Sobre subclases y variantes de los grafos perfectos

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SOBRE SUBCLASES Y VARIANTES DE LOS GRAFOS PERFECTOS

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Abstract

On subclasses and variants of perfect graphs

Perfect graphs were defined by Claude Berge in 1960. A graph G is perfect whenever for every induced subgraph H of G, the chromatic number of H equals the cardinality of a maximum complete subgraph of H. Perfect graphs are very interesting from an algorithmic point of view: while determining the clique number and the chromatic number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs.

Since then, many variations of perfect graphs were defined and studied, including the class of clique-prefect graphs. A clique in a graph is a complete subgraph maximal under inclusion. A clique-transversal of a graph G is a subset of vertices meeting all the cliques of G. A clique-independent set is a collection of pairwise vertex-disjoint cliques. A graph G is clique-perfect if the sizes of a minimum clique-transversal and a maximum clique-independent set are equal for every induced subgraph of G. The term "clique-perfect" was introduced by Guruswami and Pandu Rangan in 2000, but the equality of these parameters had been previously studied by Berge in the context of balanced hypergraphs.

A characterization of perfect graphs by minimal forbidden subgraphs was recently proved by Chudnovsky, Robertson, Seymour and Thomas, and a polynomial time recognition algorithm for this class of graphs has been developed by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem. In this thesis, we present partial results in these directions, that is, we characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph is either a line graph, or claw-free hereditary clique-Helly, or diamond-free, or a Helly circular-arc graph. Almost all of these characterizations lead to polynomial time recognition algorithms for clique-perfection in the corresponding class of graphs.

Berge defined a hypergraph to be balanced if its vertex-edge incidence matrix is balanced, that is, if it does not contain the vertex-edge incidence matrix of an odd cycle as a submatrix. In 1998, Dahlhaus, Manuel and Miller consider this concept applied to graphs, defining a graph to be balanced when its vertex-clique incidence matrix is balanced. Balanced graphs are an interesting subclass in the intersection of perfect and clique-perfect graphs. We give two new characterizations of this class, the first one by forbidden subgraphs and the second one by clique subgraphs. Using domination properties we define four subclasses of balanced graphs. Two of them are characterized by 0-1 matrices and can be recognized in polynomial time. Furthermore, we propose polynomial time combinatorial algorithms for the stable set problem, the clique-independent set problem and the clique-transversal problem in one of these subclasses. Finally, we analyze the behavior of balanced graphs and these four subclasses under the clique graph operator.

Keywords: balanced graphs, clique graph, clique-perfect graphs, diamond-free graphs, Helly circular-arc graphs, hereditary clique-Helly claw-free graphs, K-perfect graphs, line graphs, perfect graphs.

Resumen

Sobre subclases y variantes de los grafos perfectos

Los grafos perfectos fueron definidos por Claude Berge en 1960. Un grafo G es perfecto cuando para todo subgrafo inducido H de G, el número cromático de H es igual al tamaño de un subgrafo completo máximo de H. Los grafos perfectos son de gran interés desde el punto de vista algorítmico: si bien los problemas de determinar la clique máxima y el número cromático de un grafo son NP-completos, éstos se resuelven en tiempo polinomial para grafos perfectos.

Desde entonces, fueron definidas y estudiadas gran cantidad de variantes de los grafos perfectos. Entre ellas, los grafos clique-perfectos. Una clique en un grafo es un subgrafo completo maximal con respecto a la inclusión. Un transversal de las cliques de un grafo G es un subconjunto de vértices que interseca a todas las cliques de G. Un conjunto de cliques independientes es un conjunto de cliques disjuntas dos a dos. Un grafo G es clique-perfecto si el tamaño de un transversal de las cliques mínimo coincide con el de un conjunto de cliques independientes máximo, para cada subgrafo inducido de G. El término "clique-perfecto" fue introducido por Guruswami y Pandu Rangan en 2000, pero la igualdad de esos parámetros fue estudiada previamente por Berge en el contexto de hipergrafos balanceados.

En 2002, Chudnovsky, Robertson, Seymour y Thomas demostraron una caracterización de los grafos perfectos por subgrafos prohibidos minimales, cerrando una conjetura abierta durante 40 años. También durante el año 2002 fueron presentados dos trabajos, uno de ellos de Chudnovsky y Seymour, y el otro de Cornuéjols, Liu y Vušković, que mostraban que el reconocimiento de esta clase era polinomial, resolviendo otro problema abierto formulado mucho tiempo atrás. La lista de subgrafos prohibidos minimales para la clase de grafos clique-perfectos no se conoce aún, y también es una pregunta abierta la complejidad del problema de reconocimiento. En esta tesis presentamos resultados parciales en estas direcciones, es decir, caracterizamos los grafos clique-perfectos por subgrafos prohibidos minimales dentro de ciertas clases de grafos, a saber,

grafos de línea, grafos clique-Helly hereditarios sin *claw*, grafos sin diamantes y grafos arco-circulares Helly. En casi todos los casos, estas caracterizaciones conducen a un algoritmo polinomial de reconocimiento de grafos clique-perfectos dentro de la clase de grafos correspondiente.

Berge definió los hipergrafos balanceados como aquellos tales que su matriz de incidencia es balanceada, es decir, no contiene como submatriz la matriz de incidencia de un ciclo impar. En 1998, Dahlhaus, Manuel y Miller consideran este concepto aplicado a grafos, llamando balanceado a un grafo cuya matriz de incidencia cliques-vértices es balanceada. Los grafos balanceados constituyen una interesante subclase en la intersección entre grafos perfectos y clique-perfectos. En esta tesis damos dos nuevas caracterizaciones de esta clase de grafos, una por subgrafos prohibidos y la otra por subgrafos clique. Usando propiedades de dominación definimos cuatro subclases de grafos balanceados. Dos de ellas son caracterizadas por matrices binarias y pueden ser reconocidas en tiempo polinomial. Además, proponemos algoritmos polinomiales combinatorios para los problemas de conjunto independiente máximo, conjunto de cliques independientes máximo y transversal de las cliques mínimo para una de esas subclases. Finalmente, analizamos el comportamiento del operador clique sobre la clase de grafos balanceados y sus subclases.

Palabras clave: grafo clique, grafos arco-circulares Helly, grafos balanceados, grafos clique-Helly hereditarios sin $K_{1,3}$, grafos clique-perfectos, grafos de línea, grafos K-perfectos, grafos perfectos, grafos sin diamantes.

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CHAPTER 1

Introduction

Perfect graphs were defined by Claude Berge in 1960 [4]. A graph G is perfect whenever for every induced subgraph H of G, the chromatic number of H equals the cardinality of a maximum complete subgraph of H. Many known classes of graphs are perfect, like bipartite graphs, chordal graphs, and comparability graphs. Perfect graphs are very interesting from an algorithmic point of view: while determining the clique number and the chromatic number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs [47]. For more background information on algorithms on perfect graphs, we refer to [46].

Since then, many variations of perfect graphs were defined and studied, including the class of clique-prefect graphs. A clique-transversal of a graph G is a subset of vertices meeting all the cliques of G. A clique-independent set is a collection of pairwise vertex-disjoint cliques. A graph G is clique-perfect if the sizes of a minimum clique-transversal and a maximum clique-independent set are equal for every induced subgraph of G. The term "clique-perfect" was introduced by Guruswami and Pandu Rangan in 2000 [48], but the equality of these parameters had been previously studied by Berge in the context of balanced hypergraphs [10].

A characterization of perfect graphs by minimal forbidden subgraphs was recently proved [24], and a polynomial time recognition algorithm for this class of graphs has been developed [23]. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem. In Chapter 3, we present partial results in these directions, that is, we characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph is either a line graph, or claw-free hereditary clique-Helly, or diamond-free, or a Helly circular-arc graph. In almost all the cases, these characterizations lead to polynomial time recognition algorithms for clique-perfection in the corresponding class of graphs.

Berge defined a hypergraph to be balanced if its vertex-edge incidence matrix is balanced. In [36], Dahlhaus, Manuel and Miller consider this concept applied to graphs, defining a graph to be balanced when its clique matrix is balanced. Balanced graphs are an interesting subclass in the intersection of perfect and clique-perfect graphs. In Chapter 2, we give two new characterizations of this class, one by forbidden subgraphs and the other one by clique subgraphs. Using properties of domination we define four subclasses of balanced graphs. Two of them are characterized by 0-1 matrices and can be recognized in polynomial time. Furthermore, we propose polynomial time combinatorial algorithms for the stable set problem, the clique-independent set problem and the clique-transversal problem in one of these subclasses. Finally, we analyze the behavior of balanced graphs and these four subclasses under the clique graph operator.

In the remaining part of this chapter we give some basic definitions and background properties, and in Chapter 4 we present a more detailed survey of the obtained results.

1.1 Definitions, notation, and background properties

Let G be a simple finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$ the complement of G. Given two graphs G and G' we say that G' is *smaller* than G if $|V(G')| < |V(G)|$, and that G *contains* G' if G' is isomorphic to an induced subgraph of G. When we need to refer to the non-induced subgraph containment relation, we will state this relation explicitly.

A class of graphs $\mathcal{C}$ is *hereditary* if for every $G \in \mathcal{C}$, all induced subgraphs of G also belong to $\mathcal{C}$.

Let H be a graph and let t be a natural number. The disjoint union of t copies of the graph H is denoted by tH.

Some special graphs mentioned along this thesis are shown in Figure 1.1.

Neighborhoods, completes and domination

The *neighborhood* of a vertex v in a graph G is the set $N_G(v)$ consisting of all the vertices adjacent to v. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The common neighborhood and the closed common neighborhood of an edge $e = vw$ are $N_G(e) = N_G(v) \cap N_G(w)$ and $N_G[e] = N_G[v] \cap N_G[w]$, respectively, and, in a more general way, the common neighborhood and the closed common neighborhood of a non-empty subset of vertices W are $N_G(W) = \bigcap_{w \in W} N_G(w)$ and $N_G[W] = \bigcap_{w \in W} N_G[w]$, respectively. We define $N_G(\emptyset) = N_G[\emptyset] = V(G)$.

For an induced subgraph H of G and a vertex v in $V(G) \setminus V(H)$, the *set of neighbors of v in H* is the set $N_G(v) \cap V(H)$. A subset of vertices S of G is an *homogeneous set* if for every pair of vertices v, w in S, the set of neighbors of v in $G \setminus S$ is equal to the set of neighbors of w in $G \setminus S$.

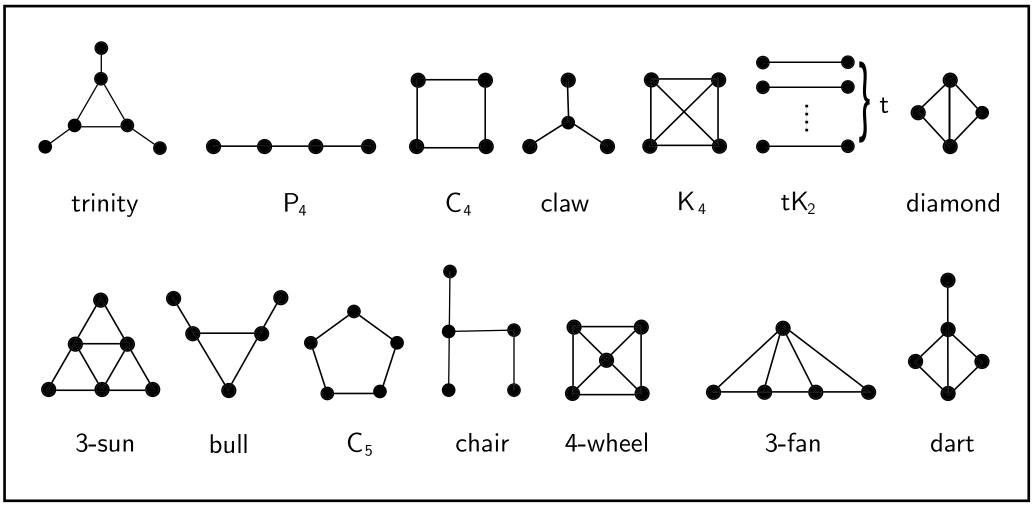

Figure 1.1: Some graphs mentioned in this thesis.

Let v, w be vertices and e, f edges of a graph G. We say that the vertex v (edge e) *dominates* vertex w (edge f) if $N_G[v] \supseteq N_G[w]$ ($N_G[e] \supseteq N_G[f]$). Similarly, the vertex v (edge e) dominates the edge f (vertex w) if $N_G[v] \supseteq N_G[f]$ ($N_G[e] \supseteq N_G[w]$). Two vertices v and w are *twins* if $N_G[v] = N_G[w]$; and u *weakly dominates* v if $N_G(v) \subseteq N_G[u]$.

A *complete set* or just a *complete* of G is a subset of pairwise adjacent vertices (in particular, an empty set is a complete set). We denote by K_n the graph induced by a complete set of size n.

Let X and Y be two sets of vertices of G. We say that X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete to* Y if no vertex of X is adjacent to a vertex of Y. Let A be a set of vertices of G, and v a vertex of G not in A. Then v is *A-complete* if it is adjacent to every vertex in A, and *A-anticomplete* if it has no neighbor in A.

A *clique* is a complete set not properly contained in any other complete set. We may also use the term "clique" to refer to the corresponding complete subgraph. The *clique number* $\omega(G)$ is the cardinality of a maximum clique of G.

A *stable set* in a graph G is a subset of pairwise non-adjacent vertices of G. The *stability number* $\alpha(G)$ is the cardinality of a maximum stable set of G.

A *diamond* is the graph isomorphic to $K_4 \setminus \{e\}$, where e is an edge of K_4. A graph is *diamond-free* if it does not contain a diamond.

A complete of three vertices is called a *triangle*, and a stable set of three vertices is called a *triad*.

A vertex v of G is *universal* if $N_G[v] = V(G)$. A vertex v is called *simplicial* if $N[v]$

induces a complete, and *singular* if $V(G) \setminus N[v]$ induces a complete. Equivalently, a vertex is singular if it does not belong to any triad. The *core* of G is the subgraph induced by the set of non-singular vertices of G. Note that a vertex belongs to exactly one clique if and only if it is simplicial.

Let v, w be vertices of G. Denote by $M(G)$ the set of cliques of G, by $M(v)$ the set of cliques of G that contain v, and by $M(v, w)$ the set of cliques of G that contain v and w.

Let G be a graph and let H be a not necessarily induced subgraph of G. The graph H is a *clique subgraph* of G if every clique of H is a clique of G.

A *clique cover* of a graph G is a subset of cliques covering all the vertices of G. The *clique covering number* of G, denoted by $k(G)$, is the cardinality of a minimum clique cover of G. It is easy to verify that $k(G) \geq \alpha(G)$ for any graph G.

The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. Equivalently, $\chi(G)$ is the cardinality of a minimum covering of the vertices of G by stable sets. An obvious lower bound for $\chi(G)$ is the clique number of G.

A *clique-transversal* of a graph G is a subset of vertices meeting all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ for any graph G.

Cutsets

Let G be a graph and let X be a subset of vertices of G. Denote by $G|X$ the subgraph of G induced by X and by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. The set X is *connected*, if there is no partition of X into two non-empty sets Y and Z, such that no edge has one endpoint in Y and the other one in Z. In this case the graph $G|X$ is also connected. The set X is *anticonnected* if it is connected in $\overline{G}$. In this case the graph $G|X$ is also anticonnected.

The set X is a *cutset* if $G \setminus X$ has more connected components than G has. Let G be a connected graph, X a cutset of G, and M_1, M_2 a partition of $V(G) \setminus X$ such that M_1, M_2 are non-empty and M_1 is anticomplete to M_2 in G. In this case we say that $G = M_1 + M_2 + X$, and $M_i + X$ denotes $G|(M_i \cup X)$, for $i = 1, 2$. When $X = \{v\}$, we simplify the notation to $M_1 + M_2 + v$ and $M_i + v$, respectively.

Let X be a cutset of G. If $X = \{v\}$ we say that v is a *cutpoint*. If X is complete, it is called a *clique cutset*. A clique cutset X is *internal* if $G = M_1 + M_2 + X$ and each M_i contains at least two vertices that are not twins.

Cycles, holes and suns

A sequence $v_1, \ldots, v_k$ of distinct vertices ($k \geq 3$) is a *cycle* in a graph G if v_1v_2, $\ldots$, $v_{k-1}v_k$, v_kv_1 are edges of G. These edges are called the edges of the cycle. The *length* of the cycle is the number k of its edges. An *odd cycle* is a cycle of odd length. In subsequent expressions concerning cycles, all index arithmetic is done modulo the length of the cycle.

A *chord* of a cycle is an edge between two vertices of the cycle that is not an edge of the cycle. A cycle is *chordless* if it contains no chords.

A *hole* is a chordless cycle of length at least 4. An *antihole* is the complement of a hole. A hole of length n is denoted by C_n. A hole or antihole on n vertices is said to be *odd* if n is odd.

A graph is *chordal* if it does not contain a hole as an induced subgraph.

An r-*sun* (or simply *sun*) is a chordal graph G on $2r$ vertices, $r \geq 3$, whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that W is a stable set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod{r}$. A sun is *odd* if r is odd. A sun is *complete* if U is a complete.

A graph is *bipartite* when it contains no cycles of odd length or, equivalently, when its vertex set can be partitioned into two stable sets.

A 4-*wheel* is a graph on five vertices $v_1, \ldots, v_5$, such that $v_1v_2v_3v_4v_1$ is a hole and v_5 is adjacent to all of v_1, v_2, v_3, v_4. A 3-*fan* is a graph on five vertices $v_1, \ldots, v_5$, such that $v_1v_2v_3v_4v_1$ induce a path and v_5 is adjacent to all of v_1, v_2, v_3, v_4.

A sequence $v_1, E_1, \ldots, v_k, E_k$ of distinct vertices $v_1, \ldots, v_k$ and distinct hyperedges $E_1, \ldots, E_k$ of a hypergraph H is a *special cycle* of length k if $k \geq 3$, $v_i, v_{i+1} \in E_i$ and $E_i \cap \{v_1, \ldots, v_k\} = \{v_i, v_{i+1}\}$, for each i, $1 \leq i \leq k$.

Intersection graphs

A family of sets S is said to satisfy the *Helly property* if every subfamily of S consisting of pairwise intersecting sets has a common element.

A graph is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G.

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being adjacent if and only if the corresponding sets intersect.

A graph G is an *interval graph* if G is the intersection graph of a finite family of intervals of the real line.

A *circular-arc* is the intersection graph of arcs on a circle. A *representation* of a circular-arc graph is a collection of circular intervals, each corresponding to a unique vertex of the graph, such that two intervals intersect if and only if the corresponding vertices are adjacent. A *Helly circular-arc* (HCA) *graph* is a circular-arc graph admitting a representation whose arcs satisfy the Helly property. In particular, in a Helly circular-arc representation of a graph, for every clique there is a point of the circle belonging to the circular intervals corresponding to the vertices in the clique, and to no others. We call such a point an *anchor* of the clique (note that an anchor may not be unique).

A *claw* is the graph isomorphic to the bipartite graph $K_{1,3}$. A graph is *claw-free* if it does not contain a claw.

The *line graph* $L(G)$ of G is the intersection graph of the edges of G. A graph F is a *line graph* if there exists a graph H such that $L(H) = F$. Clearly, line graphs are a subclass of claw-free graphs.

The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. We can define $K^j(G)$ as the j-th iterated clique graph of G, where $K^1(G) = K(G)$ and $K^j(G) = K(K^{j-1}(G))$, $j \geq 2$.

If $\mathcal{H}$ is a class of graphs, then $K(\mathcal{H})$ denotes the class of clique graphs of the graphs in $\mathcal{H}$, and $K^{-1}(\mathcal{H})$ the class of graphs whose clique graphs are in $\mathcal{H}$.

Clique graphs of several classes of graphs have been already characterized. A good survey on this topic can be found in [69].

1.2 Balanced, perfect and clique-perfect graphs

Let $M_1, \ldots, M_k$ and $v_1, \ldots, v_n$ be the cliques and vertices of a graph G, respectively. A *clique matrix* of G, denoted by A_G, is a 0-1 matrix whose entry (i, j) is 1 if $v_j \in M_i$, and 0 otherwise.

A 0-1 matrix M is *balanced* if it does not contain the vertex-edge incidence matrix of an odd cycle as a submatrix. A 0-1 matrix M is *totally balanced* if it does not contain the vertex-edge incidence matrix of a cycle as a submatrix.

Berge defined in 1969 (c.f. [37]) a hypergraph to be *balanced* if its vertex-edge incidence matrix is balanced, or equivalently, if it contains no special cycles of odd length. For further details, we refer to [6, 7]. Applying this concept to graphs, one obtains the class of *balanced graphs*, composed by those graphs having a balanced clique matrix. Note that balanced graphs are well defined, since if the clique matrix of a graph is balanced then all its clique matrices are balanced. Balanced graphs were considered in [36].

The *clique hypergraph* of a graph G has $V(G)$ as vertex set and all the cliques of G as hyperedges. Clearly, a graph G is balanced if and only if its clique hypergraph is balanced.

A graph is *strongly chordal* when it is chordal and each of its cycles of even length at least 6 has an odd chord [42]. Such a class corresponds exactly to *totally balanced* *graphs*, i.e., graphs whose clique matrices are totally balanced [1]. Clearly, strongly chordal graphs are balanced graphs.

A 0-1 matrix M is *totally unimodular* if the determinant of each square submatrix of M is 0, 1 or -1. A graph G is *totally unimodular* if its clique matrix is totally unimodular. Since the determinant of the vertex-edge incidence matrix of an odd cycle is ± 2, totally unimodular matrices are balanced matrices and then totally unimodular graphs are balanced graphs.

A graph G is *trivially perfect* if for all induced subgraphs H of G, the cardinality of the maximum stable set of H is equal to the number of cliques of H. Interval graphs and trivially perfect graphs are totally unimodular graphs [46] and, therefore, they are balanced graphs.

Perfect graphs were defined by Claude Berge in 1960 [4]. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Perfect graphs have received much attention in the last forty years, and there are many publications on this topic.

A graph is *minimally imperfect* if it is not perfect but all its proper induced subgraphs are. It is not difficult to see that odd holes and odd antiholes are not perfect. Berge conjectured in 1961 [5] that these are the only minimally imperfect graphs, that is, *a graph is perfect if and only if it does not contain odd holes or odd antiholes.* This conjecture was known as the Strong Perfect Graph Conjecture until 2002, when it was finally proved by Chudnovsky, Robertson, Seymour and Thomas.

Theorem 1.2.1 (Strong Perfect Graph Theorem). [24] *Let G be a graph. Then the following are equivalent:*

(i) *no induced subgraph of G is an odd hole or an odd antihole.*

(ii) *G is perfect.*

The second big open open question about perfect graphs was finally answered in 2003: a polynomial time recognition algorithm for perfect graphs was developed by Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [23].

A weaker result on perfect graphs, also conjectured by Berge and proved by Lóvasz in 1972 [55] and independently by Fulkerson [43] some months later, states that *a graph is perfect if and only if its complement is perfect.*

Theorem 1.2.2 (Perfect Graph Theorem). [55] *Let G be a graph. Then the following are equivalent:*

(i) $\omega(H) = \chi(H)$ *for every induced subgraph H of G.*

(ii) $\alpha(H) = k(H)$ *for every induced subgraph H of G.*

(ii) $\omega(H)\alpha(H) \geq |V(H)|$ *for every induced subgraph H of G.*

A matrix $M \in R^{k\times n}$ is *perfect* if the polyhedron $P(M) = \{x/x \in R^n, Mx \leq \mathbf{1}, x \geq 0\}$ has only integer extrema. Chvátal [27] proved the theorem below connecting perfect matrices with perfect graphs.

Theorem 1.2.3. [27] *A graph G is perfect if and only if its clique matrix is perfect.*

Since balanced matrices are perfect [44], it follows that balanced graphs are perfect graphs.

Between 1961 and 2002, many partial results related with the Strong Perfect Graph Conjecture were proved. In particular, the characterization of perfect graphs by minimal forbidden subgraphs was proved for some subclasses of graphs:

- *Circular graphs*, proved by Buckingham and Golumbic [18, 19].
- *Planar graphs*, by Tucker [72].
- *Pretty graphs*, that is, graphs in which every induced subgraph has a vertex v whose neighborhood induce a $\{P_4,2K_2\}$-free graph, by Maffray, Porto and Preissmann [59].
- *P_4-free graphs*, by Seinsche [67].
- *claw-free graphs*, by Parthasarathy and Ravindra [60].
- *diamond-free graphs*, by Tucker [75] and Conforti [30].
- *K_4-free graphs*, by Tucker [73, 74, 76].
- *C_4-free graphs*, by Cornuéjols, Conforti and Vušković [34].
- *bull-free graphs*, by Chvátal and Sbihi [29].
- *dart-free graphs*, by Sun [68].
- *chair-free*, by Sassano [65].
- *Total graphs*, by Rao and Ravindra [63]. The *total graph* $T(G)$ of $G = (V, E)$ has as vertex set $V \cup E$, where V induces G, E induces $L(G)$, and every vertex corresponding to an edge is adjacent to the vertices corresponding to its endpoints.
- *Triangular graphs*, by Le [52]. The *triangular graph* of G, $L_3(G)$ is the edge-intersection graph of the triangles of G.

A graph G is *clique-perfect* if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G. We say that a graph is *clique-imperfect* when it is not clique-perfect. A graph is *minimally clique-imperfect* if it is not clique-perfect but all its proper induced subgraphs are. Clique-perfect graphs have been implicitly studied in [2, 10, 17, 15, 21, 40, 48, 53], and the term "clique-perfect" was introduced in [48].

The two main open problems concerning this class of graphs are the following.

- find all minimal forbidden induced subgraphs for the class of clique-perfect graphs, and
- is there a polynomial time recognition algorithm for this class of graphs?

There are some partial results in these directions. In [53], clique-perfect graphs are characterized by minimal forbidden subgraphs for the class of chordal graphs, and this characterization leads to a polynomial time recognition algorithm for clique-perfect chordal graphs. In [57], minimal graphs G with $\alpha_c(G) = 1$ and $\tau_c(G) > 1$ are explicitly described.

Clique-perfect graphs are neither a subclass nor a superclass of perfect graphs. For example, antiholes of length $6k + 3$ are clique-perfect but not perfect, and antiholes of length $6k \pm 2$ are perfect but not clique-perfect.

A graph G is a *comparability graph* if there exists a partial order in $V(G)$ such that two vertices of G are adjacent if and only if they are comparable by that order. Comparability graphs are both perfect and clique-perfect. Another class in the intersection between perfect and clique-perfect graphs are balanced graphs.

1.3 Preliminary results

A graph G is *K-perfect* if its clique graph $K(G)$ is perfect. K-perfect graphs are neither a subclass nor a superclass of clique-perfect graphs. However, the following lemma establishes a connection between the parameters involved in the definition of clique-perfect graphs and those corresponding to perfect graphs.

Lemma 1.3.1. [15] *Let G be a graph. Then:*

(1) $\alpha_c(G) = \alpha(K(G))$.

(2) $\tau_c(G) \geq k(K(G))$. *Moreover, if G is clique-Helly, then* $\tau_c(G) = k(K(G))$.

The class of hereditary clique-Helly graphs can be characterized by forbidden induced subgraphs.

Theorem 1.3.2. [61] *A graph G is hereditary clique-Helly if and only if it does not contain the graphs of Figure 1.2.*

Hereditary clique-Helly graphs are of particular interest because in this case it follows from Lemma 1.3.1 that if $K(H)$ is perfect for every induced subgraph H of G, then G is clique-perfect (the converse is not necessarily true). In fact, the following proposition holds.

Proposition 1.3.1. *Let $\mathcal{L}$ be a hereditary graph class, which is HCH and such that every graph in $\mathcal{L}$ is K-perfect. Then every graph in $\mathcal{L}$ is clique-perfect.*

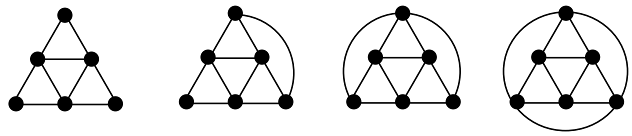

Figure 1.2: Forbidden induced subgraphs for hereditary clique-Helly graphs: (left to right) 3-sun (or 0-pyramid), 1-pyramid, 2-pyramid and 3-pyramid.

