Acotaciones con pesos para la integral fraccionaria de funciones radiales y sus aplicaciones

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Resumen

En esta tesis estudiamos acotaciones con pesos de la integral fraccionaria (también llamada potencial de Riesz)

$$(T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n$$

en el caso en que $v(x) = v_0(|x|)$ es una función radial de $\mathbb{R}^n$. En particular, demostramos que restringiendo el operador a funciones radiales, el rango de pesos potencia admisibles para que valga una desigualdad del tipo

$$\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

es estrictamente mayor del que se obtiene cuando se consideran funciones cualesquiera de $L^p(\mathbb{R}^n, |x|^{\alpha p})$.

Luego mostramos que este resultado tiene aplicaciones directas en problemas de ecuaciones diferenciales, y en el estudio de otros operadores clásicos del análisis armónico. Más precisamente, nos concentramos en tres aplicaciones:

- Obtenemos un resultado de compacidad para la inmersión de las funciones radiales de los espacios de Sobolev fraccionarios $H^s(\mathbb{R}^n)$ en espacios $L^q(\mathbb{R}^n, |x|^c)$ de utilidad en el estudio de sistemas Hamiltonianos con pesos en $\mathbb{R}^n$.

- Obtenemos mejoras para desigualdades de tipo Caffarelli-Kohn-Nirenberg y de trazas en el caso de funciones radiales.

- Obtenemos estimaciones con pesos potencias para multiplicadores de tipo transformada de Laplace para desarrollos en funciones de Laguerre y Hermite, y mostramos cómo se pueden obtener estimaciones con pesos de tipo $A_{p,q}$ para estos multiplicadores en el caso de Laguerre para ciertos valores de α.

Palabras clave: Integral fraccionaria, potencial de Riesz, funciones radiales, estimaciones con pesos, multiplicadores de Laguerre, sistemas hamiltonianos, desigualdad de Caffarelli-Kohn-Nirenberg, desigualdad de trazas.

Weighted inequalities for fractional integrals of radial functions and applications

Abstract

This thesis deals with weighted estimates for the fractional integral (also known as Riesz potential)

$$(T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n$$

in the case when $v(x) = v_0(|x|)$ is a radial function in $\mathbb{R}^n$. In particular, we prove that if we restrict the operator to the subspace of radially symmetric functions, the range of admissible power weights for the inequality

$$\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

is strictly larger than that obtained when considering arbitrary functions in $L^p(\mathbb{R}^n, |x|^{\alpha p})$.

We then show that this result has direct applications in problems in partial differential equations, and in the study of other classical operators in harmonic analysis. More precisely, we concentrate on three applications:

- We obtain a compactness result for the imbedding of radial functions of fractional Sobolev spaces $H^s(\mathbb{R}^n)$ in weighted spaces $L^q(\mathbb{R}^n, |x|^c)$, useful in the study of Hamiltonian elliptic systems with weights in $\mathbb{R}^n$.

- We obtain improvements for Caffarelli-Kohn-Nirenberg-type and trace inequalities in the case of radial functions.

- We obtain estimates with power weights for multipliers of Laplace transform type for Laguerre and Hermite expansions, and show how $A_{p,q}$-type weighted estimates can be obtained for these multipliers in the Laguerre setting for certain values of α.

Key words: fractional integrals, Riesz potentials, radial functions, weighted estimates, Laguerre multipliers, Hamiltonian systems, Caffarelli-Kohn-Nirenberg inequalities, trace inequalities.

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Introducción

Acotaciones con pesos para la integral fraccionaria

El estudio de acotaciones para la integral fraccionaria (también llamada potencial de Riesz)

$$(T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n.$$

es un problema clásico de análisis, de gran importancia por sus aplicaciones, ya que este operador permite dar una representación integral a las potencias negativas del Laplaciano, e interviene en la demostración clásica de los teoremas de inmersión de Sobolev (ver, por ejemplo, [41]).

La teoría de desigualdades con pesos para este operador se remonta al trabajo de G. H. Hardy y J. E. Littlewood [20] en en caso unidimensional, mientras que la generalización a $\mathbb{R}^n, n \geq 1$, corresponde a E. M. Stein y G. Weiss, quienes en [42] obtuvieron el siguiente teorema:

Sean $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq 0$ y $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$. Si $p \leq q < \infty$ entonces la desigualdad

$$\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

vale para toda $v \in L^p(\mathbb{R}^n, |x|^{p\alpha}dx)$, donde C es una constante independiente de v.

Desigualdades para pesos más generales fueron después estudiadas por diferentes autores, hasta la obtención por parte de E. T. Sawyer y R. L Wheeden en [37] de la caracterización de tipo $A_{p,q}$ para los pesos admisibles. De esta teoría puede deducirse, en particular, que para pesos potencia el resultado de Stein y Weiss no puede ser mejorado en general.

Sin embargo, si nos restringimos al espacio de funciones con simetría radial, es posible obtener un rango más amplio de exponentes para los cuales la integral fraccionaria es continua con pesos potencia. Más precisamente, probamos en esta tesis que vale el siguiente resultado:

Teorema 0.1. *Sean $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$ y $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$. Si $p \leq q < \infty$ entonces la desigualdad*

$$\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

vale para toda función radial $v \in L^p(\mathbb{R}^n, |x|^{p\alpha} dx)$, donde C es una constante independiente de v. Y el mismo resultado vale en el caso $p = 1$ reemplazando la condición sobre $\alpha + \beta$ por $\alpha + \beta > (n-1)(\frac{1}{q} - 1)$.

Con posterioridad a la publicación del resultado contenido en esta tesis, encontramos el mismo resultado en el trabajo [33] de B. S. Rubin. Sin embargo, dicho trabajo no contempla el caso $p = 1$ y su método de demostración es completamente distinto del nuestro, ya que se basa en propiedades de ciertas funciones hipergeométricas, mientras que nuestra prueba utiliza simplemente la desigualdad de Young en el grupo multiplicativo $(\mathbb{R}_+, \cdot)$ con la medida de Haar dx/x en combinación con estimaciones elementales sobre el comportamiento del núcleo.

Otros resultados previos en el caso de funciones con simetría radial (posteriores a [33] pero que también parecen haber desconocido ese trabajo) son los de M. C. Vilela, quien realizó una demostración para el caso $p < q$ y $\beta = 0$ en [47]; y el trabajo de K. Hidano y Y. Kurokawa [22], quienes demostraron la acotación en el caso $p < q$ con la restricción adicional $\beta < \frac{1}{q}$. Esta restricción, junto con las condiciones adicionales en α y β implican nuestro mismo teorema para el caso $n - 1 < \gamma < n$, mientras que nuestro resultado contempla todo el rango $0 < \gamma < n$. Esto se debe a que la prueba de Hidano y Kurokawa reduce el problema al caso unidimensional del teorema de Stein y Weiss, mientras que como ya dijimos nuestro método de prueba es diferente, y más simple que el de [22], sobre todo en el caso $n = 2$.

Aplicaciones al estudio de sistemas hamiltonianos con pesos

Una consecuencia inmediata del Teorema 0.1 es la obtención de un resultado de inmersión compacte de las funciones radiales de los espacios de Sobolev fraccionarios $H^s(\mathbb{R}^n)$ en espacios $L^q(\mathbb{R}^n, |x|^c)$ apropiados, generalizando el trabajo de P.L Lions [25] donde se prueba un resultado análogo sin pesos.

La idea de obtener mejores propiedades para la inmersión (y en particular compacidad) restringiéndose al subespacio de funciones radiales se remonta a los trabajos de W. Strauss [45], W. M. Ni [29] y W. Rother [33], y fue generalizada en diferentes direcciones por W. Sickel y L. Skrzypczak [38] y P. L. Lions [25].

Más precisamente, la primer aplicación de las estimaciones con pesos para la integral fraccionaria de funciones radiales será la demostración del siguiente teorema:

Teorema 0.2. *Sean* $0 < s < \frac{n}{2}$, $2 < q < 2_c^* := \frac{2(n+c)}{n-2s}$.

Entonces se tiene la inmersión compacta

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c)$$

siempre que $-2s < c < \frac{(n-1)(q-2)}{2}$.

Notemos que en el caso sin pesos ($c = 0$) el teorema da el resultado de inmesión de Sobolev clásico en el caso particular de las funciones con simetría radial. Para este caso, la compacidad de la inmersión $H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ bajo las condiciones $0 < s < \frac{n}{2}$ y $2 < q < \frac{2n}{n-2s}$ fue demostrada por P. L. Lions en [25]. El caso $s = 1$ del resultado con pesos puede en cambio encontrarse en el trabajo de W. Rother [32].

La compacidad de la inmersión en combinación con un teorema abstracto de minimax debido a T. Bartsch y D. G. de Figueiredo [3], nos permite demostrar la existencia de infinitas soluciones radiales del siguiente sistema elíptico en $\mathbb{R}^n$:

$$\begin{cases} -\Delta u + u &= |x|^a |v|^{p-2} v \\ -\Delta v + v &= |x|^b |u|^{q-2} u \end{cases} \qquad \text{(Ec. 1)}$$

Más precisamente, demostramos el siguiente teorema:

Teorema 0.3. *Si*

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1 \qquad \text{(Ec. 2)}$$

$$0 < a < \frac{(n-1)(p-2)}{2}, \quad 0 < b < \frac{(n-1)(q-2)}{2} \qquad \text{(Ec. 3)}$$

$$\frac{n+a}{p} + \frac{n+b}{q} > n - 2 \qquad \text{(Ec. 4)}$$

y

$$q < \frac{2(n+b)}{n-4}, \quad p < \frac{2(n+a)}{n-4} \quad \textit{si} \;\; n \geq 5. \qquad \text{(Ec. 5)}$$

Entonces, (Ec. 1) *admite infinitas soluciones débiles con simetría radial.*

Cabe destacar que si bien en esta tesis consideraremos el sistema modelo (Ec. 1), las mismas técnicas se pueden extender a sistemas elípticos hamiltonianos más generales en $\mathbb{R}^n$, de la forma:

$$\begin{cases} -\Delta u + u &= H_v(|x|, u, v) \\ -\Delta v + v &= H_u(|x|, u, v) \end{cases} \qquad \text{(Ec. 6)}$$

con hipótesis adecuadas sobre el hamiltoniano H (análogas a las que aparecen en [3]). Como veremos, una característica importante de esta clase de sistemas es su estructura, que permite encontrar las soluciones débiles como puntos críticos de un funcional en un espacio apropiado.

Para un dominio acotado $\Omega \subset \mathbb{R}^n$, T. Bartsch y D. G. de Figueiredo demostraron en [3] que el sistema asociado

$$\begin{cases} -\Delta u &= |x|^a |v|^{p-2} v \quad \text{en} \quad \Omega \\ -\Delta v &= |x|^b |u|^{q-2} u \quad \text{en} \quad \Omega \end{cases} \tag{Ec. 7}$$

con condiciones de Dirichlet ($u = v = 0$ en $\partial\Omega$), admite infinitas soluciones no triviales en el caso sin pesos $a = b = 0$, (Ec. 7) si

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1$$

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{n} \tag{Ec. 8}$$

$$q < \frac{2n}{n-4}, \quad p < \frac{2n}{n-4} \quad \text{si } n \geq 5.$$

También en [3] se demuestra la existencia de infinitas soluciones radiales para el sistema sin pesos en $\mathbb{R}^n$ (es decir, (Ec. 1) con $a = b = 0$).

En [14], D. G. de Figueiredo, I. Peral y J. Rossi extendieron estos resultados al problema con pesos no triviales en un dominio $\Omega \subset \mathbb{R}^n$ acotado, tal que $0 \in \Omega$. Nuestro resultado es la existencia de infinitas soluciones radiales del sistema con pesos (Ec. 1) en el espacio $\mathbb{R}^n$ con restricciones apropiadas en los pesos.

La principal diferencia con el caso acotado radica precisamente en el teorema de inmersión con pesos, ya que en el caso de un dominio acotado, la inmersión necesaria puede ser obtenida aplicando la inmersión clásica junto con la desigualdad de Hölder, mientras que en el caso de $\mathbb{R}^n$ no es posible hacer lo mismo ya que los pesos $|x|^r$ no son integrables. Además, nuestra demostración precisa del teorema abstracto de minimax mencionado anteriormente en lugar de un teorema más simple que puede ser usado en el caso acotado (ver Teorema 3.1 de [14]) ya que la prueba en el caso acotado utiliza el hecho de que el Laplaciano tiene espectro discreto en un dominio acotado, cosa que no sucede en $\mathbb{R}^n$ para la parte lineal de (Ec. 1).

Aplicaciones a desigualdades de tipo Caffarelli-Kohn-Nirenberg

La segunda aplicación del Teorema 0.1 que presentaremos en esta tesis es una mejora de las desigualdades de tipo Caffarelli-Kohn-Nirenberg en el caso de funciones radiales. Para precisar qué entendemos con esto, recordemos primero la con conocida desigualdad de intepolación de primer orden:

Teorema ([6]). *Sean $p, q \geq 1$, $r > 0$, $0 \leq a \leq 1$ tales que*

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0,$$

siendo

$$\gamma = a\sigma + (1-a)\beta.$$

Existe una constante C positiva tal que la desigualdad

$$\||x|^{\gamma} u\|_{L^r} \leq C \||x|^{\alpha} |\nabla u|\|_{L^p}^{a} \||x|^{\beta} u\|_{L^q}^{1-a} \tag{Ec. 9}$$

vale para toda $u \in C_0^{\infty}(\mathbb{R}^n)$ si y sólo si valen las siguientes relaciones:

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

$$0 \leq \alpha - \sigma \quad si \quad a > 0$$

y

$$\alpha - \sigma \leq 1 \quad si \quad a > 0 \quad y \quad \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}$$

Recordando que si $f \in C_0^{\infty}(\mathbb{R}^n)$, entonces vale que

$$|f(x)| \leq C(n) \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy = C(n) T_{n-1}(|\nabla f|)$$

se deduce que a partir de estimaciones con pesos para la integral fraccionaria se pueden obtener estimaciónes del tipo (Ec. 9).

Sin embargo, se puede ver que el rango óptimo de exponentes para los cuales vale una desigualdad asociada a (Ec. 9) en la que interviene la integral fraccionaria es distinto del que se obtiene si se considera la desigualdad para la función y el gradiente directamente. Para explicar este fenómeno consideremos, por simplicidad, el caso $a = 1$. En este caso, es fácil ver que a partir de la desigualdad asociada

$$\||x|^{\gamma} T_{n-1} f\|_{L^r} \leq C \||x|^{\alpha} f\|_{L^p} \tag{Ec. 10}$$

se obtiene la desigualdad de Caffarelli-Kohn-Nirenberg, pero con la restricción adicional $\alpha < \frac{n}{p'}$, que no es necesaria en (Ec. 9). Por lo tanto, una vez demostrada la desigualdad asociada (Ec. 10), es necesario probar que cuando f es un gradiente la desigualdad admite una automejora que permite deshacerse de ciertas restricciones.

Si bien la demostración original de (Ec. 9) es elemental (aunque técnica y separada en un gran número de casos) y nuestra aplicación precisa de acotaciones con pesos para la integral fraccionaria, la ventaja de nuestro enfoque es que permite extender inmediatamente el rango de exponentes para los que la desigualdad vale en el caso de funciones radiales (dado que si f es radial, $|\nabla f|$ también lo es). Por ejemplo, en el caso $a = 1$ considerado anteriormente, podemos reemplazar la restricción $\alpha - \sigma \geq 0$ por $\alpha - \sigma \geq (n-1)(\frac{1}{r} - \frac{1}{p})$.

Otras desigualdades que se conocen con el nombre de *desigualdades de tipo Caffarelli-Kohn-Nirenberg* son las desigualdades de trazas del tipo

$$\||x|^{-\beta} u(x,0)\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^{\alpha} \nabla u(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}_+)} \qquad \text{(Ec. 11)}$$

donde $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

Como en el caso anterior, demostraremos que estas desigualdades tienen un operador asociado, que está dado por

$$Tf(x) = \int_{\mathbb{R}^n \times \mathbb{R}_+} \frac{f(y,z)}{[(x-y)^2 + z^2]^{n/2}} \, dy \, dz.$$

Como veremos, este operador goza de propiedades análogas a a las de la integral fraccionaria que, en particular, en el caso de las funciones radiales nos permitirán demostrar el siguiente teorema:

Teorema 0.4. *Sean $n \geq 1$, $1 < p < \infty$, $-\frac{n}{q'} < \beta < \frac{n}{q}$ y $\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1$. Si $p \leq q < \infty$ entonces la desigualdad*

$$\||x|^{-\beta} Tf(x,0)\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^{\alpha} f(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}_+)}$$

vale para toda función radial $f \in L^p(\mathbb{R}^n \times \mathbb{R}_+, |(y,z)|^{p\alpha} dydz)$, donde C es una constante independiente de f.

Como en el caso de la desigualdad (Ec. 9) también probaremos que ciertas restricciones del Teorema 0.4 no son necesarias en el caso en que la función f sea un gradiente, dado que la desigualdad (Ec. 11) también admite una automejora en este caso.

Aplicaciones a multiplicadores de tipo transformada de Laplace para desarrollos de Laguerre y Hermite

En el último capítulo de esta tesis mostraremos que la técnicas que utilizamos en la demostración de las acotaciones con pesos para la integral fraccionaria de funciones radiales pueden ser utilizadas también para el estudio de acotaciones $L^p - L^q$ con pesos de ciertos multiplicadores para desarrollos en funciones de Laguerre.

Recordemos que las funciones de Laguerre, para $\alpha > -1$ fijo, están dadas por

$$l_k^\alpha(x) = \left(\frac{k!}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-x/2} L_k^\alpha(x) \ , \quad k \in \mathbb{N}_0$$

donde L_k^α son los polinomios de Laguerre. Las funciones $l_k^\alpha(x)$ son autofunciones de autovalor $\lambda_{\alpha,k} = k + (\alpha+1)/2$ del operador diferencial de Laguerre

$$L = -\left(x\frac{d^2}{dx^2} + (\alpha+1)\frac{d}{dx} - \frac{x}{4}\right),$$

y constituyen una base ortonormal de $L^2(\mathbb{R}_+, x^\alpha)$.

Entonces, dada $f \in L^p(\mathbb{R}_+, x^\gamma)$ con $\gamma < p(\alpha+1) - 1$ podemos asociarle su serie de Laguerre

$$f(x) \sim \sum_{k=0}^\infty a_{\alpha,k}(f) l_k^\alpha(x), \quad a_{\alpha,k}(f) = \int_0^\infty f(x) l_k^\alpha(x) x^\alpha dx \qquad \text{(Ec. 12)}$$

Esta serie es conocida como desarrollo de Laguerre de tipo convolución, ya que existe una estructura de convolución generalizada asociada que será la que nos permitirá explotar en este contexto las técnicas mencionadas anteriormente. Sin embargo, cabe aclarar que existen otros tipos de desarrollos de Laguerre. Un estudio exhaustivo puede encontrarse en el libro de S. Thangavelu [46].

Si $m = (m_k)$ es una sucesión acotada, podemos definir el multiplicador $M_{\alpha,m}$ asociado en $L^2(\mathbb{R}_+, x^\alpha)$ como

$$M_{\alpha,m} f(x) \sim \sum_{k=0}^\infty a_{\alpha,k}(f) m_k l_k^\alpha(x) \qquad \text{(Ec. 13)}$$

y diremos que $M_{\alpha,m}$ es un multiplicador del tipo transformada de Laplace si $m_k = m(k)$ donde la función m está dada por la transformada de Laplace-Stieljtes de alguna función $\psi(t)$ de variación acotada en $\mathbb{R}_+$, o sea, si

$$m(s) = \mathfrak{L}\psi(s) := \int_0^\infty e^{-st} d\psi(t). \qquad \text{(Ec. 14)}$$

Los multiplicadores de este tipo aparecen de forma bastante natural y, en efecto, una definición ligeramente distinta de la usaremos en esta tesis fue dada por E. M. Stein en [40] y estudiada en el caso sin pesos por E. Sasso en [41]. Más recientemente, B. Wróbel [50] demostró estimaciones L^p con pesos para los mismos multiplicadores y ciertos valores de α. También cabe destacar que T. Martínez ha estudiado mutliplicadores de tipo transformada de Laplace para expansiones ultraesféricas en [27].

