	0

$\to$	1	a_1	t	a_2	a_3	0
1	1	a_1	t	a_2	a_3	0
a_1	1	1	a_1	a_1	t	a_3
t	1	1	1	a_1	a_1	a_2
a_2	1	1	1	1	a_1	t
a_3	1	1	1	1	1	a_1
0	1	1	1	1	1	1



Since I is simple maximum we can consider I_6 and $R_{[0,1]}$ as subalgebras of I. In view of this and using the nilpotence order we have that $1/2 < t < 3/4$ since I is a chain. Therefore we can consider $u = \bigvee_{R_{[0,1]}}\{x \in R_{[0,1]} : x < t\}$ and $v = \bigwedge_{R_{[0,1]}}\{x \in R_{[0,1]} : x > t\}$ and it is clear that $u, v \in R_{[0,1]}$ since $R_{[0,1]}$ is a complete algebra. Thus $u < t < v$. This contradicts the fact that the order of $R_{[0,1]}$ is dense. Consequently $\mathcal{IMTL}$ has only trivial injectives. □

3.10 Injectives in NM-algebras

Definition 3.10.1 A nilpotent minimum algebra (or *NM-algebra*) [19] is an IMTL-algebra satisfying the equation (W).

The variety of NM-algebras is noted by $\mathcal{NM}$. As an example we consider $N_{[0,1]} = \langle [0,1], \odot, \to, \wedge, \vee, 0, 1\rangle$ such that $[0,1]$ is the real unit segment, $\wedge$, $\vee$ are the natural meet and join on $[0,1]$ and $\odot$ and $\to$ are defined as follows:

$$x \odot y = \begin{cases} x \wedge y, & \text{if } 1 < x + y \\ 0, & \text{otherwise,} \end{cases}$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \\ max(y, 1-x) & \text{otherwise .} \end{cases}$$

Note that $\{0, \frac{1}{2}, 1\}$ is the universe of a subalgebra of $N_{[0,1]}$, that we denote by L_3. The subvariety of $\mathcal{NM}$ generated by L_3 coincides with the variety $\mathcal{L}_3$ of three-valued Lukasiewicz algebras (see [37, 9]).

Proposition 3.10.2 *L_3 is the maximum simple algebra in $\mathcal{NM}$, and it is self-injective.*

Proof: Let I be a simple algebra such that $Card(I) > 2$. By Theorem 3.5.2 I has a coatom u satisfying $\neg x = u$ for each $0 < x < 1$. Thus $x = \neg\neg x = \neg u = u$ for each $0 < x < 1$. Consequently $Card(I) = 3$ and $I = L_3$. □

Corollary 3.10.3 $Sem(\mathcal{NM}) = \mathcal{L}_3$. □

Proposition 3.10.4 *Injectives in $\mathcal{NM}$ coincide with complete Post algebras of order* 3.

Proof: By Proposition 3.5.2, Theorem 2.1.5 and Theorem 3.9.4 injectives in $\mathcal{NM}$ are semisimple since $N_{[0,1]}$ is an algebra $Test_I$. Thus by Proposition 3.10.3 and [37], [9, Theorem 3.7], complete Post algebras of order 3 are the injectives in $\mathcal{NM}$. □

Variety	Equations	Injectives
$\mathcal{RL}$		Trivial
$\mathcal{DRL}$	$\mathcal{RL} + x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	Trivial
$\mathcal{GM}$	$\mathcal{RL} + \neg\neg x = x$	Trivial
$\mathcal{DGM}$	$\mathcal{GM} + x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	Trivial
$\mathcal{MTL}$	$\mathcal{RL} + (x \to y) \vee (y \to x) = 1$	Trivial
$\mathcal{WNM}$	$\mathcal{MTL} + \neg(x \odot y) \vee ((x \wedge y) \to (x \odot y)) = 1$	Trivial
$\mathcal{IMTL}$	$\mathcal{MTL} + \neg\neg x = x$	Trivial
$\mathcal{BL}$	$\mathcal{MTL} + x \wedge y = x \odot (x \to y)$	Retracts of powers of $R_{[0,1]}$
$\mathcal{MV}$	$\mathcal{BL} + \neg\neg x = x$	Retracts of powers of $R_{[0,1]}$
$\mathcal{SRL}$	$\mathcal{RL} + x \wedge \neg x = 0$	Complete boolean algebras
$\mathcal{SMTL}$	$\mathcal{MTL} + x \wedge \neg x = 0$	Complete boolean algebras
$\Pi\mathcal{SMTL}$	$\mathcal{SMTL} + \neg\neg z \odot ((x \odot z) \to (y \odot z)) \leq (x \to y)$	Complete boolean algebras
$\mathcal{PL}$	$\Pi\mathcal{SMTL} + x \wedge y = x \odot (x \to y)$	Complete boolean algebras
$\mathcal{HL}$	$\mathcal{BL} + x \wedge y = x \odot y$	Complete boolean algebras
$\mathcal{NM}$	$\mathcal{WNM} + \neg\neg x = x$	Complete Post algebras of order 3

Table 3.1: Injectives in Varieties of Residuated Lattices

Chapter 4

Injectives in Pocrims and Hoops

4.1 Absolute retracts in pocrims

Proposition 4.1.1 *Let A be a pocrim and $\bot$ be a new symbol not belonging to A. We can consider $\bot \oplus A = A \cup \{\bot\}$ with the following operation:*

$$x \odot_\bot y = \begin{cases} x \odot y, & \text{if } x, y \in A \\ \bot, & \text{if } x = \bot \text{ or } y = \bot \end{cases}$$

$$x \to_\bot y = \begin{cases} x \to y, & \text{if } x, y \in A \\ \bot, & \text{if } x \in A \text{ and } y = \bot \\ 1, & \text{if } x = \bot \end{cases}$$

Then $\langle \bot \oplus A, \odot_\bot, \to_\bot, 1 \rangle$ is a pocrim with smallest element $\bot$, and A is a subalgebra of $\bot \oplus A$.

Proof: Immediate □

Definition 4.1.2 Let $\mathcal{A}$ be a relative subvariety of $\mathcal{M}$. Then we say that $\mathcal{A}$ is *$(\bot \oplus)$-closed* iff for all $A \in \mathcal{A}$, $\bot \oplus A \in \mathcal{A}$

Theorem 4.1.3 *If $\mathcal{A}$ is a $(\bot \oplus)$-closed relative subvariety of $\mathcal{M}$, then absolute retracts in $\mathcal{A}$ are trivial algebras.*

Proof: Suppose that there exists a non-trivial absolute retract A in $\mathcal{A}$. Let $i : A \to \bot \oplus A$ be the monomorphism such that $i(x) = x$. Then there exists an epimorphism $f :\perp \oplus A \to A$ such that the composition $fi = 1_A$. Let $0 = f(\perp)$. Since for all $x \in A$, $0 = f(\perp) \leq f(i(x)) = x$, we have that 0 is the smallest element of A. In $\perp \oplus A$ we have that $0 \to\perp=\perp$. Therefore $f(0 \to\perp) = f(\perp) = 0$. On the other hand, since $i(0) = 0$, $f(0) \to f(\perp) = 0 \to 0 = 1$. Hence $0 = f(0 \to\perp)$ but $f(0) \to f(\perp) = 1$, which is a contradiction. Consequently $\mathcal{A}$ have only trivial absolute retracts. □