Proof. Let G be a graph in $\mathcal{L}$. Let H be an induced subgraph of G. Since $\mathcal{L}$ is hereditary, H is a graph in $\mathcal{L}$, so it is K-perfect. Since $\mathcal{L}$ is an HCH class, H is clique-Helly and then, by Lemma 1.3.1, $\alpha_C(H) = \alpha(K(H)) = k(K(H)) = \tau_C(H)$, and the result follows. $\square$

We will also use the following results on perfect graphs, cutsets and clique graphs (some of the results below are immediate, and in these cases we do not give a proof or a reference; we state these results for future reference).

Lemma 1.3.3. *Let G be a graph and v be a simplicial vertex of G. Then G is perfect if and only if $G \setminus \{v\}$ is.*

Theorem 1.3.4. [9] *Let G be a graph and X be a clique cutset of G, such that $G = M_1 + M_2 + X$. Then the graph G is perfect if and only if the graphs $M_1 + X$ and $M_2 + X$ are.*

Theorem 1.3.5. [76] *Let G be a perfect graph and let $e = v_1v_2$ be an edge of G. Assume that no vertex of G is a common neighbor of v_1 and v_2. Then $G \setminus e$ is perfect.*

Let P be an induced path of a graph G. The *length* of P is the number of edges in P. The *parity* of P is the parity of its length . We say that P is *even* if its length is even, and *odd* otherwise.

Theorem 1.3.6. *Let G be a graph, and let $u, v \in V(G)$ non-adjacent and such that $\{u, v\}$ is a cutset of G, $G = M_1 + M_2 + \{u, v\}$. For $i = 1, 2$, let G_i be a graph obtained from $M_i + \{u, v\}$ by joining u and v by an even induced path. If G_1 and G_2 are perfect, then G is perfect.*

Proof. Suppose G_1 and G_2 are perfect, and G contains an odd hole or an odd antihole, denote it by A. Since no odd antihole of length at least 7 has a one- or two-vertex cutset, if A is an odd antihole of length at least 7, then A is contained either in G_1 or in G_2, a contradiction. So A is an odd hole, and it is not contained in $M_i + \{u, v\}$ for $i = 1, 2$, thus $\{u, v\}$ is a cutset for A. Let A_1, A_2 be the two subpaths of A joining u and v. Then both A_1, A_2 have length at least two, and one of them, say A_1, is odd. But then, if A_1 is contained in $M_i + \{u, v\}$, the graph G_i contains an odd hole, a contradiction. $\square$

Theorem 1.3.7. [5] *Let G be a graph and let U be a homogeneous set in G. Let G' be the graph obtained from G by deleting all but one vertex of U. Then G is perfect if and only if both G' and $G|U$ are.*

Theorem 1.3.8. *Let G be a graph, and let $u, v \in V(G)$ such that u weakly dominates v. Then G is perfect if and only if both $G \setminus \{u\}$ and $G \setminus \{v\}$ are.*

Proof. The "only if" part is clear, so it is enough to prove that if $G \setminus \{u\}$ and $G \setminus \{v\}$ are perfect, then so is G. Since neither odd holes nor odd antiholes contain a pair of vertices such that one of them weakly dominates the other one, the result follows from Theorem 1.2.1. □

Lemma 1.3.9. *Let G be a graph and H a clique subgraph of G. Then $K(H)$ is an induced subgraph of $K(G)$.*

Lemma 1.3.10. *If G is disconnected, then so is $K(G)$, and G is K-perfect if and only if each connected component is.*

Lemma 1.3.11. *If G admits twins u, v, then $K(G) = K(G \setminus \{v\})$.*

Theorem 1.3.12. [41] *If G is a clique-Helly graph then $K^2(G)$ is the subgraph of G obtained by identifying twin vertices and then removing dominated vertices.*

Theorem 1.3.13. [15] *Let G be an HCH graph such that $K(G)$ is not perfect.*

(1) *If $K(G)$ contains $\overline{C_7}$ as induced subgraph, then G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain $\overline{C_7}$, and such that $K(H) = \overline{C_7}$.*

(2) *If $K(G)$ contains C_{2k+1} as induced subgraph, for some $k \geq 2$, then G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain C_{2k+1}, and such that $K(H) = C_{2k+1}$.*

Theorem 1.3.14. [62] *Let G be a claw-free graph with no induced 3-fan, 4-wheel or odd hole. Then $K(G)$ is bipartite.*

CHAPTER 2

On Balanced Graphs

Berge defined a hypergraph to be balanced if its vertex-edge incidence matrix is balanced. In [36], Dahlhaus, Manuel and Miller consider this concept applied to graphs, calling a graph to be balanced when its clique matrix is balanced. Balanced graphs are an interesting subclass lying in the intersection of perfect and clique-perfect graphs.

This chapter is organized as follows.

In Section 2.1 we describe background properties of balanced graphs.

In Section 2.2 new characterizations of balanced graphs are presented. The first one is by forbidden subgraphs and the second one is by clique subgraphs.

In Section 2.3 four subclasses of balanced graphs are introduced using simple properties of domination. We analyze the inclusion relations between them. Two of these classes are characterized using 0-1 matrices and these characterizations lead to polynomial time recognition algorithms. In the final part of this section, we present a combinatorial algorithm for the maximum stable set problem in one of these subclasses.

Finally, in Section 2.4 we study the clique graphs of balanced graphs and these four subclasses. As a corollary of these results, we deduce the existence of combinatorial algorithms for the maximum clique-independent set and the minimum clique-transversal problems for one of these subclasses of balanced graphs.

The results of this chapter appear in [16].

2.1 Preliminary results

Hereditary clique-Helly graphs can be characterized by means of their clique matrix, as the following result due to Prisner shows.

Theorem 2.1.1. [61] *A graph G is hereditary clique-Helly if and only if A_G does not contain a vertex-edge incidence matrix of a 3-cycle as a submatrix.*

This theorem implies the following result.

Corollary 2.1.1.1. *Let G be a balanced graph. Then G is hereditary clique-Helly.*

In [61] it is also proved that no connected hereditary clique-Helly graph has more cliques than edges, implying the following result.

Corollary 2.1.1.2. *Let G be a connected balanced graph. Then the number of cliques of G is at most the number of edges of G.*

There exists an algorithm which calculates all the cliques of a graph in $O(mnk)$ time where m is the number of edges, n the number of vertices and k the number of cliques [71] (the algorithm sequentially generates each clique in $O(mn)$ time). So a clique matrix of a hereditary clique-Helly graph can be computed in polynomial time in the size of the graph. On the other hand, Conforti, Cornuéjols, and Rao formulated a polynomial time recognition algorithm for balanced 0-1 matrices [32]. These two algorithms and the fact that hereditary clique-Helly graphs have no more than m cliques imply the following result.

Corollary 2.1.1.3. [36] *There is a polynomial time recognition algorithm for balanced graphs.*

Let A be a 0-1 matrix. We say that the row i is *included* in the row k if for every column j, $A(i,j) = 1$ implies $A(k,j) = 1$. It is not difficult to see that the clique matrix of a graph G and the clique matrix of an induced subgraph of G are related.

Lemma 2.1.2. *Let G be a graph and H an induced subgraph of G. Then A_H is the submatrix of A_G obtained by keeping the columns corresponding to the vertices of H and removing the included rows.*

On the other hand, if G is a hereditary clique-Helly graph, the clique matrix of G and the clique matrix of a clique subgraph of G are related.

Theorem 2.1.3. [61] *Let G be a hereditary clique-Helly graph and S a subset of its cliques. Let H be the subgraph of G formed by the vertices and edges of S. Then H is a clique subgraph of G and A_H is the submatrix of A_G obtained by taking the rows corresponding to the cliques in S and the columns corresponding to the vertices of these cliques.*

Since a submatrix of a balanced matrix is also balanced, these results imply that balanced graphs are closed under induced subgraphs and clique subgraphs.

Fulkerson, Hoffman and Oppenheim [44] proved the following result which implies that balanced matrices are perfect matrices.

Theorem 2.1.4. [44] *If M is a balanced matrix, then the polyhedra $P(M) = \{x/x \in R^n, Mx \leq \mathbf{1}, x \geq 0\}$ and $Q(M) = \{x/x \in R^n, Mx \geq \mathbf{1}, x \geq 0\}$ have only integer extrema.*

By Theorem 2.1.4 and Theorem 1.2.3, balanced graphs are perfect graphs.

A 0-1 matrix A is *k-colorable* if there exists a k-coloring of its columns such that for every row i that has at least two 1s in columns corresponding to colors J and L, there are entries $A(i,j) = A(i,l) = 1$, where column j has color J and column l has color L. Berge proved the following theorem.

Theorem 2.1.5. [8] *A 0-1 matrix A is balanced if and only if every submatrix of A is k-colorable for every k.*

Based on the proof of Theorem 2.1.5 and using the bicoloring algorithm of Cameron and Edmonds [20], a balanced matrix can be efficiently k-colored [33]. It is not difficult to verify that for a graph G a $\chi(G)$-coloring of A_G gives an $\chi(G)$-coloring of G. Moreover, for a balanced graph G, a $\chi(G)$-coloring of G is equivalent to a $\omega(G)$-coloring of G and $\omega(G)$ can be easily calculated, hence there exists a polynomial time combinatorial algorithm to find an optimal coloring of a balanced graph [31].

Berge and Las Vergnas proved in [10] a theorem about balanced hypergraphs which can be formulated in terms of graphs in the following way:

Theorem 2.1.6. [10] *If G is a balanced graph then $\tau_c(G) = \alpha_c(G)$.*

Corollary 2.1.6.1. *Balanced graphs are clique-perfect.*

Moreover, the clique-transversal number $\tau_c(G)$ (and hence the clique-independence number $\alpha_c(G)$) of a balanced graph G can be polynomially determined by linear programming [36].

2.2 New characterizations of balanced graphs

Some subclasses of balanced graphs are characterized by forbidden subgraphs, as the two following theorems show.

Theorem 2.2.1. [42] *A strongly chordal graph is balanced if and only if it does not contain suns.*

Theorem 2.2.2. [53] *A chordal graph is balanced if and only if it does not contain odd suns.*

In this section, two new characterizations of balanced graphs are presented. The first one, by forbidden subgraphs and the second one, by clique subgraphs.

An *extended odd sun* is an odd cycle C and a subset of pairwise adjacent vertices $W_e \subseteq N_G(e) \setminus C$ for each edge e of C, such that $N_G(W_e) \cap N_G(e) \cap C = \emptyset$ and $|W_e| \leq |N_G(e) \cap C|$. Clearly, odd suns are extended odd suns. The smallest extended odd sun is the Hajós graph (Figure 2.1).

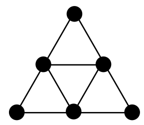

Figure 2.1: Hajós graph, also called 3-sun or 0-pyramid.

Figure 2.2 presents other examples of extended odd suns. Note that the subsets W_e and W_f, corresponding to the edges e and f respectively, may overlap.

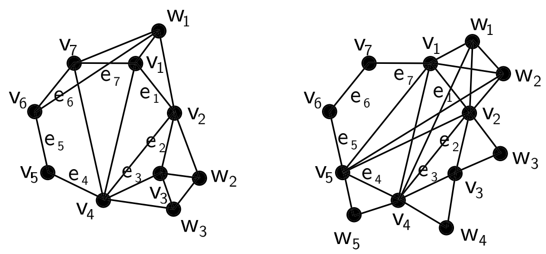

Figure 2.2: Two examples of graphs that are not balanced. In the first one, $W_{e_1} = W_{e_7} = \{w_1\}, W_{e_2} = \{w_2\}, W_{e_3} = \{w_3\}$ and $W_{e_4} = W_{e_5} = W_{e_6} = \emptyset$. In the second one, $W_{e_1} = \{w_1, w_2\}, W_{e_2} = \{w_3\}, W_{e_3} = \{w_4\}, W_{e_4} = \{w_5\}$ and $W_{e_5} = W_{e_6} = W_{e_7} = \emptyset$.

Theorem 2.2.3. *A graph is balanced if and only if it does not contain an extended odd sun.*

Proof. Let G be a graph. Suppose that G has the following extended odd sun: an odd cycle $C = \{v_1, \ldots, v_{2k+1}\}$ and a subset of pairwise adjacent vertices $W_i \subseteq N_G(e_i) \setminus C$ for each edge $e_i = v_i v_{i+1}$ of C, such that $N_G(W_i) \cap N_G(e_i) \cap C = \emptyset$.

Let $e_i = v_i v_{i+1}$ be an edge of C. Then $\{v_i, v_{i+1}\} \cup W_i$ is contained in a clique M_i of G, and $M_i \cap C = \{v_i, v_{i+1}\}$ because $N_G(e_i) \cap N_G(W_i) \cap C = \emptyset$.

Now, if we choose the rows of A_G corresponding to $M_1, \ldots, M_{2k+1}$ and the columns of A_G corresponding to $v_1, \ldots, v_{2k+1}$, we have a vertex-edge incidence matrix of an odd cycle as a submatrix of A_G. So, A_G is not balanced, and thus G is not balanced.

Conversely, suppose that G is not a balanced graph, and then A_G is not a balanced matrix. So, we have the following submatrix A' in A_G, where $M_1, \ldots, M_{2k+1}$ are cliques of G and $v_1, \ldots, v_{2k+1}$ are vertices of G:

	v_1	v_2	v_3	$\dots$	v_{2k+1}
M_1	1	1	0	$\dots$	0
M_2	0	1	1	$\dots$	0
M_3	0	0	1	$\dots$	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
M_{2k+1}	1	0	0	$\dots$	1

Figure 2.3: Vertex-edge incidence matrix of an odd cycle.

Thus $v_1, \dots, v_{2k+1}$ is an odd cycle C of G and M_i is a clique such that $M_i \cap C = \{v_i, v_{i+1}\}$. Let e_i be the edge $v_i v_{i+1}$. Then either $N_G(e_i) \cap C = \emptyset$ and then we define W_i to be the empty set, or for each $v \in N_G(e_i) \cap C$ there is a vertex w in M_i non-adjacent to v, and those vertices form a subset of pairwise adjacent vertices $W_i \subseteq N_G(e_i) \setminus C$ such that $N_G(W_i) \cap N_G(e_i) \cap C = \emptyset$ and $|W_i| \leq |N_G(e_i) \cap C|$. □

Remark 2.2.1. *Extended odd suns are not necessarily minimal. The Hajós graph is an induced subgraph of the extended odd sun of Figure 2.4.*

Theorem 2.2.4. *A graph G is balanced if and only if G is hereditary clique-Helly and no clique subgraph of G contains an odd hole.*

Proof. $\Rightarrow$) Let G be a balanced graph. By Corollary 2.1.1.1, G is HCH. Let H be a clique subgraph of G. Since balancedness is hereditary for clique subgraphs, H is balanced. Since induced subgraphs of H are also balanced, H cannot contain an odd chordless cycle of length ≥ 5.

$\Leftarrow$) Suppose that G is not a balanced graph, thus A_G is not a balanced matrix. If A_G contains the vertex-edge incidence matrix of a 3-cycle as a submatrix, then G is not HCH. Otherwise, G is HCH and A_G contains the vertex-edge incidence matrix of an odd hole as a submatrix A' (Figure 2.3, with $k \geq 2$). Let H be the subgraph of G formed by the vertices and edges of the cliques of G corresponding to the rows of A', and let H' be the subgraph of H induced by the vertices corresponding to the columns of A' (these vertices are vertices of H by the construction of A'). By Theorem 2.1.3, H is a clique subgraph of G and the clique matrix A_H is the submatrix of A_G obtained by keeping the rows of A' and then removing the null columns. Now, by Lemma 2.1.2, the clique matrix $A_{H'}$ of H' is A'. Thus H' is an odd hole. □

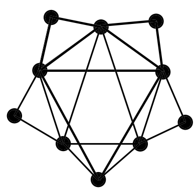

Figure 2.4: An extended odd sun which is not minimal.

2.3 Graph Classes: VE, EE, VV and EV

In this section we define and study four classes of graphs, that arise from simple domination properties. These graphs form natural subclasses of balanced graphs.

We define a graph G to be a VE graph if any odd cycle of G contains a vertex that dominates some edge of the cycle, where the edge is non-incident to the vertex.

We define a graph G to be an EV graph if any odd cycle of G contains an edge that dominates some vertex of the cycle.

Finally, we define a graph G to be a VV (resp. EE) graph if any odd cycle of it contains a vertex (resp. edge) that dominates some other vertex (resp. edge) of the cycle.

2.3.1 Inclusion relations

We now analize inclusion relations between these graph classes.

Theorem 2.3.1. *Let G be an EV graph. Then G is an EE graph and a VV graph.*

Proof. Let $C = \{v_1, \ldots, v_{2j+1}\}$ be an odd cycle of G. By hypothesis, as G is an EV graph, there is an edge $e = v_i v_{i+1}$ of C that dominates a vertex v_k of C. Then $e = v_i v_{i+1}$ dominates $e_1 = v_{k-1} v_k$ and $e_2 = v_k v_{k+1}$, and at least one of these edges is not equal to e. So, G is an EE graph. On the other hand, v_i and v_{i+1} dominate v_k, and at least one of them is different from v_k. In consequence, G is a VV graph too. □

Theorem 2.3.2. *Let G be an EE graph. Then G is a VE graph.*

Proof. Let $C = \{v_1, \ldots, v_{2j+1}\}$ be an odd cycle of G. By hypothesis, as G is an EE graph, there is an edge $e = v_i v_{i+1}$ that dominates an edge $f = v_k v_{k+1}$ of C ($e \neq f$). We may suppose that $v_i \neq v_{k+1}$, so v_i dominates $f = v_k v_{k+1}$, which implies that G is a VE graph. □

Theorem 2.3.3. *Let G be a VV graph. Then G is a VE graph.*

Proof. Let $C = \{v_1, \ldots, v_{2j+1}\}$ be an odd cycle of G. By hypothesis, as G is a VV graph, there is a vertex v_i that dominates a vertex v_k ($v_i \neq v_k$). We may suppose that $v_k \neq v_{i-1}$, so v_i dominates $f = v_k v_{k+1}$, which implies that G is a VE graph. □

Finally, we can determine that these classes of graphs are included in the class of balanced graphs.

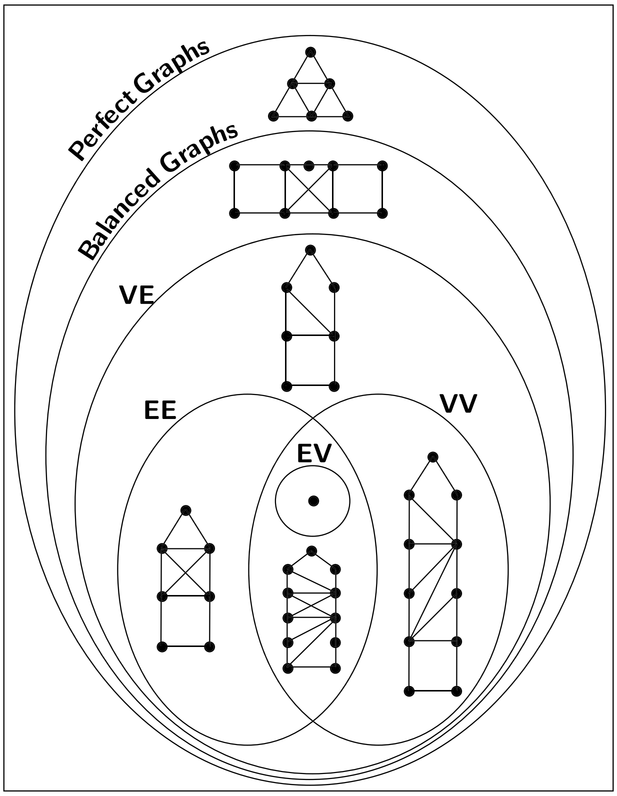

Figure 2.5: Intersection between all the classes.

Theorem 2.3.4. *Let G be a VE graph. Then G is a balanced graph.*

Proof. Suppose that A_G is not a balanced matrix. So, we have the matrix of Figure 2.3 as a submatrix A' in A_G, where $M_1, \ldots, M_{2k+1}$ are cliques of G and $v_1, \ldots, v_{2k+1}$ are vertices of G. Then $v_1, \ldots, v_{2k+1}$ is an odd cycle of G and M_i is a clique that contains the edge $v_i v_{i+1}$ ($M_i \in M(v_i, v_{i+1})$). But M_i does not contain another vertex v_j of the cycle, otherwise there would be a 1 in the position (i, j) of A'. So $M_i \notin M(v_j)$ for $j \neq i, i+1$. This fact implies that $N_G[v_i v_{i+1}] \not\subseteq N_G[v_j]$ for $j \neq i, i+1$, for any edge $v_i v_{i+1}$ of the cycle, thus G is not a VE graph. □

Corollary 2.3.4.1. *VE, EE, VV and EV graphs are perfect graphs.*

Note: Figure 2.5 shows examples of minimal graphs belonging to the possible intersections defined by the inclusions among these classes. The examples can be checked with no difficulty. We can see in this figure that the inclusions are proper.

Remark 2.3.1. *Bipartite graphs are EV graphs.*

Remark 2.3.2. *VE, EE, VV and EV graphs are hereditary classes of graphs.*

2.3.2 Matrix characterizations

Let $e_1, \dots, e_m$ and $v_1, \dots, v_n$ be the edges and vertices of a graph G, respectively. Denote by w_{1i} and w_{2i} the endpoints of the edge e_i. We define two matrices in $\{0,1\}^{m \times n}$:

- $A_{VE}(G)$, whose entry (i,j) is 1 if $N_G[e_i] \subseteq N_G[v_j]$, and 0 otherwise.
- $A_{VV}(G)$, whose entry (i,j) is 1 if $N_G[w_{1i}] \subseteq N_G[v_j]$ or $N_G[w_{2i}] \subseteq N_G[v_j]$, and 0 otherwise.

Clearly, both matrices can be constructed in polynomial time.

Theorem 2.3.5. *A graph G is a VE graph if and only if $A_{VE}(G)$ is a balanced matrix.*

Proof. $\Rightarrow$) Suppose that $A_{VE}(G)$ is not a balanced matrix. So, we have the following submatrix A' in $A_{VE}(G)$, where $e_1, \dots, e_{2k+1}$ are edges of G and $v_1, \dots, v_{2k+1}$ are vertices of G:

	v_1	v_2	v_3	$\dots$	v_{2k+1}
e_1	1	1	0	$\dots$	0
e_2	0	1	1	$\dots$	0
e_3	0	0	1	$\dots$	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
e_{2k+1}	1	0	0	$\dots$	1

Figure 2.6: Vertex-edge incidence matrix of an odd cycle.

Let $1 \leq i \leq 2k+1$. Since $N_G[e_i] \subseteq N_G[v_i] \cap N_G[v_{i+1}]$, v_i and v_{i+1} are adjacent, and then $v_1, \dots, v_{2k+1}$ is an odd cycle of G. Let f_i be the edge $v_i v_{i+1}$. Then $N_G[e_i] \subseteq N_G[f_i]$. So, if the vertex v_j dominates the edge f_i, then it also dominates the edge e_i and, therefore, there must be a 1 in the position (i,j) of A'. So the vertex v_j does not dominate the edge f_i for $j \neq i, i+1$, for any edge f_i of the cycle. Thus G is not a VE graph.

$\Leftarrow$) Suppose that G is not a VE graph. Then there is an odd cycle $C = \{v_1, \dots, v_{2k+1}\}$ such that, for any $e_i = v_i v_{i+1}$ and any $j \neq i, i+1$, $N_G[e_i] \not\subseteq N_G[v_j]$.

Now, if we choose the rows of $A_{VE}(G)$ corresponding to $e_1, \dots, e_{2k+1}$ and the columns of $A_{VE}(G)$ corresponding to $v_1, \dots, v_{2k+1}$, we have a vertex-edge incidence matrix of an odd cycle as a submatrix of $A_{VE}(G)$, so it is not a balanced matrix. $\square$

Corollary 2.3.5.1. *There is a polynomial time recognition algorithm for VE graphs.*

Theorem 2.3.6. *A graph G is a VV graph if and only if $A_{VV}(G)$ is a balanced matrix.*

Proof. $\Rightarrow$) Suppose that $A_{VV}(G)$ is not a balanced matrix. So, we have the matrix of Figure 2.6 as a submatrix A' in $A_{VV}(G)$, where $e_1, \dots, e_{2k+1}$ are edges of G and $v_1, \dots, v_{2k+1}$ are vertices of G.

Let $1 \leq i \leq 2k+1$. By definition of $A_{VV}(G)$, $N_G[e_i] \subseteq N_G[v_i] \cap N_G[v_{i+1}]$, and therefore v_i and v_{i+1} are adjacent. Then $v_1, \ldots, v_{2k+1}$ is an odd cycle of G.

Note that, if the vertex v_j dominates the vertex v_i, there must be a 1 in the position (i, j) of A' and a 1 in the position $(i-1, j)$ of A' (the sums must be understood modulo 2k+1). However, the latter does not occur. So the vertex v_j does not dominate the vertex v_i for any $j \neq i$. Thus G is not a VV graph.

$\Leftarrow$) Suppose that G is not a VV graph. Then there is an odd cycle $C = \{v_1, \ldots, v_{2k+1}\}$ such that, for any $i \neq j$, $N_G[v_i] \not\subseteq N_G[v_j]$. If we choose the rows of $A_{VV}(G)$ corresponding to $e_1, \ldots, e_{2k+1}$ and the columns of $A_{VV}(G)$ corresponding to $v_1, \ldots, v_{2k+1}$, we have a vertex-edge incidence matrix of an odd cycle as a submatrix of $A_{VV}(G)$, so it is not a balanced matrix. □

Corollary 2.3.6.1. *There is a polynomial time recognition algorithm for VV graphs.*

2.3.3 A combinatorial algorithm for the maximum stable set in VV graphs

The maximum stable set problem can be solved in polynomial time for perfect graphs by a linear programming-based algorithm [47] (and in consequence for balanced graphs and its subclasses too). We present here a purely combinatorial polynomial time algorithm (i.e., non LP-based) for the problem of determining the maximum stable set in VV graphs.

Lemma 2.3.7. *Let G be a graph and v, w two vertices of G such that v dominates w. Then there exists a maximum stable set S of G such that v does not belong to S.*

Proof. Let S be a maximum stable set in G. If v does not belong to S, the lemma holds. Otherwise, w cannot belong to S because it is adjacent to v. As v dominates w, $S \setminus \{v\} \cup \{w\}$ is a maximum stable set that does not contain v. □

Theorem 2.3.8. *There exists a polynomial time combinatorial algorithm to find a maximum stable set for VV graphs.*

Proof. Let G be a VV graph. If there exists a vertex v that dominates another vertex w, then remove v. This procedure is repeated until no more dominating vertices exist. We obtain an induced subgraph G' that can be constructed in polynomial time. As VV graphs are hereditary, G' lies in this class. So, G' has no odd cycle (and in consequence it is a bipartite graph). By Lemma 2.3.7, a maximum stable set in G' is a maximum stable set in G. Such a set can be found in $O(n^{5/2})$ time [50]. □

2.4 Clique graphs of balanced graphs

Clique graphs of several classes of graphs have already been characterized. Trees, interval graphs, chordal graphs, block graphs, clique-Helly graphs and Helly circular-arc graphs are some of them [69]. In this section we show that the class of balanced graphs and the class of totally unimodular graphs are fixed classes under the clique operator, i.e., $K(BALANCED) = BALANCED$ and $K(TOTALLY\ UNIMODULAR) = TOTALLY\ UNIMODULAR$, and finally we present a characterization of clique graphs of VE, EE, VV and EV graphs.

Some previous definitions and lemmas are needed. To this end, let A^t_G denote the transpose matrix of A_G. Then it holds the following lemma.

Lemma 2.4.1. [15] *Let G be a clique-Helly graph. Then $A_{K(G)}$ is the submatrix of A^t_G obtained by removing the included rows.*

Define the graph $H(G)$ where $V(H(G)) = \{q_1, \ldots, q_k, w_1, \ldots, w_n\}$, each q_i corresponds to the clique M_i of G, and each w_i corresponds to the vertex v_i of G. The vertices $q_1, \ldots, q_k$ induce the graph $K(G)$, the vertices $w_1, \ldots, w_n$ induce a stable set and w_j is adjacent to q_i if and only if v_j belongs to the clique M_i in G.

Theorem 2.4.2. [49] *Let G be a clique-Helly graph and $H(G)$ as defined above. Then the cliques of $H(G)$ are induced by $N_G[w_i]$ for each i, w_i is a simplicial vertex of $H(G)$ for every i, and $K(H(G)) = G$.*

Let $A \in R^{n\times m}$ and $B \in R^{n\times k}$ be two matrices. We define the matrix $A|B \in R^{n\times(m+k)}$ by $(A|B)(i,j) = A(i,j)$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $(A|B)(i, m+j) = B(i,j)$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$. Let I_n be the $n \times n$ identity matrix.