Otros tipos de multiplicadores para expansiones de Laguerre también han sido considerados, por ejemplo en los trabajos [16, 44, 46], donde se estudian criterios de acotación en términos de operadores en diferencias. En esta tesis sólo pediremos hipótesis mínimas sobre la función ψ, que son más naturales en nuestro contexto y más fácilmente verificables en los ejemplos que consideraremos. Más precisamente, probaremos el siguiente teorema

Teorema 0.5. *Sea ψ tal que:*

(H1)

$$\int_0^\infty |d\psi(t)| < +\infty$$

(H2) Existen $\delta > 0$, $C > 0$ y $0 < \sigma < \alpha + 1$ tales que

$$|\psi(t)| \leq Ct^\sigma \quad \textit{for}\, 0 \leq t \leq \delta$$

Si además $\alpha \geq 0$, $1 < p \leq q < \infty$, $a < \frac{\alpha+1}{p'}, b < \frac{\alpha+1}{q}$, $2a + 2b \geq \left(\frac{1}{q} - \frac{1}{p}\right)(2\alpha + 1)$ y

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1} \tag{Ec. 15}$$

entonces $M_{\alpha,m}$ se extiende a un operador acotado de $L^p(\mathbb{R}_+, x^{\alpha+ap})$ en $L^q(\mathbb{R}_+, x^{\alpha-bq})$ y vale la estimación

$$\|M_{\alpha,m}f\|_{L^q(\mathbb{R}_+,x^{\alpha-bq})} \leq C\|f\|_{L^p(\mathbb{R}_+,x^{\alpha+ap})}.$$

Un caso particular de estos multiplicadores, que ha sido objeto de estudio de distintos autores, es el de la integral fraccionaria de Laguerre, que corresponde a la elección $m_k = (k+1)^{-\sigma}$. Este operador fue introducido por G. Gasper, K. Stempak y W. Trebels en [16] como un análogo de la integral fraccionaria clásica para el caso de Laguerre. Estos autores demostraron, además, una desigualdad con pesos que corresponde al Teorema 0.5 en el caso particular $a + b \geq 0$. Posteriormente, en el trabajo de G. Gasper y W. Trebels [17], este resultado fue demostrado con otra técnica, obteniendo el mismo rango de exponentes admisibles del Teorema 0.5.

En [30], A. Nowak y K. Stempak demostraron un resultado similar para desarrollos de Laguerre multidimensionales aprovechando la relación entre desarrollos de Laguerre y desarrollos de Hermite. Su definición de la integral fraccionaria de Laguerre es ligeramente diferente, ya que está dada por potencias negativas del operador L. Sin embargo, las acotaciones para ambos operadores son equivalentes, gracias a un resultado profundo sobre multiplicadores, por lo que el teorema de [30] contiene como caso particular el resultado de [16] (en el caso unidimensional).

La demostración de nuestro teorema recupera algunas ideas del método original de [16], extendiéndolo para multiplicadores más generales que la integral fraccionaria y obteniendo un rango mejor de exponentes, que en particular permite redemostrar el resultado de [17] para la integral fraccionaria de Laguerre. En efecto, el Teorema 0.5 se puede aplicar a los ejemplos anteriormente mencionados eligiendo

$$m_k = (k + c)^{-\sigma}, \quad \eta(t) = \frac{1}{\Gamma(\sigma)} t^{\sigma-1} e^{-ct} \quad (c > 0)$$

(el caso $c = 1$ corresponde a la definición de integral fraccionaria de [16], mientras que el caso $c = \frac{\alpha+1}{2}$ corresponde a la definición de [30]).

Además, nuestra prueba es más simple que la de [16] en muchos detalles técnicos, gracias a que, como mencionamos anteriormente, la estructura de convolución generalizada asociada a los desarrollos de Laguerre que consideraremos está fuertemente relacionada con la integral fraccionaria (usual) de funciones radiales. Más aún, para ciertos valores de α, esta relación nos permite obtener para los multiplicadores mencionados estimaciones con pesos de tipo $A_{p,q}$ radiales, mientras que las acotaciones conocidas hasta el momento se limitan exclusivamente a pesos potencia.

Por último, de manera análoga al caso de Laguerre, consideraremos multiplicadores de tipo transformada de Laplace para desarrollos en funciones de Hermite.

Para esto, recordemos que dada $f \in L^2(\mathbb{R})$, su serie de Hermite está dada por

$$f \sim \sum_{k=0}^{\infty} c_k(f) h_k$$

donde $c_k(f) = \langle f, h_k \rangle$ y h_k son las funciones de Hermite, que se definen como

$$h_k(x) = \frac{(-1)^k}{(2^k k! \pi^{1/2})^{1/2}} H_k(x) e^{-x^2/2},$$

siendo H_k los polinomios de Hermite. Estas funciones son autofunciones normalizadas del oscilador armónico $H = -\frac{d^2}{dx^2} + |x|^2$.

Entonces, dada una sucesión acotada $\{m_k\}$ podemos definir, como antes, el multiplicador de Hermite asociado

$$M_{H,m} f \sim \sum_{k=0}^{\infty} c_k(f) m_k h_k$$

y decimos que este es un multiplicador de tipo transformada de Laplace si vale (Ec. 14). Gracias a las relaciones que existen entre las funciones de Hermite y las funciones Laguerre, veremos que vale el siguiente teorema análogo al Teorema 0.5:

Teorema 0.6. *Sea ψ tal que:*

(H1h)

$$\int_0^{\infty} |d\psi(t)| < +\infty$$

(H2h) Existen $\delta > 0$, $C > 0$ *y* $0 < \sigma < \frac{1}{2}$ *tales que*

$$|\psi(t)| \leq Ct^{\sigma} \quad \textit{for}\ 0 \leq t \leq \delta$$

Si además $1 < p \leq q < \infty$, $a < \frac{1}{p'}$, $b < \frac{1}{q}$, $a + b \geq 0$ *y*

$$\frac{1}{q} \geq \frac{1}{p} - (2s - a - b) \tag{Ec. 16}$$

entonces $M_{H,m}$ *se extiende a un operador acotado de* $L^p(\mathbb{R}, x^{\alpha+ap})$ *en* $L^q(\mathbb{R}, x^{\alpha-bq})$ *y vale la estimación*

$$\|M_{H,m}f\|_{L^q(\mathbb{R},x^{\alpha-bq})} \leq C\|f\|_{L^p(\mathbb{R},x^{\alpha+ap})}$$

Introduction

Weighted inequalities for fractional integrals

The study of weighted inequalities for fractional integrals (also called Riesz potentials)

$$(T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n.$$

is a classical problem in analysis, of great importance because of its applications, since this operator provides an integral representation of the negative powers of the Laplacian, and plays a key role in the classical proof of Sobolev's imbedding theorems (see, for example, [41]).

The study of weighted inequalities for this operator goes back to the the work of G. H. Hardy and J. E. Littlewood [20] in the one-dimensional case, while the generalization to $\mathbb{R}^n, n \geq 1$, is due to E. M. Stein and G. Weiss, who obtained the following theorem in [42]:

Let $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq 0$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$. *If* $p \leq q < \infty$ *then the inequality*

$$\| |x|^{-\beta} T_\gamma v \|_{L^q(\mathbb{R}^n)} \leq C \| |x|^\alpha v \|_{L^p(\mathbb{R}^n)}$$

holds for all $v \in L^p(\mathbb{R}^n, |x|^{p\alpha})$, *where* C *is a constant independent of* v.

Inequalities for more general weights were later studied by several authors, until the achievement of E. T. Sawyer and R. L Wheeden in [37] of an $A_{p,q}$-type characterization of the admissible weights. From this theory it can be deduced, in particular, that for power weights the result of Stein and Weiss cannot be improved in general.

However, if we restrict ourselves to the subspace of radially symmetric functions, it is possible to obtain a wider range of exponents for which the fractional integral is continuous with power weights. More precisely, we show in this thesis that the following result holds:

Theorem 0.7. *Let* $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$

and $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$*. If* $p \leq q < \infty$ *the the inequality*

$$\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

holds for all radially symmetric $v \in L^p(\mathbb{R}^n, |x|^{p\alpha})$*, where* C *is a constant independent ov. The same result holds for* $p = 1$ *replacing the condition on* $\alpha+\beta$ *by* $\alpha+\beta > (n-1)(\frac{1}{q}-1)$*.*

After the publication of the result contained in this thesis, we found the same result in the work [33] by B. S. Rubin. However, his work does not consider the case $p = 1$ and his method of proof if completely different from ours, since it is based on on properties of certain hypergeometric functions, while our proof is based only on the use of Young's inequality in the multiplicative group $(\mathbb{R}_+, \cdot)$ with Haar measure dx/x in combination with elementary estimates on the behavior of the kernel involved.

Other previous results for the case of radially symmetric functions (posterior to [33] but who also seem to have been unaware of that work) are those of M. C. Vilela, who made a proof for the case $p < q$ and $\beta = 0$ in [47]; and the work of K. Hidano and Y. Kurokawa [22], who proved the inequality in the case $p < q$ under the additional assumption $\beta < \frac{1}{q}$. This restriction, together with the other conditions on α and β implies our result for the case $n - 1 < \gamma < n$, while our result holds for the whole range $0 < \gamma < n$. This is due to the fact that the proof of Hidano and Kurokawa reduces the problem to the one-dimensional case of Stein and Weiss' theorem while, as we already said, our method of proof is different, and simpler than that of [22], particularly when $n = 2$.

Applications to the study of hamiltonian systems with weights

An immediate consequence of Theorem 0.7 is the existence of a compact imbedding of the subspace of radially symmetric functions of fractional order Sobolev spaces $H^s(\mathbb{R}^n)$ into appropriate $L^q(\mathbb{R}^n, |x|^c)$ spaces, generalizing the result of P.L. Lions [25], where he obtains an analogous result in the unweighted case.

The idea of obtaining better properties for the imbedding (and especially compactness) by restricting us to the subspace of radilly symmetric functions goes back to the works of W. Strauss [45], W. M. Ni [29] y W. Rother [33], and was generalized in different directions by W. Sickel and L. Skrzypczak [38] and P. L. Lions [25].

More precisely, the first application of the weighted estimates for the fractional integral of radial functions will be the proof of the following theorem:

Theorem 0.8. *Let* $0 < s < \frac{n}{2}$, $2 < q < 2^*_c := \frac{2(n+c)}{n-2s}$.

Then, we have a compact imbedding

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c)$$

provided that $-2s < c < \frac{(n-1)(q-2)}{2}$.

It is worth noting that the unweighted case $c = 0$ corresponds to the classical Sobolev imbedding theorem for the particular case of radially symmetric functions. In this case, the compactness of the imbedding $H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ under the conditions $0 < s < \frac{n}{2}$ and $2 < q < \frac{2n}{n-2s}$ was proved by P. L. Lions in [25]. The case $s = 1$ of the weighted case can be found in the work of W. Rother [32].

The compactness of the imbedding in combination with an abstract minimax theorem due to T. Bartsch and D. G. de Figueiredo [3], will allow us to prove the existence of infinitely many radially symmetric functions of the following elliptic system in $\mathbb{R}^n$:

$$\left\{ \begin{array}{rcl} -\Delta u + u &=& |x|^a |v|^{p-2} v \\ -\Delta v + v &=& |x|^b |u|^{q-2} u \end{array} \right. \tag{Eq. 1}$$

More precisely, we will prove the following theorem:

Theorem 0.9. *Let*

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1 \tag{Eq. 2}$$

$$0 < a < \frac{(n-1)(p-2)}{2}, \quad 0 < b < \frac{(n-1)(q-2)}{2} \tag{Eq. 3}$$

$$\frac{n+a}{p} + \frac{n+b}{q} > n-2 \tag{Eq. 4}$$

and

$$q < \frac{2(n+b)}{n-4}, \quad p < \frac{2(n+a)}{n-4} \qquad si \quad n \geq 5. \tag{Eq. 5}$$

Then, (Eq. 1) admits infinitely many radially symmetric weak solutions.

It is worth noting that although in this thesis we will consider only the model system (Eq. 1), the same techniques can be extended to more general Hamiltonian elliptic systems in $\mathbb{R}^n$ of the form:

$$\left\{ \begin{array}{rcl} -\Delta u + u &=& H_v(|x|, u, v) \\ -\Delta v + v &=& H_u(|x|, u, v) \end{array} \right. \tag{Eq. 6}$$

with appropriate hypotheses on H (analogous to those of [3]). As we will see, an important characteristic of this class of systems is their Hamiltonian structure, that allows us to find weak solutions as critical points of a functional in an appropriate space.

For a bounded domain $\Omega \subset \mathbb{R}^n$, T. Bartsch and D. G. de Figueiredo proved in [3] that the associated system

$$\left\{ \begin{array}{rcll} -\Delta u &=& |x|^a |v|^{p-2} v & \text{in } \Omega \\ -\Delta v &=& |x|^b |u|^{q-2} u & \text{in } \Omega \end{array} \right. \tag{Eq. 7}$$

with Dirichlet conditions ($u = v = 0$ in $\partial\Omega$), admits infinitely many non-trivial solutions in the unweighted case $a = b = 0$, (Eq. 7) if

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1$$
$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{n} \tag{Eq. 8}$$
$$q < \frac{2n}{n-4}, \quad p < \frac{2n}{n-4} \quad \text{if } n \geq 5.$$

They also prove in [3] the existence of infinitely many radially symmetric functions for the unweighted system in $\mathbb{R}^n$ (that is, (Eq. 1) with $a = b = 0$).

In [14], D. G. de Figueiredo, I. Peral and J. Rossi extended these results to the weighted problem in a bounded domain $\Omega \subset \mathbb{R}^n$, such that $0 \in \Omega$. Our result is the existence of infinitely many radially symmetric solutions of the weighted system (2.1) in the space $\mathbb{R}^n$ with appropriate restrictions on the weights.

The main difference with the bounded case lies precisely in the weighted imbedding theorem, since in the case of a bounded domain the necessary imbedding can be obtained by applying the classical imbedding together with Hölder's inequality, while in the case of the whole space $\mathbb{R}^n$ it is not possible to do the same since the weights $|x|^r$ are not integrable. Moreover, our proof needs the abstract minimax theorem mentioned before instead of a simpler theorem that can be used in the bounded case (see Theorem 3.1 from [14]) since the proof in the bounded case uses the fact that the Laplacian has discrete spectrum in a bounded domain, which is not the case in $\mathbb{R}^n$ for the linear part of (Eq. 1).

Applications to inequalities of Caffarelli-Kohn-Nirenberg type

The second application of Theorem 0.7 that we will present in this Thesis is an improvement of Caffarelli-Kohn-Nirenberg type inequalities in the case of radial functions. To make this precise, recall first the well-known first order interpolation inequality:

Theorem ([6]). *Let $p, q \geq 1$, $r > 0$, $0 \leq a \leq 1$ such that*

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0,$$

where

$$\gamma = a\sigma + (1-a)\beta.$$

Then, there exists a positive constant C such that the inequality

$$\||x|^\gamma u\|_{L^r} \leq C\||x|^\alpha |\nabla u|\|_{L^p}^a \||x|^\beta u\|_{L^q}^{1-a} \tag{Eq. 9}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$ if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

$$0 \leq \alpha - \sigma \quad \textit{if} \quad a > 0$$

and

$$\alpha - \sigma \leq 1 \quad \textit{if} \quad a > 0 \quad \textit{and}\ \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}$$

Using the fact that if $f \in C_0^\infty(\mathbb{R}^n)$, then

$$|f(x)| \leq C(n) \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy = C(n) T_{n-1}(|\nabla f|)$$

we can deduce that estimates of the form (Eq. 9) can be obtained from weighted estimates for the fractional integral.

However, one can see that the optimal range of exponents for which an inequality associated to (Eq. 9) and involving the fractional integral holds, is different from that obtained by considering the inequality for the function and the gradient directly. To explain this phenomenon, let us consider, for simplicity, the case $a = 1$. In this case, it is easy to see that from the associated inequality

$$\||x|^\gamma T_{n-1} f\|_{L^r} \leq C \||x|^\alpha f\|_{L^p} \tag{Eq. 10}$$

we can obtain the Caffarelli-Kohn-Nirenberg inequality in some cases, but with the additional restriction $\alpha < \frac{n}{p'}$, unnecessary for (Eq. 9). Therefore, once we have proved the associated inequality (Eq. 10), it will be necessary to prove that when f is a gradient, the inequality admits a self-improvement that allows us to get rid of certain restrictions.

Even if the original proof of (Eq. 9) is elementary (though technical and split in several different cases) and our proof requires weighted estimates for fractional integrals, the advantage of our approach is that it allows us to immediately extend the range of admissible exponents in the case of radially symmetric functions (since if f is radial, $|\nabla f|$ is radial also). For example, in the case $a = 1$ considered before, we can replace the restriction $\alpha - \sigma \geq 0$ by $\alpha - \sigma \geq (n-1)(\frac{1}{r} - \frac{1}{p})$.

Other inequalities also known by the name of *Caffarelli-Kohn-Nirenberg type inequalities* are trace inequalities like

$$\||x|^{-\beta} u(x,0)\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^\alpha \nabla u(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}_+)} \tag{Eq. 11}$$

where $u \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

As in the previous case, we will prove that these inequalities have an associated operator, given by

$$Tf(x) = \int_{\mathbb{R}^n \times \mathbb{R}_+} \frac{f(y,z)}{[(x-y)^2 + z^2]^{n/2}}\, dy\, dz.$$

As we will see, this operator enjoys properties analogous to those of the fractional integral which, in particular, in the case of radially symmetric functions will allow us to prove the following theorem:

Theorem 0.10. *Let $n \geq 1$, $1 < p < \infty, -\frac{n}{q'} < \beta < \frac{n}{q}$ and $\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1$. If $p \leq q < \infty$ then the inequality*

$$\||x|^{-\beta} Tf(x,0)\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^{\alpha} f(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)}$$

holds for all radially symmetric $f \in L^p(\mathbb{R}^n \times \mathbb{R}_+, |(y,z)|^{p\alpha} dydz)$, where C is a constant independent of f.

As in the case of inequality (Eq. 9) we will also prove that certain restrictions of Theorem 0.10 are not necessary when the function f is a gradient, since inequality (Eq. 11) also admits a self-improvement in this case.

Applications to multipliers of Laplace transform type for Laguerre and Hermite expansions

In the last chapter of this Thesis we will show how the techniques used in the proof of the weighted inequalities for the fractional integral of radial functions can also be used for the study of weighted $L^p - L^q$ bounds for certain multipliers for Laguerre expansions.

Recall that Laguerre functions, for fixed $\alpha > -1$, are given by

$$l_k^{\alpha}(x) = \left(\frac{k!}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-x/2} L_k^{\alpha}(x) \ , \quad k \in \mathbb{N}_0$$

where L_k^{α} are the Laguerre polynomials. The functions $l_k^{\alpha}(x)$ are eigenfunctions with eigenvalues $\lambda_{\alpha,k} = k + (\alpha+1)/2$ of the Laguerre differential operator

$$L = -\left(x\frac{d^2}{dx^2} + (\alpha+1)\frac{d}{dx} - \frac{x}{4}\right),$$

and form an orthonormal basis of $L^2(\mathbb{R}_+, x^{\alpha})$.