Corollary 4.1.4 *If $\mathcal{A}$ is a $(\perp \oplus)$-closed relative subvariety of $\mathcal{M}$, then $\mathcal{A}$ has only trivial injectives.* □

Corollary 4.1.5 *$\mathcal{M}$, $\mathcal{HO}$, $\mathcal{HO}(k)$, $\mathcal{BH}$ have only trivial absolute retract and trivial injectives.* □

4.2 Injectives in quasivarieties of bounded pocrims

Proposition 4.2.1 *$\mathcal{M}_0$, has only trivial absolute retract and trivial injectives.*

Proof: It follows from the same argument used in Theorem 4.1.3 □

Proposition 4.2.2 *Let $\mathcal{A}$ be a $(\perp \oplus)$-closed relative subvariety of $\mathcal{M}_0$. If B is injective in $\mathcal{A}$ then $Ds(B) \cap Idp(B) = \{1\}$.*

Proof: Let B be an injective in $\mathcal{A}$. If there is an element $a \in Ds(B) \cap Idp(B)$ with $a < 1$, then $\{0, a, 1\}$ would be a subalgebra of B such that $Ds(B) = B\backslash\{0\}$. Extend it to a maximal totally ordered subalgebra C of B such that $Ds(C) = C\backslash\{0\}$, and let $i_C : C \to B$ be defined by $i_C(x) = x$. In the algebra $\perp \oplus C$ we have $\perp< 0$. To avoid confusion, we define $\alpha := 0$. Now we define $f : C \to\perp \oplus C$ such that $f(0) =\perp$ and for each $x > 0$, $f(x) = x$. It easy to verify that f is a monomorphism. Since B is injective there exist a morphism $g :\perp \oplus C \to B$ such thar $gf = i_C$ since B is an injective object. We derive from this the following asertions:

1. $g(\alpha) \in C$ (since C is a maximal subchain of B with the property $Ds(C) = C - \{0\}$),
2. $g(\alpha) \neq 0$ (since $\neg g(\alpha) = g(\neg\alpha) = g(\perp) = 0$),
3. $g(\alpha) < 1$ (since $\alpha < a$ and then $g(\alpha) \leq g(a) = a < 1$).

Now we have that for all $x \in C - \{0\}$, $x \to g(\alpha) = g(x) \to g(\alpha) = g(x \to \alpha) = g(\alpha) < 1$ by item (3). Thus $g(\alpha) < x$. Hence by item (1) we obtain $g(\alpha) < g(\alpha)$ which is an obvious contradiction. Therefore we conclude that $Ds(B) \cap Idp(B) = \{1\}$. □

Proposition 4.2.3 *Let $\mathcal{A}$ be a $(\perp \oplus)$-closed relative subvariety of $\mathcal{M}_0$. If B is injective in $\mathcal{A}$ then $Ds(B) = \{1\}$.*

Proof: Let B be an injective in $\mathcal{A}$. We assume that there is an element $a \in Ds(B)$ with $a < 1$. For all natural number $n \geq 1$, $\neg(a^n) = 0$ since $\neg(a^n) = a^n \to 0 = a^{n-1} \to (a \to 0) = a^{n-1} \to 0 = \cdots = a \to 0 = 0$. Thus $a^n > 0$ for all $n \geq 1$, and then the principal implicative filter $\langle a \rangle$ is proper. Let $A = \langle a \rangle \cup \{0\}$. A is closed by $\neg$ since if $x = 0$ then $\neg x = 1$ and for $x \in \langle a \rangle$ there is exist $n \geq 1$ sucht that $x \geq a^n$ and then $\neg x \leq \neg(a^n) = 0$. Since $\langle a \rangle$ is an implicative filter, this proves that $A \in \mathcal{M}_0$. Let $A_\perp = \perp \oplus A$ and let $g : A \to A_\perp$ be the monomorphism such that $g(0) = \perp$ and $g(x) = x$ if $x \in \langle a \rangle$. Since B is injective, there is exist a morphism $f : A_\perp \to B$ such that:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & B \\ g \downarrow & \equiv & \nearrow f \\ A_\perp & & \end{array}$$

$f(0) \in Ds(B)$ since $\neg f(0) = f(\neg 0) = f(0 \to \perp) = f(\perp) = 0$, and $f(0) < 1$ since $f(0) \leq f(a) = 1_A(a) = a < 1$. Moreover $f(0) \in Idp(B)$ since $f(0) \odot f(0) = f(0 \odot 0) = f(0)$. Thus $f(0) \in Ds(B) \cap Idp(B)$ which is a contadition by Proposition 4.2.2. Therefore $Ds(B) = \{1\}$. □

Theorem 4.2.4 *Let $\mathcal{A}$ be $(\perp \oplus)$-closed relative subvariety of $\mathcal{M}_0$. Then A is injective in $\mathcal{A}$ iff A is injective in $\mathcal{DF}(\mathcal{A})$.*

Proof: If A is injective in $\mathcal{A}$ then by Proposition 1.2.14 $Ds(A) = \{1\}$, thus $A \in \mathcal{DF}(\mathcal{A})$ and A is injective in $\mathcal{A}/Ds$. Conversaly by Propositions 1.2.13 since $\mathcal{DF}(\mathcal{A})$ is a reflective subcategory of $\mathcal{A}$ and the reflector preserves monomorphism. It is well-known that if $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$ such that the reflector preserves monomorphisms then an injective object in $\mathcal{B}$ is also injective in $\mathcal{A}$ [2, I.18]. Thus A is injective in $\mathcal{DF}(\mathcal{A})$ then A is injective in $\mathcal{A}$. □

4.3 Injectives in varieties of bounded hoops

Definition 4.3.1 A *bounded hoop* is a bounded pocrim $\langle A, \odot, \rightarrow, 0, 1\rangle$ such that $\langle A, \odot, \rightarrow, 1\rangle$ is a hoop. It is clear that the class $\mathcal{HO}_0$ of bounded hoops is a variety contained in $\mathcal{M}_0$ whose homomorphism satisfying $\varphi(0) = 0$.

Important subvarieties of $\mathcal{HO}_0$ are $\mathcal{BL}$, $\mathcal{SBL}$, $\mathcal{PL}$, $\mathcal{H}$, $\mathcal{BA}$.

Lemma 4.3.2 *Let A be a bounded hoop, then the following assertions are valid:*

1. $x \odot \neg x = 0$,
2. $\neg(\neg\neg x \rightarrow x) = 0$ *i.e.* $\neg\neg x \rightarrow x \in Ds(A)$,
3. $x = \neg\neg x \odot (\neg\neg x \rightarrow x)$.

Proof: 1) $x \odot \neg x = x \odot (x \rightarrow 0) = x \wedge 0 = 0$. 2) Is the same argument used in [13, Lemma 1.3]. 3) $x \leq \neg\neg x$ since $x \odot \neg x = 0$, then $x = x \wedge \neg\neg x = x \odot (\neg\neg x \rightarrow x)$. □

Lemma 4.3.3 *Let A be a residuated lattice, then the following assertions are equivalent*

1. *A is a MV-algebra.*
2. *A is Girard-monoid which satisfy the equations* $x \wedge y = x \odot (x \rightarrow y)$.