As a corollary of Theorem 2.4.2, we have the following result.

Corollary 2.4.2.1. *Let G be a clique-Helly graph and $|V(G)| = n$. Then $A_{H(G)} = A^t_G|I_n$.*

From Lemma 2.4.1 we can deduce the following result, also proved in [7].

Theorem 2.4.3. *If G is a balanced graph then $K(G)$ is also balanced.*

Theorem 2.4.4. *A graph G is balanced if and only if G is clique-Helly and $H(G)$ is balanced.*

Proof. $\Rightarrow$) If G is a balanced graph, then by Corollary 2.1.1.1, G is a clique-Helly graph. So, we have that $A_{H(G)} = A^t_G|I_n$ (Corollary 2.4.2.1), and A_G is balanced, so A^t_G is balanced. On the other hand, all the columns of the vertex-edge incidence matrix of an odd cycle have two nonzero entries, so $A_{H(G)}$ is balanced.

$\Leftarrow$) If G is a clique-Helly graph and $H(G)$ is balanced, $G = K(H(G))$ (Theorem 2.4.2) and then G is balanced (Theorem 2.4.3). $\square$

The following corollary, mentioned in [56], follows from Theorem 2.4.3, Corollary 2.1.1.1 and Theorem 2.4.4.

Corollary 2.4.4.1. *The class of balanced graphs is fixed under K, that is, $K(BALANCED) = BALANCED$.*

Next, we show that similar results hold for the class of totally unimodular graphs.

Theorem 2.4.5. *If G is a totally unimodular graph then $K(G)$ is also totally unimodular.*

Proof. If G is a totally unimodular graph then G is a balanced graph and then G is a clique-Helly graph (Corollary 2.1.1.1). So Lemma 2.4.1 holds. If A_G is a totally unimodular matrix, then A_G^t is totally unimodular too, since for every square matrix M, $det(M) = det(M^t)$. And every submatrix of a totally unimodular matrix is totally unimodular. So, $A_{K(G)}$ is a totally unimodular matrix. □

Theorem 2.4.6. *A graph G is totally unimodular if and only if G is clique-Helly and $H(G)$ is totally unimodular.*

Proof. $\Rightarrow$) If G is a totally unimodular graph then G is a balanced graph and consequently G is a clique-Helly graph (Corollary 2.1.1.1). We have that $A_{H(G)} = A_G^t|I_n$ (Corollary 2.4.2.1), and A_G is totally unimodular, so A_G^t is totally unimodular. Every square submatrix M of $A_{H(G)}$ can be written as $M = M_1|M_2$, where M_1 is a submatrix of A_G^t and M_2 is a submatrix of I_n. So, using determinant properties, M is singular or $det(M) = \pm det(M_3)$, where M_3 is a square submatrix of M_1. Then, in both cases, $det(M) = 0$ or ± 1. Therefore $H(G)$ is totally unimodular.

$\Leftarrow$) If G is a clique-Helly graph and $H(G)$ is totally unimodular, $G = K(H(G))$ (Theorem 2.4.2) and then G is totally unimodular (Theorem 2.4.5). □

Corollary 2.4.6.1. *The class of totally unimodular graphs is fixed under K, i.e., $K(TOTALLY\ UNIMODULAR) = TOTALLY\ UNIMODULAR$.*

Finally, we present a characterization of clique graphs of VE, EE, VV and EV graphs.

Let $S = \{M_1, \ldots, M_{2k+1}\}$ be an odd set of cliques of G, where M_r intersects M_{r+1} for $r = 1, \ldots, 2k+1$ (all the index sums must be understood modulo $2k+1$).

A graph G is a *dually EE* graph (DEE graph) if for any such a set S there exist i, j, $1 \leq i, j \leq 2k+1$, $i \neq j$, such that $M_i \cap M_{i+1} \subseteq M_j \cap M_{j+1}$.

A graph G is a *dually VE* graph (DVE graph) if for any such a set S there exist i, j, $1 \leq i, j \leq 2k+1$, $i \neq j$, $i+1 \neq j$, such that $M_i \cap M_{i+1} \subseteq M_j$.

Theorem 2.4.7. *Let G be a DEE graph. Then G is a DVE graph.*

Proof. Let $S = \{M_1, \ldots, M_{2k+1}\}$ a set of cliques of G, where M_i intersects M_{i+1} for $i = 1, \ldots, 2k+1$. By hypothesis, as G is a DEE graph, there are cliques $M_i, M_{i+1}, M_j, M_{j+1}$ such that $M_i \cap M_{i+1} \subseteq M_j \cap M_{j+1}$ ($i \neq j$). So $M_i \cap M_{i+1} \subseteq M_j$, and if $i+1 = j$ then $i \neq j+1, i+1 \neq j+1$ and $M_i \cap M_{i+1} \subseteq M_{j+1}$, which implies that G is a DVE graph. □

Theorem 2.4.8. *Let G be a DVE graph. Then G is a balanced graph.*

Proof. Suppose that A_G is not a balanced matrix. So, we have the matrix of Figure 2.3 as a submatrix A' in A_G, where $M_1, \ldots, M_{2k+1}$ are cliques of G and $v_1, \ldots, v_{2k+1}$ are vertices of G. Then $\{M_1, \ldots, M_{2k+1}\}$ is an odd set of cliques of G where M_i intersects M_{i+1} for $i = 1, \ldots, 2k+1$. On the other hand, v_i is a vertex that belongs to $M_i \cap M_{i+1}$ but v_i does not belong to another clique M_j of the set, otherwise there would be a 1 in the position (j,i) of A'. So $v_i \notin M_j$ for $j \neq i, i+1$. This fact implies that $M_i \cap M_{i+1} \not\subseteq M_j$ for $j \neq i, i+1$, for any $i = 1, \ldots, 2k+1$, thus G is not a DVE graph. □

Theorem 2.4.9. *Let G be a graph.*

- *If G is a DVE graph then $K(G)$ is VE.*
- *If G is a DEE graph then $K(G)$ is EE.*
- *If G is a VE graph then $K(G)$ is DVE.*
- *If G is a EE graph then $K(G)$ is DEE.*

Proof. Let G be a graph. Classes DVE, DEE, VE and EE are subclasses of balanced graphs, and balanced graphs are clique-Helly. So, if G belongs to some of these classes, then G is a clique-Helly graph. The vertices of $K(G)$ are the cliques of G, and by Lemma 2.4.1 we know that the cliques of $K(G)$ are some $M(v)$ with $v \in V(G)$.

Let $\{M_1, \ldots, M_{2k+1}\}$ be an odd cycle in $K(G)$, then M_i intersects M_{i+1} in G, for $i = 1, \ldots, 2k+1$.

If G is a DVE graph, there are cliques M_i, M_{i+1}, M_j such that $M_i \cap M_{i+1} \subseteq M_j$ $(i, i+1 \neq j)$. Let $M(v)$ be a clique of $K(G)$ that contains M_i and M_{i+1}. Then, in G, v lies in $M_i \cap M_{i+1}$ implying that v is in M_j and therefore $M(v)$ contains M_j too. So, in $K(G)$, the vertex M_j dominates the edge M_iM_{i+1} and, as a consequence, $K(G)$ is in VE.

If G is a DEE graph, there are cliques $M_i, M_{i+1}, M_j, M_{j+1}$ such that $M_i \cap M_{i+1} \subseteq M_j \cap M_{j+1}$ $(i \neq j)$. Let $M(v)$ be a clique of $K(G)$ that contains M_i and M_{i+1}, then, in G, v lies in $M_i \cap M_{i+1}$ implying that v is in $M_j \cap M_{j+1}$ and therefore $M(v)$ contains M_j and M_{j+1} too. So, in $K(G)$, the edge M_jM_{j+1} dominates the edge M_iM_{i+1} and, in consequence, $K(G)$ is in EE.

Now, let $\{M(v_1), \ldots, M(v_{2k+1})\}$ be an odd set of cliques in $K(G)$, where $M(v_i)$ intersects $M(v_{i+1})$ for $i = 1, \ldots, 2k+1$. Then for each i there exists a clique M_i of G such that v_i and v_{i+1} belong to M_i, and then v_i and v_{i+1} are adjacent in G, so $v_1, \ldots, v_{2k+1}$ is an odd cycle in G.

If G is in VE, there is a vertex v_j of the cycle that dominates the edge v_iv_{i+1} with $j \neq i, i+1$. Let M be a vertex of $K(G)$, M lies in $M(v_i) \cap M(v_{i+1})$ in $K(G)$, v_i and

v_{i+1} belong to M in G, and therefore v_j belongs to M too. So $M \in M(v_j)$, and in consequence $M(v_i) \cap M(v_{i+1}) \subseteq M(v_j)$. Then $K(G)$ is a DVE graph.

If G is in EE, there is an edge v_jv_{j+1} of the cycle that dominates the edge v_iv_{i+1} with $j \neq i$. Let M be a vertex of $K(G)$, M lies in $M(v_i) \cap M(v_{i+1})$ in $K(G)$, v_i and v_{i+1} belong to M in G, and therefore v_j and v_{j+1} belong to M too. So $M \in M(v_j) \cap M(v_{j+1})$, and in consequence $M(v_i) \cap M(v_{i+1}) \subseteq M(v_j) \cap M(v_{j+1})$. Then $K(G)$ is a DEE graph. □

Theorem 2.4.10. *Let G be a clique-Helly graph.*

- *G is a DVE graph if and only if $H(G)$ is VE.*
- *G is a DEE graph if and only if $H(G)$ is EE.*
- *G is a VE graph if and only if $H(G)$ is DVE.*
- *G is a EE graph if and only if $H(G)$ is DEE.*

Proof. Let G be a clique-Helly graph and $H(G)$ as defined in Theorem 2.4.2, with $V(H(G)) = \{q_1, \ldots, q_k, w_1, \ldots, w_n\}$, each q_i corresponds to the clique M_i of G, and each w_i corresponds to the vertex v_i of G. By Theorem 2.4.2, the cliques of $H(G)$ are $N_{H(G)}[w_i]$ for each i. Then w_i and all its incident edges are dominated between themselves, and every vertex in $N_{H(G)}(w_i)$ dominates w_i and all its incident edges.

Let C be an odd cycle in $H(G)$. If there is a vertex w_i in C, then C contains an edge that dominates another edge, and a vertex that dominates an edge non incident to it.

If there is not such a vertex, C is an odd cycle $\{q_{r_1}, \ldots, q_{r_{2s+1}}\}$ that corresponds to an odd set of cliques $\{M_{r_1}, \ldots, M_{r_{2s+1}}\}$ of G, such that M_{r_i} intersects $M_{r_{i+1}}$ for $i = 1, \ldots, 2s+1$.

If G is a DVE graph, there are cliques $M_{r_i}, M_{r_{i+1}}, M_{r_j}$ such that $M_{r_i} \cap M_{r_{i+1}} \subseteq M_{r_j}$ $(i, i+1 \neq j)$. Let $N_{H(G)}[w_l]$ be a clique of $H(G)$ that contains q_{r_i} and $q_{r_{i+1}}$. Then, in G, v_l lies in $M_{r_i} \cap M_{r_{i+1}}$ implying that v_l is in M_{r_j} and therefore, in $H(G)$, $N_{H(G)}[w_l]$ contains q_{r_j} too. So, in $H(G)$, the vertex q_{r_j} dominates the edge $q_{r_i}q_{r_{i+1}}$ and, in consequence, $H(G)$ is VE.

If G is a DEE graph, there are cliques $M_{r_i}, M_{r_{i+1}}, M_{r_j}, M_{r_{j+1}}$ such that $M_{r_i} \cap M_{r_{i+1}} \subseteq M_{r_j} \cap M_{r_{j+1}}$ $(i \neq j)$. Let $N_{H(G)}[w_l]$ be a clique of $H(G)$ that contains M_{r_i} and $M_{r_{i+1}}$. Then, in G, v_l belongs to $M_{r_i} \cap M_{r_{i+1}}$ implying that v_l belongs to $M_{r_j} \cap M_{r_{j+1}}$. Therefore, in $H(G)$, $N_{H(G)}[w_l]$ contains q_{r_j} and $q_{r_{j+1}}$ too. So, in $H(G)$, the edge $q_{r_j}q_{r_{j+1}}$ dominates the edge $q_{r_i}q_{r_{i+1}}$ and, in consequence, $H(G)$ is EE.

Now, let $\{N_{H(G)}[w_{r_1}], \ldots, N_{H(G)}[w_{r_{2s+1}}]\}$ be an odd set of cliques in $H(G)$, where $N_{H(G)}[w_{r_i}]$ intersects $N_{H(G)}[w_{r_{i+1}}]$ for $i = 1, \ldots, 2s+1$. Then for each i there exists a vertex $q \in N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}]$. So v_{r_i} and $v_{r_{i+1}}$ belong to the corresponding clique M of G, and then v_{r_i} and $v_{r_{i+1}}$ are adjacent in G, so $v_{r_1}, \ldots, v_{r_{2s+1}}$ is an odd cycle in G.

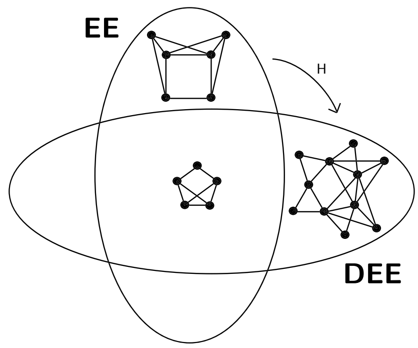

Figure 2.7: Intersection between the dual classes EE and DEE.

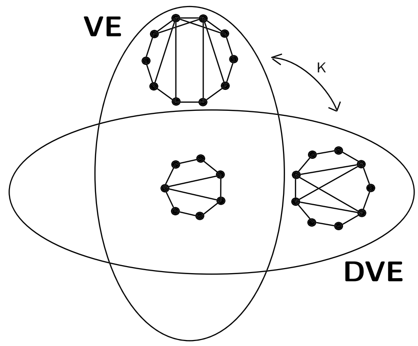

Figure 2.8: Intersection between the dual classes VE and DVE.

If G is a VE graph, there is a vertex v_{r_j} of the cycle that dominates the edge $v_{r_i}v_{r_{i+1}}$ with $j \neq i, i+1$. Let q_l be a vertex of $H(G)$, q_l lies in $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}]$ in $H(G)$, v_i and v_{i+1} belong to M_l in G, and therefore v_j belongs to M_l too. So q_l belongs to $N_{H(G)}[w_{r_j}]$, and in consequence $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}] \subseteq N_{H(G)}[w_{r_j}]$. Then $H(G)$ is DVE.

If G is a EE graph, there is an edge $v_{r_j}v_{r_{j+1}}$ of the cycle that dominates the edge $v_{r_i}v_{r_{i+1}}$ with $j \neq i$. Let q_l be a vertex of $H(G)$, q_l lies in $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}]$ in $H(G)$, v_{r_i} and $v_{r_{i+1}}$ belong to M_l in G, and therefore v_{r_j} and $v_{r_{j+1}}$ belong to M_l too. So q_l belongs to $N_{H(G)}[w_{r_j}] \cap N_{H(G)}[w_{r_{j+1}}]$ in $H(G)$, and in consequence $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}] \subseteq N_{H(G)}[w_{r_j}] \cap N_{H(G)}[w_{r_{j+1}}]$. Then $H(G)$ is DEE.

The converse properties follow from Theorem 2.4.2 and Theorem 2.4.9 applied to $H(G)$. □

Corollary 2.4.10.1. *$K(DEE) = EE$ and $K(EE) = DEE$.*

Corollary 2.4.10.2. *$K(DVE) = VE$ and $K(VE) = DVE$.*

Theorem 2.4.11. *Let G be a graph. If G is a VV graph then $K^2(G)$ is a bipartite graph.*

Proof. If G is a VV graph then G is clique-Helly (Corollary 2.1.1.1). Every odd cycle of G has a dominated vertex, and therefore, by Theorem 1.3.12, $K^2(G)$ is a bipartite graph. □

Theorem 2.4.12. *Let G be a graph. Then $K(G)$ is a bipartite graph if and only if G is a clique-Helly graph and $H(G)$ is an EV graph.*

Proof. ⇒) Let G be a graph, $V(G) = \{v_1, \ldots, v_n\}$ and $M(G) = \{M_1, \ldots, M_k\}$. Since $K(G)$ is a bipartite graph, G is clique-Helly because any set of pairwise intersecting cliques has at most two elements. Clearly, $V(H(G)) = V(K(G)) \cup \{w_1, \ldots, w_n\}$ as in the definition of $H(G)$. Also, $K(G)$ is a bipartite graph and by the definition of $H(G)$, every odd cycle C of $H(G)$ must contain a vertex w_i from $\{w_1, \ldots, w_n\}$. By Theorem 2.4.2, w_i is a simplicial vertex, so the edges of C incident to w_i dominate the vertex w_i, and then $H(G)$ is an EV graph.

⇐) If G is a clique-Helly graph and $H(G)$ is an EV graph, it is a VV graph too. So by Theorem 2.4.11, $K^2(H(G)) = K(G)$ is a bipartite graph. □

Corollary 2.4.12.1. $K^2(VV) = K^2(EV) =$ *the class of bipartite graphs.*

Proof. We will prove that $K^2(EV) \subseteq K^2(VV) \subseteq \mathit{BIPARTITE} \subseteq K^2(EV)$ and therefore the three classes are the same. The first inclusion holds because $EV \subseteq VV$. The second inclusion follows from Theorem 2.4.11. Now, for every bipartite graph G we have that $K(H(G)) = G$ and by Theorem 2.4.12 applied to $H(G)$, $H^2(G)$ is an EV graph and $K^2(H^2(G)) = G$. So the third inclusion holds too. □

The class $K^{-1}(\mathit{BIPARTITE})$ has been analyzed and characterized by forbidden subgraphs in [62].

Corollary 2.4.12.2. $K(VV) = K(EV) = K^{-1}(\mathit{BIPARTITE})$.

Proof. Let G be a VV graph. By the last corollary, $K^2(G) = K(K(G))$ is bipartite so $K(G)$ belongs to $K^{-1}(\mathit{BIPARTITE})$. Therefore $K(EV) \subseteq K(VV) \subseteq K^{-1}(\mathit{BIPARTITE})$. On the other hand, let G be a graph belonging to $K^{-1}(\mathit{BIPARTITE})$, then by Theorem 2.4.12 $H(G)$ is EV and $G = K(H(G))$. So $K^{-1}(\mathit{BIPARTITE}) \subseteq K(EV) \subseteq K(VV) \subseteq K^{-1}(\mathit{BIPARTITE})$ and we have that the three sets are equal. □

As a consequence of this result, we deduce the existence of non LP-based algorithms to find a maximum clique-independent set and a minimum clique-transversal for VV graphs.

Corollary 2.4.12.3. *There exists a polynomial time combinatorial algorithm to find a maximum clique-independent set and a minimum clique-transversal for VV graphs.*

Proof. Let G be a VV graph. Then $K(G)$ belongs to $K^{-1}(\mathit{BIPARTITE})$ and can be constructed in polynomial time. Moreover, a maximum clique-independent set of G can

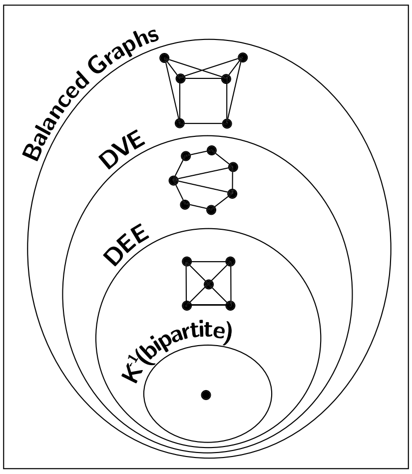

Figure 2.9: Inclusion between the classes.

be obtained from a maximum stable set of $K(G)$, and a minimum clique-transversal of G can be constructed from a minimum clique covering of $K(G)$. Since the graphs $K^{-1}(BIPARTITE)$ are claw-free [62] there exists a polynomial time combinatorial algorithm for maximum stable set in these graphs [66]. As $K(G)$ is also perfect, we can use the polynomial time combinatorial algorithm for minimum clique covering in claw-free perfect graphs [51]. So, the result holds. $\square$

To close the section, we verify that $K^{-1}(BIPARTITE)$ graphs are a subclass of DEE.

Theorem 2.4.13. *$K^{-1}(BIPARTITE) \subseteq DEE$.*

Proof. Let $G \in K^{-1}(BIPARTITE)$. Suppose that there exists an odd set $S = \{M_1, \ldots, M_{2k+1}\}$ of cliques of G, where M_i intersects M_{i+1} for $i = 1, \ldots, 2k$ and M_{2k+1} intersects M_1. Then the corresponding vertices in $K(G)$ form an odd cycle, but $K(G)$ is a bipartite graph, so such a set does not exist, and G is DEE. $\square$

Note 1. *Figure 2.9 shows that all these inclusions are proper.*

CHAPTER 3

Partial characterizations of clique-perfect graphs

A graph G is clique-perfect if the cardinality of a maximum clique-independent set of H equals the cardinality of a minimum clique-transversal of H, for every induced subgraph H of G. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. In this chapter, we present partial results in this direction, that is, we characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph belongs to certain classes.

This chapter is organized as follows.

In Section 3.1 we present some families of clique-perfect and clique-imperfect graphs.

In Subsection 3.2.1 we characterize clique-perfect diamond-free graphs by forbidden induced subgraphs. In Subsections 3.2.2, 3.2.3, and 3.2.4 we characterize clique-perfect graphs by minimal forbidden induced subgraphs, when the graph is a line graph, claw-free hereditary clique-Helly, or a Helly circular-arc graph, respectively.

Finally, in Section 3.3 we present polynomial time recognition algorithms for clique-perfection in these last three classes of graphs.

Extended abstracts of the results in this chapter appear in [11] and [14]. The full versions were recently submitted [12, 13].

3.1 Some families of clique-perfect and clique-imperfect graphs

Some known classes of clique-perfect graphs are dually chordal graphs [17], comparability graphs [2] and balanced graphs [10].

Proposition 3.1.1. *Complements of acyclic graphs are clique-perfect.*

Proof. Let G be a complement of an acyclic graph. If $\overline{G}$ contains a vertex v of degree zero, then every clique of G contains v, so $\alpha_c(G) = \tau_c(G) = 1$. Otherwise, since G does not contain a universal vertex, $\tau_c(G) > 1$ and since $\overline{G}$ is acyclic, $\overline{G}$ contains a vertex w of degree 1. Let z be the neighbor of w in $\overline{G}$. Every clique of G not containing z must contain w by maximality. So $\tau_c(G) = 2$. On the other hand, since every connected component of $\overline{G}$ is a tree with at least two vertices, we can obtain two disjoint maximal stable sets in $\overline{G}$, thus $\alpha_c(G) = 2$. Since the class of acyclic graphs is hereditary, the equality between α_c and τ_c holds for every induced subgraph of G. □

A *generalized sun* is defined as follows. Let G be a graph and C be a cycle of G not necessarily induced. An edge of C is *non proper* (or *improper*) if it forms a triangle with some vertex of C. An *r-generalized sun*, $r \geq 3$, is a graph G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with all its non proper edges $\{e_j\}_{j\in J}$ (J is allowed to be an empty set) and a stable set $U = \{u_j\}_{j\in J}$, such that for each $j \in J$, u_j is adjacent to the endpoints of e_j only. An r-generalized sun is said to be *odd* if r is odd. Clearly, odd holes and odd suns are odd generalized suns.

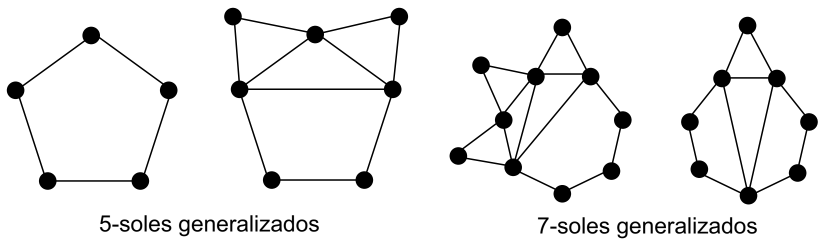

Figure 3.1: Some examples of odd generalized suns.

Theorem 3.1.1. [15] *Odd generalized suns and antiholes of length $t = 1, 2$ mod 3 ($t \geq 5$) are not clique-perfect.*

Unfortunately, not every odd generalized sun is minimally clique-imperfect (with respect to taking induced subgraphs). Nevertheless, odd holes and complete odd suns are minimally clique-imperfect, and we will distinguish two other kinds of minimally clique-imperfect odd generalized suns in order to state a characterization of HCA clique-perfect graphs by minimal forbidden induced subgraphs.

A *viking* is a graph G such that $V(G) = \{a_1, \ldots, a_{2k+1}, b_1, b_2\}$, $k \geq 2$, $a_1 \ldots a_{2k+1}a_1$ is a cycle with only one chord a_2a_4; b_1 is adjacent to a_2 and a_3; b_2 is adjacent to a_3 and a_4, and there are no other edges in G.

A 2-*viking* is a graph G such that $V(G) = \{a_1, \ldots, a_{2k+1}, b_1, b_2, b_3\}$, $k \geq 2$, $a_1 \ldots a_{2k+1}a_1$ is a cycle with only two chords, a_2a_4 and a_3a_5; b_1 is adjacent to a_2 and a_3; b_2 is adjacent to a_3 and a_4; b_3 is adjacent to a_4 and a_5, and there are no other edges in G.

Proposition 3.1.2. *Vikings and* 2*-vikings are clique-imperfect.*

Proof. Vikings and 2-vikings are odd generalized suns, where in both cases the odd cycle is $a_1 \ldots a_{2k+1}a_1$, and the stable sets are $\{b_1, b_2\}$ and $\{b_1, b_2, b_3\}$, respectively. □

We now present two new families (neither odd generalized suns nor antiholes) of minimal clique-imperfect graphs.

For $k \geq 2$, define the graph S_k as follows: $V(S_k) = \{a_1, \ldots, a_{2k+1}, b_1, b_2, b_3\}$, $a_1 \ldots a_{2k+1}a_1$ is a cycle with only one chord a_3a_5; b_1 is adjacent to a_1 and a_2; b_2 is adjacent to a_4 and a_5; b_3 is adjacent to a_1, a_2, a_3 and a_4, and there are no other edges in S_k.

For $k \geq 2$, define the graph T_k as follows: $V(T_k) = \{a_1, \ldots, a_{2k+1}, b_1, \ldots, b_5\}$, $a_1 \ldots a_{2k+1}a_1$ is a cycle with only two chords, a_2a_4 and a_3a_5; b_1 is adjacent to a_1 and a_2; b_2 is adjacent to a_1, a_2 and a_3; b_3 is adjacent to a_1, a_2, a_3, a_4, b_2 and b_4; b_4 is adjacent to a_3, a_4 and a_5; b_5 is adjacent to a_4 and a_5, and there are no other edges in T_k.

Proposition 3.1.3. *Let $k \geq 2$. Then S_k and T_k are clique-imperfect.*

Proof. Every clique of S_k contains at least two vertices of $a_1, \ldots, a_{2k+1}$, so $\alpha_c(S_k) \leq k$. The same holds for T_k, so $\alpha_c(T_k) \leq k$. On the other hand, consider in S_k the family of cliques $\{a_1, a_2, b_1\}$, $\{a_2, a_3, b_3\}$, $\{a_3, a_4, b_3\}$, $\{a_4, a_5, b_2\}$ and either $\{a_5, a_1\}$, if $k = 2$, or $\{a_5, a_6\}$, …, $\{a_{2k+1}, a_1\}$, if $k > 2$. No vertex of S_k belongs to more than two of these $2k + 1$ cliques, so $\tau_c(S_k) \geq k + 1$. Analogously, consider in T_k the family of cliques $\{a_1, a_2, b_1\}$, $\{a_2, a_3, b_2, b_3\}$, $\{a_3, a_4, b_3, b_4\}$, $\{a_4, a_5, b_5\}$ and either $\{a_5, a_1\}$, if $k = 2$, or $\{a_5, a_6\}$, …, $\{a_{2k+1}, a_1\}$, if $k > 2$. No vertex of T_k belongs to more than two of these $2k + 1$ cliques, so $\tau_c(T_k) \geq k + 1$. □

The minimality of vikings, 2-vikings, S_k and T_k ($k \geq 2$) will be proved as a corollary of the main theorem of Subsection 3.2.4.

A *drum* is a graph G on $2r$ vertices whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that $w_1 \ldots w_r$ and $u_1 \ldots u_r$ are cycles, every chord of these cycles belongs to a triangle, and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod r$. A drum is *complete* if U and W are completes. Denote by D_r the complete drum on $2r$ vertices.