Then, given $f \in L^p(\mathbb{R}_+, x^\gamma)$ with $\gamma < p(\alpha+1)-1$ we can associate to it its Laguerre series expansion

$$f(x) \sim \sum_{k=0}^{\infty} a_{\alpha,k}(f) l_k^\alpha(x), \quad a_{\alpha,k}(f) = \int_0^\infty f(x) l_k^\alpha(x) x^\alpha dx \tag{Eq. 12}$$

This series is known as Laguerre expansion of convolution type, since there exists an associated generalized convolution structure that will allow us to exploit in this context the techniques mentioned before. However, there are also other types of Laguerre expansions. An exhaustive study can be found in the book by S. Thangavelu [46].

If $m = (m_k)$ is a bounded sequence, we can define the associated multiplier operator $M_{\alpha,m}$ in $L^2(\mathbb{R}_+, x^\alpha dx)$ by

$$M_{\alpha,m} f(x) \sim \sum_{k=0}^{\infty} a_{\alpha,k}(f) m_k l_k^\alpha(x) \tag{Eq. 13}$$

and we will say that $M_{\alpha,m}$ is a multiplier of Laplace transform type if $m_k = m(k)$ where the function m is given by the Laplace-Stieljtes transform of some function $\psi(t)$ of bounded variation in $\mathbb{R}_+$, that is, if

$$m(s) = \mathfrak{L}\psi(s) := \int_0^\infty e^{-st} d\psi(t). \tag{Eq. 14}$$

Multipliers of this kind are quite natural to consider and, indeed, a slightly different definition from the one we will give in this Thesis was given by E. M. Stein in [40] and studied in the unweighted case by E. Sasso in [41]. More recently, B. Wróbel [50] proved L^p weighted bounds for the same kind of multipliers and certain values of α. It is also worth noting that T. Martínez has studied multipliers of Laplace transform type for ultraspherical expansions in [27].

Other kind of multipliers for Laguerre expansions have also been considered, for instance in the works of [16, 44, 46], where boundedness criteria are given in terms of difference operators. In this Thesis we will only require minimal assumptions on the functions ψ, which are more natural in our context and easier to verify in the examples that we will consider. More precisely, we will prove the following theorem:

Theorem 0.11. *Sea ψ tal que:*

(H1)

$$\int_0^\infty |d\psi(t)| < +\infty$$

(H2) There exist $\delta > 0$, $C > 0$ and $0 < \sigma < \alpha + 1$ such that

$$|\psi(t)| \leq Ct^{\sigma} \quad \textit{for } 0 \leq t \leq \delta$$

If $\alpha \geq 0$, $1 < p \leq q < \infty$, $a < \frac{\alpha+1}{p'}, b < \frac{\alpha+1}{q}$, $2a + 2b \geq \left(\frac{1}{q} - \frac{1}{p}\right)(2\alpha + 1)$ and

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1} \tag{Eq. 15}$$

then $M_{\alpha,m}$ can be extended to a bounded operator from $L^p(\mathbb{R}_+, x^{\alpha+ap})$ to $L^q(\mathbb{R}_+, x^{\alpha-bq})$ and the following estimate holds:

$$\|M_{\alpha,m} f\|_{L^q(\mathbb{R}_+, x^{\alpha-bq})} \leq C\|f\|_{L^p(\mathbb{R}_+, x^{\alpha+ap})}$$

A special case of these multipliers, that has been studied by several authors, is that of the Laguerre fractional integral, that corresponds to the choice $m_k = (k+1)^{-\sigma}$. This operator was introduced by G. Gasper, K. Stempak and W. Trebels in [16] as an analogue of the classical fractional integral in the setting of Laguerre expansions. They also prove a weighted estimate that corresponds to Theorem 0.11 in the particular case $a + b \geq 0$. Afterwards, in the work of G. Gasper and W. Trebels [17], this result was proved by a different method, obtaining the same range of exponents as Theorem 0.11.

In [30], A. Nowak and K. Stempak proved a similar result for multidimensional Laguerre expansions using the relation between Laguerre and Hermite expansions. Their definition of the Laguerre fractional integral is slightly different, since it is given by negative powers of the operator L. However, the bounds for both operators can be seen to be equivalent using a deep result on multipliers, therefore, the theorem from [30] contains as a special case the result of [16] (in the one-dimensional case).

The proof of our theorem recovers some of the ideas of the original method of [16], extending it to cover more general multipliers than the Laguerre fractional integral and obtaining a better range of exponents, that in particular allows us to give a different proof of the result in [17] for the Laguerre fractional integral. Indeed, Theorem 0.11 can be applied to the examples above choosing

$$m_k = (k+c)^{-\sigma}, \quad \eta(t) = \frac{1}{\Gamma(\sigma)} t^{\sigma-1} e^{-ct} \quad (c > 0)$$

(the case $c = 1$ corresponds to the definition of the fractional integral in [16], while the case $c = \frac{\alpha+1}{2}$ corresponds to the definition in [30]).

Moreover, our proof is simpler than that of [16] in many technical details thanks to the fact that, as mentioned before, the structure of generalized convolution associated to the Laguerre expansions that we will consider is strongly related to the (usual) fractional integral of radial functions. Also, for certain values of α, this relation allows us to obtain weighted $A_{p,q}$-type estimates for the multipliers considered above, while the previously known results are limited to power weights only.

Finally, analogously to the Laguerre case, we will consider multipliers of Laplace transform type for Hermite function expansions.

To this end, recall that given $f \in L^2(\mathbb{R})$, its Hermite series is given by

$$f \sim \sum_{k=0}^{\infty} c_k(f) h_k$$

where $c_k(f) = \langle f, h_k \rangle$ and h_k are the Hermite functions, given by

$$h_k(x) = \frac{(-1)^k}{(2^k k! \pi^{1/2})^{1/2}} H_k(x) e^{-x^2/2},$$

where H_k are the Hermite polynomials. This functions are normalized eigenfunctions of the harmonic oscillator $H = -\frac{d^2}{dx^2} + |x|^2$.

Then, given a bounded sequence $\{m_k\}$ we can define, as before, the associated Hermite multiplier

$$M_{H,m} f \sim \sum_{k=0}^{\infty} c_k(f) m_k h_k$$

and we say that this is a multiplier of Laplace transform type if (4.5) holds. Thanks to the well-known relations between Laguerre and Hermite polynomials, we will see that the following analogue of Theorem 0.11 holds:

Theorem 0.12. *Let ψ be such that:*

(H1h)

$$\int_0^{\infty} |d\psi(t)| < +\infty$$

(H2h) There exist $\delta > 0$, $C > 0$ and $0 < \sigma < \frac{1}{2}$ such that

$$|\psi(t)| \le C t^{\sigma} \quad \textit{for } 0 \le t \le \delta$$

If $1 < p \le q < \infty$, $a < \frac{1}{p'}$, $b < \frac{1}{q}$, $a + b \ge 0$ and

$$\frac{1}{q} \ge \frac{1}{p} - (2s - a - b) \qquad \text{(Eq. 16)}$$

then $M_{H,m}$ can be extended to a bounded operator from $L^p(\mathbb{R}, x^{\alpha+ap})$ to $L^q(\mathbb{R}, x^{\alpha-bq})$ and there holds the estimate

$$\|M_{H,m} f\|_{L^q(\mathbb{R}, x^{\alpha-bq})} \le C \|f\|_{L^p(\mathbb{R}, x^{\alpha+ap})}$$

Chapter 1

Weighted inequalities for fractional integrals of radial functions

In this chapter we prove the announced weighted bounds for the fractional integral

$$(T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma}\, dy, \quad 0 < \gamma < n$$

in the case when $v(x) = v_0(|x|)$ is a radial function in $\mathbb{R}^n$.

As we have explained in the introduction, the theory of weighted inequalities for fractional integrals has received considerable attention over the years, beginning with the work [20] of G. H. Hardy and E. Littlewood in 1928, where they consider admissible power weights in the one-dimensional case of the operator; and reaching a high-point with the achievement of E. T. Sawyer and R. L. Wheeden [42] in 1992 of an $A_{p,q}$-type characterization of the necessary and sufficient conditions for two weight inequalities in the n-dimensional case of the operator, both in the Euclidean case and in the more general context of homogeneous spaces.

However, the fact that the operator admits a larger class of weights when restricted to the subspace of radial functions seems to have passed almost unnoticed, even though this fact, interesting in itself, has also direct applications both in the field of partial differential equations and in the study of other classical operators in analysis, such as the Laguerre fractional integral.

As said before, in this thesis we shall restrict ourselves to the study of admissible power weights only. The proof we present here will appear in [11].

1.1 Statement of results and structure of this chapter

The main theorem we prove in this chapter is:

Theorem 1.1. *Let $n \geq 1$, $0 < \gamma < n, 1 < p < \infty, \alpha < \frac{n}{p'}, \beta < \frac{n}{q}, \alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$, and $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$. If $p \leq q < \infty$, then the inequality*

$$\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^{\alpha} v\|_{L^p(\mathbb{R}^n)}$$

holds for all radially symmetric $v \in L^p(\mathbb{R}^n, |x|^{p\alpha})$, where C is independent of v.

Remark 1.2. *If $p = 1$, then the result of Theorem 1.1 holds for $\alpha + \beta > (n-1)(\frac{1}{q} - 1)$ as may be seen from the proof of the Theorem.*

Remark 1.3. *When $\gamma \leq n - 1$, the condition $\frac{1}{q} = \frac{1}{p} + \frac{\gamma+\alpha+\beta}{n} - 1$ automatically implies $\alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$.*

Remark 1.4. *It is worth noting that if $n = 1$ or $p = q$, Theorem 1.1 gives the same range of exponents as those obtained in the case of non-necessarily radial functions by E. M. Stein and G. Weiss in [42].*

The key point in our proof is to write the desired estimate as a convolution inequality in the multiplicative group $(\mathbb{R}_+, \cdot)$ with Haar measure dx/x. The theorem will then follow from a combination of good estimates of the involved kernel with an improved version on Young's inequality.

The remainder of this chapter is organized as follows:

In Section 1.2 we prove Theorem 1.1 in the case $n = 1$. As we have already pointed out, in this case our range of weights coincides with that of Stein and Weiss (and Hardy and Littlewood) and, therefore, the assumption that v be radially symmetric (i.e., even) is unnecessary. Also, in this case we shall consider the multiplicative group $(\mathbb{R}^*, \cdot)$ with the correspondig Haar measure $dx/|x|$ instead of $(\mathbb{R}_+, \cdot)$, but the proof is useful to explain some of the ideas that we will use to prove n-dimensional theorem. Section 1.3 is devoted to the proof of Theorem 1.1 in the general case, and we show, by means of an example when $n = 3$, that the condition on $\alpha + \beta$ is sharp.

1.2 The 1-dimensional case

As mentioned in the introduction to this chapter, in this case we aim to write the desired estimate as a convolution inequality in $(\mathbb{R}^*, \cdot)$ and then use the following improved version of Young's inequality, that we recall for the sake of completeness:

Theorem 1.5. *[19, Theorem 1.4.24] Let G be a locally compact group with left Haar measure μ that satisfies $\mu(A) = \mu(A^{-1})$ for all measurable $A \subseteq G$, and let $*$ denote the convolution with respect to the group operation, that is*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\, d\mu(y),$$

(y^{-1} stands for the inverse of y).

Assume $1 < p, q, s < \infty$ satisfy

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{s}.$$

Then, there exists a constant $B_{pqs} > 0$ such that for all $f \in L^p(G,\mu)$ and $g \in L^{s,\infty}(G,\mu)$ we have

$$\|f * g\|_{L^q(G,\mu)} \leq B_{pqr}\|f\|_{L^p(G,\mu)}\|g\|_{L^{s,\infty}(G,\mu)}. \tag{1.1}$$

For the case $p = 1$, we shall use instead the classical Young's inequality in the locally compact group $(\mathbb{R}^*, \cdot)$; this accounts for the strict inequality $\alpha + \beta > (n-1)(\frac{1}{q} - 1)$ in Remark 1.2.

Now, recall that we want to prove

$$\||x|^{-\beta}T_\gamma f\|_{L^q(\mathbb{R})} \leq C\|f|x|^\alpha\|_{L^p(\mathbb{R})} \tag{1.2}$$

Letting $\mu = \frac{dx}{|x|}$, this inequality can be rewritten as

$$\||x|^{-\beta+\frac{1}{q}}T_\gamma f\|_{L^q(\mu)} \leq C\||x|^{\alpha+\frac{1}{p}}f\|_{L^p(\mu)}.$$

But now,

$$|x|^{-\beta+\frac{1}{q}}T_\gamma f(x) = \int_{-\infty}^{\infty} \frac{|x|^{-\beta+\frac{1}{q}}f(y)|y|^{\alpha+\frac{1}{p}}}{|y|^{\gamma-1+\alpha+\frac{1}{p}}|1-\frac{x}{y}|^\gamma}\frac{dy}{|y|} = (h * g)(x)$$

where $h(x) = f(x)|x|^{\alpha+\frac{1}{p}}$, $g(x) = \frac{|x|^{-\beta+\frac{1}{q}}}{|1-x|^\gamma}$, and we have used that $\gamma - 1 + \alpha + \frac{1}{p} = -\beta + \frac{1}{q}$. Using Young's inequality we obtain

$$\||x|^{-\beta+\frac{1}{q}}T_\gamma f\|_{L^q(\mu)} \leq C\||x|^{\alpha+\frac{1}{p}}f\|_{L^p(\mu)}\|g\|_{L^{s,\infty}(\mu)},$$

where

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{s},$$

(and taking $\|g\|_{L^s(\mu)}$ instead of the weak norm if $p = 1$).

Therefore, it suffices to check that $\|g\|_{L^{s,\infty}(\mu)} < \infty$ (respectively, $\|g\|_{L^s(\mu)} < \infty$). For this purpose, consider $\varphi \in C^\infty(\mathbb{R})$, supported in $[\frac{1}{2}, \frac{3}{2}]$ and such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $(\frac{3}{4}, \frac{5}{4})$. We split $g = \varphi g + (1-\varphi)g := g_1 + g_2$.

Clearly, $g_2 \in L^s(\mu)$, since the integrability condition at the origin for $|g_2|^s$ (with respect to the measure μ) is $\beta < \frac{1}{q}$, and the integrability condition when $x \to \infty$ is $\frac{1}{q} - \beta - \gamma < 0$, which, under our assumptions on the exponents, is equivalent to $\alpha < \frac{1}{p'}$.

Therefore,

$$\begin{aligned}
\mu(\{g_1 + g_2 > \lambda\}) &\le \mu\left(\left\{g_1 > \frac{\lambda}{2}\right\}\right) + \left(\frac{\|g_2\|_{L^s(\mu)}}{\lambda}\right)^s \\
&\le \mu\left(\left\{g_1 > \frac{\lambda}{2}\right\}\right) + \frac{C}{\lambda^s}
\end{aligned}$$

but,

$$\begin{aligned}
\mu\left(\left\{g_1 > \frac{\lambda}{2}\right\}\right) &\le \mu\left(\left\{\frac{C}{|1-x|^\gamma} > \lambda\right\}\right) \\
&= \mu\left(\left\{\frac{C}{\lambda^{\frac{1}{\gamma}}} > |x-1|\right\}\right) \\
&\le \frac{C}{\lambda^{\frac{1}{\gamma}}} \le \frac{C}{\lambda^s}
\end{aligned}$$

as long as $s\gamma \le 1$, that is, $\gamma \le 1 + \frac{1}{q} - \frac{1}{p}$, which is equivalent to $\alpha + \beta \ge 0$. Hence, $g \in L^{s,\infty}(\mu)$ and this concludes the proof if $p \ne 1$. When $p = 1$ it is easy to see that $g \in L^s(\mu)$ provided that $\alpha + \beta > 0$.

1.3 Proof of the weighted HLS theorem for radial functions

In this Section we prove Theorem 1.1. The main idea, as in the one-dimensional case, will be to write the fractional integral operator acting on a radial function as a convolution in the multiplicative group $(\mathbb{R}_+, \cdot)$ with Haar measure $\mu = \frac{dx}{x}$. For this purpose, we shall need the following lemma.

Lemma 1.6. *Let $x \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and consider an integral of the form:*

$$I(x) = \int_{S^{n-1}} f(x \cdot y)\, dy$$

(the integral is taken with respect to the surface measure on the sphere), where $f : [-1,1] \to \mathbb{R}, f \in L^1([-1,1],(1-t^2)^{(n-3)/2})$. *Then,* $I(x)$ *is a constant independent of* x *and moreover*

$$I(x) = \omega_{n-2} \int_{-1}^{1} f(t)(1-t^2)^{\frac{n-3}{2}} dt$$

where ω_{n-2} *denotes the area of* S^{n-2}.

Proof. First, observe that $I(x)$ is constant for all $x \in S^{n-1}$. Indeed, given $\tilde{x} \in S^{n-1}$, there exists a rotation $R \in O(n)$ such that $\tilde{x} = Rx$ and, therefore,

$$I(\tilde{x}) = \int_{S^{n-1}} f(\tilde{x} \cdot y)\, dy = \int_{S^{n-1}} f(Rx \cdot y)\, dy = \int_{S^{n-1}} f(x \cdot R^{-1}y)\, dy = I(x).$$

So, taking $x = e_n$, it suffices to compute $I(e_n) = \int_{S^{n-1}} f(y_n)\, dy$. To this end, we split the integral in two and consider first the integral on the upper-half sphere $(S^{n-1})^+$. Since $(S^{n-1})^+$ is the graph of the function $g : \{x \in \mathbb{R}^{n-1} : |x| < 1\} \to (S^{n-1})^+, g(x) = \sqrt{1-|x|^2}$, we obtain

$$\int_{(S^{n-1})^+} f(y_n)\, dy = \int_{\{|x|<1\}} f(\sqrt{1-|x|^2}) \frac{1}{\sqrt{1-|x|^2}}\, dx$$

using polar coordinates, this is

$$\int_{S^{n-2}} \int_0^1 f(\sqrt{1-r^2}) \frac{1}{\sqrt{1-r^2}} r^{n-2}\, dr\, dy = \omega_{n-2} \int_0^1 f(t)(1-t^2)^{\frac{n-3}{2}}\, dt.$$

Analogously, one obtains

$$\int_{(S^{n-1})^-} f(y_n)\, dy = \omega_{n-2} \int_{-1}^{0} f(t)(1-t^2)^{\frac{n-3}{2}}\, dt.$$

This completes the proof. $\square$

Now we can proceed to the proof of our main theorem.

Using polar coordinates,

$$y = ry', r = |y|, y' \in S^{n-1}$$

$$x = \rho x', \rho = |x|, x' \in S^{n-1}$$

and the identity

$$|x-y|^2 = |x|^2 - 2|x||y|x' \cdot y' + |y|^2$$

we write the fractional integral of a radial function $v(x) = v_0(|x|)$ as

$$T_\gamma v(x) = \int_0^\infty \int_{S^{n-1}} \frac{v_0(r) r^{n-1} dr dy'}{(r^2 - 2r\rho x' \cdot y' + \rho^2)^{\gamma/2}}.$$

Using lemma 1.6, we have that:

$$T_\gamma v(x) = \omega_{n-2} \int_0^\infty v_0(r) r^{n-1} \left\{ \int_{-1}^1 \frac{(1-t^2)^{(n-3)/2}}{(\rho^2 - 2\rho r t + r^2)^{\gamma/2}}\, dt \right\} dr.$$

Now, we may write the inner integral as:

$$\int_{-1}^1 \frac{(1-t^2)^{(n-3)/2}}{(\rho^2 - 2\rho r t + r^2)^{\gamma/2}}\, dt = \int_{-1}^1 \frac{(1-t^2)^{(n-3)/2}}{r^\gamma \left[1 - 2\left(\frac{\rho}{r}\right) t + \left(\frac{\rho}{r}\right)^2\right]^{\gamma/2}}\, dt.$$

Therefore,

$$T_\gamma v(x) = \omega_{n-2} \int_0^\infty v_0(r) r^{n-\gamma} I_{\gamma,k}\left(\frac{\rho}{r}\right) \frac{dr}{r}$$

where $k = \frac{n-3}{2}$, and, for $a \geq 0$,

$$I_{\gamma,k}(a) = \int_{-1}^1 \frac{(1-t^2)^k}{(1-2at+a^2)^{\gamma/2}}\, dt.$$

Notice that the denominator of this integral vanishes if $a = 1$ and $t = 1$ only. Therefore, $I_{\gamma,k}(a)$ is well defined and is a continuous function for $a \neq 1$.