Proof: See [27, IV Lemma 2.14] and [28, VI Lemma 2.3] □

Proposition 4.3.4 *If* $A \in \mathcal{HO}_0$ *then* $\mathcal{DF}(A)$ *is a Girard-monoid.*

Proof: Let $A \in \mathcal{A}$ and $[x] \in A/Ds(A)$. By lemma 4.3.2 we have that $[x] = [\neg\neg x] \odot [\neg\neg x \rightarrow x]$ and $\neg\neg x \rightarrow x \in Ds(A)$, thus $[\neg\neg x \rightarrow x] = [1]$ then $[x] = [\neg\neg x]$ i.e. $A/Ds(A)$ is a Girard-monoid. □

Corollary 4.3.5 1. $\mathcal{DF}(\mathcal{HO}_0) = \mathcal{DF}(\mathcal{BL}) = \mathcal{MV}$.

2. $\mathcal{DF}(\mathcal{SBL}) = \mathcal{DF}(\mathcal{H}) = \mathcal{DF}(\mathcal{HL}) = \mathcal{BA}$.

Proof: $\mathcal{DF}(\mathcal{HO}_0)$ and $\mathcal{DF}(\mathcal{BL})$ is $\mathcal{MV}$ since their elements are Girard-monoid satisfying the equation $x \wedge y = x \odot (x \to y)$ (Lemma 4.3.3). The other equalities are immediate. □

Corollary 4.3.6 *1. A is injective in $\mathcal{HO}_0$ or $\mathcal{BL}$ iff A is a retract of a power of the MV-algebra $R_{[0,1]}$.*

2. A is injective in $\mathcal{SBL}$, $\mathcal{H}$ or $\mathcal{HL}$ iff A is a complete boolean algebra.

Proof: Since all these classes are $(\perp \oplus)$-closed, the results follows from Theorem 1.2.14, Corolary 4.3.5 and the well-known characterization of injective MV-algebras (see [25, Corollary 2.11]) and injective boolean algebras [38]. □

In the last corollary we characterize injectives in $\mathcal{BL}$, $\mathcal{SBL}$, $\mathcal{H}$ or $\mathcal{HL}$ by arguments different of those used in section 3.8. We can give another proof about the injectives in $\mathcal{HO}_0$ using arguments of chapter 2 and chapter 3. We need a previous result:

Definition 4.3.7 A *Wajsberg hoop* [5] is a hoop that satisfies the following equation

$$(T) \qquad (x \to y) \to y = (y \to x) \to x.$$

Each Wajsberg hoop is a lattice, in which the join operation is given by $x \vee y = (x \to y) \to y$.

Proposition 4.3.8 *A simple hoop with smallest element* 0 *is a simple MV-algebra.*

Proof: Let I be a simple hoop. Then by [5, Corollary 2.3] it is a totally ordered Wajsberg hoop. If 0 is the smallest element in I then by the equation (T), $\neg\neg x = (x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$. Hence it is an MV-algebra. Since the MV-congruences are in correspondence with implicative filters, I is a simple MV-algebra. □

Proposition 4.3.9 *Let I, J be simple hoops with smallest elements $0_I, 0_J$ respectively. If $\varphi : I \to J$ is a hoop homomorphism then φ is also an MV-homomorphism, i.e., $\varphi(0_I) = 0_J$.*

Proof: Suppose that $\varphi(0_I) = a$. Since J is simple, there exists a natural number n such that $a^n = 0_J$. Thus we have, $\varphi(0_I) = \varphi(0_I^n) = (\varphi(0_I))^n = a^n = 0_J$. □

The following two results are obtained in the same way as Theorems 3.1.2 and 3.1.4 respectively.

Theorem 4.3.10 *Let $\mathcal{A}$ be a subvariety of $\mathcal{HO}_0$. If A is a non-semisimple absolute retract in $\mathcal{A}$, then $Ds(A)$ has a least element ϵ i.e, $Ds(A) = [\epsilon)$ and $\{0, \epsilon, 1\}$ is a subalgebra of A isomorphic to the three element Heyting algebra H_3.* □

Theorem 4.3.11 *Let $\mathcal{A}$ be a subvariety of $\mathcal{HO}_0$. If $\mathcal{A}$ has a non-trivial injectives and contains the Heyting algebra H_4 then injectives are semisimple.* □

Corollary 4.3.12 *Injectives in $\mathcal{HO}_0$ are exactly the retracts of powers of the MV-algebra $R_{[0,1]}$.*

Proof: By Proposition 4.3.8, semisimple bounded hoops are MV-algebras. Therefore $R_{[0,1]}$ is the maximum simple algebra and it is self injective by Proposition 4.3.9. Thus by Theorem 2.1.5 retracts of powers of the MV-algebra $R_{[0,1]}$ are injectives in $\mathcal{HO}_0$. By Theorem 4.3.11 they are the only injectives, because H_4 lies in $\mathcal{HO}_0$. □

Variety	Equations	Injectives
$\mathcal{M}$		Trivial
$\mathcal{M}_0$	$\mathcal{M} + 0 \to x = 1$	Trivial
$\mathcal{HO}$	$\mathcal{M} + (x \to y) \odot x = (y \to x) \odot y$	Trivial
$\mathcal{HO}(k)$	$\mathcal{HO} + x^k = x^{k+1}$	Trivial
$\mathcal{HO}_0$	$\mathcal{HO} + 0 \to x = 1$	Retracts of powers of $R_{[0,1]}$
$\mathcal{BH}$	$\mathcal{HO} + (x \to y) \vee (y \to x) = 1$	Trivial

Table 4.1: Injectives in Pocrims and Hoops

Chapter 5

The Cantor-Bernstein-Schröder Theorem

5.1 Basic notions

We recall from [36] other notions of lattice theory that will play an important role in what follows. Let $L = \langle L, \vee, \wedge \rangle$ be a lattice. Given a, b, c in L, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, (a,b,c)D^* hold for all permutations of a, b, c. In this case we say that $\{a, b, c\}$ is a **distributive triple**. An element z of a lattice L is called a **neutral element** iff for all elements $a, b \in L$ we have $(a, b, z)T$. An element z of a bounded lattice is called a **central element** iff z is a neutral element having a complement, which we shall denote by $\neg z$. The set of all central elements of L is called the **center** of L and is denoted by $Z(L)$. An interval $[a,b]$ of a lattice A is defined as the set $\{ x \in A : a \leq x \leq b\}$. A sequence $(a_n)_{n \in \omega}$ of elements of a lattice L with 0 is called **orthogonal** iff $a_n \wedge a_m = 0$ whenever m, n are distinct elements. In particular, L is called **orthogonally σ-complete** iff, for all orthogonal sequences $(a_n)_{n \in \omega}$, $\bigvee_{n \in \omega} a_n$ exists . A subset S of L is called a **σ-sublattice** of L when it contains with any countable subset X of S also $\bigwedge X$ and $\bigvee X$.

Proposition 5.1.1 *For each bounded lattice L, its center $Z(L)$ is a boolean sublattice of L.* □

Notation: The supremum (infimum) in $Z(L)$ of a family $(a_i)_{i\in I}$ of $Z(L)$, if it exists, will be denoted by $\sqcup_{i\in I} a_i$ ($\sqcap_{i\in I} a_i$), to distinguish it from the supremum $\bigvee_{i\in I} a_i$ (infimum $\bigwedge_{i\in I} a_i$) in L, which need not belong to $Z(L)$.