Proposition 3.1.4. *Drums on $2r$ vertices with $r = 1, 2 \bmod 3$ ($r \geq 4$) are clique-imperfect.*

Proof. Let G be a drum on $2r$ vertices, $r \geq 4$, as defined above. Every clique of G contains at least three vertices, so $\alpha_c(G) \leq \lfloor \frac{2r}{3} \rfloor$. On the other hand, consider the $2r$ cliques of G having nonempty intersection with U and W. Every vertex of G belongs to three of these cliques, so $\tau_c(G) \geq \lceil \frac{2r}{3} \rceil$. It follows that if $r = 1, 2 \ mod \ 3$ then $\tau_c(G) > \alpha_c(G)$. $\square$

Remark 3.1.1. *It is not difficult to check that D_3 is clique-perfect and drums on 8 and 10 vertices are minimally clique-imperfect if and only if they are complete. On the other hand, complete drums on $2r$ vertices with $r \geq 6$ are clique-imperfect since they contain the graph $D_6 \setminus \{w_1, w_4\}$, which is minimally clique-imperfect.*

In [57] the minimal graphs G such that $K(G)$ is complete (i.e. $\alpha_c(G) = 1$) and no vertex of G is universal (i.e., $\tau_c(G) > 1$) are characterized. The graph Q_n, $n \geq 3$, is defined as follows: $V(Q_n) = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ is a set of $2n$ vertices; $v_1, \ldots, v_n$ induce $\overline{C_n}$; for each $1 \leq i \leq n$, $N_{Q_n}[u_i] = V(Q_n) - \{v_i\}$.

The following result will be useful for our purposes.

Theorem 3.1.2. [57] *For $k \geq 1$, $\alpha_c(Q_{2k+1}) = 1$ and $\tau_c(Q_{2k+1}) = 2$. Moreover, if G is a graph such that $\alpha_c(G) = 1$ and $\tau_c(G) > 1$, then G contains Q_{2k+1} for some $k \geq 1$.*

As a corollary of Theorem 3.1.2, graphs Q_{2k+1}, where $k \geq 1$, are not clique-perfect. Note that Q_n contains $\overline{C_n}$, so Q_n is neither clique-perfect nor minimally clique-imperfect for $n = 1, 2$ mod 3, $n \geq 5$. On the other hand, Q_3 is the 3-sun, so it is minimally clique-imperfect.

Proposition 3.1.5. *Let $k \geq 1$. Then Q_{6k} is clique-perfect and Q_{6k+3} is minimally clique-imperfect.*

Proof. Let $k \geq 1$. By Theorem 3.1.2, Q_{6k+3} is clique-imperfect. On the other hand, in Q_{6k}, the set $\{v_1, u_1\}$ is a clique-transversal, and $A = \{v_i : i \text{ is odd}\} \cup \{u_i : i \text{ is even}\}$ and $B = \{v_i : i \text{ is even}\} \cup \{u_i : i \text{ is odd}\}$ are two disjoint cliques. Let $n = 3t$, with $t \geq 1$. In order to prove the minimality of Q_{6k+3} as well as the clique perfection of Q_{6k}, it remains to show that the equality of τ_c and α_c holds for every proper induced subgraph of Q_n. Please note that $Q_n \setminus \{v_i\}$ is the complement of an acyclic graph, so it is clique-perfect by Proposition 3.1.1, and we have to consider only the induced subgraphs of Q_n containing all the vertices $v_1, \ldots, v_n$. In $\overline{C_{3t}}$, we have $\tau_c(\overline{C_{3t}}) = \alpha_c(\overline{C_{3t}}) = 3$, so suppose there are some vertices from $u_1, \ldots, u_n$, but no all of them. Without loss of generality, let H be an induced subgraph of Q_n such that $v_1, \ldots, v_n$ and u_1 belong to H and u_n does not. Then $\{v_1, u_1\}$ is a clique-transversal of H. If n is even, $A = \{v_i : i \text{ is odd}\} \cup \{u_i \in H : i \text{ is even}\}$ and $B = \{v_i : i \text{ is even}\} \cup \{u_i \in H : i \text{ is odd}\}$ are two disjoint cliques of H. If n is odd, $A = \{v_i : i < n \text{ and } i \text{ is odd}\} \cup \{u_i \in H : i \text{ is even}\}$ and $B = \{v_i : i \text{ is even}\} \cup \{u_i \in H : i \text{ is odd}\}$ are two disjoint cliques of H. That concludes the proof. $\square$

3.2 Partial characterizations

For some classes of graphs, it is enough to exclude the families of clique-imperfect graphs presented in Section 3.1 in order to guarantee that the graph is clique-perfect.

Theorem 3.2.1. [53] *Let G be a chordal graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd sun.*

The main results in this chapter are the following four theorems, which will be proved in the next subsections.

Theorem 3.2.2. *Let G be a diamond-free graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd generalized sun.*

Theorem 3.2.3. [12] *Let G be a line graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd hole or a 3-sun.*

Theorem 3.2.4. [12] *Let G be an HCH claw-free graph. Then G is clique-perfect if and only if no induced subgraph of G is an odd hole or an antihole of length seven.*

Theorem 3.2.5. *Let G be an HCA graph. Then G is clique-perfect if and only if it does not contain a 3-sun, an antihole of length seven, an odd hole, a viking, a 2-viking or one of the graphs S_k or T_k.*

3.2.1 Diamond-free graphs

In this subsection we prove Theorem 3.2.2, which states that if a graph G is diamond-free, then G is clique-perfect if and only if it does not contain odd generalized suns. To accomplish this, we first prove that diamond-free graphs with no odd generalized suns are K-perfect.

Theorem 3.2.6. *Let G be a diamond-free graph. If G does not contain odd generalized suns, then $K(G)$ is perfect.*

Proof. By Theorem 1.2.1, it suffices to prove that $K(G)$ contains no odd holes or odd antiholes. By [22], G being diamond-free implies that $K(G)$ is diamond-free, and hence $K(G)$ contains no antihole of length at least 7. Suppose $K(G)$ contains an odd hole $k_1k_2 \dots k_{2n+1}$, where $k_1, \dots, k_{2n+1}$ are cliques of G. Then G contains an odd cycle $v_1v_2 \dots v_{2n+1}v_1$, where v_i belongs to $k_i \cap k_{i+1}$ and no other k_j. Since G contains no odd generalized suns, we may assume that some edge of this cycle, say, v_1v_2 is in a triangle with another vertex of the cycle, say v_m. Now v_1, v_2 both belong to k_2, and v_m does not. Since k_2 is a clique, it follows that v_m has a non-neighbor w in k_2. But now $\{v_1, v_2, v_m, w\}$ induces a diamond, a contradiction. □

We are now in position to prove the characterization of clique-perfect diamond-free graphs.

Proof of Theorem 3.2.2. By Theorem 3.1.1, if G is clique-perfect then no induced subgraph of G is an odd generalized sun. As a direct corollary of Theorem 1.3.2, it follows that diamond-free graphs are HCH. Thus, since the class of diamond-free graphs with no odd generalized suns is hereditary, the converse follows from Theorem 3.2.6 and Proposition 1.3.1. □

3.2.2 Line graphs

The purpose of this subsection is to prove Theorem 3.2.3, which states that if G is a line graph, then G is clique-perfect if and only if it does not contain odd holes or a 3-sun. We start by analyzing line graphs with no odd holes or induced 3-suns.

Graphs such that its line graph is perfect were characterized by Trotter.

Theorem 3.2.7. [70] *Let H be a graph. The graph $G = L(H)$ is perfect if and only if H contains no odd cycle of length at least five.*

As a corollary of Theorem 3.2.7, a line graph G is perfect if and only if it contains no odd hole. In [58] Maffray gave a third equivalent statement.

Theorem 3.2.8. [58] *Let $G = L(H)$ be the line graph of a graph H. Then the following three conditions are equivalent:*

(i) *G is a perfect graph.*

(ii) *H does not contain any odd cycle of length at least five.*

(iii) *Any connected subgraph H' of H satisfies at least one of the following properties:*
- *H' is a bipartite graph;*
- *H' is a complete of size four;*
- *H' consists of exactly $p+2$ vertices $x_1, \ldots, x_p, a, b$, such that $\{x_1, \ldots, x_p\}$ is a stable set, and $\{x_j, a, b\}$ is a triangle for each $j = 1, \ldots, p$.*
- *H' has a cutpoint.*

Theorem 3.2.9. *If G is a line graph and G does not contain odd holes, then $K(G)$ is perfect.*

Proof. Let $G = L(H)$. By Lemma 1.3.10, we may assume H is connected. Since G has no odd holes, it follows that all the odd cycles of H are triangles. So by Theorem 3.2.8 either H is a bipartite graph, or H is a complete of size four, or H consists of exactly $p + 2$ vertices $x_1, \ldots, x_p, a, b$, such that $\{x_1, \ldots, x_p\}$ is a stable set, and $\{x_j, a, b\}$ is a triangle for each $j = 1, \ldots, p$, or H has a cutpoint.

If H is bipartite then $G = K(H)$ and $K(G) = K^2(H)$ is an induced subgraph of H (Theorem 1.3.12), so it is bipartite and hence perfect.

If H is a complete of size four, then $K(L(H))$ is the complement of $4K_2$, and so it is perfect (it is the complement of a bipartite graph).

If H consists of exactly $p+2$ vertices $x_1, \ldots, x_p, a, b$, such that $\{x_1, \ldots, x_p\}$ is a stable set, and $\{x_j, a, b\}$ is a triangle for each $j = 1, \ldots, p$, then all the cliques of G contain the vertex corresponding to the edge ab of H, so $K(G)$ is a complete graph, and hence perfect.

Suppose H has a cutpoint x, and let M_x be the complete subgraph of G induced by the vertices corresponding to the edges of H incident to x. Since x is a cutpoint of H, M_x is a clique of G, and let v be the vertex of $K(G)$ corresponding to M_x.

If $H = H_1 + H_2 + x$ and both H_1 and H_2 have at least one edge, then v is a cutpoint of $K(G)$, and $K(G) = M_1 + M_2 + v$, where M_i is the clique graph of the line graph of the subgraph of H formed by H_i and the edges incident to x with their respective endpoints. So the property follows from Theorem 1.3.4 by an inductive argument.

Otherwise, x is adjacent to at least one vertex y of degree one in H. Let M'_x be the complete subgraph of $L(H \setminus \{y\})$ induced by the vertices corresponding to the edges of $H - \{y\}$ incident to x. If M'_x is still a clique of $L(H \setminus \{y\})$, then $K(G) = K(L(H \setminus \{y\}))$, and the property holds by an inductive argument.

If M'_x is not a maximal complete in $L(H \setminus \{y\})$, then x has degree 3 in H, and the other two neighbors z and w of x in H are adjacent. The cliques intersecting M_x in G pairwise intersect (all of them contain the vertex corresponding to the edge wz of H), so v is simplicial in $K(G)$. On the other hand, $K(L(H \setminus \{y\})) = K(G) \setminus \{v\}$, so the property follows from Theorem 1.3.3 by an inductive argument. □

Theorem 3.2.3 is an immediate corollary of the following result.

Theorem 3.2.10. *Let G be a line graph. Then the following are equivalent:*

(i) *no induced subgraph of G is and odd hole, or a 3-sun.*

(ii) *G is clique-perfect.*

(iii) *G is perfect and it does not contain a 3-sun.*

Proof. The equivalence between (i) and (iii) is a corollary of Theorem 3.2.7. From Theorem 3.1.1 it follows that (ii) implies (i).

It therefore suffices to prove that (i) implies (ii). This proof is again by induction on $|V(G)|$. The class of line graphs with no odd holes or induced 3-suns is hereditary, so we only have to prove that for every graph in this class τ_C equals α_C. By Theorem 3.2.9, every such graph is K-perfect. So, if G is an HCH, by Lemma 1.3.1, $\tau_C(G) = k(K(G)) = \alpha(K(G)) = \alpha_C(G)$. Let $G = L(H)$ and suppose that G is not HCH. Then G contains a 0-,1-,2- or 3-pyramid.

A 0-pyramid is a 3-sun. A 2-pyramid is not a line graph, and therefore is not an induced subgraph of G.

Assume first that H contains a complete set of size four, say K. By Lemma 1.3.10 we may assume H is connected. We analyze how vertices of $V(H) \setminus K$ attach to K. If a vertex v is adjacent to two different vertices of K, then H contains an odd cycle as a subgraph and G contains an odd hole. If two different vertices v, w are adjacent to two different vertices of K, then H contains a trinity as a subgraph and so G contains a 3-sun. These cases can be seen in Figure 3.2.

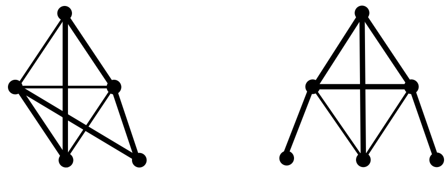

Figure 3.2: How the remaining vertices of H can be attached to the K_4.

So only one of the four vertices x_1, x_2, x_3, x_4 of K may have neighbors in $H \setminus K$, say x_1. Let v, w, z_1, z_2, z_3 and z_4 be the vertices of G corresponding to the edges x_1x_2, x_3x_4, x_1x_3, x_1x_4, x_2x_4 and x_2x_3 of H, respectively. The vertex w is adjacent in G only to z_1, z_2, z_3 and z_4, which induce a hole of length 4 and are adjacent also to v. So $G \setminus \{w\}$ is a clique subgraph of G (every clique of $G \setminus \{w\}$ is a clique of G). On the other hand, since x_2 has no neighbors in $H \setminus K$, all the neighbors of v are vertices corresponding to edges of H containing x_1, and they are a complete in G. This situation can be seen in Figure 3.3.

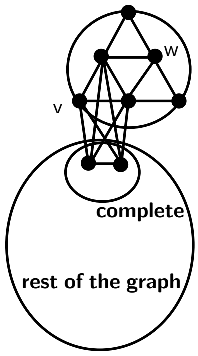

Figure 3.3: Structure of G when H has a K_4.

By the inductive hypothesis, $G \setminus \{w\}$ is clique-perfect. Let A be a maximum clique-independent set and T be a minimum clique-transversal of $G \setminus \{w\}$. By maximality and by the structure of G, A has exactly one clique containing v. Adding w, four new cliques appear, each one disjoint from a different one of the four cliques containing v, and adding w to T we have a clique-transversal of G, so $\alpha_C(G) = \alpha_C(G \setminus \{w\}) + 1 = \tau_C(G \setminus \{w\}) + 1 = \tau_C(G)$. So we may assume that H contains no complete set of size four.

Since if G contains a 3-pyramid, then H contains a complete set of size four, it follows that the only remaining case is when G contains a 1-pyramid. Since G contains a 1-pyramid, H contains as a subgraph a graph on five vertices $v_1, \ldots, v_5$ where v_1 is adjacent to v_2, v_3 and v_4, v_2 is adjacent to v_3 and v_4, and v_3 is adjacent to v_5 (Figure 3.4). Moreover, v_3 and v_4 are not adjacent because H does not contain a complete set of size four, v_1 and v_2 are not adjacent to v_5, otherwise H contains an odd cycle as a subgraph, and v_1 and v_2 do not have other neighbors, otherwise H contains a trinity as a subgraph. Then v_1 and v_2 form a cutset in H, because if there is a path v_3Pv_4 in

$H \setminus \{v_1, v_2\}$, then either $v_3Pv_4v_1v_3$ or $v_3Pv_4v_1v_2v_3$ is an odd cycle in H.

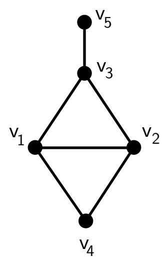

Figure 3.4: Subgraph of H when H contains no K_4 and G contains a 1-pyramid.

Let $w_1, \ldots, w_5$ be the vertices of G corresponding to the edges v_1v_3, v_2v_3, v_1v_4, v_2v_4 and v_1v_2 of H, respectively. Then $w_1w_2w_4w_3w_1$ is a hole of length four in G, w_5 is adjacent only to w_1, w_2, w_3, w_4 and w_2, w_3, w_5 is a cutset of G. The remaining neighbors of w_1 or w_2 are adjacent to both w_1 and w_2, and form a non-empty complete in G (they are the vertices corresponding to the edges of H containing v_3 and not v_1 or v_2, and there exists at least one such edge, namely the edge v_3v_5). Similarly, the neighbors of w_3 or w_4 are adjacent to both w_3 and w_4, and form a (possibly empty) complete in G. The structure of G in this case can be seen in Figure 3.5.

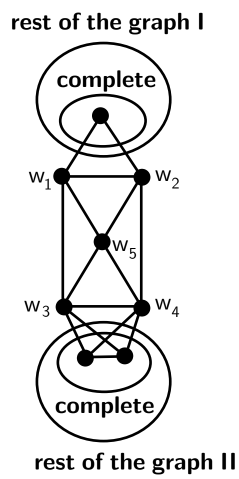

Figure 3.5: Structure of G when H has no K_4.

We show that $\alpha_C(G) = \alpha_C(G')$ and $\tau_C(G) = \tau_C(G')$, where G' is the line graph of the graph H', obtained from H by deleting the edges v_2v_3 and v_1v_4. So $G' = G \setminus \{w_2, w_3\}$.

Since every clique transversal of G' either contains w_5, or contains both w_1 and w_4, it follows that every clique transversal of G' is a clique transversal of G. On the other hand, starting with a clique transversal T of G and replacing the vertices w_2 and w_3 by w_1 and w_4 respectively, if w_2 or w_3 belong to T, produces a clique transversal of G'. Therefore $\tau_C(G) = \tau_C(G')$.

We claim that there is a maximum clique-independent set of G not containing either of the cliques $\{w_1, w_3, w_5\}$, $\{w_2, w_4, w_5\}$. Suppose the claim is false. Let I be a clique independent set of G, we may assume I contains the clique $\{w_1, w_3, w_5\}$. Then I does not contain any other clique containing w_1 or w_5; and since the only clique containing w_2 and not w_1 is $\{w_2, w_4, w_5\}$, it follows that every clique in I is disjoint from $\{w_1, w_2, w_5\}$. But now the set obtained from I by removing the clique $\{w_1, w_3, w_5\}$ and adding the clique $\{w_1, w_2, w_5\}$ has the desired property. This proves the claim.

Let I a maximum clique independent set of G not containing either of the cliques $\{w_1, w_3, w_5\}$, $\{w_2, w_4, w_5\}$. Let I' be a set of cliques of G', obtained from I by replacing the clique $\{w_1, w_2, w_5\}$ by $\{w_1, w_5\}$ if $\{w_1, w_2, w_5\} \in I$, and the clique $\{w_3, w_4, w_5\}$ by $\{w_4, w_5\}$ if $\{w_3, w_4, w_5\} \in I$. Conversely, every clique independent set of G' gives rise to a clique independent set of G, and therefore $\alpha_C(G) = \alpha_C(G')$.

But now, since G' is a proper induced subgraph of G, it follows inductively that $\alpha_c(G') = \tau_C(G')$, and therefore $\alpha_c(G') = \tau_C(G')$. This completes the proof of Theorem 3.2.10. □

3.2.3 Hereditary clique-Helly claw-free graphs

The main purpose of this subsection is to prove Theorem 3.2.4, which states that if a graph G is HCH claw-free, then G is clique-perfect if and only if it does not contain odd holes or an antihole of length seven.

To simplify the notation along this subsection, let us call a graph *interesting* if it does not contain odd holes or an antihole of length seven. We will use Proposition 1.3.1 to prove the characterization for HCH claw-free graphs, so first we will prove the following result.

Theorem 3.2.11. *Let G be an interesting HCH claw-free graph. Then $K(G)$ is perfect.*

To prove Theorem 3.2.11 we need some previous results.

We start with some definitions in order to state some useful structure theorems for claw-free graphs.

A graph G is *prismatic* if for every triangle T of G, every vertex of G not in T has a unique neighbor in T. A graph G is *antiprismatic* if its complement graph $\overline{G}$ is prismatic.

Construct a graph G as follows. Take a circle C, and let $V(G)$ be a finite set of points of C. Take a set of intervals from C (an *interval* means a proper subset of C homeomorphic to $[0, 1]$) such that there are not three intervals covering C; and say that $u, v \in V(G)$ are adjacent in G if the set of points $\{u, v\}$ of C is a subset of one of the intervals. Such a graph is called *circular interval graph*. When the set of intervals does not cover C, the graph is called *linear interval graph*.

Circular interval graphs form a subclass of Helly circular-arc graphs.

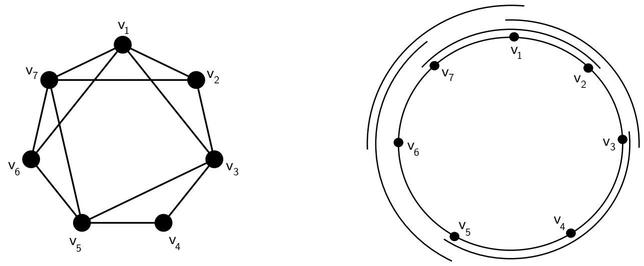

Figure 3.6: Example of a circular interval graph and its circular interval representation.

Let G be a graph and A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. If, in addition, B is empty, then A is called a *homogeneous set*. Let (A, B) be a homogeneous pair such that A, B are both completes, and A is neither complete nor anticomplete to B. In these circumstances the pair (A, B) is called a *W-join*. Note that there is no requirement that $A \cup B \neq V(G)$. The pair (A, B) is *non-dominating* if some vertex of $G \setminus (A \cup B)$ has no neighbor in $A \cup B$, and it is *coherent* if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a complete.

Suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are non-empty and there are no edges between V_1 and V_2. The pair (V_1, V_2) is called a *0-join* in G. Thus G admits a 0-join if and only if it is not connected.

Suppose now that V_1, V_2 is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- for $i = 1, 2$, A_i is a complete, and A_i, $V_i \setminus A_i$ are both non-empty
- A_1 is complete to A_2
- every edge between V_1 and V_2 is between A_1 and A_2.

In these circumstances, the pair (V_1, V_2) is called a *1-join*.

Suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- for $i = 1, 2$, A_i, B_i are completes, $A_i \cap B_i = \emptyset$, and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all non-empty
- A_1 is complete to A_2, and B_1 is complete to B_2, and there are no other edges between V_1 and V_2
- V_0 is a complete, and for $i = 1, 2$, V_0 is complete to $A_i \cup B_i$ and anticomplete to $V_i \setminus (A_i \cup B_i)$.

The triple (V_0, V_1, V_2) is called a *generalized 2-join*, and if $V_0 = \emptyset$, the pair (V_1, V_2) is called a *2-join*. This is closely related to, but not the same as, what has been called a 2-join in some papers, like [23].

The last decomposition is the following. Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are completes A_i, B_i, $C_i \subseteq V_i$ with the following properties:

- For $i = 1, 2$ the sets A_i, B_i, C_i are pairwise disjoint and have union V_i
- V_1 is complete to V_2 except that there are no edges between A_1 and A_2, between B_1 and B_2, and between C_1 and C_2
- V_1, V_2 are both non-empty.

In these circumstances it is said that G is a *hex-join* of $G|V_1$ and $G|V_2$. Note that if G is expressible as a hex-join as above, then the sets $A_1 \cup B_2$, $B_1 \cup C_2$ and $C_1 \cup A_2$ are three completes with union $V(G)$, and consequently no graph G with $\alpha(G) > 3$ is expressible as a hex-join.

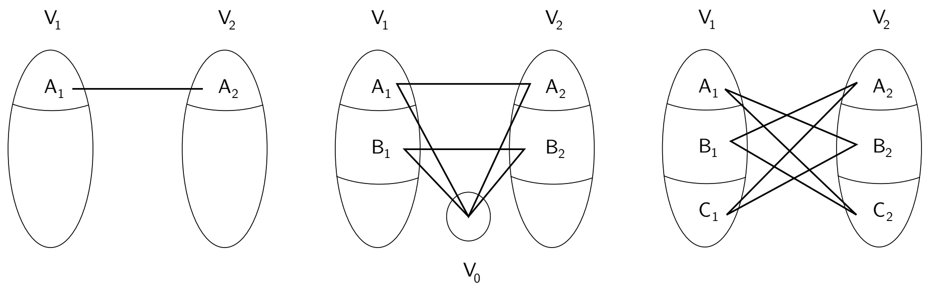

Figure 3.7: Scheme for 1-join, 2-join and hex-join.

Finally, the classes $\mathcal{S}_0, \ldots, \mathcal{S}_6$ are defined as follows.

- $\mathcal{S}_0$ is the class of all line graphs.
- The *icosahedron* is the unique planar graph with twelve vertices all of degree five. For $0 \leq k \leq 3$, *icosa*$(-k)$ denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. A graph $G \in \mathcal{S}_1$ if G is isomorphic to *icosa*(0), *icosa*(-1) or *icosa*(-2). As it can be seen in Figure 3.8, all of them contain odd holes.
- Let H_1 be the graph with vertex set $\{v_1, \ldots, v_{13}\}$, with adjacency as follows: $v_1 v_2 \ldots v_6 v_1$ is a hole in G of length 6; v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to v_3, v_4, v_6, v_1, v_9, v_{10}; v_{12} is adjacent to v_2, v_3,

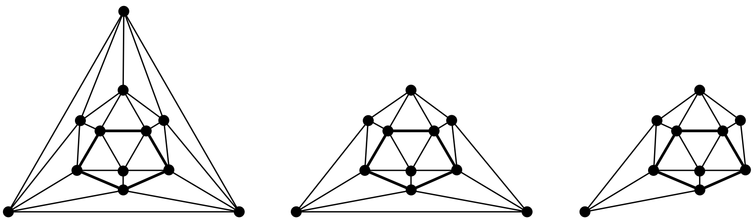

Figure 3.8: Graphs $icosa(0)$, $icosa(-1)$ and $icosa(-2)$.

v_5, v_6, v_9, v_{10}; and v_{13} is adjacent to v_1, v_2, v_4, v_5, v_7, v_8. A graph $G \in \mathcal{S}_2$ if G is isomorphic to $H_1 \setminus X$, where $X \subseteq \{v_{11}, v_{12}, v_{13}\}$. Please note that vertices $v_3v_4v_5v_6v_9v_3$ induce a hole of length five in G.

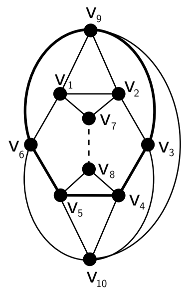

Figure 3.9: Graph $H_1 \setminus \{v_{11}, v_{12}, v_{13}\}$. Every graph in $\mathcal{S}_2$ contains it as an induced subgraph.

- $\mathcal{S}_3$ is the class of all circular interval graphs.

- Let H_2 be the graph with seven vertices $h_0, \ldots, h_6$, in which $h_1, \ldots, h_6$ are pairwise adjacent and h_0 is adjacent to h_1. Let H_3 be the graph obtained from the line graph $L(H_2)$ of H_2 by adding one new vertex, adjacent precisely to the members of $V(L(H_2)) = E(H_2)$ that are not incident with h_1 in H_2. Then H_3 is claw-free. Let $\mathcal{S}_4$ be the class of all graphs isomorphic to induced subgraphs of H_3. Note that the vertices of H_3 corresponding to the members of $E(H_2)$ that are incident with h_1 in H_2, form a complete in H_3. So every graph in $\mathcal{S}_4$ is either a line graph or it has a singular vertex.

- Let $n \geq 0$. Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three pairwise disjoint completes. For $1 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j$, and let c_i be adjacent to a_j, b_j if and only if $i \neq j$. Let d_1, d_2, d_3, d_4, d_5 be five more vertices, where d_1 is $(A \cup B \cup C)$-complete; d_2 is complete to $A \cup B \cup \{d_1\}$; d_3 is complete to $A \cup \{d_2\}$; d_4 is complete to $B \cup \{d_2, d_3\}$; d_5 is adjacent to d_3, d_4; and there are no more edges. Denote by H_4 the graph just constructed. A graph $G \in \mathcal{S}_5$ if (for some n) G is isomorphic to $H_4 \setminus X$ for some $X \subseteq A \cup B \cup C$. Note that vertex d_1 is adjacent to all the vertices but the triangle formed by d_3, d_4 and d_5, so it is a singular vertex in G (see Figure 3.10).

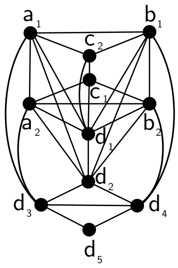

Figure 3.10: Graph H_4, for $n = 2$.