This formula shows in a explicit way that $T_\gamma v$ is a radial function, and can be therefore thought of as a function of ρ. Furtheremore, we observe that as consequence of this formula, $\rho^{\frac{n}{q}-\beta} T_\gamma v$ has the structure of a convolution on the multiplicative group $(\mathbb{R}^+, \cdot)$:

$$\begin{aligned}
\rho^{\frac{n}{q}-\beta} T_\gamma v(x) &= \omega_{n-2} \int_0^\infty v_0(r) r^{n-\gamma+\frac{n}{q}-\beta} \frac{\rho^{\frac{n}{q}-\beta}}{r^{\frac{n}{q}-\beta}} I_{\gamma,k}\left(\frac{\rho}{r}\right) \frac{dr}{r} \\
&= \omega_{n-2} \left(v_0 r^{n-\gamma+\frac{n}{q}-\beta}\right) * \left(r^{\frac{n}{q}-\beta} I_{\gamma,k}(r)\right).
\end{aligned}$$

Hence, using Theorem 1.5 we get that

$$\begin{aligned}
\||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} &= \left(\omega_{n-1} \int_0^\infty |T_\gamma v(\rho)|^q \rho^{n-\beta q}\, \frac{d\rho}{\rho}\right)^{1/q} \\
&= \omega_{n-1}^{1/q}\, \|T_\gamma v(\rho) \rho^{\frac{n}{q}-\beta}\|_{L^q(\mu)} \\
&\leq \omega_{n-1}^{1/q} \omega_{n-2} \|v_0(r) r^{n-\gamma+\frac{n}{q}-\beta}\|_{L^p(\mu)}\, \|r^{\frac{n}{q}-\beta} I_{\gamma,k}(r)\|_{L^{s,\infty}(\mu)}
\end{aligned}$$

provided that:

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{s} \tag{1.3}$$

(and with the obvious modification in the case $p=1$).

Using polar coordinates once again:

$$\begin{aligned}\omega_{n-1}^{1/p}\,\|v_0(r)r^{n-\gamma+\frac{n}{q}-\beta}\|_{L^p(\mu)} &= \omega_{n-1}^{1/p}\left(\int_0^\infty |v_0(r)|^p r^{\left(n-\gamma+\frac{n}{q}-\beta\right)p-n} r^n\frac{dr}{r}\right)^{1/p}\\ &= \|v_0|x|^{n-\gamma+\frac{n}{q}-\beta-\frac{n}{p}}\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

But, by the conditions of our theorem,

$$n-\gamma+\frac{n}{q}-\beta-\frac{n}{p}=\alpha.$$

Therefore, it suffices to prove that

$$\|r^{\frac{n}{q}-\beta}I_{\gamma,k}(r)\|_{L^{s,\infty}(\mu)}<+\infty. \tag{1.4}$$

For this purpose, consider $\varphi\in C^\infty(\mathbb{R})$, supported in $[\frac{1}{2},\frac{3}{2}]$ and such that $0\le\varphi\le 1$ and $\varphi\equiv 1$ in $(\frac{3}{4},\frac{5}{4})$. We split $r^{\frac{n}{q}-\beta}I_{\gamma,k}=\varphi r^{\frac{n}{q}-\beta}I_{\gamma,k}+(1-\varphi)r^{\frac{n}{q}-\beta}I_{\gamma,k}:=g_1+g_2$.

We claim that $g_2\in L^s(\mu)$. Indeed, since $I_{\gamma,k}(r)$ is a continuous function for $r\neq 1$, to analyze the behavior (concerning integrability) of g_2 it suffices to consider the behavior of $r^{(\frac{n}{q}-\beta)s}|I_{\gamma,k}(r)|^s$ at $r=0$, and when $r\longrightarrow+\infty$.

Since $I_{\gamma,k}(r)$ has no singularity at $r=0$ ($I_{\gamma,k}(0)$ is finite) the local integrability condition at $r=0$ is $\beta<\frac{n}{q}$.

When $r\longrightarrow+\infty$, we observe that

$$I_{\gamma,k}(r)=\frac{1}{r^\gamma}\int_{-1}^{1}\frac{(1-t^2)^k}{(r^{-2}-2r^{-1}t+1)^{\gamma/2}}\,dt$$

and using the bounded convergence theorem, we deduce that

$$I_{\gamma,k}(r)\sim\frac{C_k}{r^\gamma}\quad\text{as } r\longrightarrow+\infty\quad(\text{with } C_k=\int_{-1}^{1}(1-t^2)^k dt).$$

It follows that the integrability condition at infinity is $\frac{n}{q}-\beta-\gamma<0$, which, under our conditions on the exponents, is equivalent to $\alpha<\frac{n}{p'}$.

We proceed now to g_1. To analyze its behavior near $r=1$, we shall need the following lemma:

Lemma 1.7. *For $r \sim 1$ and $k > -1$, we have that*

$$|I_{\gamma,k}(r)| \leq \begin{cases} C_{\gamma,k} & \text{if} \quad \gamma < 2k+2 \\ C_{\gamma,k} \log \frac{1}{|1-r|} & \text{if} \quad \gamma = 2k+2 \\ C_{\gamma,k}|1-r|^{-\gamma+2k+2} & \text{if} \quad \gamma > 2k+2 \end{cases}$$

Remark 1.8. *Notice that since in the proof of our theorem $k = \frac{n-3}{2}$, the conditions relating γ and k above correspond to conditions on γ and n which cover all the range $0 < \gamma < n$.*

Proof. Assume first that $k \in \mathbb{N}_0$ and $-\frac{\gamma}{2} + k > -1$. Then,

$$I_{\gamma,k}(1) \sim \int_{-1}^{1} \frac{(1-t^2)^k}{(2-2t)^{\frac{\gamma}{2}}}\, dt \sim C \int_{-1}^{1} \frac{(1-t)^k}{(1-t)^{\frac{\gamma}{2}}}\, dt.$$

Therefore, $I_{\gamma,k}$ is bounded.

If $-\frac{\gamma}{2} + k = -1$, then

$$I_{\gamma,k}(r) \sim \int_{-1}^{1} (1-t^2)^k \frac{d^k}{dt^k} \left\{ (1-2rt+r^2)^{-\frac{\gamma}{2}+k} \right\} dt.$$

Integrating by parts k times (the boundary terms vanish),

$$I_{\gamma,k}(r) \sim \left| \int_{-1}^{1} \frac{d^k}{dt^k} \left\{ (1-t^2)^k \right\} (1-2rt+r^2)^{-\frac{\gamma}{2}+k}\, dt \right|.$$

But $\frac{d^k}{dt^k}\left\{(1-t^2)^k\right\}$ is a polynomial of degree k and therefore is bounded in $[-1,1]$ (in fact, it is up to a constant the classical Legendre polynomial). Therefore,

$$I_{\gamma,k}(r) \sim \frac{1}{2r} \log \left(\frac{1+r}{1-r} \right)^2 \leq C \log \frac{1}{|1-r|}.$$

Finally, if $-\frac{\gamma}{2} + k < -1$, then integrating by parts as before,

$$I_{\gamma,k}(r) \leq C_k \int_{-1}^{1} (1-2rt+r^2)^{-\frac{\gamma}{2}+k}\, dt.$$

Thus,

$$I_{\gamma,k}(r) \sim (1-2rt+r^2)^{-\frac{\gamma}{2}+k+1} \Big|_{t=-1}^{t=1} \leq C_{k,\gamma} |1-r|^{-\gamma+2k+2}.$$

This finishes the proof if $k \in \mathbb{N}_0$.

Consider now the case $k = m + \nu$ with $m \in \mathbb{N}_0$ and $0 < \nu < 1$. Then,

$$I_{k,\gamma}(r) = \int_{-1}^{1} (1-t^2)^{\nu(m+1)}(1-2rt+r^2)^{-\frac{\nu\gamma}{2}}(1-t^2)^{(1-\nu)m}(1-2rt+r^2)^{-\frac{(1-\nu)\gamma}{2}}\, dt$$

therefore, by Hölder's inequality with exponent $\frac{1}{\nu}$,

$$\begin{aligned} I_{\gamma,k}(r) &\leq \left(\int_{-1}^{1}(1-t^2)^{(m+1)}(1-2rt+r^2)^{-\frac{\gamma}{2}}\, dt\right)^{\nu}\left(\int_{-1}^{1}(1-t^2)^{m}(1-2rt+r^2)^{-\frac{\gamma}{2}}\, dt\right)^{1-\nu} \\ &= I^{\nu}_{m+1,\gamma}(r) I^{1-\nu}_{m,\gamma}(r) \end{aligned}$$

If $\gamma < 2m+2$, by the previous calculation

$$|I_{\gamma,k}(r)| \leq C.$$

If $\gamma > 2(m+1)+2$, then, by the previous calculation

$$|I_{k,\gamma}(r)| \leq C|1-r|^{\nu(-\gamma+2(m+1)+2)}|1-r|^{(1-\nu)(-\gamma+2m+2)} = C|1-r|^{-\gamma+2k+2}.$$

For the case $2m+2 < \gamma < 2m+4$, notice that we can always assume $r < 1$, since $I_{\gamma,k}(r) = r^{-\gamma} I_{\gamma,k}(r^{-1})$. Then, as before, we can prove that

$$I'_{\gamma,k}(r) \leq \gamma(1-r) I_{\gamma+2,k}(r)$$

But now we are in the case $\gamma+2 > 2(m+1)+2$ and, therefore, $|I_{k,\gamma+2}(r)| \leq C|1-r|^{-\gamma+2k}$.

Therefore, if $-\gamma+2k+1 \neq -1$

$$I_{\gamma,k}(r) = \int_0^r I'_{\gamma,k}(s)\, ds \leq C\int_0^r (1-s)^{-\gamma+2k+1}\, ds \leq C|1-r|^{-\gamma+2k+2},$$

and if $-\gamma+2k+1 = -1$

$$I_{\gamma,k}(r) \leq C\int_0^r \frac{1}{1-s}\, ds = C\log\frac{1}{|1-r|}.$$

It remains to check the case $k \in (-1,0)$,

$$\begin{aligned} I_{\gamma,k}(r) &= \int_{-1}^{0} \frac{(1-t^2)^k}{(1-2rt+r^2)^{\frac{\gamma}{2}}}\, dt + \int_0^1 \frac{(1-t^2)^k}{(1-2rt+r^2)^{\frac{\gamma}{2}}}\, dt \\ &= I + II \end{aligned}$$

Since $\gamma > 0$ and $k+1>0$,

$$I \le \int_{-1}^{0} (1+t)^k \, dt = C$$

$$\begin{aligned} II &\le \int_0^1 \frac{(1-t)^k}{(1-2rt+r^2)^{\frac{\gamma}{2}}} \, dt = -\frac{1}{k+1} \int_0^1 \frac{\frac{d}{dt}[(1-t)^{k+1}]}{(1-2rt+r^2)^{\frac{\gamma}{2}}} \, dt \\ &= \frac{2r}{k+1} \int_0^1 \frac{(1-t)^{k+1}}{(1-2rt+r^2)^{\frac{\gamma}{2}+1}} \, dt \le C I_{\gamma+2,k+1}. \end{aligned}$$

Since now $k+1>0$, $I_{\gamma,k}$ can be bounded as before. □

Now we can go back to the study of g_1. We shall split the proof into three cases, depending on whether γ is less than, equal to or greater than $n-1$.

i. Assume first that $0<\gamma<n-1$. Then $|r|^{(-\beta+\frac{n}{q})s}|I_{\gamma,k}(r)|^s$ is bounded when $r \sim 1$, and, therefore, $\|g_1\|_{L^s(\mu)} < +\infty$.

ii. Consider now the case $\gamma = n-1$. Since in this case

$$|I_{\gamma,k}(r)| \le C \log \frac{1}{|1-r|},$$

we conclude, as before, that $\|g_1\|_{L^s(\mu)} < +\infty$.

iii. Finally, we have to consider the case $n-1<\gamma<n$. In this case,

$$|I_{\gamma,k}(r)| \le C|1-r|^{-\gamma+2k+2} = C|1-r|^{-\gamma+n-1}.$$

Therefore,

$$\begin{aligned} \mu\left(\left\{ g_1 > \frac{\lambda}{2} \right\}\right) &\le \mu\left(\left\{ \frac{C}{|1-x|^{\gamma-n+1}} > \lambda \right\}\right) \\ &= \mu\left(\left\{ \frac{C}{\lambda^{\frac{1}{\gamma-n+1}}} > |1-x| \right\}\right) \\ &\le \frac{C}{\lambda^{\frac{1}{\gamma-n+1}}} \le \frac{C}{\lambda^s} \end{aligned}$$

as long as $s(\gamma-n+1) \le 1$, which is equivalent to $\alpha+\beta \ge (n-1)(\frac{1}{p}-\frac{1}{q})$. Therefore, $\|g_1\|_{L^{s,\infty}(\mu)} < +\infty$ (and if $p=1$, the strong norm is bounded provided the condition on $\alpha+\beta$ holds with strict inequality).

Remark 1.9. *The following example shows that for $n=3$ the condition $\alpha+\beta \geq (n-1)(\frac{1}{q}-\frac{1}{p})$ is necessary.*

Assume that $\alpha+\beta < (n-1)(\frac{1}{q}-\frac{1}{p})$. Then, by Remark 1.3, $\gamma > n-1$.

Since $\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1$, we obtain $\gamma-n+1>\frac{1}{s}$ and, therefore, by Lemma 1.7, for $n=3$ and $r\sim 1$, $I_{\gamma,k}(r)\sim \frac{1}{|1-r|^{\frac{1}{s}+\varepsilon}}$ for some $\varepsilon>0$.

Fix η such that $\eta p>1$ and let

$$f(r)=\frac{\chi_{[\frac{1}{2},\frac{3}{2}]}(r)}{|1-r|^{\frac{1}{p}}\log(\frac{1}{|1-r|})^{\eta}}$$

Then $f\in L^p(\mu)$ and, for $r>1$,

$$\begin{aligned}(I_{\gamma,k}*f)(r) &\geq \int_r^{\frac{3}{2}} \frac{t^{\frac{1}{s}+\varepsilon}}{t^{\frac{1}{s}+\varepsilon}|1-\frac{r}{t}|^{\frac{1}{s}+\varepsilon}|1-t|^{\frac{1}{p}}\log(\frac{1}{|1-t|})^{\eta}}\,dt\\ &\geq \int_r^{\frac{3}{2}} \frac{1}{(t-r)^{\frac{1}{s}+\varepsilon}(t-1)^{\frac{1}{p}}(\log\frac{1}{|1-r|})^{\eta}}\,dy\\ &\geq \frac{1}{(\log\frac{1}{|1-r|})^{\eta}}\int_r^{\frac{3}{2}}\frac{dy}{(t-1)^{\frac{1}{s}+\frac{1}{p}+\varepsilon}}\\ &\sim \frac{1}{(\log\frac{1}{|1-r|})^{\eta}|1-r|^{\frac{1}{q}+\varepsilon}} \notin L^q.\end{aligned}$$

Recall now that for a radial function,

$$\rho^{\frac{n}{q}-\beta}T_{\gamma}f_0(\rho)=f_0r^{\frac{n}{p}+\alpha}*r^{\frac{n}{q}-\beta}I_{\gamma,k}(r)$$

Therefore, defining $f_0=f(|x|)|x|^{-\frac{n}{p}-\alpha}$ we have, $\|f_0|x|^{\alpha}\|_{L^p}<\infty$ but $T_{\gamma}f|x|^{-\beta}\notin L^q$.

Chapter 2

Application to the study of Hamiltonian elliptic systems with weights

In this chapter, we will use the weighted estimates in the previous chapter to obtain a weighted imbedding theorem for radial functions and apply this theorem to the study of the existence of non-trivial, radially symmetric solutions of the following Hamiltonian elliptic system in $\mathbb{R}^n$:

$$\left\{ \begin{array}{rcl} -\Delta u + u & = & |x|^a |v|^{p-2} v \\ -\Delta v + v & = & |x|^b |u|^{q-2} u \end{array} \right. \tag{2.1}$$

The compactness of our weighted imbedding will be of fundamental importance, since it will allow us to prove a suitable form of the Palais-Smale compactness condition (see Lemma 2.8). The other important ingredient of our proof of the main theorem in this chapter (Theorem 2.2) is an abstract minimax theorem from T. Bartsch and D. G. de Figueiredo [3] that we recall for easy-reference (see Theorem 2.4 below).

The results of this chapter were published in [12]. However, the proof of the existence of the compact imbedding in that paper was obtained with a different technique, using the Fourier transform definition of fractional order Sobolev spaces and a theorem on $L^p - L^q$ estimates for the Fourier-Bessel transform due to L. De Carli [8]. The original proof of the imbedding that we obtained in [12] can be found in Appendix A.

2.1 Statement of results and stucture of this chapter

Our first result in this chapter is a weighted imbedding theorem for fractional order Sobolev spaces. More precisely, we will prove:

Theorem 2.1. *Let* $0 < s < \frac{n}{2}$, $2 < q < 2^*_c := \frac{2(n+c)}{n-2s}$. *Then we have the compact imbedding*

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c)$$

provided that $-2s < c < \frac{(n-1)(q-2)}{2}$. *Here* $H^s_{rad}(\mathbb{R}^n)$ *denotes the subspace of radially symmetric functions of the Sobolev space* $H^s(\mathbb{R}^n)$.

Then, we will proceed to apply this theorem to the study of system (2.1) and will show that the following result holds:

Theorem 2.2. *Assume that the following conditions hold:*

$$p, q > 2, \quad \frac{1}{p} + \frac{1}{q} < 1 \tag{2.2}$$

$$0 < a < \frac{(n-1)(p-2)}{2}, \quad 0 < b < \frac{(n-1)(q-2)}{2} \tag{2.3}$$

$$\frac{n+a}{p} + \frac{n+b}{q} > n-2 \tag{2.4}$$

and

$$q < \frac{2(n+b)}{n-4}, \quad p < \frac{2(n+a)}{n-4} \quad \text{if } n \geq 5. \tag{2.5}$$

Then, (2.1) admits infinitely many radially symmetric weak-solutions (see Definition 2.6 below).

The remainder of this chapter is organized as follows. In section 2.2 we prove the announced weighted imbedding theorem for fractional order Sobolev spaces. In section 2.3 we recall the abstract minimax theorem from T. Bartsch and D. G. de Figueiredo [3] (see Theorem 2.4 below) that we will use to prove existence of solutions of system (2.1). Finally, in Section 2.4 we complete the proof of Theorem 2.2 by checking that all the conditions of Theorem 2.4 hold.