Definition 5.1.2 *A variety $\mathcal{V}$ of algebras is an $\mathcal{L}$- variety* [21] *iff*

(1) there are terms of the language of $\mathcal{V}$ defining on each $A \in \mathcal{V}$ operations $\vee$, $\wedge$, 0,1 such that $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice;

(2) for all $A \in \mathcal{V}$ and for all $z \in Z(L(A))$, the binary relation Θ_z on A defined by $a\Theta_z b$ iff $a \wedge z = b \wedge z$ is a congruence on A, such that $A \cong A/\Theta_z \times A/\Theta_{\neg z}$.

For an algebra A in an $\mathcal{L}$-variety, we will write simply $Z(A)$ instead of $Z(L(A))$.

Observe that each subvariety of an $\mathcal{L}$-variety is an $\mathcal{L}$-variety.

Definition 5.1.3 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety of algebras of similarity type τ. For all $A \in \mathcal{V}$, all $z \in Z(A)$ and all operation symbols $f \in \tau$, we define $f_z(x_1, \ldots, x_n) = z \wedge f(x_1, \ldots, x_n)$, where n is the arity of f. Moreover, we define $[0, z]_A = \langle [0, z], (f_z)_{f\in\tau}\rangle$.*

Taking into account that for each $f \in \tau$ of arity n and elements $x_1, \ldots, x_n$ in A, $x_i\Theta_z(x_i \wedge z)$ for $i = 1 \ldots n$; we have $f(x_1, \ldots, x_n)\Theta_z f(x_1 \wedge z, \ldots, x_n \wedge z)$, i.e., $f(x_1 \wedge z, \ldots, x_n \wedge z) \wedge z = f(x_1, \ldots, x_n) \wedge z$. Now it is easy to prove the following result:

Proposition 5.1.4 *The correspondence $x/\Theta_z \mapsto x \wedge z$ defines an isomorphism from A/Θ_z onto $[0, z]_A$. Morever, the correspondence $x \mapsto (x \wedge z, x \wedge \neg z)$ defines an isomorphism from A onto $A/\theta_z \times A/\theta_{\neg z}$.* □

5.2 Examples of $\mathcal{L}$-varieties

Example 5.2.1 *The variety $\mathcal{L}_{01}$ of bounded lattices and its subvarieties.* In particular, the subvarieties of modular and of distributive lattices.

Example 5.2.2 A *lattice with involution* [34] is an algebra $\langle L, \vee, \wedge, \sim\rangle$ such that $\langle L, \vee, \wedge\rangle$ is a lattice and $\sim$ is a unary operation on L that fulfils the following conditions:

$$(i)\ \sim\sim x = x \quad \text{and} \quad (ii)\ \sim(x \vee y) = \sim x \wedge \sim y.$$

The variety $\mathcal{L}_i$ of bounded lattices with involution which satisfy the *Kleene equation* (iii) $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$ is an $\mathcal{L}$-variety. Indeed, suppose $L \in \mathcal{L}_i$ and let $z \in Z(L)$. It is clear that Θ_z is a lattice congruence. To see that Θ_z also preserves the operation $\sim$, observe first that $\sim z = \neg z$. Indeed, we have

$$\neg z = \neg z \wedge 1 = \neg z \wedge (\sim z \vee \sim\neg z) = (\neg z \wedge \sim z) \vee (\neg z \wedge \sim\neg z) \leq$$

$$(\neg z \wedge \sim z) \vee (z \vee \sim z) = z \vee \sim z.$$

Hence $\neg z = \neg z \wedge (z \vee \sim z) = \neg z \wedge \sim z$, and then $z \vee \sim z \geq z \vee \neg z = 1$. Consequently, taking into account properties (i) and (ii), we can conclude that $\sim z$ is the complement of z, i. e., $\sim z = \neg z$. Suppose now that $x \wedge z = y \wedge z$. Then $\sim x \vee \neg z = \sim y \vee \neg z$, which implies $z \wedge x = z \wedge y$. This shows that $\sim$ is preserved by Θ_z.

Subvarieties of $\mathcal{L}_i$ are the variety $\mathcal{OL}$ of *ortholattices* [4, 36], characterized by the equation $x \wedge \sim x = 0$, and the variety $\mathcal{K}$ of *Kleene algebras* [2], characterized by the distributive law. The intersection $\mathcal{OL} \cap \mathcal{K}$ is the variety $\mathcal{B}$ of boolean algebras. An important subvariety of $\mathcal{OL}$ is the variety $\mathcal{OML}$ of *orthomodular lattices* [4, 36].

Example 5.2.3 *The variety $\mathcal{B}_\omega$ of pseudocomplemented distributive lattices* [2]. We prove that the pseudo complement $*$ has Θ_z-compatibility. Indeed, let $B \in \mathcal{B}_\omega$, $z \in Z(B)$, and $a, b \in B$. If $a \wedge z = b \wedge z$, then $(a \wedge z) \vee \neg z = (b \wedge z) \vee \neg z$. Hence $a \vee \neg z = b \vee \neg z$ because $z \in Z(A)$. Consequently, $(a \vee \neg z)^* = (b \vee \neg z)^*$ and $a^* \wedge z = b^* \wedge z$.
The variety of Stone algebras $\mathcal{ST}$ is the subvariety of $\mathcal{B}_\omega$ characterized by the equation $(x \wedge y)^* = x^* \vee y^*$ [2].

Example 5.2.4 *Subvarieties of $\mathcal{RL}$*

Example 5.2.5 *$\mathcal{L}_n$, the varieties of Lukasiewicz and of Post algebras of order $n \geq 2$* [2], as well as the various types of Lukasiewicz - Moisil algebras which are considered in [6].

Example 5.2.6 $\mathcal{PMV}$, *the variety of pseudo MV-algebras.* A pseudo MV-algebra [30] is an algebra $\langle A, \oplus, ^-, ^\sim, 0, 1\rangle$ of type $\langle 2,1,1,0,0\rangle$ such that when defining the derived operations by $y \odot x := (x^- \oplus y^-)^\sim$, $x \vee y = x \oplus (x^\sim \odot y)$, $x \wedge y := x \odot (x^- \oplus y)$ the following axioms are satisfied

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
2. $x \oplus 0 = 0 \oplus x = x$ and $x \oplus 1 = 1 \oplus x = 1$,
3. $1^\sim = 0$ and $1^- = 0$,
4. $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
5. $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$,
6. $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,
7. $(x^-)^\sim = x$.

$L(A) = \langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice (Corollary 1.14 [24]). $\mathcal{PMV}$ is categorically equivalent to lattice ordered (not necessarily abelian) groups with a strong unit [17].

Proposition 5.2.7 [17] *Let G be a lattice ordered group with a strong unit u, we consider the interval $[0, u]$ equipped with the following operations*

$$\begin{aligned} x \oplus y &= (x + y) \wedge u, \\ x^- &= u - x, \\ x^\sim &= x - u, \end{aligned}$$

then $\Gamma(G, u) = \langle [0, u], \oplus, -, \sim, 0, u\rangle$ is a pseudo MV-algebra and for each pseudo MV-algebra A, there exist a lattice ordered group G with a strong unit u such that $A = \Gamma(G, u)$.