- Let $n \geq 0$. Let $A = \{a_0, \ldots, a_n\}$, $B = \{b_0, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three pairwise disjoint completes. For $0 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. Let the graph just constructed be H_5. A graph $G \in \mathcal{S}_6$ if (for some n) G is isomorphic to $H_5 \setminus X$ for some $X \subseteq A \cup B \cup C$, and then G is said to be 2-*simplicial of antihat type* (Figure 3.11).

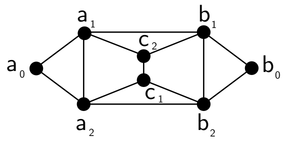

Figure 3.11: Graph H_5, for $n = 2$.

We shall use the following structure theorems for claw-free graphs.

Theorem 3.2.12. [26] *Let G be a claw-free graph. Then either $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$, or G admits twins, or a non-dominating W-join, or a coherent W-join, or a 0-join, or a 1-join, or a generalized 2-join, or a hex-join, or G is antiprismatic.*

Theorem 3.2.13. [25] *Let G be a claw-free graph admitting an internal clique cutset. Then G is either a linear interval graph or G is the 3-sun, or G admits twins, or a 0-join, or a 1-join, or a coherent W-join.*

In the remainder of this subsection we use Theorems 3.2.12 and 3.2.13 to prove that every interesting HCH claw-free graph is K-perfect. The proof is by induction on $|V(G)|$.

Circular Interval Graphs

We first prove that clique graphs of interesting HCH circular interval graphs are perfect.

Lemma 3.2.14. *Let G be a circular interval graph. Then $K(G)$ is an induced subgraph of G.*

Proof. Let G be a circular interval graph with vertices $v_1, \ldots, v_n$ in clockwise order, say. We define a homomorphism v from $V(K(G))$ to $V(G)$ (meaning that for two distinct vertices $a, b \in V(K(G))$, $v(a) \neq v(b)$; and a is adjacent to b if and only if $v(a)$ is adjacent to $v(b)$). For every clique M of G, since no three intervals in the definition of a circular interval graph cover the circle, $M = \{v_i, \ldots, v_{i+t}\}$ (where the indices are taken mod n). In this case we say that v_i is the *first vertex* of M. We define $v(M) = v_i$. Since v_i is the first vertex of a unique clique, it follows that $v(M) \neq v(M')$ if M and M' are distinct cliques of G. It remains to show that $v(M)$ is adjacent to $v(M')$ if and only if $M \cap M' \neq \emptyset$. If M and M' intersect at a vertex v_k, then the clockwise order of $v(M)$, $v(M')$ and v_k is either $v(M)$, $v(M')$, v_k or $v(M')$, $v(M)$, v_k and in both cases $v(M)$ and $v(M')$ are adjacent. On the other hand, if there are two cliques such that $v(M)$ and $v(M')$ are adjacent, we may assume $v(M)$ appears first clockwise in the circular interval which contains both $v(M)$ and $v(M')$. Then since $v(M)$ is the first vertex of the clique M, it follows that $v(M')$ belongs to M, so M and M' intersect. $\square$

Proposition 3.2.1. *Let G be an HCH interesting circular interval graph. Then $K(G)$ is perfect.*

Proof. By Lemma 3.2.14, $K(G)$ is an induced subgraph of G. Since G is HCH and interesting, it contains no odd hole and no antihole of length at least seven, and therefore it is perfect by Theorem 1.2.1. $\square$

Decompositions

We now show that if an interesting HCH claw-free graph admits one of the decompositions of Theorem 3.2.12, then either it is K-perfect or we can reduce the problem to a smaller one.

Theorem 3.2.15. *Let G be an interesting HCH claw-free graph. If G admits a 1-join, then $K(G)$ has a cutpoint v, $K(G) = H_1 + H_2 + v$, and $H_i + v$ is the clique graph of a smaller interesting HCH claw-free graph.*

Proof. Since G admits a 1-join, it follows that $V(G)$ is the disjoint union of two nonempty sets V_1 and V_2, each V_i contains a complete M_i, such that $M_1 \cup M_2$ is a complete and there are no other edges from V_1 to V_2. So $M_1 \cup M_2$ is a clique in G. Let v be the vertex of K(G) corresponding to $M_1 \cup M_2$. Every other clique of G is either contained in V_1 or in V_2, and no clique of the first type intersects a clique of the second type. So

v is a cutpoint of $K(G)$, and $K(G) = H_1 + H_2 + v$. Let G_i be the graph obtained from $G|V_i$ by adding a vertex v_i complete to M_i and with no other neighbors in G_i. Then G_i is isomorphic to an induced subgraph of G, so it is interesting, HCH and claw-free, and for $i = 1, 2$, $H_i + v$ is isomorphic to $K(G_i)$ (where the vertex v is mapped to the vertex of $K(G_i)$ corresponding to the clique $M_i \cup \{v_i\}$ of G_i). This proves Theorem 3.2.15. □

Theorem 3.2.16. *Let G be an interesting HCH claw-free graph. If G admits a generalized 2-join and no twins, 0-join or 1-join, then there exist two clique graphs of smaller interesting HCH claw-free graphs, H_1 and H_2, such that if H_1 and H_2 are perfect, then so is $K(G)$.*

Proof. Since G admits a generalized 2-join, it follows that $V(G)$ is the disjoint union of three sets V_0, V_1 and V_2, for $i = 1, 2$ each V_i contains two completes A_i, B_i such that A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all non-empty, $A_1 \cup A_2 \cup V_0$ and $B_1 \cup B_2 \cup V_0$ are completes and there are no other edges from V_1 to V_2 or from V_0 to $V_1 \cup V_2$. Since G admits no twins, it follows that $|V_0| \le 1$.

So $A_1 \cup A_2 \cup V_0$ and $B_1 \cup B_2 \cup V_0$ are cliques of G, and they correspond to vertices w_1, w_2 of $K(G)$. Every other clique of G is either contained in V_1 or in V_2, and no clique of the first type intersects a clique of the second type. So $\{w_1, w_2\}$ is a cutset in $K(G)$.

If V_0 is non-empty, then w_1 is adjacent to w_2 and $\{w_1, w_2\}$ is a clique cutset in $K(G)$. Let $V_0 = \{v_0\}$. Now $K(G) = M_1 + M_2 + \{w_1, w_2\}$, where, for $i = 1, 2$, $H_i = M_i + \{w_1, w_2\}$ is the clique graph of the subgraph of G induced by $V_i \cup \{v_0\}$. By Theorem 1.3.4, $K(G)$ is perfect if and only if H_1 and H_2 are. So we may assume that V_0 is empty, and therefore w_1 is non-adjacent to w_2.

We start with the following easy observation:

(*) Let S be a graph which is either a claw, or an odd hole, or $\overline{C_7}$, or a 0-,1-,2-, or 3-pyramid, and suppose there exists a vertex $s \in V(S)$, whose neighborhood is the union of two non-empty completes with no edges between them. Then S is and odd hole.

Since G admits no 0-join or 1-join, for $i = 1, 2$ there exist a_i in A_i and b_i in B_i joined by an induced path with interior in $V_i \setminus (A_i \cup B_i)$. (The *interior* of a path are the vertices different from the endpoints; the interior may be empty, if a_i and b_i are adjacent.)

Then, since G contains no odd hole, for every a_i in A_i and b_i in B_i, all induced paths from a_1 to b_1 with interior in $V_1 \setminus (A_1 \cup B_1)$ and all induced paths from a_2 to b_2 with interior in $V_2 \setminus (A_2 \cup B_2)$ have the same parity.

<u>Case 1</u>: This parity is even.

Note that in this case A_i is anticomplete to B_i. Let H be the graph obtained from $K(G)$ by adding the edge w_1w_2. Since A_i is anticomplete to B_i, there is no clique in G intersecting both $A_1 \cup A_2$ and $B_1 \cup B_2$. So w_1 and w_2 have no common neighbor in $K(G)$. By Theorem 1.3.5, if H is perfect then $K(G)$ is.

Construct graphs G_i with vertex set $V_i \cup \{v_i\}$, where $G_i|V_i = G|V_i$ and v_i is complete to $A_i \cup B_i$ and has no other neighbors in G_i. Now, $H = M_1 + M_2 + \{w_1, w_2\}$, with $M_i + \{w_1, w_2\} = K(G_i)$, and $\{w_1, w_2\}$ is a clique cutset in H. By Theorem 1.3.4, it follows that if $K(G_1)$ and $K(G_2)$ are perfect then H is perfect and thus $K(G)$ is perfect.

We claim that for $i = 1, 2$ the graphs G_i are claw-free, HCH and interesting. Suppose that G_1, say, is not. So G_1 contains an induced subgraph S isomorphic to a claw, an odd hole, $\overline{C_7}$, or a 0-,1-,2- or 3-pyramid. If $V(S)$ does not contain v_1, then S is isomorphic to an induced subgraph of G, a contradiction. If $V(S)$ contains v_1 but has empty intersection with A_1 or B_1, say B_1, then S is isomorphic to an induced subgraph of G, obtained by replacing v_1 by any vertex of A_2, a contradiction. So $V(S)$ meets both A_1 and B_1, and therefore the neighborhood of v_1 in S can be partitioned into two non-empty completes A_S, B_S, such that A_S is anticomplete to B_S. By (*), S is an odd hole. Let $a_1 \in A_1$ and $b_1 \in B_1$ be the neighbors of v_1 in S. Then $S \setminus \{v_1\}$ is an induced odd path from a_1 to b_1 with interior in $V_1 \setminus (A_1 \cup B_1)$, a contradiction.

Case 2: This parity is odd.

Construct graphs G_i with vertex set $V_i + \{v_{A,i}, v_{B,i}\}$, where $G_i|V_i = G|V_i$, $v_{A,i}$ is complete to A_i, $v_{B,i}$ is complete to B_i, $v_{A,i}$ is adjacent to $v_{B,i}$, and there are no other edges in G_i. Now, $K(G) = M_1 + M_2 + \{w_1, w_2\}$, and $K(G_i)$ is obtained from $M_i + \{w_1, w_2\}$ by joining w_1 and w_2 by an induced path of length two. By Theorem 1.3.6, if $K(G_1)$ and $K(G_2)$ are perfect, so is $K(G)$.

We claim that both G_i are claw-free, interesting and HCH. Suppose that G_1 contains an induced subgraph S isomorphic to a claw, an odd hole, $\overline{C_7}$, or a 0-,1-,2-,or 3-pyramid.

If $V(S)$ does not contain $v_{A,1}$ or $v_{B,1}$, say $v_{B,1}$, then S is isomorphic to an induced subgraph of G, obtained by replacing $v_{A,1}$ by any vertex of A_2, a contradiction. If $V(S)$ contains $v_{A,1}$ and $v_{B,1}$ but has empty intersection with A_1 or B_1, say B_1, then S is isomorphic to an induced subgraph of G, obtained by replacing $v_{A,1}$ and $v_{B,1}$ by two adjacent vertices a_2, c_2 of V_2 such that $a_2 \in A_2$ and $c_2 \in V_2 \setminus A_2$ (such a pair of vertices exist because there is at least one path from A_2 to B_2 in G), a contradiction. So $V(S)$ meets both A_1 and B_1, and the neighborhood of $v_{A,1}$ in S can be partitioned into two non-empty completes with no edges between them, namely $A_S = A_1 \cap V(S)$ and $\{v_{B,1}\}$. By (*) S is an odd hole. Let $a_1 \in A_1$ and $b_1 \in B_1$ be the neighbors of $v_{A,1}$ and $v_{B,1}$ in $V(S) \cap V_1$, respectively. Then $S \setminus \{v_{A,1}, v_{B,1}\}$ is an induced even path from a_1 to b_1 with interior in $V_1 \setminus (A_1 \cup B_1)$, a contradiction. This concludes the proof of Theorem 3.2.16. □

Lemma 3.2.17. *Let G be an HCH graph such that $\overline{G}$ is a bipartite graph. Then $K(G)$ is perfect.*

Proof. In this proof we use the vertices of $K(G)$ and the cliques of G interchangeably. By Theorem 1.2.1, if $K(G)$ is not perfect then it contains an odd hole or an odd antihole.

Let A, B be two disjoint completes of G such that $A \cup B = V(G)$. If there exists a vertex v of G adjacent to every other vertex in G, then v belongs to every clique of G and $K(G)$ is a complete graph, and therefore perfect. So we may assume that no vertex of A is complete to B and no vertex of B is complete to A. Then A and B are cliques of G, and every other clique of G meets both A and B. The degrees of A and B in $K(G)$ is $|V(K(G))| - 1$, so they cannot be part of an odd hole or an odd antihole in $K(G)$.

It is therefore enough to show that there is no odd hole or antihole in the graph obtained from $K(G)$ by deleting the vertices A and B. We prove a stronger statement, namely that there is no induced path of length two in this graph. Since every hole and antihole of length at least five contains a two edge path, the result follows.

Suppose for a contradiction that there are three cliques X, Y and Z in G, each meeting both A and B, and such that X is disjoint from Z, and both $X \cap Y$ and $Y \cap Z$ are non-empty. From the symmetry we may assume that $X \cap Y$ contains a vertex $a_{xy} \in A$.

Suppose first that there is a vertex $a_{yz} \in A \cap Y \cap Z$. Let b_y be a vertex in $Y \cap B$. Since no vertex of B is complete to A, there is a vertex a in A non-adjacent to b_y. Since a_{yz} does not belong to X, there is a vertex b_x in X non-adjacent to a_{yz}, and since A is a complete, b_x belongs to B. Analogously, since a_{xy} does not belong to Z, there is a vertex b_z in $B \cap Z$ non-adjacent to a_{xy}. But now $\{a_{xy}, a_{yz}, b_y, b_z, b_x, a\}$ induce a 1-, 2- or 3-pyramid, a contradiction.

So $A \cap Y \cap Z$ is empty, and therefore $B \cap Y \cap Z$ is non-empty, and, by the argument of the previous paragraph with A and B exchanged, $B \cap X \cap Y$ is empty. Choose b_{yz} in $B \cap Y \cap Z$. Choose a_z in $Z \cap A$, then $a_z \notin X \cup Y$. Since a_z does not belong to X, there is a vertex $b_x \in X$ non-adjacent to a_z, and since A is a complete, b_x is in B. Since b_{yz} does not belong to X and B is a complete, there is a vertex $a_x \in A \cap X$ non-adjacent to b_{yz}; and since a_{xy} does not belong to Z and A is a complete, there is a vertex $b_z \in B \cap Z$ non-adjacent to a_{xy}. But now $\{a_z, a_{xy}, b_{yz}, a_x, b_x, b_z\}$ induces a 2- or a 3-pyramid, a contradiction. This proves Lemma 3.2.17. □

Theorem 3.2.18. *Let G be a connected interesting HCH claw-free graph, and suppose G admit no twins. Assume that G admits a coherent or a non-dominating W-join (A, B). Then either $K(G)$ is perfect, or there exist induced subgraphs $G_1, \ldots, G_k$ of G, each smaller than G, such that if $K(G_i)$ is perfect for every $i = 1, \ldots, k$, then $K(G)$ is perfect.*

Proof. Choose a coherent or non-dominating W-join (A, B) with $A \cup B$ minimal. Let C be the vertices complete to A and anticomplete to B, D be the vertices complete to B and anticomplete to A, E be the vertices complete to $A \cup B$, and F be the vertices anticomplete to $A \cup B$. Since the W-join (A, B) is either coherent or non-dominating, it follows that either E is a complete, or F is non-empty.

3.2.18.1 $A \cup C$, $B \cup D$ are both completes, and E is anticomplete to F.

Suppose not. Assume first that there exist two nonadjacent vertices c_1, c_2 in C. Choose

a in A and b in B such that a is adjacent to b, now $\{a, c_1, c_2, b\}$ is a claw, a contradiction. So C is a complete, and since A is a complete, it follows that $A \cup C$ is a complete. From the symmetry it follows that $B \cup D$ is a complete.

Next assume that there are two adjacent vertices e in E and f in F. Choose a in A and b in B such that a is not adjacent to b. Then $\{e, a, b, f\}$ is a claw, a contradiction. This proves 3.2.18.1.

Let E_1 be a clique of $G|E$. Let $\mathcal{L}$ be the set of all cliques of $G|(A \cup B)$. Let

$$U = \{E_1 \cup L \;\; : \;\; L \in \mathcal{L} \;\; and \;\; L \neq A, B\}.$$

Since E is anticomplete to F, and every member of U meets both A and B, it follows that the members of U are cliques of G.

3.2.18.2 We may assume that $|U| \geq 2$.

Suppose $|U| \leq 1$. Since in G there is at least one edge between A and B, it follows that there is a unique clique L in $G|(A \cup B)$ meeting both A and B, and $|U| = 1$. Let $A' = A \cap L$, $B' = B \cap L$. Then A' is complete to B', $A \setminus A'$ is anticomplete to B and $B \setminus B'$ is anticomplete to A. Since G does not admit twins, each of A', $A \setminus A'$, B', $B \setminus B'$ has size at most 1, and by the minimality of $A \cup B$ at most one of $A \setminus A'$, $B \setminus B'$ is non-empty. By the symmetry, we may assume that $B \setminus B'$ is empty and $|A'| = |B'| = |A \setminus A'| = 1$. Let $A' = \{a_1\}$, $B' = \{b_1\}$ and $A \setminus A' = \{a_2\}$.

If $K(G \setminus \{a_2\}) = K(G)$ then the theorem holds, so we may assume not. Therefore there exists a subset E' of E such that $M = A \cup E'$ is a clique of G. It follows, in particular, that no vertex of C is complete to E.

Assume first that E is a complete, consider the cliques $M_1 = \{a_1, b_1\} \cup E$ and $M_2 = \{a_1, a_2\} \cup E$ of G. Since every clique of G containing a_2 also contains a_1, it follows that every clique of G that has a non-empty intersection with M_2, meets M_1. Therefore the vertex w_1 of $K(G)$, corresponding to M_1, weakly dominates the vertex w_2 of $K(G)$, corresponding to M_2. Since $K(G) \setminus \{w_1\}$ is an induced subgraph of $K(G \setminus \{a_1\})$ and $K(G) \setminus \{w_2\} = K(G \setminus \{a_2\})$, by Theorem 1.3.8, $K(G)$ is perfect if $K(G \setminus \{a_1\})$ and $K(G \setminus \{a_2\})$ are, and the theorem holds. So we may assume that E is not a complete.

Next we claim that D is empty. Since E is not a complete, there are two non-adjacent vertices e_1, e_2 in E, and let d in D. If d is non-adjacent to both of e_1 and e_2, then $\{b_1, e_1, e_2, d\}$ is a claw, a contradiction. But then, $\{b_1, e_1, e_2, d, a_1, a_2\}$ induces a 1- or 2-pyramid, a contradiction. This proves that D is empty.

Since D is empty, every clique disjoint from F contains the vertex a_1, and, since every clique containing a vertex of F is disjoint from A, B and E, it follows that the vertices of $K(G)$ corresponding to the cliques $\{a_1, b_1\} \cup E'$, with E' a clique of $G|E$, are simplicial in $K(G)$. By Lemma 1.3.3, $K(G)$ is perfect if and only if $K(G \setminus \{b_1\})$ is. This proves 3.2.18.2.

3.2.18.3 We may assume that no vertex of B is complete to A, and no vertex of A is complete to B.

Suppose there is a vertex $b \in B$ complete to A. Since A is not complete to B, there is a vertex $b' \in B \setminus \{b\}$. By 3.2.18.2, $|A| > 1$. But now $(A, B \setminus \{b\})$ is a coherent or non-dominating W-join in G, contrary to the minimality of $A \cup B$. This proves 3.2.18.3.

In view of 3.2.18.2 and 3.2.18.3, we henceforth assume that $|U| \geq 2$, no vertex of A is complete to B, and no vertex of B is complete to A.

3.2.18.4 E is a complete.

Since no vertex of B is complete to A, and there is at least one edge between A and B, there is a vertex $a_1 \in A$ with a neighbor b_1 and a non-neighbor b_2 in B. Since b_1 is not complete to A, there is a vertex $a_2 \in A$, non-adjacent to b_1. Since A, B are both cliques, a_1 is adjacent to a_2 and b_1 to b_2. If there exist two non-adjacent vertices e_1 and e_2 in E, now $\{a_1, a_2, b_1, b_2, e_1, e_2\}$ induces a 2- or a 3-pyramid in G, a contradiction. This proves 3.2.18.4.

3.2.18.5 Every vertex of $K(G) \setminus U$ with a neighbor in U is complete to U.

Throughout the proof of 3.2.18.5 we use cliques of G and vertices of $K(G)$ interchangeably.

It follows from 3.2.18.4 that $E_1 = E$. Let w be a vertex of $K(G) \setminus U$ with a neighbor in U. Since w has a neighbor in U, it follows that w meets one of A, B, E. If w meets E, then w is complete to U and the result follows. If w includes one of A, B, then since every member of U meets each of A, B, we again deduce that w is complete to U and the result follows. So we may assume that w is disjoint from E, and the sets $w \cap (A \cup B)$, $A \setminus \{w\}$, and $B \setminus \{w\}$ are all non-empty.

Assume first that w meets both A and B. Since w is a clique of G, $C \cup F$ is anticomplete to B and $D \cup F$ is anticomplete to B, it follows that $w \subseteq A \cup B \cup E$. But now, since w is a clique, it follows that w includes E and w belongs to U, a contradiction. So we may assume that w is disjoint from at least one of A and B.

By the symmetry we may assume that w is disjoint from B, and therefore w meets A. Since $F \cup D$ is anticomplete to A, it follows that w is a subset of $A \cup C \cup E$, and since w is a clique, w includes A, a contradiction. This proves 3.2.18.5.

3.2.18.6 U is a homogeneous set in $K(G)$ and the graph $K(G)|U$ is perfect.

It follows from 3.2.18.5 that U is a homogeneous set in $K(G)$. The graph $K(G)|U$ is isomorphic to the graph obtained from $K(G|(A \cup B \cup E))$ by deleting the vertices corresponding to the cliques $A \cup E$ and $B \cup E$. Since $\overline{G|(A \cup B \cup E)}$ is bipartite, it follows from Theorem 3.2.17 that $K(G)|U$ is perfect. This proves 3.2.18.6.

Choose $u \in U$.

3.2.18.7 If there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$, such that a_1 is adjacent to b_1 and not to b_2, and a_2 is adjacent to b_2 and not to b_1, then either $K(G)$ is perfect, or there is an induced subgraph G' of G, such that $K(G) \setminus (U \setminus \{u\}) = K(G')$.

If there exist non-adjacent $c \in C$ and $e \in E$, then $\{a_1, a_2, e, c, b_1, b_2\}$ induces a 1-pyramid, a contradiction, so C is complete to E, and similarly D is complete to E. By 3.2.18.4, E is a complete. Since G admits no twins, $|E| \leq 1$. If $C \cup D$ is empty, then, since G is connected, F is empty, and G is the complement of a bipartite graph. By Lemma 3.2.17, $K(G)$ is perfect. So we may assume that C is non-empty, and in particular, $A \cup E$ is not a clique of G. But now $K(G) \setminus (U \setminus \{u\}) = K(G \setminus ((A \cup B) \setminus \{a_1, b_1, b_2\}))$. This proves 3.2.18.7.

To finish the proof, let $a_1 \in A$ and $b_1 \in B$ be adjacent. By 3.2.18.3, there exist a vertex $b_2 \in B$, non-adjacent to a_1 and a vertex $a_2 \in A$ non-adjacent to b_1. If a_2 is adjacent to b_2, then the theorem follows from 3.2.18.6, 3.2.18.7 and Theorem 1.3.7. So we may assume that a_2 is non-adjacent to b_2. Let $G' = G \setminus ((A \cup B) \setminus \{a_1, b_1, a_2, b_2\})$. We deduce from 3.2.18.2 that G' is smaller than G. Moreover, G' is an induced subgraph of G. But $K(G) \setminus (U \setminus \{u\}) = K(G')$, and, together with 3.2.18.6 and Theorem 1.3.7, this implies that the theorem holds. This proves Theorem 3.2.18. □

Theorem 3.2.19. *Let G be an interesting HCH claw-free graph. Suppose G admits a hex-join and no twins and every vertex of G is in a triad. Then $G = C_6$.*

Proof. Since G admits a hex-join, there exist six completes A_1, A_2, A_3, B_1, B_2, B_3 in G such that A_i is anticomplete to B_i and complete to B_j for i different from j; $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$ are non-empty; and $V(G) = A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3$. Since every vertex of G is in a stable set of size three and no stable set of size three meets both $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$, it follows that A_i, B_i are all non-empty.

Suppose there is an edge $a_1a'_2$ with a_1 in A_1 and a'_2 in A_2. Since every vertex is a stable set of size three, there exists a stable set $\{b_1, b_2, b_3\}$ with b_i in B_i and a stable set $\{a_1, a_2, a_3\}$ with a_i in A_i. Since G is interesting, $a_1a'_2b_1a_3b_2a_1$ is not a hole in G, so a'_2 is adjacent to a_3. But now $\{a'_2, a_1, a_2, a_3\}$ is a claw in G, a contradiction. So A_1 is anticomplete to A_2, A_3. Since the vertices of A_1 are not twins in G, it follows that $|A_1| = 1$. From the symmetry, $|B_i| = |A_i| = 1$ for all i, and $G = C_6$. This proves Theorem 3.2.19. □

Theorem 3.2.20. *Let G be an interesting HCH graph. Assume that G admits no twins and no coherent or non-dominating W-join, and contains no stable set of size three. Then $K(G)$ is perfect.*

Proof. We may assume G contains either a 4-wheel or a 3-fan, otherwise, by Theorem 1.3.14, $K(G)$ is bipartite.

<u>Case 1</u>: G contains a 4-wheel. Let $a_1a_2a_3a_4a_1$ be a hole and let c be adjacent to all a_i. We claim every vertex in G is adjacent to c. Suppose v is non-adjacent to c. Then since G contains no stable set of size three, from the symmetry we may assume v is adjacent to a_1, a_2. But now $\{a_1, a_2, a_3, a_4, c, v\}$ induces a 1-,2-, or 3-pyramid, a contradiction.

So every clique in G contains c, then $K(G)$ is a complete graph and the result follows. This proves Case 1.

Case2: G contains a 3-fan and no 4-wheel.

Let $A_1, \ldots, A_k$ be anticonnected sets in G, pairwise complete to each other, with $k > 2$, $|A_1| > 1$, and subject that with maximal union, say A. (Such sets exist because there is a 3-fan. Let $a_1a_2a_3a_4$ be a path and let c be adjacent to all a_i. Then $A_1 = \{a_1, a_3\}$, $A_2 = \{a_2\}$, $A_3 = \{c\}$ make a family of sets with the desired properties.)

Suppose $|A_2| > 1$. Then, since A_1, A_2 are both anticonnected, each of A_1, A_2 contains a non-edge, say a_ib_i. Choose a_3 in A_3. Now $\{a_1, a_2, b_1, b_2, a_3\}$ is a 4-wheel, a contradiction. So for $2 \leq i \leq k$, $|A_i| = 1$, and let $A_i = \{a_i\}$.

(*) No vertex in $V(G) \setminus A$ is complete to more than one of $A_1, \ldots, A_k$.

Let v be a vertex in $V(G) \setminus A$ and define $I = \{i : 1 \leq i \leq k$ and v is complete to $A_i\}$ and $J = \{j : 1 \leq j \leq k$ and v has a non-neighbor in $A_j\}$. Suppose $|I| > 1$. Define $A'_t = A_t$ for $t \in I$ and $A'_J = \bigcup_{j \in J} A_j \cup \{v\}$. Then $\{A'_i\}_{i \in I}, A'_J$ is a collection of at least three anticonnected sets, pairwise complete to each other, but their union is a proper superset of A, contrary to the maximality of A. This proves (*).

(**) There is no C_4 in A_1.

Otherwise, G contains a 4-wheel with center a_2, a contradiction. This proves (**).

Since $|A_1| > 1$ and A_1 is anticonnected, A_1 contains a non-edge, and so, since there is no stable set of size three in G, every vertex of $V(G) \setminus A$ has a neighbor in A_1. Let $A' = A \setminus A_1$. If no vertex of $V(G) \setminus A$ has a neighbor in A', then the vertices of A' are twins, a contradiction.