2.2 A weighted imbedding theorem

In this section we prove Theorem 2.1. To this end, we recall the definition of the fractional order Sobolev space

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (-\Delta)^{s/2} u \in L^2(\mathbb{R}^n)\} \quad (s \geq 0).$$

Then, if $u \in H^s_{rad}(\mathbb{R}^n)$, $f := (-\Delta)^{s/2} u \in L^2(\mathbb{R}^n)$, and, recalling the relation between the negative powers of the Laplacian and the fractional integral (see, e.g., [45, Chapter V]), we obtain

$$T_{n-s} f = C(-\Delta)^{-s/2} f = Cu.$$

Then, it follows from Theorem 1.1 that

$$\||x|^{\frac{c}{q}}u\|_{L^{2^*_c}(\mathbb{R}^n)} = C\||x|^{\frac{c}{q}}T_{n-s}f\|_{L^{2^*_c}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

Therefore, writing $q = 2\nu + (1-\nu)2^*_c$, and using Hölder's inequality, we obtain

$$\||x|^{\frac{c}{q}}u\|_{L^q(\mathbb{R}^n)} \leq \||x|^{\frac{c}{q}}u\|^{\nu}_{L^{2^*_c}(\mathbb{R}^n)}\|u\|^{1-\nu}_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

It remains to prove that the imbedding $H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c)$ is compact. To this end, it suffices to show that if $u_n \to 0$ weakly in $H^s_{rad}(\mathbb{R}^n)$, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^n, |x|^c)$. Since

$$2 < q < 2^*_c = \frac{2(n+c)}{n-2s}$$

by hypothesis, we claim that it is possible to choose r and $\tilde{q}$ so that $2 < r < q < \tilde{q} < 2^*_c$ and the following conditions hold

$$\tilde{q} < \frac{2(n+\tilde{c})}{n-2s}, \quad -2s < \tilde{c} < \frac{(n-1)(\tilde{q}-2)}{2}.$$

Indeed, assume first that $c > 0$. Then, if $\theta \in (0,1)$ is sufficently small, taking $\tilde{c} = \frac{c}{1-\theta}$, we see that $\tilde{c} < \frac{(q-2)(n-1)}{2}$. Since $\tilde{c} > c$, it is clear that $q < \frac{2(n+\tilde{c})}{n-2s}$, and we can choose $\tilde{q} > q$ such that $\tilde{q} < \frac{2(n+\tilde{c})}{n-2s}$ and $\theta > \frac{\tilde{q}-q}{\tilde{q}-2}$.

On the other hand, if $-2s < c < 0$ and $\tilde{c} = \frac{c}{1-\theta}$ with $\theta \in (0,1)$ sufficiently small, we have that $-2s < \tilde{c}$ and $q < \frac{2(n+\tilde{c})}{n-2s}$. Since now $\tilde{c} < c$, it is immediate that $\tilde{c} < \frac{(q-1)(n-1)}{2}$, and we can choose $\tilde{q} > q$ such that $\theta > \frac{\tilde{q}-q}{\tilde{q}-2}$ and $\tilde{c} < \frac{(\tilde{q}-1)(n-1)}{2}$.

In either case, if we let r satisfy $q = \theta r + (1-\theta)\tilde{q}$, it follows from the previous conditions that $2 < r < q$ and, using Hölder's inequality, we have that

$$\int_{\mathbb{R}^n} |x|^c|u_n|^q\,dx \leq \left(\int_{\mathbb{R}^n} |u_n|^r\,dx\right)^{\theta}\left(\int_{\mathbb{R}^n} |x|^{\tilde{c}}|u_n|^{\tilde{q}}\,dx\right)^{1-\theta}. \tag{2.6}$$

Therefore, by the imbedding that we have already established:

$$\left(\int_{\mathbb{R}^n} |x|^{\tilde{c}}|u_n|^{\tilde{q}}\,dx\right)^{1/\tilde{q}} \leq C\|u_n\|_{H^s} \leq C'.$$

Since the imbedding $H^s_{rad}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ is compact by Lions theorem [25], we have that $u_n \to 0$ in $L^r(\mathbb{R}^n)$. From (2.6) we conclude that $u_n \to 0$ strongly in $L^q(\mathbb{R}^n, |x|^c)$, which shows that the imbedding in our theorem is also compact. This concludes the proof.

2.3 An abstract critical point theorem

In order to prove Theorem 2.2 we will use an abstract critical point result from [3]. For the reader's convenience, we will try to keep the notation from that paper. We start by recalling the specific form of the Palais-Smale-Cerami compactness condition used in [3]:

Definition 2.3. We consider a Hilbert space E and a functional $\Phi \in C^1(E, \mathbb{R})$. Given a sequence $(X_n)_{n\in\mathbb{N}}$ of finite dimensional subspaces of X, with $X_n \subset X_{n+1}$ and $\overline{\bigcup X_n} = E$, the functional Φ is said to satisfy *condition* $(PS)_c^{\mathcal{F}}$ *at level* c if every sequence $(z_j)_{j\in\mathbb{N}}$ with $z_j \in X_{n_j}$, $n_j \to +\infty$ and such that

$$\Phi(z_j) \to c \quad \text{and} \quad (1 + \|z_j\|)(\Phi|_{X_{n_j}})'(z_j) \to 0$$

(a so-called $(PS)_c^{\mathcal{F}}$ sequence) has a subsequence which converges to a critical point of Φ.

Theorem 2.4 (Fountain Theorem, Theorem 2.2 from [3]). *Assume that the Hilbert space E splits as a direct sum $E = E^+ \oplus E^-$, and that $E_1^\pm \subset E_2^\pm \subset \ldots \subset E_n^\pm \subset$ are strictly increasing sequences of finite dimensional subspaces such that $\overline{\bigcup_{n=1}^\infty E_n^\pm} = E^\pm$ and let $E_n = E_n^+ \oplus E_n^-$. Furthermore, assume that the functional Φ satisfies the following assumptions:*

(Φ_1) $\Phi \in C^1(E, \mathbb{R})$ *and satisfies* $(PS)_c^{\mathcal{F}}$ *with respect to* $\mathcal{F} = (E_n)_{n\in\mathbb{N}}$ *and every* $c > 0$.

(Φ_2') *There exists a sequence* $r_k > 0$ $(k \in \mathbb{N})$ *such that*

$$b_k = \inf\{\Phi(z) : z \in E^+,\ z \perp E_{k-1}\ \|z\| = r_k\}$$

satisfy $b_k \to +\infty$.

(Φ_3') *There exist a sequence of isomorphisms* $T_k : E \to E$ $(k \in \mathbb{N})$ *with* $T_k(E_n) = E_n$ *for all* k *and* n, *and there exists a sequence* $R_k > 0$ $(k \in \mathbb{N})$ *such that, for* $z = z^+ + z^- \in E_k^+ \oplus E^-$ *with* $\max(\|z^+\|, \|z^-\|) = R_k$ *one has*

$$\|T_k\| > R_k \quad \textit{and} \quad \Phi(T_k z) < 0.$$

(Φ_4') $d_k = \sup\{\Phi(T_k(z^+ + z^-)) : z^+ \in E_k^+,\ z^- \in E^-,\ \|z^+\|, \|z^-\| \le R_k\} < +\infty.$

(Φ_5) Φ *is even, i.e.* $\Phi(-z) = \Phi(z)\ \forall\, z \in E$.

Then Φ has an unbounded sequence of critical values.

In our application, we will also use Remark 2.2 from [3], that we state here as a lemma for the sake of completeness:

Lemma 2.5. *Let E be a Hilbert space, and $E_1 \subset E_2 \subset E_3 \subset \ldots$ be a sequence of finite dimensional subspaces of E such that $E = \overline{\bigcup_{n=1}^{\infty} E_n}$. Assume that we have a compact imbedding $E \subset X$, where X is a Banach space.*

Let $\Phi \in C^1(E, R)$ be a functional of the form $\Phi = P - \Psi$ where

$$P(z) \geq \alpha \|z\|_E^p \;\forall\; z \in E$$

and

$$|\Psi(x)| \leq \beta(1 + \|z\|_X^q) \;\forall\; z \in E$$

where α, β and $q > p$ are positive constants. Then, there exist $r_k > 0$ ($k \in \mathbb{N}$) such that

$$b_k = \inf\{\Phi(z) : z \in E,\ z \perp E_{k-1}\ \|z\| = r_k\} \to +\infty$$

i.e. condition (Φ_2') in theorem 2.4 holds.

2.4 Proof of Theorem 2.2

2.4.1 The Functional Setting

Using conditions (2.2), (2.4) and (2.5) we may choose s, t such that $0 < s, t < \frac{n}{2}$, $s + t = 2$ and

$$2 < p < \frac{2(n+a)}{n-2t}, \quad 2 < q < \frac{2(n+b)}{n-2s}.$$

From Theorem 2.1 we then have the compact imbeddings

$$H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^b), \quad H^t_{rad}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, |x|^a). \tag{4.1}$$

Recalling that $s + t = 2$, and following the ideas of [13], we consider the functional associated to (2.1) given by:

$$\Phi(u, v) = \int_{\mathbb{R}^n} A^s u \cdot A^t v - \int_{\mathbb{R}^n} H(x, u, v) \tag{4.2}$$

in the subspace $E = H^s_{rad}(\mathbb{R}^n) \times H^t_{rad}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \times H^t(\mathbb{R}^n)$, with the pseudo-differential operator $A^s u = (-\Delta + I)^{s/2}$ given in terms of the Fourier transform:

$$\widehat{A^s u}(\omega) = (1 + |\omega|^2)^{s/2} \hat{u}(\omega),$$

and where H is the Hamiltonian:

$$H(x, u, v) = \frac{|x|^b |u|^q}{q} + \frac{|x|^a |v|^p}{p}.$$

The imbeddings (4.1) imply that Φ is well defined in E and $\Phi \in C^1(E, \mathbb{R})$ (see the appendix of [33]).

Definition 2.6. We say that $z = (u, v) \in H^s_{rad} \times H^t_{rad}$ is an (s,t)-*weak solution* of the system (2.1) if z is a critical point of the functional (4.2).

Remark 2.7. The functional Φ is not well-defined in $H^s(\mathbb{R}^n) \times H^t(\mathbb{R}^n)$ because the imbedding (4.1) is not valid in general for non-radial functions. However, the functional is well-defined in $\tilde{E} = (H^s(\mathbb{R}^n) \cap L^q(\mathbb{R}^n, |x|^b dx)) \times (H^t(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, |x|^a dx))$. Moreover, since Φ is invariant with respect to radial symmetries, the critical points of Φ in E are also critical points in $\tilde{E}$ thanks to the Symmetric Criticality Principle (see Theorem A 5.4 of [31]).

Next, consider the bilinear form $B : E \times E \to \mathbb{R}$ given by:

$$B[z, \eta] := \int (A^s u A^t \phi + A^s \psi A^t v) \text{ where } z = (u, v),\ \eta = (\psi, \phi).$$

Asociated with B we have the quadratic form

$$Q(z) = \frac{1}{2} B(z, z) = \int A^s u A^t v.$$

It is well-known that the operator $L : E \to E$ defined by $\langle Lz, \eta \rangle = B[z, \eta]$ has exactly two eigenvalues $+1$ and -1 and that the corresponding eigenspaces are given by

$$E^+ = \{(u, A^{-t} A^s u) : u \in E^s\}, \qquad E^- = \{(u, -A^{-t} A^s u) : u \in E^s\}.$$

Then, we have that

$$\Phi(z) = \frac{1}{2} \langle Lz, z \rangle - \Psi(z) \tag{4.3}$$

where:

$$\Psi(z) = \int H(x, u, v).$$

We now define the sequence of finite dimensional subspaces that we need to apply Theorem 2.4. For this purpose, choose an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of $H^s_{rad}(\mathbb{R}^n)$. By density, we can choose $e_j \in \mathcal{S}(\mathbb{R}^n)$ (the Schwarz class). Then $f_j = A^{-t} A^s e_j$ form an orthonormal basis of $H^t_{rad}(\mathbb{R}^n)$, $f_j \in \mathcal{S}(\mathbb{R}^n)$, and we may define the following finite dimensional subspaces:

$$E^s_n = \langle e_j : j = 1...n \rangle \subset H^s_{rad}(\mathbb{R}^n)$$

$$E^t_n = \langle f_j : j = 1...n \rangle \subset H^t_{rad}(\mathbb{R}^n)$$

$$E_n = E^s_n \oplus E^t_n.$$

2.4.2 The Palais-Smale condition

In what follows, we will prove that the functional Φ satisfies conditions $(\Phi_1), (\Phi_2') - (\Phi_4'), (\Phi_5)$ in Theorem 2.4. We begin by checking the compactness condition $(PS)_c^{\mathcal{F}}$:

Lemma 2.8. *Condition* (Φ_1) *holds.*

Proof. Using the imbedding in Theorem 2.1, it follows from standard arguments (see for example [33]) that Φ is well defined, and morover $\Phi \in C^1(E, \mathbb{R})$. It remains to show that Φ satisfies the $(PS)_c^{\mathcal{F}}$ condition. Assume that we have a sequence $z_j \in E_{n_j}$ such that $\Phi(z_j) \to c, (1 + \|z_j\|)(\Phi|_{E_{n_j}})'(z_j) \to 0$.

We observe that since the functional Φ has the form (4.3) where $L : E \to E$ is a linear Fredholm operator of index zero and $\nabla\Psi : E \to E$ is completely continuous (due to the compactness of the imbeddings (4.1)), then by Remark 2.1 of [3], it is enough to prove that z_j is bounded.

Since $(\Phi|_{E_{n_j}})'(z_j) \to 0$, in particular we have that

$$|\Phi'(z_j)(w)| \leq C\|w\|_E \text{ for all } w \in E_{n_j}. \tag{4.4}$$

If $z_j = (u_j, v_j)$, taking $w_j = \frac{pq}{p+q}\left(\frac{1}{p}u_j, \frac{1}{q}v_j\right)$, we have that

$$\begin{aligned} C(1 + \|w_j\|_E) &\geq \Phi(z_j) - \Phi'(z_j)(w_j) \\ &= \int A^s u_j A^t v_j - \int H(x, u_j, v_j) \\ &\quad - \Bigg[\frac{q}{p+q}\int A^s u_j A^t v_j + \frac{p}{p+q}\int A^s u_j A^t v_j \\ &\quad - \frac{q}{p+q}\int H_u(x, u_j, v_j)u_j - \frac{p}{p+q}\int H_v(x, u_j, v_j)v_j\Bigg] \\ &= \left(\frac{pq}{p+q} - 1\right)\int H(x, u_j, v_j). \end{aligned}$$

Using (2.2) and Theorem 2.1 we obtain

$$\int H(u_j, v_j, x)\, dx = \int \frac{|x|^b |u_j|^q}{q} + \frac{|x|^a |v_j|^p}{p} \leq C(1 + \|u_j\|_{H^s} + \|v_j\|_{H^t}). \tag{4.5}$$

Now, considering $w = (\psi, 0), \psi \in E^s_{n_j} \subset H^s_{rad}(\mathbb{R}^n)$ in (4.4)

$$|Q'(z_j)(w)| = \left|\int A^s\psi A^t v_j\right| \leq \int |H_u(u_j, v_j, x)\psi|\, dx + C\|\psi\|_{H^s}$$

$$= \int |x|^b |u_j|^{q-2} u_j \psi + C\|\psi\|_{H^s} \leq \|u_j\|^{q-1}_{L^q(|x|^b)} \|\psi\|_{L^q(|x|^b)} + C\|\psi\|_{H^s}$$

and using Theorem 2.1 we conclude that

$$\left| \int A^s \psi A^t v_j \right| \leq C \left(\|u_j\|^{q-1}_{L^q(|x|^b)} + 1 \right) \|\psi\|_{H^s}.$$

Using a duality argument (and the fact that $\int A^s \psi A^t v_j = 0 \ \forall \psi \in (E^s_{n_j})^\perp$), this implies that

$$\|v_j\|_{H^t} \leq C \left(\|u_j\|^{q-1}_{L^q(|x|^b)} + 1 \right). \tag{4.6}$$

Similarly, taking $w = (0, \psi), \psi \in E^t_{n_j}$ in (4.4), we obtain

$$|Q'(z_j)(w)| = \left| \int A^s u_j A^t \psi \right| \leq C \left(\|v\|^{p-1}_{L^p(|x|^a)} + 1 \right) \|\psi\|_{H^t}$$

hence,

$$\|u_j\|_{H^s} \leq C \left(\|v_j\|^{p-1}_{L^p(|x|^a)} + 1 \right). \tag{4.7}$$

Therefore, replacing (4.6) and (4.7) into (4.5), we obtain

$$\left(\frac{1}{C}\|u_j\|_{H^s} - 1 \right)^{q/(q-1)} + \left(\frac{1}{C}\|v_j\|_{H^t} - 1 \right)^{p/(p-1)} \leq C \left(1 + \|u_j\|_{H^s} + \|v_j\|_{H^t} \right).$$

Since $p, q > 1$, we conclude that z_j is bounded in E, as we have claimed. It follows that Φ satisfies the $(PS)^{\mathcal{F}}_c$ condition.

Lemma 2.9. *Condition* $(\Phi_2\prime)$ *holds.*

Proof. We follow the proof of Lemma 3.2 of [3]. We apply Lemma 2.5 in E^+, with

$$P(z) := Q(z) = \int A^s u A^t v, \quad \Psi(z) := \int H(x, u, v)\, dx.$$

Since $z \in E^+$,

$$Q(z) = \int A^s u A^t (A^{-t} A^s u)\, dx = \int |A^s u|^2\, dx = \|u\|^2_{H^s} = \frac{1}{2}\|z\|^2_E.$$

Using the imbeddings (4.1), we have that

$$\left| \int H(x, u, v) \right| \leq C \left(\|u\|^q_{H^s} + \|v\|^p_{H^t} \right) \leq C \left(\|z\|^q_E + \|z\|^p_E \right) \leq \|z\|^{\max(p,q)}_E.$$

Thus, we have $\Phi = P - \psi$ with $P(z) \geq \frac{1}{2}\|z\|^2_E$ and $|\psi(z)| = |\int H(x, u, v)| \leq C(1 + \|z\|^{\max(p,q)}_E)$, with $\max(p, q) > 2$. Therefore, by Lemma 2.5, condition $(\Phi_2\prime)$ holds.

2.4.3 The Geometry of the Functional Φ

In the next two lemmas, we check the requiered conditions on the geometry of the functional Φ:

Lemma 2.10. *Condition* $(\Phi_3\prime)$ *holds.*

Proof. We follow the proof of Lemma 5.1 of [3]. We want to prove that there exist isomorphisms $T_k : E \to E$ $(k \in \mathbb{N})$ such that $T_k(E_n) = E_n$ for all k, n and that there exist $R_k > 0 (k \in \mathbb{N})$ such that, if $z = z^+ + z^- \in E_k^+ \oplus E_k^-$ with $R_k = \max(\|z^+\|, \|z^-\|)$, then $\|T_k z\| > r_k$ and $\phi(T_k z) < 0$ (r_k being the same as that in condition $(\Phi_2\prime)$).