□

Lemma 5.2.8 $\mathcal{PMV}$ *is an $\mathcal{L}$- variety.*

Proof: Let $A = \Gamma(G, u) \in \mathcal{PMV}$. Through this proof, z will denote an element of $Z(A)$, and a, b elements of A. We have to prove that the operations $\oplus$, $-$ and $\sim$ are Θ_z-compatible. Note first that $z \wedge (a + b) \leq (z \wedge a) + (z \wedge b)$ ([4, Page 296, Ex.3]), thus $z \wedge (a \oplus b) \leq (z \wedge a) \oplus (z \wedge b)$. On the other hand, $(z \wedge a) \oplus (z \wedge b) = u \wedge ((z \wedge a) + (z \wedge b)) = u \wedge (z + z) \wedge$

$(z+a) \wedge (z+b) \wedge (a+b) \leq (u \wedge (z+z)) \wedge (u \wedge (a+b)) = (z \oplus z) \wedge (a \oplus b)$ $= z \wedge (a \oplus b)$, because $z \oplus z = z$ by [30, Lemma 3.2]). Hence $z \wedge (a \oplus b) = (z \wedge a) \oplus (z \wedge b)$, and $\oplus$ is Θ_z-compatible. To prove that - is Θ_z-compatible, note that if $a \wedge z = b \wedge z$ then $(a \wedge z)^- = (b \wedge z)^-$ i.e., $u - (a \wedge z) = u - (b \wedge z)$. By [24, Proposition 1.16], we have $u + (-a \vee -z) = u + (-b \vee -z)$ and we obtain $(u-a) \vee (u-z) = (u-b) \vee (u-z)$ i.e, $a^- \vee z^- = b^- \vee z^-$. Thus $(a^- \vee z^-) \wedge z = (b^- \vee z^-) \wedge z$. Since $L(A)$ is distributive and by [30, Corollary 3.3] z^- is the complement of z, and $z^- z^{\sim}$, we infer that $a^- \wedge z = b^- \wedge z$. Similarly we verify that $\sim$ has Θ_z – *compatibility.* □

5.3 The CBS property

The aim of this section is to give a formulation of the CBS theorem for algebras in $\mathcal{L}$-varieties. We begin by proving some technical results.

Proposition 5.3.1 *Let L be a bounded lattice. Then following assertions hold for all $z \in Z(L)$:*

1. $Z([0,z]) = Z(L) \cap [0,z]$.

2. *If $x \in Z([0,z])$ then the complement of x relative to $[0,z]$ is $\neg_z x = z \wedge \neg x$.*

Proof: Let $x \in Z([0,z])$.We first prove that, if x is a neutral element in $[0,z]$, then x is a neutral element in L. Let $a, b \in L$.

a $(a,b,x)D$: $x \wedge (a \vee b) = (x \wedge (a \vee b)) \wedge (z \vee \neg z) = (x \wedge (a \vee b) \wedge z) \vee (x \wedge (a \vee b) \wedge \neg z) = (x \wedge (a \vee b) \wedge z) \vee 0 = x \wedge ((a \wedge z) \vee (b \wedge z)) = (x \wedge (a \wedge z)) \vee (x \wedge (b \wedge z)) = (x \wedge a) \vee (x \wedge b)$. By the same argument it is possible to check $(b,a,x)D$.

b $(x,b,a)D$: $a \wedge (x \vee b) = (a \wedge (x \vee b)) \wedge (z \vee \neg z) = (a \wedge (x \vee b) \wedge z) \vee (a \wedge (x \vee b) \wedge \neg z) = ((a \wedge z) \wedge ((x \vee b) \wedge z)) \vee (a \wedge ((x \wedge \neg z) \vee (b \wedge \neg z))) = ((a \wedge z) \wedge ((x \wedge z) \vee (b \wedge z))) \vee (a \wedge (0 \vee (b \wedge \neg z))) = ((a \wedge z) \wedge (x \vee (b \wedge z))) \vee (a \wedge b \neg z) = ((a \wedge z \wedge x) \vee (a \wedge b \wedge z)) \vee (a \wedge b \neg z) = (a \wedge x) \vee ((a \wedge b \wedge z) \vee (a \wedge b \neg z)) = (a \wedge x) \vee ((a \wedge b) \vee (z \vee \neg z)) = (a \wedge x) \vee (a \wedge b)$. By the same argument it is possible to check $(b,x,a)D$, $(x,a,b)D$ and $(a,x,b)D$.

c $(a,b,x)D^*$:$x \vee (a \wedge b) = (x \vee (a \wedge b)) \wedge (z \vee \neg z) = ((x \vee (a \wedge b)) \wedge z) \vee ((x \vee (a \wedge b)) \wedge \neg z) = ((x \wedge z) \vee (a \wedge b \wedge z)) \vee ((x \wedge \neg z) \vee (a \wedge b \wedge \neg z)) =$

$(x\vee(a\wedge b\wedge z))\vee(0\vee(a\wedge b\wedge\neg z)) = (x\vee((a\wedge z)\wedge(b\wedge z)))\vee(a\wedge b\wedge\neg z) = ((x\vee(a\wedge z))\wedge(x\vee(b\wedge z)))\vee(a\wedge b\wedge\neg z) = ((x\vee a)\wedge(x\vee z)\wedge(x\vee b)\wedge(x\vee z))\vee(a\wedge b\wedge\neg z) = ((x\vee a)\wedge(x\vee b)\wedge z)\vee(a\wedge b\wedge\neg z) = ((x\vee a)\wedge(x\vee b))\vee(a\wedge b\wedge\neg z))\wedge(z\vee(a\wedge b\wedge\neg z)) = (x\vee a)\wedge(x\vee b)\wedge(z\vee(a\wedge b)) = (x\vee a)\wedge(x\vee b)\wedge(z\vee a)\wedge(z\vee b) = (x\vee a)\wedge(x\vee b)$. By the same argument it is possible to check that $(b,a,x)D^*$.

d $(x,b,a)D^*$: $a\vee(x\wedge b) = (a\vee(x\wedge b))\wedge(z\vee\neg z) = ((a\vee(x\wedge b)\wedge z)\vee((a\vee(x\wedge b)\wedge\neg z) = ((a\wedge z)\vee(x\wedge b\wedge z))\vee((a\wedge\neg z)\vee(x\wedge b\wedge\neg z)) = ((a\wedge z)\vee x)\wedge((a\wedge z)\vee(b\wedge z)))\vee((a\wedge\neg z)\vee 0) = ((a\vee x)\wedge(a\vee b)\wedge z)\vee(a\wedge\neg z) = (((a\vee x)\wedge(a\vee b))\vee(a\wedge\neg z))\wedge(z\vee(a\wedge\neg z)) = (a\vee x)\wedge(a\vee b)\wedge(z\vee a) = (a\vee x)\wedge(a\vee b)$. By the same argument it is possible to check $(b,x,a)D^*$, $(x,a,b)D^*$ and $(a,x,b)D^*$.