So there exists v in $V(G) \setminus A$ with a neighbor in A_1 and a neighbor a' in A'. By (*) v has a non-neighbor a'' in A'. If v has two non-adjacent neighbors in A_1, say x, y then $xvya''x$ is a 4-hole and a' is complete to it, so G contains a 4-wheel, a contradiction. So the neighbors of v in A_1 are a complete. Since G has no stable set of size three, the non-neighbors of v in A_1 are a complete. Thus $G|A_1$ is complement bipartite, and since it is anticonnected the bipartition is unique, say X, Y, both X and Y are non-empty, and every vertex of $V(G) \setminus A$ with a neighbor in A' is either complete to X and anticomplete to Y, or complete to Y and anticomplete to X. Let X' be the vertices with a neighbor in A' and complete to X, Y' be the vertices with a neighbor in A' and complete to Y. Then, $X' \cup Y'$ is non-empty, and since there is no stable set of size three in G, X', Y' are both completes.

For $i = 2, \ldots, k$ let X_i be the vertices of X' adjacent to a_i, and let Y_i be defined similarly. By (*), $A_i \cap A_j = \emptyset$ for $i \neq j$, and the same holds for B_i, B_j. If there is an edge from X to Y then there is no edge from X_i to Y_i, or else G contains a 4-wheel with center a_i. Let Z be the vertices of G with no neighbor in A'. Then, since G contains no triad, Z is a complete.

3.2.20.1 Every vertex in Z is complete to $X' \cup Y'$ and to one of X, Y.

If some vertex z in Z has a non-neighbor x_2 in X_2, then z, x_2, a_3 is a stable set of size three, a contradiction, so Z is complete to X', and similarly Y'. Next suppose some vertex z in Z has a non-neighbor x in X and a non-neighbor y in Y. Then x is adjacent to y, and there is an odd antipath Q from x to y in $X \cup Y$. Thus $xQyzx$ is an antihole, so Q has length 1 mod 3. But then Q has length at least 4, and so $X \cup Y$ contains a C_4, contrary to (**). This proves 3.2.20.1.

Let Z_x be the vertices of Z complete to X, and let $Z_y = Z \setminus Z_x$.

3.2.20.2 $k \leq 4$ and $X' = X_i$, $Y' = Y_j$ for some i different from j.

Suppose both X_2, X_3 are non-empty, choose x_2 in X_2 and x_3 in X_3. Then $a_2x_2x_3a_3a_2$ is a hole of length four, and every x in X is complete to it, so G contains a 4-wheel, a contradiction. So we may assume that $X' = X_2$ and, similarly, $Y' = Y_j$ for some j. If Y_2 is non-empty, then since x_2, y_2, a_3 is not a stable set of size three, x_2 is adjacent to y_2. Since A_1 is anticonnected, there exist non-adjacent vertices $x \in X$ and $y \in Y$. But now $xx_2y_2ya_3x$ is a hole of length five, a contradiction. So Y_2 is empty and therefore i is different from j, say $j = 3$. Since a_4, a_5 are not twins, $k \leq 4$. This proves 3.2.20.2.

By 3.2.20.2 we may assume that $X' = X_2$, $Y' = Y_3$. Let M_1 be the vertices in X with a neighbor in Z_y, $M_2 = X \setminus M_1$. Let N_1 be the vertices in Y with a neighbor in Z_x, $N_2 = X \setminus N_1$.

3.2.20.3 If Z, X', Y' are all non-empty then the theorem holds.

We may assume Z_x is non-empty. Since $a_2x_2zy_3a_3a_2$ (where $z \in Z$, $x_2 \in X_2$ and $y_3 \in Y_3$) is not a hole of length five, X_2 is complete to Y_3. Suppose z in Z_x has a neighbor y in Y. Since A_1 is anticonnected, y has a non-neighbor x in X. But now $a_3za_2y_3xyx_2a_3$ (with x_2 in X_2 and y_3 in Y_3) is an antihole of length seven, a contradiction. So Z_x is anticomplete to Y. Choose z in Z_x and non-adjacent x in X and y in Y. Then zxa_2yy_3z is a hole of length five, a contradiction. This proves 3.2.20.3.

3.2.20.4 If Z is empty then the theorem holds.

The pairs (X, Y) and (X_2, Y_3) are coherent homogeneous pairs, and since G does not admit twins or a coherent W-join, all four of these sets have size ≤ 1. Every vertex of G is adjacent to a_3, except the vertex x_2 of X_2, if X' is non-empty. So every clique of G contains either a_3 or x_2, and therefore $K(G)$ is perfect (it is either a complete graph, or the complement of a bipartite graph). This proves 3.2.20.4.

In view of 3.2.20.4, we henceforth assume that $Z \neq \emptyset$. By 3.2.20.3 we may assume X' is empty, and so Y' is non-empty. By 3.2.20.2 we may assume $Y' = Y_3$. Since the vertices of Y_3 are not twins, $Y_3 = \{y_3\}$.

3.2.20.5 Z is complete to Y.

Suppose not. Choose z in Z, with a non-neighbor y is in Y. Then z in Z_x. Since A_1 is anticonnected, y has a non-neighbor x in X. But now zxa_2yy_3z is a hole of length five, a contradiction. This proves 3.2.20.5.

Let M be the set of vertices in X with a neighbor in Z. Suppose some z in Z has adjacent neighbors x in X and y in Y. Then xya_3 is a triangle, z is adjacent to x, y and not to a_3; y_3 is adjacent to a_3, y and not to x. Choose a non-neighbor x' of y in X. Then x' is adjacent to a_3, x. But now the graph induced by $\{x, x', y, y_3, a_3, z\}$ is a 1- or 2-pyramid, a contradiction. This proves that M is anticomplete to Y. Now (Z, M) is a coherent homogeneous pair, and the same for $(X \setminus M, Y)$. Since G admits no twins and no coherent W-join, all four of these sets have size ≤ 1. Also, since a_2 and a_4 are not twins, $k = 3$. Let $Z = \{z\}$. Every vertex of G different from z is adjacent to a_3. So every clique of G contains either a_3 or z, and then $K(G)$ is perfect (it is the complement of a bipartite graph). This completes the proof of Theorem 3.2.20. □

Theorem 3.2.21. *Let G be an interesting HCH claw-free graph, and suppose that G is connected, does not admit a coherent or non-dominating W-join, a 1-join or twins. If G contains a stable set of size three and a singular vertex, then $K(G)$ is perfect.*

Proof. The proof is by induction on $|V(G)|$. Assume that for every smaller graph G' satisfying the hypotheses of the theorem, $K(G')$ is perfect. Let v be a singular vertex in G with maximum number of neighbors. Let A be the set of neighbors of v and B be the set of its non-neighbors. Since v is singular, B is a complete.

Since G contains a stable set of size three, and every such set meets both A and B (because B is a clique, and G is claw-free), there exist vertices in B that are non singular. Let U be the set of all such vertices.

3.2.21.1 If U is anticomplete to A then $K(G)$ is perfect.

Let $V = B \setminus U$, so every vertex of V is singular, and since G is connected, V is non-empty. Let a_1, a_2 be two non-adjacent vertices in A. If $b \in V$ is non-adjacent to both a_1, a_2, then $\{b, a_1, a_2\}$ is a stable set of size three, and if b is adjacent to both a_1, a_2 then $\{b, a_1, a_2, u\}$ is a claw for every $u \in U$; in both cases we get a contradiction. So every vertex in V is adjacent to exactly one of a_1, a_2. Suppose there exist v_1, v_2 in V with v_i adjacent to a_i. Then $v_1v_2a_2va_1v_1$ is a hole of length five, a contradiction. So one of a_1, a_2 is anticomplete to V, and therefore the other one is complete to V. Let A_1 be the vertices in A complete to V, A_2 be the vertices in A anticomplete to V and $A_3 = A \setminus (A_1 \cup A_2)$. It follows from the previous argument that $A_1 \cup A_3$ and $A_2 \cup A_3$ are both completes. If A_3 is non-empty, then $|V| > 1$ and (A_3, V) is a coherent W-join, a contradiction. So we may assume A_3 is empty. Now (A_1, A_2) is a coherent homogeneous pair, and all the vertices of each of U, V are twins. So all these sets have size at most 1 and $K(G)$ is the clique graph of an induced subgraph of a 4-edge path, and hence perfect. This proves 3.2.21.1.

So we may assume that there exists a non-singular vertex u in B with a neighbor in A. Let M be the set of neighbors of u in A, N the set of non-neighbors. Since u is non-singular, N contains two non-adjacent vertices x, y. Choose m in M. If m is adjacent to both x, y then $\{m, x, y, u\}$ is a claw. If m is non-adjacent to both x, y then $\{v, x, y, m\}$ is a claw. So every vertex in M is adjacent to exactly one of x, y. So there is no complement of an odd cycle in $G|N$, and therefore the complement of $G|N$ is bipartite and N is the union of two completes.

Let M_1 be the vertices in M adjacent to x, M_2 those adjacent to y, then $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \emptyset$.

If there exists m_1 in M_1 and m_2 in M_2 such that m_1 is adjacent to m_2, then the graph induced by $\{m_1, m_2, v, x, y, u\}$ is 3-sun, a contradiction. So there are no edges between M_1 and M_2, M_1 is anticomplete to y and M_2 is anticomplete to x. Since $\{v, m, m', y\}$ is not a claw for m, m' in M_1, it follows that M_1 is a complete, and the same holds for M_2.

Case 1: M_1 and M_2 are both non-empty.

Since A contains no stable set of size three (for otherwise there would be a claw in G), every vertex in N is complete to one of M_1, M_2. Let N_3 be the vertices complete to $M_1 \cup M_2$, N_1 the vertices of $N \setminus N_3$ complete to M_1 and N_2 vertices of $N \setminus N_3$ complete to M_2. So $x \in N_1$ and $y \in N_2$. Since $\{m, n, n', u\}$ is not a claw for m in M_1 and n, n' in $N_1 \cup N_3$, it follows that $N_1 \cup N_3$ is a complete. Similarly $N_2 \cup N_3$ is a complete. Suppose N_3 is non-empty, and choose $n \in N_3$. Then n is complete to $(A \cup \{v\}) \setminus \{n\}$, and therefore is singular (for its non-neighbors are a subset of B); and by the choice of v, n and v are twins. Since G admits no twins, it follows that N_3 is empty. Suppose some n_1 in N_1 is adjacent to n_2 in N_2. Choose m'_1 in M_1 non-adjacent to n_2 and m'_2 in M_2 non-adjacent to n_1. Then $m'_1 n_1 n_2 m'_2 u m'_1$ is a hole of length five, a contradiction. So N_1 is anticomplete to N_2. Suppose n_1 in N_1 has a neighbor m'_2 in M_2. Then $\{m'_2, n_1, y, u\}$ is a claw, a contradiction. So N_1 is anticomplete to M_2, and, similarly, N_2 is anticomplete to M_1.

For $i = 1, 2$ choose m'_i in M_i, and assume that m'_i has a non-neighbor b_i in B. If m'_1 and m'_2 have a common non-neighbor $b \in B$, then $\{u, m'_1, m'_2, b\}$ is a claw, a contradiction. So there are two vertices b_1 and b_2 in B such that b_1 is non-adjacent to m'_1 and adjacent to m'_2, and b_2 is non-adjacent to m'_2 and adjacent to m'_1. But then $m'_1 b_2 b_1 m'_2 v m'_1$ is a hole of length five, again a contradiction. So, exchanging M_1 and M_2 if necessary, we may assume that M_1 is complete to B, and since G admits no twins, $|M_1| = 1$, say $M_1 = \{m_1\}$.

Let b be a vertex of B with a neighbor in N_1. We claim that b is complete to M_2 and anticomplete to N_2. For if b has a non-neighbor m_2 in M_2, then $n_1 b u m_2 v n_1$ is a hole of length five; and if b has a neighbor n_2 in N_2, then $\{b, n_1, n_2, u\}$ is a claw; in both cases a contradiction. This proves the claim.

So every vertex of B is either anticomplete to N_1, or complete to M_2 and anticomplete to N_2. Let B_1 be the set of vertices of B with a neighbor in N_1. Then (B_1, N_1) is a non-dominating homogeneous pair, and since G does not admit a non-dominating W-join or twins, it follows that $|B_1| \leq 1$ and $|N_1| = 1$, say $N_1 = \{n_1\}$.

Assume that B_1 is non-empty, let $B_1 = \{b_1\}$. Let $B_2 = B \setminus B_1$. We claim that in this case B_2 is complete to M_2. If b_2 in B_2 has a non-neighbor m_2 in M_2, then $b_2 \neq b_1$ and $\{b_1, n_1, m_2, b_2\}$ is a claw, a contradiction. This proves the claim. But now the vertices of M_2 are all twins, and since G does not admit twins, $|M_2| = 1$. Moreover, (B_2, N_2) is a non-dominating homogeneous pair, and since G does not admit a non-dominating W-join or twins, it follows that $|B_2| = |N_2| = 1$, so $B_2 = \{u\}$ and $N_2 = \{n_2\}$. But now every clique of G contains either v or b_1, and hence $K(G)$ is the complement of a bipartite graph, and therefore perfect. This finishes the case when B_1 is non-empty.

If B_1 is empty, $(B, M_2 \cup N_2)$ is a non-dominating homogeneous pair, and since G does not admit a non-dominating W-join or twins, it follows that $|B| = |M_2 \cup N_2| = 1$, a contradiction because both M_2 and N_2 are non-empty. This finishes the case when both M_1 and M_2 are non-empty.

Case 2: One of M_1, M_2 is empty.

We may assume that M_2 is empty, and so M is complete to x and anticomplete to y. Let N_1 be the set of vertices in N complete to M, N_2 the set of vertices in N that are anticomplete to M and let $N_3 = N \setminus (N_1 \cup N_2)$.

We claim that $N_1 \cup N_3$ and $N_2 \cup N_3$ are both completes. Choose two different vertices n_3 in $N_3 \cup N_1$ and n_1 in N_1, and let m be a neighbor of n_3 in M. Since $\{m, u, n_1, n_3\}$ is not a claw, n_1 is adjacent to n_3; and therefore N_1 is a complete and N_1 is complete to N_3. Next, choose two different vertices n_3 in $N_3 \cup N_2$ and n_2 in N_2, and let m be a non-neighbor of n_3 in M. Since $\{v, m, n_2, n_3\}$ is not a claw, n_2 is adjacent to n_3; and therefore N_2 is a complete and N_2 is complete to N_3. Finally, suppose there exist two non-adjacent vertices n_3 and n_3' in N_3. Since $\{m, u, n_3, n_3'\}$ is not a claw for any $m \in M$, it follows that no vertex of M is adjacent to both n_3 and n_3'. Let m be a neighbor of n_3 in M and m' be a neighbor of n_3' in M. Then m is non-adjacent to n_3' and m' is non-adjacent to n_3, and the graph induced by $\{v, m, m', u, n_3, n_3'\}$ is a 3-sun, a contradiction. So N_3 is a complete. This proves the claim. Since there exist two non-adjacent vertices in N, both N_1 and N_2 are non-empty.

3.2.21.2 Let b in B adjacent to n_3 in N_3 and to m in M. Then n_3 is non-adjacent to m.

Suppose they are adjacent. Let m' be a non-neighbor of n_3 in M, and let n_2 be in N_2. Then n_3mv is a triangle, b is adjacent to n_3, m; n_2 is adjacent to v and n_3; m' is adjacent to v and m, and this is a 0-, 1- or 2-pyramid, a contradiction. This proves 3.2.21.2.

3.2.21.3 Every vertex in N_1 has a non-neighbor in N_2.

Suppose some vertex n_1 of N_1 is complete to N_2. Then the set of non-neighbors of n_1 is included in B, and therefore n_1 is singular; and it is complete to $A \setminus \{n_1\}$. From the choice of v, n_1 has no neighbor in B, but now n_1 and v are twins, a contradiction. This proves 3.2.21.3.

3.2.21.4 M is complete to B.

Let B_1 be the set of vertices in B that are complete to M. Suppose there exists b_2 in $B \setminus B_1$, and let m be a non-neighbor of b_2 in M.

3.2.21.4.1 $|N_2| = 1$, N_2 is anticomplete to B, and consequently all stable sets of size three using u share a vertex in A.

Let n be in N_2. Since nb_2umvn is not a hole of length five, it follows that n is non-adjacent to b_2, and the same holds for every vertex of $B \setminus B_1$. So n is anticomplete to $B \setminus B_1$. Since $\{b_1, b_2, m, n\}$ is not a claw for $b_1 \in B_1$, it follows that n is anticomplete to B_1, and the same holds for every vertex of N_2. Therefore N_2 is anticomplete to B. But now $\{v\} \cup N_1 \cup N_3$ is a clique cutset separating N_2 from $M \cup B$. By Theorem 3.2.13, G is either a linear interval graph or G is the 3-sun, or G admits twins, or a 0-join, or a 1-join, or a coherent W-join, or it is not an internal clique cutset; and it follows from the hypotheses of the theorem and from Theorem 3.2.1, that we may assume that the last alternative holds, and $|N_2| = 1$, say $N_2 = \{n_2\}$. Now, since M, B and $N_1 \cup N_3$ are all completes, it follows that n_2 belongs to every stable set of size three using u. This proves 3.2.21.4.1.

3.2.21.4.2 N_1 is anticomplete to n_2.

Follows from 3.2.21.3.

3.2.21.4.3 We may assume that every vertex of B has a neighbor in A.

Suppose not. Let b be a vertex of B anticomplete to A.

We claim that in this case $K(G)$ is perfect if and only if $K(G \setminus \{b\})$ is. Since every vertex of $G \setminus B$ has a non-neighbor in B, B is a clique of G. b is a simplicial vertex and B is the only clique containing b. Let v_B be the vertex of $K(G)$ corresponding to B. There are two possibilities: either $B \setminus \{b\}$ is a clique of $G \setminus \{b\}$, and then $K(G \setminus \{b\}) = K(G)$, or there is a vertex m_B in A complete to $B \setminus \{b\}$ in G, and then $K(G \setminus \{b\}) = K(G) \setminus \{v_B\}$. The vertex m_B belongs to M because, in particular, it is adjacent to u. We claim that every clique of G different from B and having non-empty intersection with B contains the vertex m_B. Otherwise, there is a clique of G containing a vertex of B, say b_3, and a vertex a of A non-adjacent to m_B. But now $\{b_3, b, m_B, a\}$ is a claw, a contradiction. Thus v_B is simplicial in $K(G)$, and Lemma 1.3.3 completes the proof of the claim. But now, since $K(G \setminus \{b\})$ is perfect, so is $K(G)$. This proves 3.2.21.4.3.

We henceforth assume that every vertex of B has a neighbor in A.

3.2.21.4.4 Let $b \in B$ be a vertex non-adjacent to some $n_3 \in N_3$; and let m be in M. Then n_3 is adjacent to m.

Suppose not. Then b is in a stable set of size three $\{b, n_3, m\}$ and b has a neighbor in A; and by 3.2.21.4.1 applied to b instead of u, $\{b, n_2\} \cup N_1$ does not contain a stable set of size three. So b is complete to N_1. But now $\{n_1, b, m, n_3\}$ is a claw for every $n_1 \in N_1$, a contradiction. This proves 3.2.21.4.4.

3.2.21.4.5 B is anticomplete to N_3.

Suppose a vertex $b \in B$ has a neighbor $n \in N_3$. By the definition of N_3, n has a neighbor m in M. By 3.2.21.2, m is non-adjacent to b. By 3.2.21.4.4 n is adjacent to m. But now $\{n, n_2, b, m\}$ is a claw, a contradiction. This proves 3.2.21.4.5.

Now $M \cup N_1$ is a clique cutset separating $\{v\} \cup N_2 \cup N_3$ from B. Since $|B| > 1$ and $|\{v\} \cup N_2 \cup N_3| > 1$, it follows from Theorem 3.2.13, that G is a linear interval graph, and therefore $K(G)$ is perfect by Theorem 3.2.1. This completes the proof of 3.2.21.4.

By 3.2.21.4, for every non-singular vertex in B, the set of its neighbors in A is complete to B.

3.2.21.5 B is anticomplete to N_3.

Suppose some vertex $b \in B$ has a neighbor $n_3 \in N_3$. By the definition of N_3, n_3 has a neighbor in M, and this contradicts 3.2.21.2. This proves 3.2.21.5.

3.2.21.6 N_3 is empty and $|M| = 1$.

If N_3 is non-empty then $|M| > 1$ and (N_3, M) is a coherent homogeneous pair. So N_3 is empty, but now the vertices of M are twins, so $|M| = 1$. This proves 3.2.21.6.

It follows from 3.2.21.6 that every singular vertex in B has at most one neighbor in A, and since M is complete to B and has size 1, every singular vertex in B is complete to M and anticomplete to $A \setminus M$. Therefore the vertices of U are all twins, and since G admits no twins, $U = \{u\}$. Let $B_2 = B \setminus U$.

3.2.21.7 B_2 is non-empty.

Otherwise (N_1, N_2) is a coherent homogeneous pair, so each of them has size 1 and $K(G)$ is a three-edge path. This proves 3.2.21.7.

3.2.21.8 If n_1 in N_1 is non-adjacent to n_2 in N_2, then every b in B_2 is adjacent to exactly one of n_1, n_2.

Let b_2 in B_2. Since b_2 in B_2 is singular, b_2 is adjacent to at least one of n_1, n_2. Since $\{b_2, n_1, n_2, u\}$ is not a claw, b_2 is non-adjacent to at least one of n_1, n_2. This proves 3.2.21.8.

3.2.21.9 No vertex of N_1 has a neighbor and a non-neighbor in B_2.

Suppose n_1 in N_1 has a neighbor b_1 in B_2 and a non-neighbor b_2 in B_2. By 3.2.21.3 n_1 has a non-neighbor n_2 in N_2. By 3.2.21.8 n_2 is adjacent to b_2 and not to b_1. But now $b_1 n_1 v n_2 b_2 b_1$ is a hole of length five, a contradiction. This proves 3.2.21.9.

Let N_{11} be the vertices of N_1 complete to B_2, $N_{12} = N_1 \setminus N_{11}$. So N_{12} is anticomplete to B. It follows from 3.2.21.8 every vertex of N_2 is either complete to N_{11} or to N_{12}. Let N_{22} be the set of vertices in N_2 with a non-neighbor in N_{11}. Then N_{22} is complete to N_{12}. Let N_{21} be the vertices in N_2 with a non-neighbor in N_{12}. Then N_{21} is complete to N_{11}. Let $N_{23} = N_2 \setminus (N_{21} \cup N_{22})$. So N_{23} is complete to N_1. By 3.2.21.8 B_2 is anticomplete to N_{22} and complete to N_{21}. Now (B_2, N_{23}) is a coherent homogeneous pair, and all the vertices of $N_{11}, N_{12}, N_{22}, N_{21}$ are twins, so all these sets have size at most 1.

Now, every clique of G contains either v or b_2, so $K(G)$ is the complement of a bipartite graph, and hence it is perfect. This completes the proof of Theorem 3.2.21. $\square$

Basic classes

We finally show that if an interesting HCH claw-free graph belongs to one of the basic classes of Theorem 3.2.12, then its clique graph is perfect.

Theorem 3.2.22. *If G is interesting HCH, antiprismatic and every vertex of G is in a triad, then $K(G)$ is perfect.*

Proof. We prove that G contains no 4-wheel or 3-fan, and then, by Theorem 1.3.14, $K(G)$ is bipartite.

Suppose G contains a 4-wheel. Let $a_1a_2a_3a_4a_1$ be a hole and let c be adjacent to all a_i. Since every vertex is in a triad, there are two vertices c_1, c_2 different from a_1, a_2, a_3, a_4 such that $\{c, c_1, c_2\}$ is a stable set. Since G is antiprismatic, every other vertex in G is adjacent exactly to two of $\{c, c_1, c_2\}$. In particular, each a_i is adjacent either to c_1 or to c_2. If two consecutive vertices of the hole, for instance a_1, a_2, are adjacent to the same c_j, then $\{a_1,\ a_3,\ a_2,\ a_4,\ c,\ c_j\}$ induces a 1-,2- or 3-pyramid, a contradiction because G is HCH. So, without loss of generality, we may assume that a_1 and a_3 are adjacent to c_1 and not to c_2, while a_2 and a_4 are adjacent to c_2, and not to c_1. But then $\{a_1, a_2, a_3, c_2\}$ is a claw, a contradiction. This proves that G does not contain a 4-wheel.

Suppose now that G contains a 3-fan. Let $a_1a_2a_3a_4$ be an induced path and let c be adjacent to all a_i. Since every vertex is in a triad, there are two vertices c_1, c_2 different from a_1, a_2, a_3, a_4 such that $\{c, c_1, c_2\}$ is a stable set. Since G is antiprismatic, each a_i is adjacent either to c_1 or to c_2. If a_2 and a_3, are adjacent to the same c_j, then $\{a_1,\ a_3,\ a_2,\ a_4,\ c,\ c_j\}$ induces a 0-,1- or 2-pyramid, a contradiction because G is HCH. So, without loss of generality, we may assume that a_2 is adjacent to c_1 and not c_2, while a_3 is adjacent to c_2 and not c_1. Since $\{a_3, a_2, c_2, a_4\}$ is not a claw, a_4 is adjacent to c_2, and, analogously, a_1 is adjacent to c_1. By the same argument applied to the 3-fan induced by the path $a_2ca_4c_2$ and the vertex a_3, there is a vertex d adjacent to a_4 and c_2 but not adjacent to a_2, c or a_3, and so $d \notin \{a_1, a_2, a_3, a_4, c, c_1, c_2\}$ (see Figure 3.12).

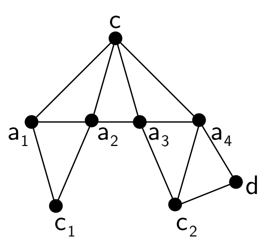

Figure 3.12: Situation for the second part of the proof of Theorem 3.2.22.

Since $c_1a_2a_2a_4dc_1$ is not a hole of length five, d is non-adjacent to c_1. Thus c_1, c and d form a triad, but the vertex c_2 is adjacent only to one of them, a contradiction because G is antiprismatic. This concludes the proof of Theorem 3.2.22. $\square$

Theorem 3.2.23. *Let $G \in \mathcal{S}_6$ be a connected interesting HCH graph such that every vertex of G is in a triad. Then $K(G)$ is perfect.*

Proof. Let A, B and C be the sets of vertices of the graph H_5 in the definition of the class $\mathcal{S}_6$, and let A_G, B_G and C_G be those sets intersected with $V(G)$. Every triad in G is of the form $\{a_i, b_j, c_k\}$, since A_G, B_G and C_G are complete sets. Moreover, either $i = j = 0$ or $k = i$ and $j = 0$ or $k = j$ and $i = 0$. Since every vertex of G is in a triad, it follows that A_G, B_G and C_G are non-empty and if $i \neq 0$ and $a_i \in A_G$, then $b_0 \in B_G$ and $c_i \in C_G$. Analogously, if $i \neq 0$ and $b_i \in B_G$, then $a_0 \in A_G$ and $c_i \in C_G$. Let $I_A = \{i > 0 : a_i \in A_G\}$, $I_B = \{i > 0 : b_i \in B_G\}$ and $I_C = \{i > 0 : c_i \in C_G\}$. Then $I_A \cup I_B \subseteq I_C$.

Assume first that $I_C \setminus (I_A \cup I_B)$ is non-empty. Since every vertex is in a triad, it follows that a_0 and b_0 belong to G. Since the set $C' = \{c_i \ : \ i \in C \setminus (I_A \cup I_B)\}$ is complete to $V(G) \setminus (C' \cup \{a_0, b_0\})$, and the only cliques containing a_0 or b_0 are A_G and B_G, respectively, it follows that every pair of cliques of G, except for the pair A_G, B_G, has non-empty intersection. Thus $K(G)$ is a split graph (that is, $V(K(G))$ is the union of a stable set and a complete), and hence $K(G)$ is perfect [46].

So we may assume that $I_A \cup I_B = I_C$. If $|I_A \cup I_B| \geq 3$, we may assume by switching A and B if necessary that $1, 2 \in I_A$, and then the graph induced by $\{a_1, a_2, c_1, c_2, c_3, a_0\}$ is a 1-pyramid, a contradiction because G is HCH. On the other hand, since G is connected, both I_A and I_B are non-empty and $|I_A \cup I_B| \geq 2$. So, without loss of generality, we consider three cases: $I_A = I_B = \{1, 2\}$; $I_A = \{1, 2\}$ and $I_B = \{2\}$; $I_A = \{1\}$ and $I_B = \{2\}$. Graphs obtained in each case are depicted in Figure 3.13, with their corresponding clique graphs, which are all perfect. That concludes this proof. $\square$

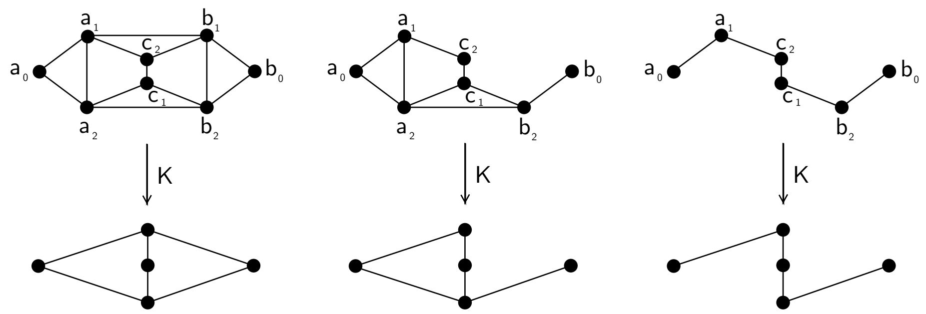

Figure 3.13: Last three cases for the proof of Theorem 3.2.23.