We want to see that there exists λ_k such that the above condition holds with $T_k = T_{\lambda_k}$ and $R_k = \lambda_k$, where

$$T_{\lambda_k}(u, v) = (\lambda_k^\mu u, \lambda_k^\nu v) \text{ with } \mu = \frac{m-q}{q}, \ \nu = \frac{m-p}{p}, \ m > \max(p, q).$$

Clearly, $T_k : E \to E$ is isomorphism for all k. Moreover, $T_{\lambda_k} E_n = E_n$ for all k and, for all $\lambda > 0$, we have that

$$\int_{\mathbb{R}^n} H(x, T_\lambda z) \geq C \left(\lambda^{\mu q} \int_{\mathbb{R}^n} |u|^q |x|^b + \lambda^{\nu p} \int_{\mathbb{R}^n} |v|^p |x|^a \right). \tag{4.8}$$

For $z = z^+ + z^- \in E_k^+ \oplus E_k^-$, let $z^- = z_1^- + z_2^-$ with $z_1^- \in E_k^-$ and $z_2^- \perp E_k^-$, and let $\bar{z} = z^+ + z_1^-$. If $z = (u, v)$, we extend these definitions to u and have that $\bar{u} = u^+ + u_1^-$ and, therefore, $u_2^- \perp \bar{u}$ in L^2. Then,

$$\|\bar{u}\|_{L^2}^2 = |\langle \bar{u}, \bar{u} \rangle_{L^2}| = |\langle \bar{u} + u_2^-, \bar{u} \rangle_{L^2}| = |\langle u, \bar{u} \rangle_{L^2}|$$

$$= \int_{\mathbb{R}^n} |u||\bar{u}||x|^{b/q}|x|^{-b/q} \leq \|u\|_{L^q(|x|^b)} \|\bar{u}\|_{L^{q'}(|x|^{-b/(q-1)})}.$$

But, since $\bar{u} \in E_s^k \subset \mathcal{S}(\mathbb{R}^n)$, we have that

$$\|\bar{u}\|_{L^{q'}(|x|^{-b/(q-1)})} = \left(\int_{\mathbb{R}^n} |\bar{u}|^{q'} |x|^{-b/(q-1)} \right)^{1/q'} < +\infty \text{ since } b < n(q-1)$$

and, thanks to the equivalence of the norms $\|\bar{u}\|_{L^2}$ and $\|\bar{u}\|_{L^{q'}(|x|^{-b/(q-1)})}$ (in the finite dimensional subespace E_k^s), we obtain

$$\|u\|_{L^q(|x|^b)} \geq \gamma_k \|\bar{u}\|_{H^s} \ \forall\, u \in E_k^s$$

for some $\gamma_k > 0$. Similarly there exists $\overline{\gamma}_k > 0$ such that

$$\|v\|_{L^p(|x|^a)} \geq \overline{\gamma}_k \|\bar{v}\|_{H^t} \ \forall\, v \in E_k^t.$$

It then follows from (4.8) that

$$\int_{\mathbb{R}^n} H(x, T_\lambda z) \geq C\left(\lambda^{\mu q}\ \gamma_k^q \|\overline{u}\|_{H^s}^q + \lambda^{\nu p}\ \overline{\gamma}_k^p \|\overline{v}\|_{H^t}^p\right)$$

and (as in lemma 4.2 of [3]) we get a lower bound of the form:

$$\int_{\mathbb{R}^n} H(x, T_\lambda z) \geq c \min\left\{\frac{1}{2^q}\lambda^{\mu q}\gamma_k^q \lambda^q, \frac{1}{2^p}\lambda^{\nu p}\overline{\gamma}_k^p \lambda^p\right\} \geq \sigma_k \lambda^m$$

provided that $\|z^+\|_E = \lambda$.

On the other hand,

$$Q(T_\lambda z) = \lambda^{\nu+\mu}(\|z^+\|_E^2 - \|z^-\|_E^2) \leq \lambda^{\nu+\mu+2}$$

for $\|z^+\|_E = \lambda$. As a consequence, we have that

$$\Phi(T_\lambda z) \leq \lambda^{\nu+\mu+2} - \sigma_k \lambda^m.$$

Since $m > \nu + \mu + 2$, it follows that there is a $\lambda_0(k)$ such that $T_{\lambda_k}(z) < 0$ if $\lambda_k > \lambda_0(k)$. Also we have that

$$\|T_\lambda z\|_E \geq \lambda^{\min(\nu,\mu)} \|z\|_E^2$$

which implies that

$$\|T_\lambda z\|_E \geq \lambda_k^{\min(\mu,\nu)+2} \text{ for } \max(\|z^+\|_E, \|z^-\|_E) = \lambda_k$$

Therefore, it is possible to select $\lambda_k > 0$ such that

$$\Phi(T_{\lambda_k} z) \leq 0 \quad \text{and} \quad \|T_{\lambda_k} z\|_E \geq r_k$$

for any given r_k.

Finally, we observe that condition (Φ_5) holds trivially. Therefore, all the conditions of Theorem 2.4 are fulfilled, and hence the proof of Theorem 2.2 is complete.

Chapter 3

Application to Caffarelli-Kohn-Nirenberg type inequalities

In this chapter we will use the weighted estimates for the fractional integral of radial functions from Chapter 1 to obtain an improvement of the Caffarelli-Kohn-Nirenberg inequality [6] in the case of radially symmetric functions. More precisely, the improvement that we will obtain will be a direct consequence of Theorem 1.1 together with the well-known inequality relating $u \in C_0^\infty(\mathbb{R}^n)$ with the fractional integral of its gradient

$$|u(x)| \le C \int_{\mathbb{R}^n} \frac{|\nabla u|(y)}{|x-y|^{n-1}}\, dy =: T_{n-1}(|\nabla u|)(x), \tag{3.1}$$

and the observation the Caffarelli-Kohn-Nirenberg inequality enjoys a certain self-improving property. It is worth noting that this method of proof is different from that of the original proof in [6], and also from a different approach developed by F. Catrina and Z-Q. Wang in [7].

We then use similar ideas of to show that also certain trace inequalities admit better power weights when restricted to radially symmetric functions. In this case the operator associated to the inequalities we will prove is no longer the fractional integral, but, as we shall see, the analogous of Theorem 1.1 for this operator can be obtained with similar ideas.

The results appearing in this chapter are the subject of the article [9].

3.1 Statement of results and structure of this chapter

Recall the Cafarelli-Kohn-Nirenberg first order interpolation inequality

Theorem ([6]). *Assume*

$$p, q \geq 1, \quad r > 0, \quad 0 \leq a \leq 1 \tag{3.2}$$

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0, \tag{3.3}$$

where

$$\gamma = a\sigma + (1-a)\beta. \tag{3.4}$$

Then, there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\||x|^\gamma u\|_{L^r} \leq C \||x|^\alpha \nabla u\|_{L^p}^a \||x|^\beta u\|_{L^q}^{1-a} \tag{3.5}$$

if and only if the following relations hold:

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right) \tag{3.6}$$

$$0 \leq \alpha - \sigma \qquad \text{if } a > 0, \tag{3.7}$$

and

$$\alpha - \sigma \leq 1 \qquad \text{if } a > 0 \quad \text{and} \quad \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}. \tag{3.8}$$

As explained before, although the conditions of the above theorem cannot be improved in general, we will prove that if we require u to be radially symmetric, inequality (3.5) holds true for certain negative values of $\alpha - \sigma$ also. Indeed, the first theorem we will prove in this chapter is:

Theorem 3.1. *Assume conditions* (3.2), (3.3), (3.4) *and* (3.6) *hold. Then there exists a positive constant C such that inequality* (3.5) *holds for all radially symmetric $u \in C_0^\infty(\mathbb{R}^n)$ and all*

$$\frac{1-a}{q} \leq \frac{1}{r} \leq \frac{a}{p} + \frac{1-a}{q} \tag{3.9}$$

provided that, if $a > 0$,

$$(n-1)\left[\frac{1}{a}\left(\frac{1}{r} - \frac{1}{q}\right) + \frac{1}{q} - \frac{1}{p}\right] \leq \alpha - \sigma \leq 0 \tag{3.10}$$

and

$$-\frac{\sigma}{n} < \frac{1}{a}\left(\frac{1}{r} - \frac{1}{q}\right) + \frac{1}{q}, \tag{3.11}$$

with strict inequality in (3.10) *if $p = 1$.*

Remark 3.2. *If $\sigma > 0$ condition* (3.11) *trivially holds because of* (3.9)*, and thus our result admits a simpler statement in this case.*

We then show that also improved trace inequalities can be obtained in a similar way, but looking at weighted estimates for another operator instead of the fractional integral. However, we will see that the behavior of this operator when restricted to radially symmetric functions can be analyzed with ideas similar to those used in Chapter 1. By doing this, we will prove the following theorem:

Theorem 3.3. *Let $x \in \mathbb{R}^n$ and*

$$Tf(x) := \int_{\mathbb{R}^n \times \mathbb{R}_+} \frac{f(y,z)}{[(x-y)^2 + z^2]^{\frac{n}{2}}}\, dy\, dz. \tag{3.12}$$

Assume $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ is such that $f(y,z) = f_0(|y|, z)$. Then, the inequality

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \tag{3.13}$$

holds provided that

$$1 \leq p \leq q < \infty \tag{3.14}$$

$$\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1 \tag{3.15}$$

and

$$-\frac{n}{q'} < \beta < \frac{n}{q}. \tag{3.16}$$

Once this theorem is proved, we will use the weighted estimates to obtain the following trace inequality:

Theorem 3.4. *Let $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ be a radially symmetric function in the first n variables. Then, the following inequality holds*

$$\|f(x,0)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^\alpha \nabla f(y,z)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)} \tag{3.17}$$

provided that:

$$-\frac{n}{q'} \leq \alpha + \beta \leq \frac{1}{p'}, \tag{3.18}$$

$$\alpha > -\frac{n+1}{p} + 1, \tag{3.19}$$

and

$$\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1. \tag{3.20}$$

Remark 3.5. *Using condition* (3.20)*, condition* (3.18) *can be seen to be equivalent to $1 \leq p \leq q < \infty$.*

The above theorem can be easily seen to be a refinement for radially symmetric functions (in the first n variables) of the following known trace inequality:

Theorem ([2])**.** *Let $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$. Then, the following inequality holds*

$$\||x|^{-\beta} f(x,0)\|_{L^q(\mathbb{R}^n)} \leq C \||(y,z)|^\alpha \nabla f(y,z)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^+)}$$

provided that:

$$0 \leq \alpha + \beta \leq \frac{1}{2}, \tag{3.21}$$

$$\alpha > -\frac{n+1}{2} + 1, \tag{3.22}$$

and

$$\frac{n}{q} - \frac{n+1}{2} = \alpha + \beta - 1. \tag{3.23}$$

The remainder of this chapter is organized as follows:

In Section 3.2 we prove Theorem 3.1. In Section 3.3 we explain the relation between the operator Tf defined by (3.12) and the weighted trace inequalities we are interested in, and find a convenient expression for this operator when acting on radially symmetric functions (in the first n variables). In Section 3.4 we prove Theorem 3.3 and, finally, in Section 3.4, we use Theorem 3.3 to prove Theorem 3.4.

3.2 Proof of Theorem 3.1

Clearly, when $a = 0$ the theorem is completely trivial. Therefore, we will split the proof into two cases, namely, when $a = 1$ and when $0 < a < 1$.

3.2.1 Case $a = 1$

Notice that in this case, $\sigma = \gamma$ by (3.4).

Observing that for $u \in C_0^\infty(\mathbb{R}^n)$

$$|u(x)| \leq C \int_{\mathbb{R}^n} \frac{|\nabla u|(y)}{|x-y|^{n-1}}\, dy := T_{n-1}(|\nabla u|)(x)$$

we see that

$$\||x|^\gamma u\|_{L^r} \leq C \||x|^\gamma T_{n-1}(|\nabla u|)\|_{L^r}$$

but, since we are assuming that u is a radial function, then so is $|\nabla u|$ and we can use Theorem 1.1 to deduce that

$$\||x|^\gamma T_{n-1}(|\nabla u|)\|_{L^r} \le C\||x|^\alpha \nabla u\|_{L^p}$$

provided that

$$1 \le p \le r < \infty \tag{3.24}$$

$$\frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha - 1}{n} \tag{3.25}$$

$$\alpha < \frac{n}{p'} \tag{3.26}$$

$$-\gamma < \frac{n}{r} \tag{3.27}$$

and

$$(n-1)\left(\frac{1}{r} - \frac{1}{p}\right) \le \alpha - \gamma, \tag{3.28}$$

with strict inequality in (3.28) if $p = 1$.

Clearly, the scaling condition (3.25) equals condition (3.6) when $a = 1$; and using (3.25), condition (3.24) can be seen to be equivalent to $\gamma - \alpha \le 1$, which holds because of hypothesis (3.10) (recall that in this case $\gamma = \sigma$). Condition (3.27) equals condition (3.11) (in this case it is also included in (3.3)); and (3.28) follows from (3.10) since $a = 1$.

We claim that condition (3.26) can be removed if we only wish to consider the inequality

$$\||x|^\gamma u\|_{L^r} \le C\||x|^\alpha \nabla u\|_{L^p} \tag{3.29}$$

(this is not the case if the operator T_{n-1} is not acting on $|\nabla u|$). Indeed, we will prove that if (3.29) holds for α and γ, then it also holds for $\alpha + 1$ and $\gamma + 1$, provided that $\alpha p \ne -1$. To this end, we apply the inequality to $|x|u$ (strictly speaking, this function is not C_0^∞, but it suffices to take a regularized distance function to the origin, see e.g. [41], and apply the same argument).

Then,

$$\||x|^{\gamma+1} u\|_r \le C\||x|^\alpha \nabla(|x|u)\|_p \sim C(\||x|^{\alpha+1}\nabla u\|_p + \||x|^\alpha u\|_p)$$

and, therefore, it suffices to see that $\||x|^{\alpha}u\|_p \leq C\||x|^{\alpha+1}\nabla u\|_p$. To this end write

$$\begin{aligned}
\||x|^{\alpha}u\|_p^p &= \int |x|^{p\alpha}|u|^p\,dx \\
&\leq C\int |\nabla|x|^{p\alpha+1}||u|^p\,dx \\
&\leq C\int |x|^{p\alpha+1}|\nabla|u|^p|\,dx \\
&\leq C\int |x|^{p\alpha+1}|u|^{p-1}|\nabla u|\,dx \\
&\leq C\left(\int |x|^{p\alpha}|u|^p\,dx\right)^{\frac{1}{p'}}\left(\int |x|^{p(\alpha+1)}|\nabla u|^p\,dx\right)^{\frac{1}{p}}
\end{aligned}$$

Thus, we have proved that

$$\||x|^{\alpha}u\|_p^p \leq C\||x|^{\alpha}u\|_p^{\frac{p}{p'}}\||x|^{\alpha+1}\nabla u\|_p$$

whence it follows immediately that

$$\||x|^{\alpha}u\|_p \leq C\||x|^{\alpha+1}\nabla u\|_p.$$

Iterating the same argument, we can see that if (3.29) holds for γ and α, then it also holds for $\gamma+k$ and $\alpha+k$ with $k \in \mathbb{N}_0$ provided that $(\alpha-k)p \neq -1$. Therefore, to see that we can remove condition (3.26), it suffices to observe that any $\alpha \geq \frac{n}{p'}$ can be written as $(\alpha-k)+k$, with $-\frac{n}{p} < \alpha-k < \frac{n}{p'}$, and $(\alpha-k)p \neq -1$. Indeed, since $\frac{n}{p'}-(-\frac{n}{p}) = n$, such a k exists except when $n=1$ and $\alpha = \frac{1}{p'}$. But this is impossible, since in that case, by (3.25) we should have $\frac{1}{r}+\gamma = \frac{1}{p}+\frac{1}{p'}-1$, that is, $\frac{1}{r}+\gamma = 0$, which contradicts (3.3).

3.2.2 Case $0 < a < 1$

Write

$$\begin{aligned}
\left(\int |x|^{\gamma r}|u|^{ra+(1-a)r}\,dx\right)^{\frac{1}{r}} &= \left(\int |x|^{r\beta(1-a)}|u|^{(1-a)r}|x|^{r\gamma(1-\frac{\beta(1-a)}{\gamma})}|u|^{ar}\,dx\right)^{\frac{1}{r}} \\
&\leq \||x|^{\beta}u\|_{L^q}^{1-a}\||x|^{\frac{\gamma}{a}(1-\frac{\beta(1-a)}{\gamma})}u\|^a_{L^{\frac{arq}{q-r(1-a)}}} \qquad (3.30) \\
&= \||x|^{\beta}u\|_{L^q}^{1-a}\||x|^{\sigma}u\|^a_{L^{\frac{arq}{q-r(1-a)}}} \qquad (3.31)
\end{aligned}$$

where in (3.30) we have used Hölder's inequality with exponent $\frac{q}{r(1-a)}$ (which is larger than 1 by (3.9)) and in (3.31) we have used the definition of σ, given in (3.4).

Applying now the result obtained in the case $a = 1$, we deduce that

$$\||x|^\gamma u\|_{L^r} \leq C\||x|^\beta u\|_{L^q}^{1-a}\||x|^\alpha \nabla u\|_{L^p}^a$$

provided that

$$1 \leq p \leq \frac{arq}{q - r(1-a)} < \infty \tag{3.32}$$

$$\frac{q - r(1-a)}{arq} + \frac{\sigma}{n} = \frac{1}{p} + \frac{\alpha - 1}{n} \tag{3.33}$$

$$-\sigma < \frac{n(q - r + ar)}{arq} \tag{3.34}$$

and

$$\alpha - \sigma \geq (n-1)\left(\frac{q - r(1-a)}{arq} - \frac{1}{p}\right), \tag{3.35}$$

where in (3.35) the inequality is strict if $p = 1$.

Clearly, condition (3.32) holds because of (3.9), and condition (3.33) is easily seen to be equivalent to (3.6) using the definition of σ given in (3.4). Finally, condition (3.34) equals conditon (3.11) while (3.35) is the same as (3.10). This concludes the proof.

3.3 The operator associated to trace inequalities

Before we can proceed to the proof of the announced trace inequality, we first need to obtain an expression analogous to (3.1) and, then, a convenient expression for the involved operator when acting on radial functions.

To this end, given u and a unitary vector ξ, consider $g(s) = u(s\xi, 0)$. Then, $g(0) = -\int_0^\infty g'(s)\,ds = -\int_0^\infty \nabla u(s\xi)\cdot\xi\,ds$.

Consider now $\varphi \in C_0^\infty(S^n)$ supported in $\mathbb{R}^n \times \mathbb{R}_+$ and such that $\int_{S^n}\varphi(\xi)\,d\sigma(\xi) = 1$. Then

$$u(0,0) = -\int_0^\infty \int_{S_n} \nabla u(s\xi)\cdot\xi\,\varphi(\xi)\,d\xi\,ds.$$

For $(y,z) \in \mathbb{R}^{n+1}$ let $\phi(y,z) = \varphi((y,z)/\|(y,z)\|)$. Therefore, $\phi(s\xi) = \varphi(\xi)$ for all $s \in \mathbb{R}^+, \xi \in S^n$, and the above identity becomes

$$\begin{aligned} u(0,0) &= -\int_0^\infty \int_{S_n} \nabla u(s\xi)\cdot s\xi\,\phi(s\xi)\frac{1}{s^{n+1}}s^n\,ds\,d\xi \\ &= -\int_{\mathbb{R}^n\times\mathbb{R}_+} \nabla u(y,z)\cdot(y,z)\,\phi(y,z)\frac{1}{\|(y,z)\|^{n+1}}\,dy\,dz \end{aligned}$$

More generally,

$$
\begin{aligned}
|u(x,0)| &\le \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla u(y,z)| \frac{1}{\|(x-y,z)\|^n}\, dy\, dz \\
&= \int_{\mathbb{R}^n\times\mathbb{R}_+} |\nabla u(y,z)| \frac{1}{[(x-y)^2+z^2]^{\frac{n}{2}}}\, dy\, dz
\end{aligned}
$$

Then, we have to study the behavior of the operator

$$
Tf(x) = \int_{\mathbb{R}^n\times\mathbb{R}_+} \frac{f(y,z)}{[(x-y)^2+z^2]^{\frac{n}{2}}}\, dy\, dz
$$

for $x \in \mathbb{R}^n$.

Since we are interested in the radial case, assume f is a radially symmetric function in the first variable (by an abuse of notation we will still call it f).

Using polar coordinates

$$
\begin{aligned}
y = ry', &\quad r = |y|, \quad y' \in S^{n-1} \\
x = \rho x', &\quad \rho = |x|, \quad x' \in S^{n-1}
\end{aligned}
$$

if $n \ge 2$ we may write:

$$
\begin{aligned}
Tf(x) &= \int_0^\infty \left[\int_0^\infty \int_{S^{n-1}} \frac{f(r,z)\, r^{n-1}}{(\rho^2 - 2\rho r x'\cdot y' + r^2 + z^2)^{\frac{n}{2}}}\, dy'\, dr \right] dz \\
&= \int_0^\infty \int_0^\infty f(r,z) r^{n-1} \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(\rho^2 - 2\rho r t + r^2 + z^2)^{\frac{n}{2}}}\, dt\, dr\, dz
\end{aligned}
$$

where the second equality can be justified integrating in the sphere as in Lemma 1.6 from Chapter 1.