Thus x is neutral in L. We proceed now to prove that if x is complemented in $[0,z]$ then x is also complemented in L. In fact, let $\neg_z x$ be the complement of x in $[0,z]$ and define x_1 by $x_1 = \neg_z x\vee\neg z$. Hence $x\vee x_1 = x\vee(\neg_z x\vee\neg z) = z\vee\neg z = 1$ and since x is a neutral element, $x\wedge x_1 = 0$. Thus x_1 is the complement of x in L. From the two preceding results, it follows that $x\in Z(L)$. On the other hand, it is easy to verify that if x is a neutral element in L then x is a neutral element in $[0,z]$. Moreover, if x has a complement $\neg x$ in L, then $\neg_z x = \neg x\wedge z$ is the complement of x in $[0,z]$. Therefore if $x\in[0,z]$ is a central element in the lattice L, then x is a central element in the lattice $[0,z]$. □

Proposition 5.3.2 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety, $A,B\in\mathcal{V}$, $\alpha{:}A\to B$ an isomorphism. Then*

(1) for all $z\in Z(A)$, $\alpha(z)\in Z(B)$, and the restriction of α to $Z(A)$ is a boolean algebra isomorphism from $Z(A)$ onto $Z(B)$;

(2) for all $z\in Z(A)$, the restriction of α to $[0,z]_A$ is an isomorphism from $[0,z]_A$ onto $[0,\alpha(z)]_B$. □

Definition 5.3.3 Let $\mathcal{V}$ be an $\mathcal{L}$-variety. We say that $A\in\mathcal{V}$ **possesses the CBS property** iff the following holds: *Given $B\in\mathcal{V}$ and $b\in Z(B)$ such that there is $a\in Z(A)$ with $A\cong[0,b]_B$ and $B\cong[0,a]_A$, it follows that $A\cong B$.*

Proposition 5.3.4 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety. The following conditions are equivalent for each $A\in\mathcal{V}$:*

(1) A possesses the CBS property.

(2) For all $b \in Z(A)$, if $A \cong [0,b]_A$, then for all $z \in Z(A)$ such that $z \geq b$ we have $A \cong [0,z]_A$.

Proof: We suppose that A possesses the CBS property. Let $z, b \in Z(A)$ be such that $z \geq b$ and $A \cong [0,b]_A$. We denote by B the $\mathcal{V}$-algebra $[0,z]_A$. By Proposition 5.3.1, $b \in Z(B)$. Now we have $A \cong [0,b]_B$ and $B \cong [0,z]_A$ (for this we use the identity $id_{[0,z]}$), and we conclude that $A \cong [0,z]_A$. For the converse, suppose that $B \in \mathcal{V}$, $a \in Z(A)$, $b \in Z(B)$ and that there are morphisms $\alpha{:}A \to [0,b]_B$ and $\beta{:}B \to [0,a]_A$. If z = $\beta(b)$, then $A \cong [0,z]_A$ and $a \geq z$. Now by the hypothesis $A \cong [0,a]_A$. This proves that $A \cong B$. □

Let $\mathcal{V}$ be an $\mathcal{L}$-variety, $A \in \mathcal{V}$, $b \in Z(A)$ and let $\alpha{:}A \to [0,b]_A$ be an isomorphism. If we consider $z \in Z(A)$ such that $z \geq b$ and the $\mathcal{V}$-algebra $B = [0,z]_A$, then there is an isomorphism $\beta{:}B \to [0,a]_A$ (for instance we can take $\beta = id_{[0,z]}$). We define recursively two sequences, $(a_n)_{n\in\omega}$ in A, $(b_n)_{n\in\omega}$ in B, called respectively the **A-sequence** and the **B-sequence** as follows:

$$\begin{aligned} a_0 &= 1_A & b_0 &= 1_B = z \\ a_1 &= \beta(z) = a & b_1 &= \alpha(a_0) = b \\ a_{n+1} &= \beta(b_n) & b_{n+1} &= \alpha(a_n) \end{aligned}$$

Then the sequence

$$(a_2 \wedge \neg a_3, a_4 \wedge \neg a_5, \ldots) = (a_{2n} \wedge \neg a_{2n+1})_{n\in\omega, n\geq 1}$$

is called a **CBS sequence**. Fixing b, z as above, then for each pair of isomorphisms $\alpha{:}A \to [0,b]_B$, $\beta{:}B \to [0,a]_A$ we have a CBS sequence, which we will denote by $\langle b, z, \alpha, \beta\rangle$.

Proposition 5.3.5 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety, $A \in \mathcal{V}$, and let $\langle b, z, \alpha, \beta\rangle$ be a CBS sequence. Then*

(1) the A, B-sequences are strictly decreasing in $Z(A)$,

(2) $\langle b, z, \alpha, \beta\rangle$ is an orthogonal sequence in $Z(A)$, and $\beta\alpha(a_{2n} \wedge \neg a_{2n+1}) = a_{2n+2} \wedge \neg a_{2n+3}$ for $n \geq 0$.

Proof: By Proposition 5.3.2 it is easy to see that $a_1 = a$, $b_1 = b$, and that all a_n, b_n are central elements. Hence $\langle b, z, \alpha, \beta\rangle$ is in Z(A). By the injectivity of α and β, $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ are strictly decreasing. Let $m, n \in \omega$ such that $m < n$. Since $(a_n)_{n\in\omega}$ is strictly decreasing, $(a_{2m} \wedge \neg a_{2m+1}) \wedge (a_{2n} \wedge \neg a_{2n+1}) \leq (a_{2m} \wedge \neg a_{2m+1}) \wedge (a_{2m+1} \wedge \neg a_{2n+1}) = 0$. Finally, $\beta\alpha(a_{2n} \wedge \neg a_{2n+1}) = \beta(\alpha(a_{2n}) \wedge \alpha(a_{2n+1})) = \beta(\alpha(a_{2n}) \wedge b \wedge \neg\alpha(a_{2n+1})) = \beta(b_{2n+1} \wedge \neg b_{2n+2}) = \beta(b_{2n+1}) \wedge a \wedge \neg\beta(b_{2n+2}) = a_{2n+2} \wedge \neg a_{2n+3}$. □

Definition 5.3.6 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety and $A \in \mathcal{V}$. Then A is called* **CBS complete** *iff for all $b \in Z(A)$ such that $A \cong_{\mathcal{V}} [0,b]_A$ and for all $z \in Z(A)$ such that $z \geq b$ there exists a CBS sequence $\langle b, z, \alpha, \beta\rangle$ which has the (boolean) supremum $\sqcup_{n\geq 1}(a_{2n} \wedge \neg a_{2n+1})$.*

Theorem 5.3.7 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety. Then the following conditions are equivalent for each $A \in \mathcal{V}$:*

(1) A is CBS complete.

(2) A possesses the CBS property.

Proof: Suppose that A is CBS complete. Let $z, b \in Z(A)$ be such that $z \geq b$, $A \cong [0,b]_A$ and $B = [0,z]_A$. We want to prove that $A \cong [0,z]_A = B$. By the hypothesis there are isomorphisms, $\alpha{:}A \to [0,b]_B$, $\beta{:}B \to [0,a]_A$ defining A, B-sequences

$$\begin{aligned} a_0 &= 1_A & b_0 &= 1_B = z \\ a_1 &= a & b_1 &= b \\ a_{n+1} &= \beta(b_n) & b_{n+1} &= \alpha(a_n) \end{aligned}$$

and the CBS sequence $\langle b, z, \alpha, \beta\rangle = (a_{2n} \wedge \neg a_{2n+1})_{n\in\omega, n\geq 1}$ with $y = \sqcup_{n\geq 1}(a_{2n} \wedge \neg a_{2n+1})$. Let $x = y \vee \neg a$. By Proposition 5.1.4 we have

$$A \cong [0, \neg x] \times [0, x]. \tag{5.1}$$

Since $y \in Z([0,a])$ by Proposition 5.3.1, we have

$$[0,a]_A \cong [0, \neg_a y] \times [0,y] = [0, a \wedge \neg y] \times [0,y]. \tag{5.2}$$

But $\neg x = a \wedge \neg y$, hence

$$[0, \neg x] = [0, a \wedge \neg y]. \tag{5.3}$$

By Proposition 5.3.2, $[0,x] \cong [0, \beta\alpha(x)] = [0, \beta\alpha(\bigsqcup_{n\in\omega} a_{2n} \wedge \neg a_{2n+1})] = [0, \bigsqcup_{n\in\omega} \beta\alpha(a_{2n} \wedge \neg a_{2n+1})]$, and by Proposition 5.3.5 , $\beta\alpha(a_{2n} \wedge \neg a_{2n+1}) = (a_{2n+2} \wedge \neg a_{2n+3})$. Thus we have

$$[0,x] \cong [0, \sqcup_{n\geq 1}(a_{2n} \wedge \neg a_{2n+1})] = [0,y]. \tag{5.4}$$

¿From (5.1),(5.2),(5.3) and (5.4) we obtain that $A \cong [0,a]$, hence $A \cong_{\mathcal{V}} B$.