Proof of Theorem 3.2.11. Let G be an interesting HCH claw-free graph. The proof is by induction on $|V(G)|$, using the decomposition of Theorem 3.2.12. Assume that for every smaller interesting HCH claw-free G', $K(G')$ is perfect. We show that $K(G)$ is perfect.

If G admits twins, then $K(G)$ is perfect by Lemma 1.3.11, and if G is not connected, then $K(G)$ is perfect by Lemma 1.3.10. If G is connected, admits a 1-join and no twins, then $K(G)$ is perfect by Theorem 3.2.15 and Lemma 1.3.4. If G admits no twins, 0- or 1-joins, but admits a 2-join, then $K(G)$ is perfect by Theorem 3.2.16. If G admits a coherent or non-dominating W-join and no twins, then $K(G)$ is perfect by Theorem 3.2.18. If G contains a singular vertex, then $K(G)$ is perfect by Theorems 3.2.20 and 3.2.21. So we may assume not. If G admits a hex-join and no twins, then by Theorem 3.2.19 $G = K(G) = C_6$, and therefore $K(G)$ is perfect.

So we may assume that G admits none of the decompositions of the previous paragraph, and by Theorem 3.2.12, G is antiprismatic, or belongs to $\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$.

If $G \in \mathcal{S}_0$, then $K(G)$ is perfect by Theorem 3.2.9. The graphs $icosa(-2)$, $icosa(-1)$ and $icosa(0)$ contain holes of length five, and therefore are not interesting, so $G \notin \mathcal{S}_1$. $G \notin \mathcal{S}_2$, because vertices v_3, v_4, v_5, v_6, v_9 induce a hole of length five in H_1 (Figure 3.9). If $G \in \mathcal{S}_3$, then by Proposition 3.2.1, $K(G)$ is perfect. If $G \in \mathcal{S}_4$ then, since G does not contain a singular vertex, G is a line graph and $K(G)$ is perfect by Theorem 3.2.9. $G \notin \mathcal{S}_5$, because the vertex d_1 in the definition of the class $\mathcal{S}_5$ is singular. If $G \in \mathcal{S}_6$, then $K(G)$ is perfect by Theorem 3.2.23, and finally, if G is antiprismatic, then $K(G)$ is perfect by Theorem 3.2.22. This completes the proof of Theorem 3.2.11. $\square$

Theorem 3.2.4 is an immediate corollary of the following:

Theorem 3.2.24. *Let G be claw-free and assume that G is HCH. Then the following are equivalent:*

(i) *no induced subgraph of G is an odd hole, or $\overline{C_7}$.*

(ii) *G is clique-perfect.*

(iii) *G is perfect.*

Proof. Since every antihole of length at least eight contains a 2-pyramid, it follows from Theorem 1.3.2 that no HCH graph contains an antihole of length at least eight. Thus the equivalence between (i) and (iii) is a corollary of Theorem 1.2.1. From Theorem 3.1.1 it follows that (ii) implies (i). Finally, by Theorem 3.2.11 and Proposition 1.3.1, we deduce that (i) implies (ii), and this completes the proof. $\square$

3.2.4 Helly circular-arc graphs

In this subsection we provide a proof of Theorem 3.2.5, which states that if a graph G is HCA, then G is clique-perfect if and only if it does not contain the graphs of Figure 3.14.

In fact, we will show that an HCA graph that does not contain any of the graphs of Figure 3.14 is K-perfect. The class of clique-perfect graphs is neither a subclass nor a superclass of the class of K-perfect graphs. But K-perfection allows us to apply similar arguments to those used in the proof of Proposition 1.3.1 in order to prove

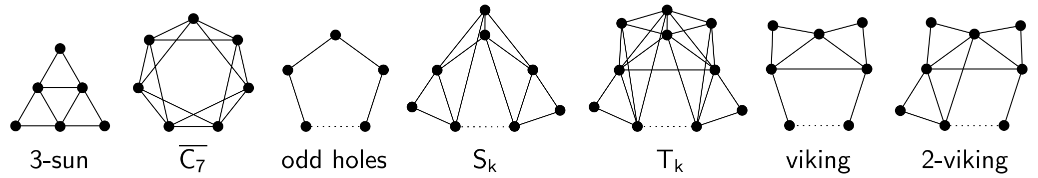

Figure 3.14: Minimal forbidden subgraphs for clique-perfect graphs inside the class of HCA graphs. Dotted lines represent any induced path of odd length at least 1.

Theorem 3.2.5 for HCA graphs that are also HCH. The graphs in $HCA \setminus HCH$ are handled separately.

We start with some straightforward results about HCA graphs.

Throughout this subsection, an arc of a circle defined by two points will be called a *sector*, in order to distinguish them from arcs corresponding to vertices of an HCA graph. For example, the bold arc in Figure 3.15 is one of the two sectors defined by the points a and b. Given a collection $\mathcal{C}$ of points on the circle, for $a, b, c \in \mathcal{C}$ we say that c is *between* a and b if the sector defined by a and b that contains c does not contain any other point of $\mathcal{C}$. For example, in Figure 3.15, the point c is between a and b but the point d is not.

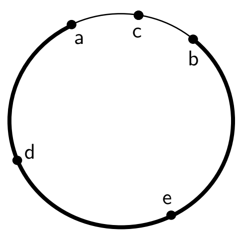

Figure 3.15: Example of notation. The bold arc is one of the two *sectors* defined by the points a and b of the circle. The point c is *between* a and b but the point d is not.

Lemma 3.2.25. *Let G be an HCA graph that has an HCA representation with no two arcs covering the circle. Then G is HCH.*

Proof. Suppose not. By Theorem 1.3.2, G contains a 0-,1-, 2-, or 3-pyramid P. Let $\{v_1, \ldots, v_6\}$ be the vertices of P, such that v_1, v_2, v_3 form a triangle; v_4 is adjacent to v_2 and v_3 but not to v_1; v_5 is adjacent to v_1 and v_3 but not to v_2; v_6 is adjacent to v_1 and v_2 but not to v_3. Since P is an induced subgraph of G, P has an HCA representation with no two arcs covering the circle. Let $\mathcal{A} = \{A_i\}_{1 \leq i \leq 6}$ be such a representation, where the arc A_i corresponds to the vertex v_i. The sets $C_1 = \{v_1, v_2, v_3\}$ and $C_2 = \{v_1, v_2, v_6\}$ are cliques of P, let a be an anchor of C_1 and b of C_2. Then a and b are distinct points of the circle. Let S_1 and S_2 be the two sectors with ends a, b. Since A_1, A_2 do not cover the circle, and a, b belong to both A_1 and A_2, we may assume that S_1 is included both in A_1 and in A_2. Since $a \in A_3$ but $b \notin A_3$, it follows that A_3 has an endpoint, say c, in $S_1 \setminus \{b\}$ (see Figure 3.16). But now, since the pairs A_1, A_3 and A_2, A_3 do not cover the circle, it follows that either $A_1 \cap A_3 \subseteq A_2$, or $A_2 \cap A_3 \subseteq A_1$. In the former case there is no anchor for the clique $\{v_1, v_3, v_5\}$, and in the later there is none for the clique $\{v_2, v_3, v_4\}$; in both cases a contradiction. □

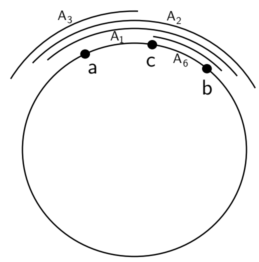

Figure 3.16: Scheme of representation of arcs A_6, A_1, A_2 and A_3, in the proof of Lemma 3.2.25.

Lemma 3.2.26. *Every HCA representation of a 4-wheel has two arcs covering the circle.*

Proof. Let $\{a_1, a_2, a_3, a_4, b\}$ be the vertices of a 4-wheel W, where $a_1a_2a_3a_4a_1$ is a cycle of length four and b is adjacent to all of a_1, a_2, a_3, a_4, and let $\mathcal{A} = \{A_1,\ A_2,\ A_3,\ A_4,\ B\}$ be an HCA representation of W. Let p_1, p_2, p_3 and p_4 be anchors of the cliques $\{a_1, a_2, b\}$, $\{a_2, a_3, b\}$, $\{a_3, a_4, b\}$, $\{a_4, a_1, b\}$, respectively. Then there are only two possible circular orders of the anchors: p_1, p_2, p_3, p_4 and the reverse one, and for $1 \leq i \leq 4$, each arc A_i passes exactly through p_i and p_{i-1} (index operations are done modulo 4). Since the arc B passes through the four points p_i, it follows that B and one of the A_i cover the circle. □

Lemma 3.2.27. *If G is an HCA graph and it has an HCA representation without two arcs covering the circle, then this representation cannot have three arcs covering the circle.*

Proof. Let $\mathcal{A}$ be a HCA representation for a HCA graph G. Suppose that there are three arcs A, B, and C in $\mathcal{A}$ covering the circle $\mathcal{C}$ but no two arcs cover it. Since $A \cup B$ do not cover the circle, there is a point c in $\mathcal{C} \setminus (A \cup B)$. Since $\mathcal{C} = A \cup B \cup C$, it follows that $c \in C$. Analogously, there exist points a and b in $A \setminus (B \cup C)$ and $B \setminus (A \cup C)$, respectively. Since the arcs are open and $A \cup B \cup C$ but no two of them cover $\mathcal{C}$, the three arcs mutually intersect. Since $\mathcal{A}$ verifies the Helly property, there is a common intersection point p of A, B and C. But since a belongs to A and neither b nor c belong to A, p cannot lie between b and c. Analogously, it cannot lie neither between a and b nor between a and c, a contradiction. □

Lemma 3.2.28. *Let S denote the unit circle. Let G be an HCA graph that has an HCA representation with no two arcs covering S, and let $\mathcal{A}$ be such a representation. Let H be a clique subgraph of G. Then H is HCA and has an HCA representation*

$\mathcal{A}'$ with no two arcs covering S. Moreover, let $M_1, \ldots, M_s$ be the cliques of H, and for $1 \le i \le s$ let a_i be an anchor of M_i in $\mathcal{A}$. Let $\varepsilon = \frac{1}{3}\, min_{1 \le i < j \le s}\, dist(a_i, a_j)$, where $dist(a_i, a_j)$ denotes the length of the shortest sector of S between a_i and a_j. For an arc $A \in \mathcal{A}$ that contains at least one of the points $a_1, \ldots, a_s$, let the derived arc *A' of A be defined as follows: let $a_{i_k}, \ldots, a_{i_m}$ be the points of $a_1, \ldots, a_s$ traversed by A in clockwise order, let u be the point of S which is at distance ε from a_{i_k} going anti-clockwise, and v the point of S which is at distance ε from a_{i_m} going clockwise. Then A' is the arc with endpoints u and v and containing all of $a_{i_k}, \ldots, a_{i_m}$. In this notation, $\mathcal{A}'$ is precisely the set of all arcs A' that are the derived arcs of some $A \in \mathcal{A}$ such that A contains at least one of $a_1, \ldots, a_s$. Please note that $\mathcal{A}'$ depends on the choice of the anchors $a_1, \ldots, a_s$.*

Proof. Let H' be the intersection graph of the arcs of $\mathcal{A}'$. We claim that H' is isomorphic to H. Since the arcs of $\mathcal{A}'$ are sub-arcs of the arcs of $\mathcal{A}$ that correspond to vertices of G that belong to $\bigcup_{i=1}^{s} M_i$, there is a one-to-one correspondence between the vertices of H' and the vertices of H, and we may assume that $V(H) = V(H')$. Moreover, for every clique M_i and every $A \in \mathcal{A}$, the derived arc of A contains a_i if and only if A does. So $M_1, \ldots, M_s$ are cliques on H', and a_i is an anchor of M_i. Since two vertices of a graph are adjacent if and only if there exists a clique containing them both, in order to show that H is isomorphic to H', it remains to check that every two adjacent vertices of H' belong to M_i for some i. But it follows from the construction of $\mathcal{A}'$ (and in particular from the choice of ε) that $A'_1 \cap A'_2 \neq \emptyset$ for $A'_1, A'_2 \in \mathcal{A}'$, if and only if $a_i \in A'_1 \cap A'_2$ for some $1 \le i \le s$, which means that the corresponding vertices of H' belong to the clique M_i. This proves that $E(H) = E(H')$ and completes the proof of the lemma. □

Figure 3.17 provides an example of the construction of Lemma 3.2.28.

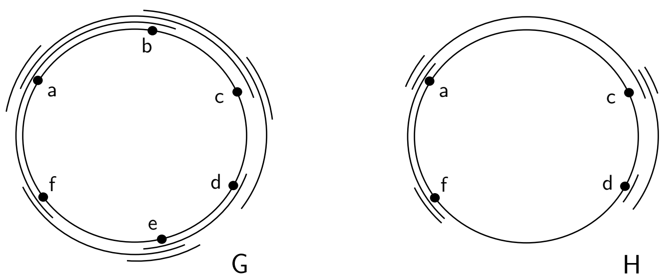

Figure 3.17: HCA representation of the clique subgraph H of G whose cliques are a, c, d and f.

Remark 3.2.1. *Let G be an HCA graph with representation $\mathcal{A}$, and let H be a clique subgraph of G with representation $\mathcal{A}'$ given by Lemma 3.2.28, with anchors $a_1, \ldots, a_s$. Let $A'_1, A'_2 \in \mathcal{A}'$ be the derived arcs of $A_1, A_2 \in \mathcal{A}$. Then $A_1 \cap A_2$ may be non-empty even if A'_1, A'_2 are disjoint, but no point of $A_1 \setminus A'_1$ or $A_2 \setminus A'_2$ belongs to $\{a_1, \ldots, a_s\}$.*

Lemma 3.2.29. *Let G be an HCA graph and let $\mathcal{A}$ be an HCA representation of G. Let $M_1, \ldots, M_k$, with $k \geq 5$, be a set of cliques of G such that $M_i \cap M_{i+1}$ is non-empty for $i = 1, \ldots, k$, and $M_i \cap M_j$ is empty for $j \neq i, i+1, i-1$ (index operations are done modulo k) . Let $S = \{v_1, \ldots, v_k\}$ such that $v_i \in M_{i-1} \cap M_i$. Let $w \in M_i \setminus S$ non-adjacent to v_{i+2}. Then the neighbors of w in S are either $\{v_i, v_{i+1}\}$, or $\{v_{i-1}, v_i, v_{i+1}\}$, or $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}\}$.*

Proof. For $1 \leq i \leq k$ let m_i be an anchor of M_i, let A_i be the arc of $\mathcal{A}$ corresponding to v_i, and let W be the arc corresponding to w. Since for every i, A_i contains m_{i-1} and m_i, and no m_j with $j \neq i-1, i$, it follows that there are only two possible circular orders of the anchors: $m_1, m_2, \ldots, m_k$ and the reverse one. Since w belongs to M_i, it is adjacent to v_i and v_{i+1}, and $m_i \in W$. Since w is non-adjacent to v_{i+2}, w does not belong to M_{i+1}, and $m_{i+1} \notin W$. Since $w \in M_i$ and M_i is disjoint from M_j for $j \neq i-1, i, i+1$, it follows that $m_j \notin W$ for $j \neq i-1, i$ (see Figure 3.18). Now, if $m_{i-1} \notin W$, then the neighbors of w in S are v_i and v_{i+1} or v_{i-1}, v_i, v_{i+1}, and if $m_{i-1} \in W$, then the neighbors of w in W are v_{i-1}, v_i, v_{i+1} or $v_{i-2}, v_{i-1}, v_i, v_{i+1}$. ☐

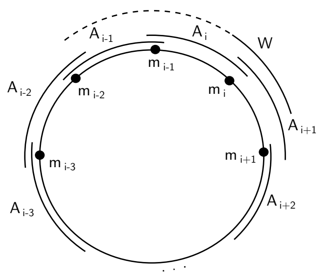

Figure 3.18: Scheme of representation of arcs $A_{i-3}, \ldots, A_{i+2}$ and W, in the proof of Lemma 3.2.29.

Theorem 3.2.30 gives a sufficient condition for the clique graph of an HCA graph to be perfect.

Theorem 3.2.30. *Let G be an HCA graph. If G does not contain any of the graphs in Figure 3.14, then $K(G)$ is perfect.*

Proof. Let G be an HCA graph which does not contain any of the graphs in Figure 3.14, and $\mathcal{A}$ be an HCA representation of G. Assume first that there are two arcs $A_1, A_2 \in \mathcal{A}$ covering the circle, and let v_1, v_2 be the corresponding vertices of G. Then the clique-transversal number of G is at most two, because every anchor of a clique of G is contained in one of A_1, A_2, and therefore every clique contains either v_1 or v_2. Since, by Lemma 1.3.1, the clique covering number of $K(G)$ is less or equal to the clique-transversal number of G, $K(G)$ is the complement of a bipartite graph, and so it is perfect.

So we may assume no two arcs of $\mathcal{A}$ cover the circle, and so by Lemma 3.2.27 no three arcs of $\mathcal{A}$ cover the circle. By Lemma 3.2.25, G is HCH, so $K(G)$ is also HCH [3].

Consequently, if $K(G)$ is not perfect, then it contains an odd hole or $\overline{C_7}$ (for every antihole of length at least eight contains a 2-pyramid, and therefore is not HCH by Theorem 1.3.2).

Suppose first that $K(G)$ contains $\overline{C_7}$. By Theorem 1.3.13, G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain $\overline{C_7}$. Consider the HCA representation $\mathcal{A}'$ of H given by Lemma 3.2.28, and let $v_1, \ldots, v_7$ be vertices inducing $\overline{C_7}$ in H, where the cliques are $\{v_1, v_3, v_5\}$, $\{v_3, v_5, v_7\}$, $\{v_5, v_7, v_2\}$, $\{v_7, v_2, v_4\}$, $\{v_2, v_4, v_6\}$, $\{v_4, v_6, v_1\}$ and $\{v_6, v_1, v_3\}$. That is essentially the unique circular order of the cliques (the other possible order is the reverse one), so the arcs $A_1, \ldots, A_7$ corresponding to $v_1, \ldots, v_7$ must appear in $\mathcal{A}'$ as in Figure 3.19.

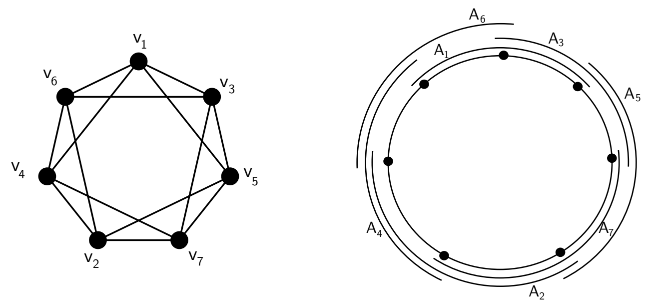

Figure 3.19: HCA representation of $\overline{C_7}$.

If some pair of non-adjacent vertices v_i, v_j in H are adjacent in G, then there are three arcs covering the circle in $\mathcal{A}$, a contradiction. Otherwise $\{v_1, \ldots, v_7\}$ induce $\overline{C_7}$ in G, a contradiction.

Next suppose that $K(G)$ contains C_{2k+1}, for some $k \geq 2$. By Theorem 1.3.13, G contains a clique subgraph H in which identifying twin vertices and then removing dominated vertices we obtain C_{2k+1}, and such that $K(H) = C_{2k+1}$. Consider the HCA representation $\mathcal{A}'$ of H given by Lemma 3.2.28 corresponding to anchors $a_1, \ldots, a_{2k+1}$, and let $v_1, \ldots, v_{2k+1}$ be vertices inducing C_{2k+1} in H, where the cliques are $v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $v_n v_1$. Then in G the graph induced by $v_1, \ldots, v_{2k+1}$ is a cycle, say C, with chords. We assume that $v_1, \ldots, v_{2k+1}$ are chosen to minimize the number N of such chords. Again, that is essentially the unique circular order of the cliques (the other possible order is the reverse one), so the arcs $A'_1, \ldots, A'_{2k+1}$ corresponding to $v_1, \ldots, v_{2k+1}$ must appear in $\mathcal{A}'$ as in Figure 3.20.

Now it is possible that two disjoint arcs $A'_i, A'_j \in \mathcal{A}'$ are derived from arcs $A_i, A_j \in \mathcal{A}$ whose intersection is non-empty, but it follows from Remark 3.2.1 that in this case $|j - i| = 2$ (throughout this proof, indices of vertices in a cycle should be read modulo the length of the cycle). The proof now breaks into cases depending on the values of k and N.

<u>Case $k = 2$</u>:
As there are no three arcs in $\mathcal{A}$ covering the circle, C can have at most one chord incident with each vertex and so $N \leq 2$. The possible HCA-representations of $G|\{v_1, \ldots, v_5\}$

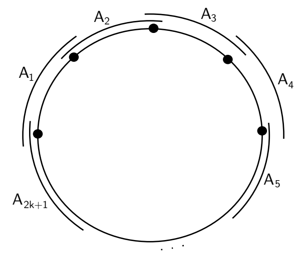

Figure 3.20: HCA representation of C_{2k+1}, $k \geq 2$.

are depicted in Figure 3.21. Let $M_1, \ldots, M_5$ be the cliques of H such that M_1 contains v_1 and v_2, M_2 contains v_2 and v_3, $\ldots$, M_5 contains v_5 and v_1, for $1 \leq i \leq 5$, a_i is an anchor of M_i, and the vertices corresponding to $M_1, M_2, \ldots, M_5$ induce C_5 in $K(G)$.

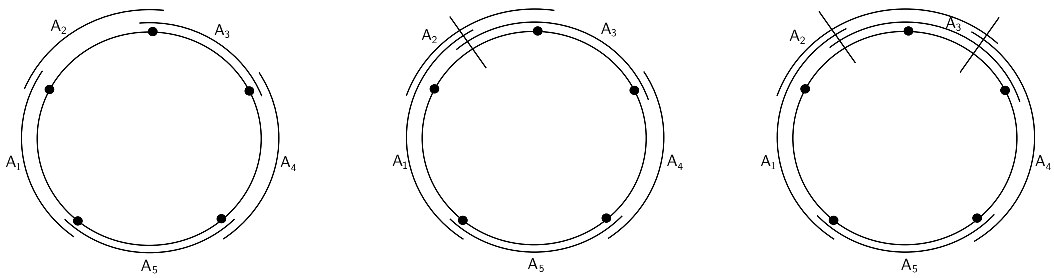

Figure 3.21: Possible cases for $k = 2$, corresponding to no chords, one chord or two chords in the cycle.

1. <u>N=0</u>: In this case G contains an odd hole, a contradiction.

2. <u>N=1</u>: Suppose that the vertices v_1 and v_3 are adjacent in G. As v_3 does not belong to M_1, there is a vertex w_1 in M_1 which is not adjacent to v_3. Analogously, there is a vertex w_2 in M_2 which is not adjacent to v_1. The vertices w_1 and w_2 are non-adjacent, otherwise $\{v_1, v_3, w_2, w_1, v_2\}$ induce a 4-wheel, which does not have an HCA representation with no three arcs covering the circle. For $i = 1, 2$, w_i can have two, three or four neighbors in C.

   2.1. If w_1 and w_2 have two neighbors each one, then $\{v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ w_1,\ w_2\}$ induce a viking.

   2.2. If w_1 and w_2 have four neighbors each one, then $\{v_1,\ w_2,\ w_1,\ v_3,\ v_5,\ v_2,\ v_4\}$ induce $\overline{C_7}$.

   2.3. If one of w_1, w_2 has three neighbors, say w_1, for the other case is symmetric, then if follows from Lemma 3.2.29 that w_1 is adjacent to v_5, v_1, v_2. But now $\{w_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ induce C_5.

2.4. If one of w_1, w_2 has two neighbors and the other one has four neighbors, we may assume that w_1 has two and w_2 has four (the other case is symmetric). The clique M_4 does not intersect M_2, so w_2 does not belong to M_4 and there is a vertex w_3 in M_4 which is not adjacent to w_2.

If the arcs corresponding to w_3 and v_3 intersect in a point of the circle that is between a_3 and a_4, then one of them passes through a point that belongs both to the arc corresponding to v_5 and to the arc corresponding to w_2, but w_3 is non-adjacent to w_2 and v_3 is non-adjacent to v_5, a contradiction. If the arcs corresponding to w_3 and v_3 intersect in a point of the circle between a_1 and a_2, then the arcs corresponding to v_3, v_4 and w_3 cover the circle. So w_3 and v_3 are not adjacent, and w_3 can be adjacent either to v_4, v_5, v_1 and v_2; or to v_4, v_5 and v_1; or only to v_4 and v_5. In the first case, the vertices v_1, w_2, w_3, v_3, v_5, v_2, v_4 induce $\overline{C_7}$. In the second case, the vertices v_1, v_2, w_2, v_4, w_3 induce C_5. In the last case, the eight vertices induce S_2.

3. <u>N=2</u>: The same vertex cannot belong to two chords, so all the cases are symmetric to the case where v_1 is adjacent to v_3 and v_2 to v_4. As v_3 does not belong to M_1, there is a vertex w_1 in M_1 which is not adjacent to v_3. Analogously, as v_2 does not belong to M_3, there is a vertex w_3 in M_3 which is not adjacent to v_2.

   Please note that if w_3 is adjacent to v_1 then their corresponding arcs must intersect in a point of the circle between a_4 and a_5, because w_3 is not adjacent to v_2. But in this case the arcs corresponding to v_1, v_3 and w_3 cover the circle, so w_3 is not adjacent to v_1. Analogously, we can prove that w_1 is not adjacent to v_4.

3.1. If w_1 and w_3 are adjacent, then their corresponding arcs must intersect in a point of the circle between a_4 and a_5, because w_1 is non-adjacent to v_3 and v_4 and w_3 is non-adjacent to v_1 and v_2. So both are adjacent to v_5, and the vertices v_1, v_4, w_1, v_3, v_5, v_2, w_3 induce $\overline{C_7}$.

3.2. If w_1 and w_3 are not adjacent but both of them are adjacent to v_5, the vertices w_1, v_2, v_3, w_3, v_5 induce C_5.

3.3. The remaining case is when w_1 and w_3 are not adjacent but at most one of them is adjacent to v_5.

For this case, we have to consider the clique M_2. Since v_1 and v_4 do not belong to M_2, there is a vertex in M_2 which is not adjacent to v_1 and there is a vertex in M_2 which is not adjacent to v_4.

3.3.1. If there is a vertex w which is non-adjacent to v_1 and v_4, then w cannot be adjacent either to w_1 or w_3, otherwise $\{v_1, v_3, w, w_1, v_2\}$ (or $\{v_2, w, w_3, v_4, v_3\}$, respectively) induce a 4-wheel, a contradiction by Lemma 3.2.26.

Therefore, if each of w_1 and w_3 has two neighbors in C, then the vertices $v_1, \ldots, v_5, w_1, w, w_3$ induce a 2-viking in G, and, if w_1 and w_3 have two and three neighbors (respectively) in C, the vertices $v_1, v_2, v_3, w_3, v_5, w_1, w$ induce a viking in G (the case when w_1 has three neighbors and w_3 has two neighbors in C is symmetric).

3.3.2. If there is no such a vertex w, every vertex of M_2 is either adjacent to v_1 or to v_4. Then there exist two vertices w_2 and w_4 in M_2, such that w_2 is adjacent to v_4 but not to v_1 and w_4 is adjacent to v_1 but not to v_4. Since by Lemma 3.2.26 G does not contain a 4-wheel, it follows that w_2 is not adjacent to w_1 and w_4 is not adjacent to w_3. If neither w_4 nor w_2 is adjacent to v_5, then the vertices v_1, w_4, w_2, v_4, v_5 induce C_5. If w_2 and w_4 are both adjacent to v_5, then the arcs corresponding to w_2, w_4 and v_5 cover the circle. Otherwise, suppose w_2 is adjacent to v_5 and w_4 is not (the other case is symmetric), so by the circular-arc representation w_2 belongs to M_3, and it is adjacent to w_3.