Making the change of variables $z = r\bar{z}$, $dz = r\, d\bar{z}$ we obtain

$$
\begin{aligned}
Tf(x) &= \int_0^\infty \int_0^\infty f(r, r\bar{z}) r^n \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{r^n \left[1 - 2\left(\frac{\rho}{r}\right) t + \left(\frac{\rho}{r}\right)^2 + \bar{z}^2\right]^{\frac{n}{2}}}\, dt\, dr\, d\bar{z} \\
&= \int_0^\infty \int_0^\infty f(r, rz) I\left(\frac{\rho}{r}, z\right)\, dr\, dz \qquad (3.36)
\end{aligned}
$$

where, for $a > 0$,

$$
I(a,z) := \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(1 - 2at + a^2 + z^2)^{\frac{n}{2}}}\, dt.
$$

Expression (3.36) will allow us to write Tf as convolution operator and to obtain Theorem 3.3, that we proceed to prove next.

3.4 Proof of Theorem 3.3

If $n = 1$ recall that we want to prove

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R})} \leq C\||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}\times\mathbb{R}^+)}$$

Since in this case (3.36) does not hold, we remark that

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R},dx)} = \||x|^{-\beta+\frac{1}{q}}Tf\|_{L^q(\mathbb{R},\frac{dx}{|x|})}$$

and write

$$\begin{aligned}
|x|^{-\beta+\frac{1}{q}}Tf(x) &= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{f(y,z)|x|^{-\beta+\frac{1}{q}}}{[(x-y)^2+z^2]^{\frac{1}{2}}}\,dz\,dy \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} \frac{f(y,|y|\bar{z})|x|^{-\beta+\frac{1}{q}}|y|}{(|\frac{x}{y}-1|^2+\bar{z}^2)^{\frac{1}{2}}}\,d\bar{z}\,\frac{dy}{|y|} \\
&= \int_0^{\infty}\int_{-\infty}^{\infty} \frac{f(y,|y|\bar{z})(\frac{|x|}{|y|})^{-\beta+\frac{1}{q}}|y|^{1-\beta+\frac{1}{q}}}{(|\frac{x}{y}-1|^2+\bar{z}^2)^{\frac{1}{2}}}\,\frac{dy}{|y|}\,d\bar{z} \\
&= \int_0^{\infty} (f(y,|y|\bar{z})|y|^{1-\beta+\frac{1}{q}}) * \left(\frac{|y|^{-\beta+\frac{1}{q}}}{(|y-1|^2+\bar{z}^2)^{\frac{1}{2}}}\right) d\bar{z}
\end{aligned}$$

where the convolution is taken with respect to the first variable in the multiplicative group $\mathbb{R}-\{0\}$ with Haar measure $dx/|x|$.

Let $g(y) = \frac{|y|^{-\beta+\frac{1}{q}}}{(|y-1|^2+\bar{z}^2)^{\frac{1}{2}}}$. Then, by Young's inequality, if

$$\frac{1}{q}+1 = \frac{1}{p}+\frac{1}{s} \tag{3.37}$$

$$\begin{aligned}
&\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R},dx)} \\
&\quad\leq \int_0^{\infty} \|f(y,|y|\bar{z})|y|^{1-\beta+\frac{1}{q}}\|_{L^p(\frac{dy}{|y|})}\|g\|_{L^s(\frac{dy}{|y|})}\,d\bar{z} \\
&\quad= \int_0^{\infty}\left(\int_{-\infty}^{\infty}|f(y,|y|\bar{z})|^p|y|^{(1-\beta+\frac{1}{q})p-1}(1+\bar{z}^2)^{\frac{\alpha p}{2}}\right)^{\frac{1}{p}}\left(\frac{\|g\|_{L^s(\frac{dy}{|y|})}}{(1+\bar{z}^2)^{\frac{\alpha}{2}}}\right)d\bar{z}
\end{aligned} \tag{3.38}$$

Observing now that

$$\begin{aligned}\||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}\times\mathbb{R}_+)} &= \int_0^\infty \int_{-\infty}^\infty (y^2+z^2)^{\frac{\alpha p}{2}} |f(y,z)|^p \, dy\, dz \\ &= \int_0^\infty \int_{-\infty}^\infty (y^2+y^2\bar{z}^2)^{\frac{\alpha p}{2}} |f(y,|y|\bar{z})|^p |y| \, dy\, d\bar{z} \\ &= \int_0^\infty \int_{-\infty}^\infty (1+\bar{z}^2)^{\frac{\alpha p}{2}} |y|^{\alpha p+1} |f(y,|y|\bar{z})|^p \, dy\, d\bar{z}\end{aligned}$$

and that $(1-\beta+\frac{1}{q})p-1=\alpha p+1$ (by (3.15)), we can apply Hölder's inequality to (3.38) to obtain

$$\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} \le \||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} \left(\int_0^\infty \frac{\|g\|^{p'}_{L^s(\frac{dy}{|y|})}}{(1+z^2)^{\frac{\alpha p'}{2}}}\, dz\right)^{\frac{1}{p'}}$$

Therefore, to conclude the proof of the one-dimensional case it suffices to see that

$$\int_0^\infty \frac{\|g\|^{p'}_{L^s(\frac{dy}{|y|})}}{(1+z^2)^{\frac{\alpha p'}{2}}}\, dz < +\infty$$

provided that (3.15), (3.16) and (3.37) hold. We omit the details since the computations are analogous to those that we will do in the higher dimensional case.

Now we proceed to the case $n \ge 2$. In this case, remark that,

$$\begin{aligned}\|Tf(x)|x|^{-\beta}\|_{L^q(\mathbb{R}^n)} &= C\left(\int_0^\infty |Tf(\rho)|^q \rho^{-\beta q+n} \frac{d\rho}{\rho}\right)^{\frac{1}{q}} \\ &= C\|\rho^{-\beta+\frac{n}{q}} Tf\|_{L^q(\frac{d\rho}{\rho})}\end{aligned}$$

We claim that $\rho^{-\beta+\frac{n}{q}}Tf$ can be written as a convolution in the multiplicative group $(\mathbb{R}_+,\cdot)$. Indeed,

$$\begin{aligned}\rho^{-\beta+\frac{n}{q}}Tf &= \int_0^\infty \int_0^\infty f(r,rz)\, I\left(\frac{\rho}{r},z\right) \rho^{-\beta+\frac{n}{q}}\, drdz \\ &= \int_0^\infty \int_0^\infty f(r,rz)\, I\left(\frac{\rho}{r},z\right)\left(\frac{\rho}{r}\right)^{-\beta+\frac{n}{q}} r^{-\beta+\frac{n}{q}+1}\, \frac{dr}{r} dz \\ &= \int_0^\infty (f(r,rz)r^{-\beta+\frac{n}{q}+1}) * (I(r,z)r^{-\beta+\frac{n}{q}})\, dz\end{aligned}$$

where $*$ denotes the convolution with respect to the Haar measure dr/r in the first variable.

Therefore, using Young's inequality, for

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{s}, \tag{3.39}$$

we obtain

$$\begin{aligned}
&\|Tf(\rho)\rho^{-\beta+\frac{n}{q}}\|_{L^q(\frac{d\rho}{\rho})}\\
&\quad\leq \int_0^\infty \|(f(r,rz)r^{-\beta+\frac{n}{q}+1})*(I(r,z)r^{-\beta+\frac{n}{q}})\|_{L^q(\frac{dr}{r})}\,dz\\
&\quad\leq \int_0^\infty \|f(r,rz)r^{-\beta+\frac{n}{q}+1}\|_{L^p(\frac{dr}{r})}\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}\,dz\\
&\quad= \int_0^\infty \left(\int_0^\infty |f(r,rz)|^p r^{(-\beta+\frac{n}{q}+1)p}\frac{dr}{r}\right)^{\frac{1}{p}}\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}\,dz\\
&\quad= \int_0^\infty \left(\int_0^\infty |f(r,rz)|^p r^{(-\beta+\frac{n}{q}+1)p}(1+z^2)^{\frac{\alpha p}{2}}\frac{dr}{r}\right)^{\frac{1}{p}}\frac{\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}}{(1+z^2)^{\frac{\alpha}{2}}}\,dz
\end{aligned}$$

Now, since

$$\begin{aligned}
\||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} &= \left(\int_0^\infty\int_0^\infty (r^2+z^2)^{\frac{\alpha p}{2}}|f(r,z)|^p r^{n-1}\,drdz\right)^{\frac{1}{p}}\\
&= \left(\int_0^\infty\int_0^\infty (r^2+r^2\bar{z}^2)^{\frac{\alpha p}{2}}|f(r,r\bar{z})|^p r^{n}\,d\bar{z}dr\right)^{\frac{1}{p}}\\
&= \left(\int_0^\infty\int_0^\infty r^{\alpha p}(1+\bar{z}^2)^{\frac{\alpha p}{2}}|f(r,r\bar{z})|^p r^{n}\,d\bar{z}dr\right)^{\frac{1}{p}},
\end{aligned}$$

observing that $n+\alpha p=p(-\beta+\frac{n}{q}+1)-1$ and applying Hölder's inequality to the above expression, we obtain

$$\begin{aligned}
&\|Tf(\rho)\rho^{-\beta+\frac{n}{q}}\|_{L^q(\frac{d\rho}{\rho})}\\
&\qquad\leq \||(y,z)|^\alpha f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\left(\int_0^\infty \frac{\|I(r,z)r^{-\beta+\frac{n}{q}}\|^{p'}_{L^s(\frac{dr}{r})}}{(1+z^2)^{\frac{\alpha p'}{2}}}\,dz\right)^{\frac{1}{p'}}
\end{aligned}$$

Therefore, to conclude the proof of the theorem it suffices to see that

$$\int_0^\infty \|I(r,z)r^{-\beta+\frac{n}{q}}\|^{p'}_{L^s(\frac{dr}{r})}(1+z^2)^{-\frac{\alpha p'}{2}}\,dz<+\infty. \tag{3.40}$$

Observe first that the denominator of

$$I(r,z)=\int_{-1}^{1}\frac{(1-t^2)^{\frac{n-3}{2}}}{(1-2rt+r^2+z^2)^{\frac{n}{2}}}\,dt$$

can be rewritten as $[(r-t)^2+(1-t^2)+z^2]^{\frac{n}{2}}$ and, therefore, it vanishes for $r=t=1$ and $z=0$ only.

To bound $\|I(r,z)r^{-\beta+\frac{n}{q}}\|_{L^s(\frac{dr}{r})}$, consider $\varphi\in C^\infty(\mathbb{R})$ such that $supp(\varphi)\subseteq[\frac{1}{2},\frac{3}{2}]$, $0\le\varphi\le 1$ and $\varphi\equiv 1$ in $(\frac{3}{4},\frac{5}{4})$. We can then split $I(r,z)r^{-\beta+\frac{n}{q}}=I(r,z)r^{-\beta+\frac{n}{q}}\varphi(r)+I(r,z)r^{-\beta+\frac{n}{q}}(1-\varphi(r))=g_1(r)+g_2(r)$ and bound both terms separately. To this end, we will study first the behavior of g_1 and g_2 and then estimate (3.40).

Consider first g_2. For $r\to 0$, we have

$$I(0,z)=(1+z^2)^{-\frac{n}{2}}\int_{-1}^{1}(1-t^2)^{\frac{n-3}{2}}\,dt\sim(1+z^2)^{-\frac{n}{2}}.$$

Therefore, $\|g_2\|_{L^s(\frac{dr}{r})}$ behaves like $(1+z^2)^{-\frac{n}{2}}$, provided that $\beta<\frac{n}{q}$.

When $r\to\infty$,

$$I(r,z)\sim\frac{1}{(r^2+z^2)^{\frac{n}{2}}}.$$

In this case, if z is bounded, say $z\le 2$, $\|g_2\|_{L^s(\frac{dr}{r})}$ is also bounded provided that $\beta>-\frac{n}{q'}$. On the other hand, when $z\to\infty$, we need to estimate

$$\begin{aligned}\left(\int_2^\infty\frac{r^{s(-\beta+\frac{n}{q})}}{(r^2+z^2)^{\frac{ns}{2}}}\frac{dr}{r}\right)^{\frac{1}{s}}&=\left(z^{s(-\beta+\frac{n}{q}-n)}\int_{\frac{2}{z}}^\infty\frac{r^{s(-\beta+\frac{n}{q})}}{(r^2+1)^{\frac{ns}{2}}}\frac{dr}{r}\right)^{\frac{1}{s}}\\&\sim z^{-\beta-\frac{n}{q'}}\end{aligned}$$

assuming again that $\beta>-\frac{n}{q'}$.

We can proceed now to $\|g_1\|_{L^s(\frac{dr}{r})}$. We consider first the case $k=\frac{n-3}{2}\in\mathbb{N}_0$, that is $n\ge 3$ and odd.

If z is sufficiently large, then $I(r,z)\sim z^{-n}$ and, therefore, $\|g_1\|_{L^s(\frac{dr}{r})}\sim z^{-n}$.

If, on the contrary, $z\to 0$, we may write

$$I(r,z)\sim\int_{-1}^{1}(1-t^2)^k\frac{d^k}{dt^k}\left\{(1-2rt+r^2+z^2)^{-\frac{n}{2}+k}\right\}\,dt$$

and integrating by parts k-times (the boundary terms vanish), we obtain

$$I(r,z)\le C_k[(1-r)^2+z^2]^{-\frac{n}{2}+k+1}.$$

Since we are assuming that $-\frac{n}{2}+k+1=-\frac{1}{2}$, we conclude that

$$\begin{aligned}\|g_1\|_{L^s(\frac{dr}{r})} &\sim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dr}{[(1-r)^2+z^2]^{\frac{s}{2}}}\right)^{\frac{1}{s}}\\ &\sim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{dr}{(|1-r|+z)^s}\right)^{\frac{1}{s}}\\ &\sim \frac{1}{z^{1-\frac{1}{s}}}\end{aligned}$$

We can consider now $k=m+\frac{1}{2}, m\in\mathbb{N}_0$. In this case

$$\begin{aligned}&\left|\frac{d}{dz}I(r,z)\right|\\ &\quad\le Cz\int_{-1}^{1}\frac{(1-t^2)^k}{(1-2rt+r^2+z^2)^{\frac{n}{2}+1}}\,dt\\ &\quad\le Cz\left(\int_{-1}^{1}\frac{(1-t^2)^m}{(1-2rt+r^2+z^2)^{\frac{n+2}{2}}}\,dt\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\frac{(1-t^2)^{m+1}}{(1-2rt+r^2+z^2)^{\frac{n+2}{2}}}\,dt\right)^{\frac{1}{2}}\end{aligned}$$

and, since now $\frac{n+2}{2}\in\mathbb{N}$, we deduce from the previous case that

$$\begin{aligned}\left|\frac{d}{dz}I(r,z)\right| &\le Cz[(1-r^2)+z^2]^{\frac{-(n+2)+2m+3}{2}}\\ &= Cz[(1-r)^2+z^2]^{-\frac{3}{2}}\\ &\le Cz[|1-r|+z]^{-3}\end{aligned}$$

Therefore,

$$I(r,z)=\int_0^z\frac{d}{dt}I(r,t)\,dt\le Cz[|1-r|+s]^{-2}|_0^z\le Cz[|1-r|+z]^{-2}$$

which implies

$$\|g_1\|_{L^s(\frac{dr}{r})}\sim\frac{1}{z^{1-\frac{1}{s}}}.$$

It remains to check the case $k=-\frac{1}{2}$ (i.e., $n=2$). To this end, we write

$$I(r,z)=\underbrace{\int_{-1}^{0}\frac{(1-t^2)^{-\frac{1}{2}}}{(1-2at+a^2+z^2)}\,dt}_{(i)}+\underbrace{\int_{0}^{1}\frac{(1-t^2)^{-\frac{1}{2}}}{(1-2at+a^2+z^2)}\,dt}_{(ii)}$$

Clearly,

$$(i) \leq \int_{-1}^{0} \frac{dt}{(1+t)^{\frac{1}{2}}} = 2$$

while

$$\begin{aligned}(ii) &\leq \int_0^1 \frac{(1-t)^{-\frac{1}{2}}}{(1-2at+a^2+z^2)}\, dt \\ &= -2\int_0^1 \frac{\frac{d}{dt}[(1-t)^{\frac{1}{2}}]}{1-2at+a^2+z^2}\, dt \\ &\leq 4a\int_0^1 \frac{(1-t^2)^{\frac{1}{2}}}{(1-2at+a^2+z^2)^2}\, dt\end{aligned}$$

and the last integral can be bounded as before (notice that it corresponds to the case $n = 4$).

We are now able to see that (3.40) holds. Indeed, by our previous calculations, we need to bound

$$\begin{aligned}&\int_0^1 \left(\frac{1}{z^{1-\frac{1}{s}}(1+z^2)^{\frac{\alpha}{2}}} + \frac{1}{(1+z^2)^{\frac{n+\alpha}{2}}} \right)^{p'} dz \\ +&\int_1^\infty \left(\frac{1}{z^n(1+z^2)^{\frac{\alpha}{2}}} + \frac{1}{z^{\beta+\frac{n}{q'}}(1+z^2)^{\frac{\alpha}{2}}} \right)^{p'} dz\end{aligned}$$

When $z \to 0$, the integrability condition is $p'(1-\frac{1}{s}) < 1$, which holds because of (3.14) and (3.39). When $z \to \infty$, since we are assuming that $\beta < \frac{n}{q}$, there holds that $n > \beta + \frac{n}{q'}$, whence the integralibity condition is $p'(\beta + \frac{n}{q'} + \alpha) > 1$, that is, $\alpha + \beta > \frac{1}{p'} - \frac{n}{q'}$. But, by (3.20) this condition is equivalent to $\frac{n}{p'} > 0$, which trivially holds. This concludes the proof of the theorem.

3.5 Proof of Theorem 3.4

As in the case of the Caffarelli-Kohn-Nirenberg interpolation inequality, if we simply apply Theorem 3.3 to $|\nabla f|$ we obtain (3.17) provided that

$$1 \leq p \leq q < \infty \tag{3.41}$$

$$\frac{n}{q} - \frac{n+1}{p} = \alpha + \beta - 1 \tag{3.42}$$

and

$$-\frac{n}{q'} < \beta < \frac{n}{q}. \tag{3.43}$$

Notice that this last condition is equivalent to $-\frac{n+1}{p} + 1 < \alpha < \frac{n+1}{p'}$ because of (3.42).

To prove Theorem 3.4 we need to see that condition $\alpha < \frac{n+1}{p'}$ is unnecessary for inequality (3.17) to hold. Indeed, with a similar argument as that used for Theorem 3.1, we will prove that if the inequality holds for α and β then it also holds for $\alpha+1$ and $\beta-1$ provided that $\alpha p \neq -1$.