Suppose now that A possesses the CBS property. Let $b \in Z(A)$ be such that we can find an isomorphism $\alpha{:}A \to [0,b]_A$ and a $z \in Z(A)$ such that $z \geq b$. By hypothesis there is an isomorphim $\beta{:}[0,z]_A \to A$. The corresponding A, $[0,z]_A$-sequences have the form

$$\begin{array}{ll}
a_0 = 1_A & b_0 = z \\
a_1 = \beta(b_0) = 1 & b_1 = \alpha(a_0) = z \\
a_2 = \beta(b_1) = \beta(z) & b_2 = \alpha(a_1) = z \\
a_3 = \beta(b_2) = \beta(z) & b_3 = \alpha(a_2) = \alpha\beta(z)
\end{array}$$

It is easy to show (by induction) that $a_{2n} = a_{2n+1}$ for all $n \geq 1$. Thus we have $\langle b, z, \alpha, \beta \rangle = (0,0,0,\ldots)$ and the boolean supremum is 0. Therefore there exists at least one CBS sequence associated with $z \geq b$ admitting the boolean supremum. Therefore A is CBS complete. □

Corollary 5.3.8 *Let $\mathcal{V}$ be an $\mathcal{L}$-variety and $A \in \mathcal{V}$. If $Z(A)$ is an orthogonally σ-complete lattice, then A possesses the CBS property.* □

Corollary 5.3.9 (Sikorski) *The σ-complete Boolean algebras possesses the CBS property.* □

Corollary 5.3.10 *Let A be a CBS complete algebra in an $\mathcal{L}$-variety $\mathcal{V}$. Then $A \cong A^2$ iff $A \cong A^n$ for all $n \geq 2$.*

Proof: It is an easy adaptation of the proof of Proposition 3.2 in [16]. □

Remark 5.3.11 It is worth noting that the σ-completeness condition for Boolean algebras is not necessary for the CBS property, as is shown by the Boolean algebra B_N of finite and cofinite subsets of N. B_N is not even orthogonally σ-complete. Indeed, $\{2n\}_{n\in N}$ is an orthogonal sequence in B_N, but $\bigvee_{n\in N}\{2n\}$ is not in B_N. By cardinality arguments it is very easy to see that $B_N \cong [\emptyset, X]_{B_N}$ iff X is a cofinite set. Thus B_N possesses the

CBS Property. On the other hand, there are Boolean algebras which do not possesses the CBS property. For instance, Hanf constructed a Boolean algebra B such that $B \cong B^3$ but $B \ncong B^2$ [32, §6.2]. This means that $B \cong [(0,0,0),(0,0,1)]_{B^3}$ but $B \ncong [(0,0,0),(0,1,1)]_{B^3}$.

5.4 Centers and σ-completeness

In general, the σ-completeness of an algebra A in an $\mathcal{L}$-variety does not imply that $Z(A)$ is an orthogonally σ-complete lattice, as the following example shows:

Example 5.4.1 Let B_N be as in Remark 5.3.11 and let H_N be the Heyting algebra of all ideals of B_N. We observe that H_N is a complete Heyting algebra such that $Z(H_N)$, which is formed by the principal ideals generated by the elements of B_N, is not orthogonally σ-complete. Indeed, the principal ideals $(\langle 2n \rangle)_{n \in N}$ form an orthogonal sequence in $Z(H_N)$, but obviously this sequence does not have a central supremum. It is worth noting that H_N possesses the CBS property, as can be shown by cardinality arguments similar to those used in Remark 5.3.11.

In what follows we give examples of $\mathcal{L}$-varieties $\mathcal{V}$ with the property that σ-completeness conditions on the algebras in $\mathcal{V}$ guarantee the corresponding σ-completeness of their centers, and then, in the light of Corollary 5.3.8, the CBS property of these algebras.

5.4.1 Orthomodular lattices

Proposition 5.4.2 *Let L be a σ-complete orthomodular lattice and $(a_n)_{n \in \omega}$ a sequence in $Z(L)$Then $\bigvee_{n \in \omega} a_n \in Z(L)$, i.e., $\bigsqcup_{n \in \omega} a_n = \bigvee_{n \in \omega} a_n$.*

Proof: The proof is an easy adaptation of the proof of (5.14) and (29.16) in [36]. □

5.4.2 Stone algebras

Proposition 5.4.3 *Let S be a Stone algebra and $(a_i)_{i \in I}$ a family of central elements such that there exist $\bigwedge_{i \in I} a_i$ and $\bigvee_{i \in I} a_i$. Then $\sqcap_{i \in I} a_i = \bigwedge_{i \in I} a_i$ (i.e. $\bigwedge_{i \in I} a_i \in Z(S)$) and $\sqcup_{i \in I} a_i = \neg\neg \bigvee_{i \in I} a_i$. Thus if S is a σ-complete, (orthogonally σ-complete) Stone algebra then $Z(S)$ is a σ-complete (orthogonally σ-complete) lattice.*

Proof: It is well-known that $Z(S) = \{x \in S : \neg\neg x = x\}$ (see [2]). Let $a = \bigwedge_{i\in I} a_i$. For all $i \in I$, if $a \leq a_i$, then $\neg\neg a \leq \neg\neg a_i = a_i$. Thus $\neg\neg a \leq \bigwedge_{i\in I} a_i = a$, and since $a \leq \neg\neg a$, we have $a \in Z(S)$. From the basic properties of the pseudocomplement it follows that $\neg\neg \bigvee_{i\in I} a_i \in Z(S)$ and it is easy to see that $\neg\neg \bigvee_{i\in I} a_i$ is the least boolean upper bound of $(a_i)_{i\in I}$.
□

5.4.3 BL-algebras

Lemma 5.4.4 [14] *For each $A \in \mathcal{BL}$, let $Idp(A) = \{x \in A : x \odot x = x\}$ be the set of all idempotent elements of A. $Idp(A)$ is a Heyting algebra, $Z(A)$ is a subalgebra of $Idp(A)$ and $z \in Idp(A)$ iff $z \odot a = z \wedge a$ for all $a \in A$.* □

Lemma 5.4.5 *Let B be a BL-algebra and $(a_i)_{i\in I}$ a sequence in B such that $\bigvee_{i\in I} a_i$ exists. Then we have*

1. $a \odot \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \odot a_i)$, $(\bigvee_{i\in I} a_i) \rightarrow b = \bigwedge_{i\in I}(a_i \rightarrow b)$, $a \wedge \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \wedge a_i)$ *and* $\neg(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} \neg a_i$;
2. *if $(a_i)_{i\in I}$ is a family in $Idp(B)$ then $\bigvee_{i\in I} a_i \in Idp(B)$.*