In this case w_2 is a twin of v_3 in H. Consider the hole $v_1v_2w_2v_4v_5v_1$ of H, say C'. In G $\{v_1, v_2, w_2, v_4, v_5\}$ induces a cycle with two chords, v_2v_4 and w_2v_5. If vertex w_3 has only two neighbors in C, then it has two neighbors in C', namely w_2 and v_4, and it is non-adjacent to v_2 and v_5, so we get a contradiction by a previous case (Case 3.3.1).

The last case is when w_3 has three neighbors in C and w_1 has only two. If w_3 belongs to M_4 then w_3 and v_4 are twins in H, but the cycle of H obtained by replacing v_4 with w_3 in C has only one chord in G, contrary to the choice of C.

If w_3 does not belong to M_4, let w_5 be a vertex of M_4, that minimizes the distance of the endpoint of its corresponding arc that lies between a_3 and a_4, to a_4. Since none of w_2, v_3, w_3 belongs to M_4, they are not adjacent to w_5. The set of neighbors of w_5 in C includes $\{v_4, v_5\}$ and, by Lemma 3.2.29, is a subset of $\{v_1, v_2, v_4, v_5\}$. If w_5 is adjacent to v_1 and v_2, then the arcs corresponding to vertices v_2, v_4 and w_5 cover the circle. If w_5 is adjacent to v_1 but not to v_2, then the vertices v_1, w_4, w_2, v_4, w_5 induce C_5. If w_5 has only two neighbors in C (v_4 and v_5), then w_1 and w_5 are non-adjacent, because w_1 is non-adjacent to v_5 and w_5 is non-adjacent to v_1. Now if w_4 and w_1 are non-adjacent, then the vertices $\{v_1, \ldots, v_5, w_1, \ldots, w_5\}$ induce T_2, otherwise, the eight vertices $v_1, w_4, v_3, v_4, v_5, w_1, w_2, w_5$ induce S_2.

Case $k \geq 3$: Let $M_1, \ldots, M_{2k+1}$ be the cliques of H such that M_1 contains v_1 and v_2, M_2 contains v_2 and v_3, ..., M_{2k+1} contains v_{2k+1} and v_1, for $1 \leq i \leq 2k+1$, a_i is an anchor of M_i, and the vertices corresponding to $M_1, M_2, \ldots, M_{2k+1}$ induce C_{2k+1} in $K(G)$. We remind the reader that if v_i is adjacent to v_j in G, then $|i - j| \leq 2$.

If $N = 0$, then G contains an odd hole, one of the forbidden subgraphs of Figure 3.14. If $N = 1$, say v_1v_3 is a chord of C, then the arcs corresponding to v_1 and v_3 intersect in some point of the circle that is between a_1 and a_2. The vertices v_1, v_2 and v_3 belong to some clique M of G, distinct from M_i for $i = 1, \ldots, 2k+1$. Every anchor of M is between a_1 and a_2, every vertex of M which is not in H is only adjacent to vertices of H belonging to M_1 or M_2 (their corresponding arcs are bounded by a_1 and a_2), and every vertex of M in H belongs to M_1 or M_2. Both M_1 and M_2 are disjoint from $M_4, \ldots, M_{2k}$, so M is disjoint from $M_4, \ldots, M_{2k}$. But the vertex v_1 belongs to $M \cap M_{2k+1}$ and vertex v_3 belongs to $M \cap M_3$, and therefore $M, M_3, M_4, \ldots, M_{2k}, M_{2k+1}$

induce C_{2k} in $K(G)$.

Repeating this argument twice (we do not use the fact that the cycle is odd, but only the fact that it has at least six vertices), if there exist two chords v_iv_{i+2} and v_jv_{j+2} in C such that v_iv_{i+1}, $v_{i+1}v_{i+2}$, v_jv_{j+1} and $v_{j+1}v_{j+2}$ are four distinct edges of G, we can reduce the problem to a smaller one, the case of an odd hole with $2k-1$ vertices induced in $K(G)$.

So we only need to consider two cases:

- $N = 1$; and
- $N = 2$, and for some i, v_i is adjacent to v_{i+2} and v_{i+1} is adjacent to v_{i+3}.

1. N=1: Suppose that the vertices v_1 and v_3 are adjacent in G. As v_3 does not belong to M_1, there is a vertex w_1 in M_1 which is not adjacent to v_3. Analogously, there is a vertex w_2 in M_2 which is not adjacent to v_1. The vertices w_1 and w_2 are non-adjacent, otherwise $\{v_1, v_3, w_2, w_1, v_2\}$ induces a 4-wheel, contrary to Lemma 3.2.26. By Lemma 3.2.29 the vertex w_1 has two, three or four neighbors in C and they are consecutive in it (v_2 and v_1; or v_2, v_1 and v_{2k+1}; or v_2, v_1, v_{2k+1} and v_{2k}, respectively). Analogously, w_2 has two, three or four neighbors in C and they are consecutive in the cycle (v_2 and v_3; or v_2, v_3 and v_4; or v_2, v_3, v_4 and v_5, respectively). In all cases w_1 and w_2 have no common neighbors in $V(C)\setminus\{v_2\}$, since $k \geq 3$.

   1.1. If w_1 and w_2 have exactly two neighbors each one in C, the vertices $v_1, \ldots, v_{2k+1}, w_1, w_2$ induce a viking.

   1.2. If w_1 and w_2 have exactly four neighbors each one in C, the vertices w_1, v_2, w_2, $v_5, \ldots, v_{2k}$ induce C_{2k-1}.

   1.3. If one of w_1, w_2 has exactly three neighbors in C (suppose w_1, the other case is symmetric), the vertices $w_1, v_2, v_3, \ldots, v_{2k+1}$ induce C_{2k+1}.

   1.4. If one of w_1, w_2 has exactly two neighbors in C and the other one has exactly four neighbors in C, suppose w_1 has two and w_2 has four (the other case is symmetric). The clique M_4 is disjoint from M_2, so w_2 does not belong to M_4 and there is a vertex w_3 in M_4 which is not adjacent to w_2.

   The arc corresponding to w_3 cannot pass through the points of the circle corresponding either to M_3 (because w_2 and w_3 are not adjacent) or to M_6 (because M_4 and M_6 are disjoint), so if the arcs corresponding to w_3 and v_3 have non-empty intersection, they must intersect at a point of the circle that is between a_3 and a_4. In this case one of them passes through a point that belongs to both the arc corresponding to v_5 and the arc corresponding to w_2, but w_3 is non-adjacent to w_2, and v_3 is non-adjacent to v_5. So w_3 and v_3 are not adjacent, and, by Lemma 3.2.29, w_3 can be adjacent either to v_4, v_5, v_6 and v_7; or to v_4, v_5 and v_6; or only to v_4 and v_5. In the first case, the vertices $v_1, v_3, v_4, w_3, v_7, \ldots, v_{2k+1}$ induce C_{2k-1}. In the second case, the vertices $v_1, v_2, w_2, v_4, w_3, v_6, \ldots, v_{2k+1}$

induce C_{2k+1}. In the last case, w_3 is non-adjacent to w_1 because both are non-adjacent to v_6, hence the $2k+4$ vertices $v_1, \ldots, v_{2k+1}, w_1, w_2, w_3$ induce S_k.

2. N=2, and for some i, v_i is adjacent to v_{i+2} and v_{i+1} is adjacent to v_{i+3}:
Without loss of generality, we may assume that $i = 1$, so the chords are v_1v_3 and v_2v_4. As v_3 does not belong to M_1, there is a vertex w_1 in M_1 which is not adjacent to v_3. As v_2 does not belong to M_3, there is a vertex w_3 in M_3 which is not adjacent to v_2. No vertex of G belongs to more than two cliques of $M_1, \ldots, M_{2k+1}$. These facts imply that the vertices w_1 and w_3 are non-adjacent, and, by Lemma 3.2.29, each of them has two, three or four consecutive neighbors in C. The vertex w_3 can be adjacent to v_3, v_4, v_5 and v_6; or to v_3, v_4 and v_5; or only to v_3 and v_4. The vertex w_1 can be adjacent to v_2, v_1, v_{2k+1} and v_{2k}; or to v_2, v_1 and v_{2k+1}; or only to v_2 and v_1.

2.1. If w_3 has four neighbors in C, then the vertices v_1, v_3, w_3, v_6, $\ldots$, v_{2k+1} induce C_{2k-1}. The case of w_1 having four neighbors is symmetric.

2.2. If w_1 and w_3 have three neighbors each one in C, then the vertices w_1, v_2, v_3, w_3, $v_5, \ldots, v_{2k+1}$ induce C_{2k+1}.

2.3. It remains to analyze the cases when w_1 and w_3 each have two neighbors in C, and when one of them has three neighbors in C and the other one has two. For these cases, we have to consider the clique M_2.

Since v_1 and v_4 do not belong to M_2, there is a vertex in M_2 which is not adjacent to v_1 and there is a vertex in M_2 which is not adjacent to v_4.

2.3.1. If there is a vertex $w \in M_2$ which is non-adjacent to v_1 and v_4, then w is non-adjacent to w_1 and w_3, for otherwise $\{v_1, v_3, w, w_1, v_2\}$ (or $\{v_2, w, w_3, v_4, v_3\}$, respectively) induces a 4-wheel, contrary to Lemma 3.2.26. Therefore, if w_1 and w_3 have two neighbors each in C, then the vertices $v_1, \ldots, v_{2k+1}, w_1, w, w_3$ induce a 2-viking in G. If w_1 and w_3 have two and three neighbors (respectively) in C, then v_1, v_2, v_3, w_3, $v_5, \ldots$, v_{2k+1}, w_1, w induce a viking in G. If w_1 has three neighbors and w_3 has two neighbors in C, then w_1, v_2, $v_3, \ldots$, v_{2k+1}, w, w_3 induce a viking in G.

2.3.2. If no such a vertex w exists, then every vertex of M_2 is either adjacent to v_1 or to v_4, and there exist two vertices w_2 and w_4 in M_2, such that w_2 is adjacent to v_4 but not to v_1 and w_4 is adjacent to v_1 but not to v_4. Since G does not contain a 4-wheel, it follows that w_2 is not adjacent to w_1 and w_4 is not adjacent to w_3. If w_4 is not adjacent to v_{2k+1} and w_2 is not adjacent to v_5, then the vertices v_1, w_4, w_2, v_4, $\ldots, v_{2k+1}$ induce C_{2k+1}. If w_4 is adjacent to v_{2k+1} and w_2 is adjacent to v_5, then the vertices $w_4, w_2, v_5, \ldots, v_{2k+1}$ induce C_{2k-1}. Otherwise, suppose w_2 is adjacent to v_5 and w_4 is not adjacent to v_{2k+1} (the other case is symmetric), so since G is a circular-arc graph, w_2 belongs to M_3, and it is adjacent to w_3. In this case w_2 is a twin of v_3 in H. Consider the hole $v_1v_2w_2v_4 \ldots v_{2k+1}v_1$,

say C', in H. The graph induced by $\{v_1, v_2, w_2, v_4, \ldots, v_{2k+1}\}$ in G is a cycle with two chords, v_2v_4 and w_2v_5. If the vertex w_3 has exactly two neighbors in C, then it has exactly two neighbors in C', namely w_2 and v_4, and it is non-adjacent to v_2 and v_5, and we get a contradiction by a previous case (Case 2.3.1).
The last case is when w_3 has three neighbors in the cycle and w_1 has only two. If w_3 belongs to M_4 then w_3 and v_4 are twins in H, but the cycle of H obtained by replacing v_4 with w_3 in C has only one chord in G, contrary to the choice of C.
If w_3 does not belong to M_4, let w_5 be a vertex of M_4, that minimizes the distance of the endpoint of its corresponding arc that lies between a_3 and a_4, to a_4. Since w_2, v_3, w_3 do not belong to M_4, they are not adjacent to w_5. The neighbor set of the vertex w_5 includes $\{v_4, v_5\}$ and, by Lemma 3.2.29, is a subset of $\{v_4, v_5, v_6, v_7\}$. If w_5 is adjacent to v_6 and v_7, then the vertices $v_1, v_3, v_4, w_5, v_7, \ldots, v_{2k+1}$ induce C_{2k-1}. If w_5 is adjacent to v_6 but not to v_7, then the vertices $v_1, w_4, w_2, v_4, w_5, v_6, \ldots, v_{2k+1}$ induce C_{2k+1}. So we may assume that v_4 and v_5 are the only neighbors of w_5 in C. But now, if w_4 and w_1 are not adjacent, then the vertices $v_1, \ldots, v_{2k+1}, w_1, \ldots, w_5$ induce T_k, and otherwise, the $2k + 4$ vertices $v_1, w_4, v_3, \ldots, v_{2k+1}, w_1, w_2, w_5$ induce S_k.

In each case we get a contradiction. This concludes the proof. □

We can now prove the characterization theorem for HCA graphs.

Proof of Theorem 3.2.5. The "only if" part follows from Theorem 3.1.1, Proposition 3.1.2 and Proposition 3.1.3. Let us prove the "if" statement. Let G be an HCA graph which does not contain any of the graphs in Figure 3.14, and let $\mathcal{A}$ be an HCA representation of G. Since the class of HCA graph is hereditary, it is enough to prove that $\tau_c(G) = \alpha_c(G)$.

Assume first that some two arcs of $\mathcal{A}$ cover the circle. Then $\tau_c(G) \leq 2$. If $\tau_c(G) = 1$ or $\alpha_c(G) = 2$, then $\alpha_c(G) = \tau_c(G)$ and the theorem holds. So we may assume that $\tau_c(G) = 2$ and $\alpha_c(G) = 1$. By Theorem 3.1.2, G contains Q_{2k+1} for some $k \geq 1$. It is not difficult to check that the 3-pyramid is not an HCA graph. Moreover, $\overline{C_{2k+1}}$ (an induced subgraph of Q_{2k+1}) contains the 3-pyramid for $k \geq 4$. So, G contains either Q_3, or Q_5, or Q_7. But Q_3 is the 3-sun, Q_5 contains C_5 and Q_7 contains $\overline{C_7}$, a contradiction.

So we may assume that no two arcs of $\mathcal{A}$ cover the circle. But now, by Lemma 3.2.25 and Theorem 3.2.30, G is clique-Helly and K-perfect, and so, by Lemma 1.3.1, $\tau_c(G) = \alpha_c(G)$. □

It is easy to check that no two graphs of the families represented in Figure 3.14 are properly contained in each other. Therefore, as a corollary of Theorem 3.2.5, we obtain the following result.

Corollary 3.2.30.1. *Vikings, 2-vikings, S_k and T_k ($k \geq 2$), are minimally clique-imperfect.*

3.3 Recognition algorithms

Chordal graphs can be recognized in polynomial time [64]. On the other hand, Theorem 2.2.2 and Theorem 3.2.1 imply that the recognition problem for clique-perfect chordal graphs can be reduced to the recognition of balanced graphs, which is solvable in polynomial time (Corollary 2.1.1.3).

The recognition problem for line graphs can be solved in polynomial time [54]. By Theorem 3.2.10, the recognition of clique-perfect line graphs can be reduced to the recognition of perfect graphs with no 3-sun, which is solvable in polynomial time [23].

By Theorem 3.2.24, the recognition of clique-perfect HCH claw-free graphs can be also reduced to the recognition of perfect graphs.

Helly circular-arc graphs can be recognized in polynomial time [45] and, given a Helly representation of an HCA graph G, both parameters $\tau_c(G)$ and $\alpha_c(G)$ can be computed in linear time [38, 39]. However, these properties do not immediately imply the existence of a polynomial time recognition algorithm for clique-perfect HCA graphs, because we need to check the equality for every induced subgraph. The characterization in Theorem 3.2.5 leads to such an algorithm, which is strongly based on the recognition of perfect graphs. The algorithm is based on the ideas applied in [35] for recognizing balanceable matrices.

Algorithm:

Input: An HCA graph $G = (V, E)$.
Output: True if G is clique-perfect, False if G is not.

1. Check if G contains a 3-sun. If G contains a 3-sun, return False.
2. (*Checking for odd holes and $\overline{C_7}$*) Check if G is perfect. If G is not perfect, return False.
3. (*Checking for vikings*) For every 7-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2$ such that the edges between those vertices in G are a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_4a_5, b_1a_2, b_1a_3, b_2a_3, b_2a_4, and possibly a_1a_5, do the following:
   (a) If a_1 is adjacent to a_5, return False.
   (b) Let G' be the graph obtained from G by removing the vertices a_2, a_3, a_4, b_1, b_2 and all their neighbors except for a_1 and a_5, and adding a new vertex c adjacent only to a_1 and a_5.
   (c) Check if G' is perfect. If G' is not perfect, return False.

4. (*Checking for* 2-*vikings*) For every 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$ such that the edges between those vertices in G are a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_3a_5, a_4a_5, b_1a_2, b_1a_3, b_2a_3, b_2a_4, b_3a_4 and b_3a_5, do the following:
   (a) If a_1 is adjacent to a_5, return FALSE.
   (b) Let G' be the graph obtained from G by removing the vertices a_2, a_3, a_4, b_1, b_2, b_3 and all their neighbors except for a_1 and a_5, and adding a new vertex c adjacent only to a_1 and a_5.
   (c) Check if G' is perfect. If G' is not perfect, return FALSE.
5. (*Checking for* S_k) For every 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$ such that the edges between those vertices in G are a_1a_2, a_2a_3, a_3a_4, a_3a_5, a_4a_5, b_1a_1, b_1a_2, b_2a_4, b_2a_5, b_3a_1, b_3a_2, b_3a_3, b_3a_4, and possibly a_1a_5, do the following:
   (a) If a_1 is adjacent to a_5, return FALSE.
   (b) Let G' be the graph obtained from G by removing the vertices a_2, a_3, a_4, b_1, b_2, b_3 and all their neighbors except for a_1 and a_5, and adding a new vertex c adjacent only to a_1 and a_5.
   (c) Check if G' is perfect. If G' is not perfect, return FALSE.
6. (*Checking for* T_k) For every 10-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5$ such that the edges between those vertices in G are a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_3a_5, a_4a_5, b_1a_1, b_1a_2, b_2a_1, b_2a_2, b_2a_3, b_2b_3, b_3a_1, b_3a_2, b_3a_3, b_3a_4, b_3b_4, b_4a_3, b_4a_4, b_4a_5, b_5a_4, b_5a_5, and possibly a_1a_5, do the following:
   (a) If a_1 is adjacent to a_5, return FALSE.
   (b) Let G' be the graph obtained from G by removing the vertices a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5 and all their neighbors except for a_1 and a_5, and adding a new vertex c adjacent only to a_1 and a_5.
   (c) Check if G' is perfect. If G' is not perfect, return FALSE.
7. Return TRUE.

Correctness: The output of the algorithm is TRUE if it finishes in step (7), otherwise the output is FALSE. Let us prove that, given as input an HCA graph G, the algorithm finishes in step (7) if and only if G does not contain the graphs of Figure 3.14. The correctness of the algorithm then follows from Theorem 3.2.5.

Let G be an HCA graph. Step (1) will output FALSE if and only if G contains a 3-sun. So henceforth suppose that G does not contain a 3-sun.

1. *Step* (2) *will output* FALSE *if and only if* G *contains an odd hole or* $\overline{C_7}$.

If G contains an odd hole or $\overline{C_7}$ then it is not perfect. Conversely, if G is not perfect it contains an odd hole or an odd antihole. Since G is HCA, it does not contain an antihole of length at least nine. So G must contain an odd hole or $\overline{C_7}$. This proves 1. So henceforth suppose that G is perfect, and, in particular, it does not contain an odd hole or $\overline{C_7}$.

2. *Step* (3) *will output* FALSE *if and only if* G *contains a viking.*

If G contains a viking H with $V(H) = \{a_1, \ldots, a_{2k+1}, b_1, b_2\}$ and adjacencies as defined in Section 3.1, at some point the algorithm will consider the 7-tuple a_1, a_2, a_3, a_4, a_5, b_1, b_2. In H, either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (3.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5a_6 \ldots a_{2k+1}a_1$. Since $a_6, \ldots, a_{2k+1}$ are non-neighbors of a_2, a_3, a_4, b_1, b_2, it follows that $ca_5a_6 \ldots a_{2k+1}a_1c$ is an odd hole in G', so the algorithm will output FALSE at step (3.c).

Conversely, if the algorithm outputs FALSE at step (3.a), then $\{a_1, \ldots, a_5, b_1, b_2\}$ induce a viking in G. If the algorithm outputs FALSE at step (3.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G'. Since it has degree two, c belongs to an odd hole $ca_5v_1 \ldots v_{2t}a_1c$ in G'. Since $v_1, \ldots, v_{2t}$ are different from and non-adjacent to a_2, a_3, a_4, b_1, b_2, it follows that $\{a_1, \ldots, a_5, v_1, \ldots, v_{2t}, b_1, b_2\}$ induce a viking in G. This proves 2. So henceforth suppose that G contains no viking.

3. *Step* (4) *will output* FALSE *if and only if* G *contains a* 2*-viking.*

If G contains a 2-viking H with $V(H) = \{a_1, \ldots, a_{2k+1}, b_1, b_2, b_3\}$ and adjacencies as defined in Section 3.1, at some point the algorithm will consider the 8-tuple a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3. In H, either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (4.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5a_6 \ldots a_{2k+1}a_1$. Since $a_6, \ldots, a_{2k+1}$ are non-neighbors of a_2, a_3, a_4, b_1, b_2, b_3, it follows that $ca_5a_6 \ldots a_{2k+1}a_1c$ is an odd hole in G', so the algorithm will output FALSE at step (4.c).

Conversely, if the algorithm outputs FALSE at step (4.a), then $\{a_1, \ldots, a_5, b_1, b_2, b_3\}$ induce a 2-viking in G. If the algorithm outputs FALSE at step (4.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G'. Since it has degree two, c belongs to an odd hole $ca_5v_1 \ldots v_{2t}a_1c$ in G'. Since $v_1, \ldots, v_{2t}$ are different from and non-adjacent to a_2, a_3, a_4, b_1, b_2, b_3, it follows that $a_1, \ldots, a_5, v_1, \ldots, v_{2t}, b_1, b_2, b_3$ induce a 2-viking in G. This proves 3. So henceforth suppose that G contains no 2-viking.

4. *Step* (5) *will output* FALSE *if and only if* G *contains* S_k *for some* $k \geq 2$.

If G contains S_k for some $k \geq 2$, with $V(S_k) = \{a_1, \ldots, a_{2k+1}, b_1, b_2, b_3\}$ and adjacencies as defined in Section 3.1, at some point the algorithm will consider the 8-tuple $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3$. In S_k, either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (5.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5a_6 \ldots a_{2k+1}a_1$. Since $a_6, \ldots, a_{2k+1}$ are non-neighbors of $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that $ca_5a_6 \ldots a_{2k+1}a_1c$ is an odd hole in G', so the algorithm will output FALSE at step (5.c).

Conversely, if the algorithm outputs FALSE at step (5.a), then vertices $\{a_1, \ldots, a_5, b_1, b_2, b_3\}$ induce S_2 in G. If the algorithm outputs FALSE at step (5.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G'. Since it has degree two, c belongs to an odd hole $ca_5v_1 \dots v_{2t}a_1c$ in G'. Since $v_1, \dots, v_{2t}$ are different from and non-adjacent to $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that vertices $\{a_1, \dots, a_5, v_1, \dots, v_{2t}, b_1, b_2, b_3\}$ induce S_{t+2} in G. This proves 4. So henceforth suppose that G does not contain S_k for $k \geq 2$.

5. *Step* (6) *will output* FALSE *if and only if* G *contains* T_k *for some* $k \geq 2$.

If G contains T_k for some $k \geq 2$, with $V(T_k) = \{a_1, \dots, a_{2k+1}, b_1, b_2, b_3, b_4, b_5\}$ and adjacencies as defined in Section 3.1, at some point the algorithm will consider the 10-tuple $a_1, \dots, a_5, b_1, \dots, b_5$. In T_k, either $k = 2$ and a_1 is adjacent to a_5 (in this case the algorithm will output FALSE at step (6.a)) or a_5 and a_1 are joined by an odd path of length at least three, $a_5a_6 \dots a_{2k+1}a_1$. Since $a_6, \dots, a_{2k+1}$ are non-neighbors of $a_2, a_3, a_4, b_1, b_2, b_3$, it follows that $ca_5a_6 \dots a_{2k+1}a_1c$ is an odd hole in G', so the algorithm will output FALSE at step (6.c).

Conversely, if the algorithm outputs FALSE at step (6.a), then vertices $\{a_1, \dots, a_5, b_1, \dots, b_5\}$ induce S_2 in G. If the algorithm outputs FALSE at step (6.c), then G' is not perfect. Since at this point we are assuming that G is perfect, the vertex c must belong to an odd hole or odd antihole in G'. Since it has degree two, c belongs to an odd hole $ca_5v_1 \dots v_{2t}a_1c$ in G'. Since $v_1, \dots, v_{2t}$ are different from and non-adjacent to $a_2, a_3, a_4, b_1, \dots, b_5$, it follows that $\{a_1, \dots, a_5, v_1, \dots, v_{2t}, b_1, \dots, b_5\}$ induce T_{t+2} in G. This proves 5, and completes the proof of correctness. □

Time complexity: The time complexity of the best known algorithm to recognize perfect graphs is $O(|V|^9)$ [23]. So the time complexity of this algorithm is given by step (6) and it is $O(|V|^{19})$.

Thus we can affirmatively answer the question of the existence of a polynomial time recognition algorithm for clique-perfect graphs within the class of HCA graphs.

CHAPTER 4

Conclusions

In this thesis we mainly work on clique-perfect graphs, a variation of perfect graphs. We study in particular a class of graphs in the intersection of perfect and clique-perfect graphs: balanced graphs.

A graph is balanced when its clique matrix is balanced. We give two new characterizations of balanced graphs, the first one by forbidden subgraphs (Theorem 2.2.3) and the second one by clique subgraphs (Theorem 2.2.4).

Using properties of domination we define four subclasses of balanced graphs: VV, VE, EE and EV graphs. We analyze the inclusion relations between them. Classes VV and VE are characterized using 0-1 matrices and the characterizations lead to polynomial time recognition algorithms. We also study the behavior of the clique graph operator on balanced graphs and these four subclasses. Some of these classes are fixed under the clique graph operator, while some others have a clique-dual class of graphs, as Table 4.1 shows.

Class $\mathcal{A}$	$K(\mathcal{A})$	**Reference**
Balanced	Balanced	[56]
DEE	EE	Cor 2.4.10.1
DVE	VE	Cor 2.4.10.2
EE	DEE	Cor 2.4.10.1
EV	K^{-1}(bipartite)	Cor 2.4.12.2
Totally Unimodular	Totally Unimodular	Cor 2.4.6.1
VE	DVE	Cor 2.4.10.2
VV	K^{-1}(bipartite)	Cor 2.4.12.2

Table 4.1: Clique graphs of subclasses of balanced graphs.

As a corollary of these results, we deduce the existence of polynomial time combinatorial algorithms for the maximum stable set, maximum clique-independent set and the minimum clique-transversal problems for VV graphs.

Results in Chapter 3 allow us to formulate partial characterizations of clique-perfect graphs by forbidden subgraphs, as Table 4.2 shows. Some of these characterizations also lead to a polynomial time recognition algorithm for clique-perfect graphs within the analyzed class.

Graph classes	**Forbidden subgraphs**	**Recognition**	**Reference**
Chordal graphs	odd suns	**P**	[53, 32]
Diamond-free graphs	odd generalized suns	**?**	Thm 3.2.2
Line graphs	odd holes, 3-sun	**P**	Thm 3.2.3
HCH claw-free graphs	odd holes, $\overline{C_7}$	**P**	Thm 3.2.4
HCA graphs	3-sun, odd holes, $\overline{C_7}$, vikings, 2-vikings, S_k, T_k	**P**	Thm 3.2.5

Table 4.2: Known partial characterizations of clique-perfect graphs by forbidden induced subgraphs, and computational complexity of the recognition problem.

Note that in the last three cases all the forbidden induced subgraphs are minimal. In the second case, however, we need to forbid every odd-generalized sun. Obviously, in this case it is enough to forbid diamond-free odd generalized suns. It is easy to see that all such suns have no improper edges but we do not yet know what the minimal diamond-free odd generalized suns are. It also remains as an open question the complexity of the recognition of clique-perfect diamond-free graphs.

Finally, it is also shown in Chapter 3 that for the graph classes in Table 4.2, clique-perfect graphs are both perfect and K-perfect, that is, their clique graphs are also perfect.

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