Proof: Item 1) follows from basic the properties of residuated lattices [27]. To prove 2), let $a = \bigvee_{i\in I} a_i$. By item 1), we have $a \odot a = a \odot \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \odot a_i) = \bigvee_{i\in I}(a \wedge a_i) = \bigvee_{i\in I} a_i =$ a. □

Lemma 5.4.6 [14] *Let B be a BL-algebra. The following conditions are equivalent:*

1. $z \in Z(B)$,
2. $z \vee \neg z = 1$,
3. *there is v in $Idp(b)$ such that $z = \neg v$.* □

Proposition 5.4.7 *Let B be a BL-algebra and $(a_i)_{i\in I}$ a sequence in $Z(B)$ such that there exist $\bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} a_i$. Then $\sqcup_{i\in I} a_i = \neg\neg \bigvee_{i\in I} a_i$ and $\sqcap_{i\in I} a_i = \bigwedge_{i\in I} a_i$.*

Proof: If $(a_i)_{n\in I}$ is a sequence in $Z(B)$ with $a = \bigwedge_{i\in I} a_i$, by Lemma 5.4.6 it suffices to show that $a \vee \neg a = 1$. According to Lemma 5.4.5 we have $a \vee \neg a$ $= (\bigwedge_{i\in I} a_i) \vee \neg a = \neg(\bigvee_{n\in I} \neg a_i) \vee \neg a = \neg((\bigvee_{i\in I} \neg a_i) \wedge a) = \neg \bigvee_{i\in I}(\neg a_i \wedge a)$

$= 1$, therefore $\sqcap_{i \in I} a_i = \bigwedge_{i \in I} a_i$. According to Lemmas 5.4.6.2, 5.4.5.3, we have $\neg\neg \bigvee_{i \in I} a_i \in Z(B)$ and $\neg\neg \bigvee_{i \in I} a_i$ is a boolean upper bound of $(a_i)_{i \in I}$. Moreover, if b is a boolean upper bound of $(a_i)_{i \in I}$ then $\bigvee_{i \in I} a_i \leq b$ hence, $\neg\neg \bigvee_{i \in I} a_i \leq b$, thus $\sqcup_{i \in I} a_i = \neg\neg \bigvee_{i \in I} a_i$. □

Corollary 5.4.8 *If B is a σ-complete (orthogonally σ-complete) BL-algebra then $Z(B)$ is a σ-complete (orthogonally σ-complete) lattice.* □

Proposition 5.4.9 *If B is a σ-complete (orthogonally σ-complete) PL-algebra or MV-algebra then $Z(B)$ is a σ-sublattice (orthogonal σ-sublattice) of $L(B)$.*

Proof: If B is a PL-algebra then according to Proposition 3.1 in [11], $Idp(B) = Z(B)$. Thus by Lemma 5.4.5.2, $\sqcup_{n \in \omega} a_n = \bigvee_{n \in \omega} a_n$ for $(a_n)_{n \in \omega}$ in $Z(B)$. If B is an MV-algebra then using Lemma 5.4.5.2 and $\neg\neg x = x$ we have the same result.

□

5.4.4 Łukasiewicz and Post algebras of order n

Proposition 5.4.10 [9, Lemma 3.1] *Let A be a Lukasiewicz algebra of order $n \geq 2$. If A is σ-complete, then $Z(A)$ is a σ-sublattice of $L(A)$.* □

5.4.5 Pseudo MV-algebra

Let A be a pseudo MV-algebra. If A is σ-complete, then A is an MV-algebra (see [17, Theorem 4.2] and [18, Proposition 2.8]). Thus by Proposition 5.4.7, $Z(A)$ is a σ-sublattice of $L(A)$ and if A is orthogonally σ-complete then $Z(A)$ is an orthogonally σ-complete lattice (see Proposition 3.4 in [30]).

5.5 CBS theorem and absolute retracts

Theorem 5.5.1 *Let A be absolute retract in an $\mathcal{L}$-variety $\mathcal{V}$. Then $Z(A)$ is a complete lattice.*

Proof: Let X be down directed subset of $Z(A)$. Suppose that X does not admit minimum element and consider the ultrapower $A^X/\mathcal{U}$ as in Remark 2.2.3. It is not very hard to see that the $\mathcal{U}$-equivalence class $[1_X]$ is a neutral element in $A^X/\mathcal{U}$, having a complement $\neg[1_X]$ given by the $\mathcal{U}$-equivalence class of the function $X \to A$ such that $x \mapsto \neg x$. Thus $[1_X] \in Z(A^X/\mathcal{U})$. The same arguments used in Theorem 2.2.4 give that $\bigwedge X \in Z(A)$. Therefore we have proved that the infimum in A of a down directed subsets of $Z(A)$ belongs to $Z(A)$. From this the result follows by a standard argument. □

Corollary 5.5.2 *Each absolute retract in an $\mathcal{L}$-variety satisfies the CBS property.*

Proof: It follows by Theorem 5.5.1 and Corollary 5.3.8. □

5.6 CBS-type theorem for posets

The category of posets and monotonic functions will be denoted by $\mathcal{P}os$. Let A be a poset and $X \subseteq A$. X is **decreasing (increasing)** iff for all $x \in X$, if $a \leq x$ $(a \geq x)$, then $a \in X$. The set of all decreasing sets in A is denoted by $O(A)$, and it is well-known that $O(A)$ has the structure of a complete Heyting algebra. Let L be a complete lattice and let $k \in L$. Then k is said to be **compact** iff for every subset S of L, if $k \leq \bigvee S$ then $k \leq \bigvee T$ for some finite subset T of S. It is easy to show that $(a]$ is compact in $O(A)$. Moreover, $X \in O(A)$ is compact iff there exist $a_1, \ldots, a_n$ in A such that $X = (a_1] \cup \ldots \cup (a_n]$. It is easy to show that $Z(O(A)) = \{ B \in O(A) : B \textit{ is an increasing set} \}$ and that $Z(O(A))$ is a complete lattice.

Lemma 5.6.1 *Let A, B be posets. If $O(A)$ and $O(B)$ are isomorphic then A and B are isomorphic.* □

Theorem 5.6.2 *Let A, B be posets and let $X \subseteq A$ and $Y \subseteq B$ be simultaneously increasing and decreasing sets. If there are isomorphisms $\alpha: A \to Y$ and $\beta: B \to X$, then $A \cong_{\mathcal{P}os} B$.*

Proof: We first prove that $O(A) \cong_{\mathcal{P}os} [\emptyset, Y]$. For all $S \in O(A)$ we have $S = \bigcup_{a \in S}(a]$ and $(\alpha(a)] \subseteq Y$, since Y is decreasing. Consequently, if $\psi: O(A) \to [\emptyset, Y]$ is such that $S = \bigcup_{a \in S}(a] \mapsto \bigcup_{a \in S}(\alpha(a)]$, then it is easy to show that ψ is an order isomorphism under $\subseteq$. Analogously, we can obtain that $O(B) \cong_{\mathcal{P}os} [\emptyset, X]$. But these $\mathcal{P}os$-isomorphisms are also Heyting isomorphisms. Then by Theorem 5.3.7, $O(A) \cong O(B)$ as Heyting algebras. Finally, in view of Lemma 5.6.1 we have $A \cong_{\mathcal{P}os} B$. □

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