To see this, consider $f(x)|x|$ (strictly speaking, we would need to replace $|x|$ by a regularized distance, to guarantee that the product is in C_0^∞). Then,

$$\begin{aligned}\|f(x,0)|x|^{-\beta+1}\|_{L^q(\mathbb{R}^n)} &\leq C\||(y,z)|^{\alpha}\nabla(|(y,z)|f(y,z))\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\\ &\leq C\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\\ &\quad+\||(y,z)|^{\alpha}f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\end{aligned}$$

Therefore, it suffices to see that

$$\||(y,z)|^{\alpha}f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)} \leq C\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}.$$

To this end, consider

$$\begin{aligned}&\||(y,z)|^{\alpha}f(y,z)\|^p_{L^p(\mathbb{R}^n\times\mathbb{R}^+)}\\ &\qquad= \int_{\mathbb{R}_+}\int_{\mathbb{R}^n}|(y,z)|^{p\alpha}|f(y,z)|^p\,dy\,dz\\ &\qquad\leq C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}|\nabla|(y,z)|^{p\alpha+1}||f(y,z)|^p\,dy\,dz\\ &\qquad\leq C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}|(y,z)|^{p\alpha+1}|\nabla|f(y,z)|^p|\,dy\,dz\\ &\qquad\leq C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}|(y,z)|^{p\alpha+1}|f(y,z)|^{p-1}|\nabla f(y,z)|\,dy\,dz\\ &\qquad= C\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}|(y,z)|^{\alpha(p-1)}|f(y,z)|^{p-1}|(y,z)|^{\alpha+1}|\nabla f(y,z)|\,dy\,dz\end{aligned}$$

Applying Hölder's inequality we see that

$$\||(y,z)|^{\alpha}f(y,z)\|_p^p \leq C\||(y,z)|^{\alpha}f(y,z)\|_p^{\frac{p}{p'}}\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_p$$

and it follows immediately that

$$\||(y,z)|^{\alpha}f(y,z)\|_p \leq C\||(y,z)|^{\alpha+1}\nabla f(y,z)\|_p$$

as we wanted to see.

Iterating the same argument we see that if inequality (3.17) holds for α and β, then it holds for $\alpha + k$ and $\beta - k$ with $k \in \mathbb{N}_0$. Therefore, to see that condition $\alpha < \frac{n+1}{p'}$ is uneccessary, it suffices to see that any $\alpha \geq \frac{n+1}{p'}$ can be written as $(\alpha - k) + k$, with $-\frac{n+1}{p} + 1 < \alpha - k < \frac{n+1}{p'}$ and $(\alpha - k)p \neq -1$.

But, $\frac{n+1}{p'} - \left(-\frac{n+1}{p} + 1\right) = n$, and therefore k can be chosen as above, except when $n = 1$ and $\alpha = \frac{n+1}{p'} = \frac{2}{p'}$ (that is, $\beta = -\frac{1}{q'}$) that cannot happen because for $n = 1$, $\alpha > \frac{2}{p'}$ (because of (3.43) and (3.42)). This completes the proof of the theorem.

Chapter 4

Application to Laplace transform type multipliers for Laguerre and Hermite expansions

In this chapter, we show that the ideas of the proof of Theorem 1.1 can be used to obtain weighted bounds for certain multiplier operators for Laguerre and Hermite expansions. More precisely, we consider multipliers that arise from a Laplace-Stieltjes transform. In doing so, we extend the weighted bounds obtained by for Laguerre fractional integrals by G. Gasper, K. Stempak and W. Trebels in [16] and simplify their method of proof, recovering in particular the improved result for Laguerre fractional integrals proved by G. Gasper and W. Trebels in [17]. We also give a rigorous interpretation of the series defining these operators, showing that their convergence can be justified in the Abel sense.

The well-known connection between Laguerre and Hermite expansions allows us then to extend the result for Laplace type multipliers for Laguerre expansions to analogous results for Laplace type multipliers for Hermite expansions.

The key point in our proofs is to exploit the generalized convolution structure for the kind of multipliers under consideration (described in [16] for the case of the Laguerre fractional integral). Indeed, we show that the generalized convolution can be bounded by a convolution in the multiplicative group $(\mathbb{R}_+, \cdot)$ with a kernel of the kind considered in Lemma 1.7. This fact suggest that there is a connection between the operators under consideration in this chapter and the fractional integral of radial functions, and as a final remark we indicate how this fact can be exploited to obtain weighted estimates of $A_{p,q}$-type for Laguerre multipliers for certain values of α.

The bounds with power weights in this section are the subject of the article [10], while the final remarks on more general weights correspond to work-in-progress that exceeds the aim of this Thesis.

4.1 Statement of results and structure of this chapter

In order to explain our results, recall first that the Laguerre functions, for a given $\alpha > -1$, are given by

$$l_k^\alpha(x) = \left(\frac{k!}{\Gamma(k+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{x}{2}} L_k^\alpha(x) \ , \quad k \in \mathbb{N}_0$$

where $L_k^\alpha(x)$ are the Laguerre polynomials. The $l_k^\alpha(x)$ are eigenfunctions with eigenvalues $\lambda_{\alpha,k} = k + (\alpha+1)/2$ of the differential operator

$$L = -\left(x\frac{d^2}{dx^2} + (\alpha+1)\frac{d}{dx} - \frac{x}{4}\right) \tag{4.1}$$

and are an orthonormal basis in $L^2(\mathbb{R}_+, x^\alpha dx)$. Therefore, for $\gamma < p(\alpha+1) - 1$ we can associate to any $f \in L^p(\mathbb{R}_+, x^\gamma\, dx)$ its Laguerre series:

$$f(x) \sim \sum_{k=0}^{\infty} a_{\alpha,k}(f) l_k^\alpha(x), \quad a_{\alpha,k}(f) = \int_0^\infty f(x) l_k^\alpha(x) x^\alpha dx \tag{4.2}$$

and, given a bounded sequence $\{m_k\}$, we can define a multiplier operator by

$$M_{\alpha,m} f(x) \sim \sum_{k=0}^{\infty} a_{\alpha,k}(f) m_k l_k^\alpha(x). \tag{4.3}$$

The main example of the kind of multipliers we are interested in is the Laguerre fractional integral, introduced by G. Gasper, K. Stempak and W. Trebels in [16] as an analogue in the Laguerre setting of the classical fractional integral of Fourier analysis, and given by

$$I_\sigma f(x) \sim \sum_{k=0}^{\infty} (k+1)^{-\sigma} a_{k,\alpha} l_k^\sigma(x). \tag{4.4}$$

In [16] the aforementioned authors obtained weighted estimates for this operator that were later improved by G. Gasper and W. Trebels in [17] using a completely different proof. In this work we recover some of the ideas of the original method of [16], but simplifying the proof in many technical details and extending it to obtain a better range of exponents that, in particular, give the same result of [17] for the Laguerre fractional integral. Moreover, we show that our proof applies to a wide class of multipliers, namely multipliers arising from a Laplace-Stieltjes transform, which are of the form (4.3) with $m_k = m(k)$ given by the Laplace-Stieljtes transform of some real-valued function $\psi(t)$, that is,

$$m(s) = \mathfrak{L}\psi(s) := \int_0^\infty e^{-st} d\psi(t). \tag{4.5}$$

We will assume that ψ is of bounded variation in $\mathbb{R}_+$, so that the Laplace transform converges absolutely in the half plane $\mathrm{Re}(s) \geq 0$ (see [49, Chapter 2]) and the definition of the operator $M_{\alpha,m}$ makes sense.

Multipliers of this kind are quite natural to consider and, indeed, a slightly different definition is given by E. M. Stein in [40] and was previously used in the unweighted setting by E. Sasso in [41]. More recently, B. Wróbel [50] has proved weighted L^p estimates for the same kind of multipliers and certain values of α (see Section 4 below for a precise comparison). Also, let us mention that T. Martínez has considered multipliers of Laplace transform type for ultraspherical expansions in [27].

Other kind of multipliers for Laguerre expansions have also been considered, see, for instance, [16, 44, 46] where boundedness criteria are given in terms of difference operators. In our case, we will only require minimal assumptions on the function ψ, which are more natural in our context, and easier to verify in the case of the Laguerre fractional integral and in other examples that we will consider later. Indeed, the main theorem we will prove for multipliers for Laguerre expansions reads as follows:

Theorem 4.1. *Assume that $\alpha > -1$ and that $M_{\alpha,m}$ is a multiplier of Laplace transform type for Laguerre expansions, given by* (4.3) *and* (4.5)*, such that:*

(H1)

$$\int_0^\infty |d\psi|(t) < +\infty;$$

(H2) there exist $\delta > 0$, $0 < \sigma < \alpha + 1$, and $C > 0$ such that

$$|\psi(t)| \leq Ct^\sigma \quad \text{for } 0 \leq t \leq \delta.$$

Then $M_{\alpha,m}$ can be extended to a bounded operator such that

$$\|M_{\alpha,m} f\|_{L^q(\mathbb{R}_+, x^{(\alpha - bq)})} \leq C\|f\|_{L^p(\mathbb{R}_+, x^{(\alpha+ap)})}$$

provided that the following conditions hold:

$$1 < p \leq q < \infty \tag{4.6}$$

$$a < \frac{\alpha+1}{p'} \tag{4.7}$$

$$b < \frac{\alpha+1}{q} \tag{4.8}$$

$$2a + 2b \geq \left(\frac{1}{q} - \frac{1}{p}\right)(2\alpha+1) \tag{4.9}$$

and

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1}. \tag{4.10}$$

The well-known connection between Laguerre and Hermite expansions will then allow us to extend the above result to an analogous result for Laplace type multipliers for Hermite expansions. To make this precise, recall that, given $f \in L^2(\mathbb{R})$, we can consider its Hermite series expansion

$$f \sim \sum_{k=0}^{\infty} c_k(f) h_k, \quad c_k(f) = \int_{-\infty}^{\infty} f(x) h_k(x) dx. \tag{4.11}$$

where h_k are the Hermite functions given by

$$h_k(x) = \frac{(-1)^k}{(2^k k! \pi^{1/2})^{1/2}} H_k(x) e^{-\frac{x^2}{2}},$$

which are the normalized eigenfunctions of the Harmonic oscillator operator $H = -\frac{d^2}{dx^2} + |x|^2$.

As before, given a bounded sequence $\{m_k\}$ we can define a multiplier operator by

$$M_{H,m} f \sim \sum_{k=0}^{\infty} c_k(f) m_k h_k \tag{4.12}$$

and we say that it is a Laplace transform type multiplier if equation (4.5) holds. Then, we have the following analogue of Theorem 4.1, which, in the case of the Hermite fractional integral (that is, for $m_k = (2k+1)^{-\sigma}$), gives the same result of [30, Theorem 2.5] in the one-dimensional case:

Theorem 4.2. *Assume that $M_{H,m}$ is a multiplier of Laplace transform type for Hermite expansions, given by* (4.12) *and* (4.5)*, such that:*

(H1h)

$$\int_0^{\infty} |d\psi|(t) < +\infty;$$

(H2h) there exist $\delta > 0$, $0 < \sigma < \frac{1}{2}$, and $C > 0$ such that

$$|\psi(t)| \leq C t^{\sigma} \quad \textit{for}\, 0 \leq t \leq \delta.$$

Then $M_{H,m}$ can be extended to a bounded operator such that

$$\|M_{H,m} f\|_{L^q(\mathbb{R}, x^{-bq})} \leq C \|f\|_{L^p(\mathbb{R}, x^{ap})}$$

provided that the following conditions hold:

$$1 < p \leq q < \infty \tag{4.13}$$

$$a < \frac{1}{p'} \tag{4.14}$$

$$b < \frac{1}{q} \tag{4.15}$$

$$a + b \geq 0 \tag{4.16}$$

and

$$\frac{1}{q} \geq \frac{1}{p} - (2\sigma - a - b). \tag{4.17}$$

The remainder of this chapter is organized as follows. In Section 4.2 we prove Theorem 4.1. For the case $\alpha \geq 0$ the proof relies on the representation of the operator as a twisted generalized convolution, already used in [16] for the Laguerre fractional integral. However, instead of using the method of that paper to obtain weighted bounds, we give a simpler proof based on the use of Young's inequality in the multiplicative group $(\mathbb{R}_+, \cdot)$, which allows us to obtain a wider range of exponents. Moreover, we obtain an estimate for the convolution kernel which simplifies and generalizes Lemma 2.1 from [16]. For the case $-1 < \alpha < 0$ the result is obtained from the previous case by means of a weighted transplantation theorem from [15]. A similar idea was used by Y. Kanjin and E. Sato in [24] to prove unweighted estimates for the Laguerre fractional integral using a transplantation theorem from [23]. In Section 4.3 we exploit the relation between Laguerre and Hermite expansions to derive Theorem 4.2 from Theorem 4.1. In Section 4.4 we present some examples of operators covered by Theorems 4.1 and 4.2 and make some further remarks. Finally, in Section 4.5 we prove weighted $A_{p,q}$-type estimates in the Laguerre case for certain values of α, .

4.2 Proof of the theorem in the Laguerre case

In this section we prove Theorem 4.1. We will divide the proof in three steps:

1. We write the operator as a twisted generalized convolution and obtain the estimate for the convolution kernel when $\alpha \geq 0$. This part of the proof follows essentially the ideas of [16], but in the more general setting of multipliers of Laplace transform type. In particular, we provide an easier proof of the analogue of [16, Lemma 2.1] in this setting (see Lemma 4.3 below).

2. We complete the proof of the theorem in the case $\alpha \geq 0$ by proving weighted estimates for the generalized euclidean convolution.

3. We extend the results to the case $-1 < \alpha < 0$ using the case $\alpha \geq 0$ and a weighted transplantation theorem from [15] (Lemma 4.6 below).

4.2.1 Step 1: representing the multiplier operator as a twisted generalized convolution when $\alpha \geq 0$

Following [28, 1] we define the twisted generalized convolution of F and G by

$$F \times G := \int_0^\infty \tau_x F(y)\, G(y)\, y^{2\alpha+1}\, dy$$

where the twisted translation operator is defined by

$$\tau_x F(y) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} \int_0^\pi F((x,y)_\theta)\mathcal{J}_{\alpha-1/2}(xy\sin\theta)(\sin\theta)^{2\alpha}\, d\theta$$

with

$$\mathcal{J}_\beta(x) = \Gamma(\beta+1)J_\beta(x)/(x/2)^\beta$$

where $J_\beta(x)$ is the Bessel function of order β and

$$(x,y)_\theta = (x^2+y^2-2xy\cos\theta)^{1/2}.$$

Then, we have (formally) that

$$M_{\alpha,m} f(x^2) = F \times G \tag{4.18}$$

where

$$F(y) = f(y^2) \quad , \quad G(y) = g(y^2)$$

and

$$g(x) \sim \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^\infty m_k L_k^\alpha(x) e^{-\frac{x}{2}}. \tag{4.19}$$

Recalling that $|\mathcal{J}_\beta(x)| \leq C_\beta$ if $\beta \geq -\frac{1}{2}$, we have that:

$$|F \times G| \leq C(|F| \star |G|) \tag{4.20}$$

where $\star$ denotes the generalized Euclidean convolution which is defined by

$$F \star G(x) := \int_0^\infty \tau_x^E F(y)\, G(y)\, y^{2\alpha+1}\, dy \tag{4.21}$$

with

$$\tau_x^E F(y) := \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} \int_0^\pi F((x,y)_\theta)(\sin\theta)^{2\alpha}\, d\theta. \tag{4.22}$$

As a consequence of (4.18) and (4.20), the operator $M_{\alpha,m}$ is pointwise bounded by a generalized euclidean convolution with the kernel G (with respect to the measure

$x^{2\alpha+1}\,dx$). Therefore, we need to obtain an appropriate estimate for $G(x) = g(x^2)$, that essentially is:

$$|g(x)| \leq Cx^{\sigma-\alpha-1} \text{ for } \alpha \geq 0 \text{ and } 0 < \sigma < \alpha + 1$$

(see Lemma 4.3 below for a precise statement).

This generalizes the result given in [16, Lemma 2.1] but, while in that paper the proof of the corresponding estimate is based on delicate pointwise estimates for the Laguerre functions, our proof is based on the following generating function for the Laguerre polynomials (see, for instance, [46]):

$$\sum_{k=0}^{\infty} L_k^\alpha(x) w^k = (1-w)^{-\alpha-1} e^{-\frac{xw}{1-w}} := Z_{\alpha,x}(w) \quad (|w| < 1) \tag{4.23}$$

To explain our ideas, we point out that if the series in (4.19) were convergent (this need not be the case) we would have:

$$\begin{aligned}
g(x) &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} m_k L_k^\alpha(x) e^{-\frac{x}{2}} \\
&= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \left(\int_0^\infty e^{-kt} d\psi(t) \right) L_k^\alpha(x) e^{-\frac{x}{2}} \\
&= \frac{1}{\Gamma(\alpha+1)} e^{-\frac{x}{2}} \int_0^\infty Z_{\alpha,x}(e^{-t})\, d\psi(t).
\end{aligned}$$

The main advantage of this formula is that it shields a rather explicit expression for g in which, thanks to (4.23), the Laguerre polynomials do not appear.

However, in general it is not clear if the series in (4.19) is convergent (not even in the special case of the Laguerre fractional integral $m(t) = t^{\sigma-1}$). Moreover, the integration of the series in $Z_{\alpha,x}(w)$ is difficult to justify since it is not uniformly convergent in the interval $[0, 1]$ (because $Z_{\alpha,x}(w)$ is not analytical for $w = 1$).

Nevertheless, we will see that the formal manipulations above can be given a rigorous meaning if we agree in understanding the convergence of the series in (4.19) in the Abel sense. For this purpose, we introduce a regularization parameter $\rho \in (0, 1)$ and consider the regularized function

$$g_\rho(x) = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} m_k \rho^k L_k^\alpha(x) e^{-\frac{x}{2}} \tag{4.24}$$

and recall that the series in (4.19) is sumable in Abel sense to the limit $g(x)$ if there exists the limit

$$g(x) = \lim_{\rho \to 1} g_\rho(x). \tag{4.25}$$

With this definition in mind, we can give a rigorous meaning to the heuristic idea described above. More precisely, we will prove the following:

Lemma 4.3. *Let g_ρ be defined by* (4.24)*. Then:*

(1) For $0<\rho<1$ the series (4.24) *converges absolutely.*

(2) The following representation formula holds:

$$g_\rho(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty Z_{\alpha,x}(\rho e^{-t})\, d\psi(t). \tag{4.26}$$

(3) If we define $g(x)$ by setting $\rho = 1$ in this representation formula, $g(x)$ is well defined and the series (4.19) *converges to $g(x)$ in the Abel sense.*

(4) If $\alpha>0$, $0<\rho_0<\rho\leq 1$ and $0<\sigma<\alpha+1$

$$|g_\rho(x)| \leq Cx^{\sigma-\alpha-1},$$

where the constant $C = C(\alpha,\sigma)$ does not depend on ρ.

Proof. (1) Observe first that hypothesis $(H1)$ implies that (m_k) is a bounded sequence. Indeed,

$$|m_k| \leq \int_0^\infty e^{-kt}|d\phi|(t) \leq \int_0^\infty |d\phi|(t) = C < +\infty.$$

Now recall that ([46, Lemma 1.5.3]), if $\nu=\nu(k) = 4k+2\alpha+2$,

$$|l_k^\alpha(x)| \leq C(x\nu)^{-\frac{1}{4}} \quad \text{if } \frac{1}{\nu} \leq x \leq \frac{\nu}{2}.$$

Therefore, if we fix x, for $k\geq k_0$, x is in the region where this estimate holds (since $\nu\to+\infty$ when $k\to+\infty$), and from Stirling's formula we deduce that

$$\frac{k!}{\Gamma(k+\alpha+1)} = \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} = O(k^{-\alpha}).$$

Then we have the following estimate for the terms of the series in (4.24)

$$|m_k\rho^k L_k^\alpha(x)|e^{-\frac{x}{2}} \leq C(x)\rho^k k^{-\sigma} \text{ for } k\geq k_0,$$

and, since $\rho<1$, this implies that the series converges absolutely.

(2) First, observe that $Z_{\alpha,x}(w)$ is continuous as a function of a real variable for $w\in[0,1]$ (if we define $Z_{\alpha,x}(1)=0$) and, therefore, it is bounded, say

$$|Z_{\alpha,x}(w)| \leq C = C(\alpha,x) \text{ for } w\in[0,1].$$

Hence, using hypothesis $(H1)$ we see that the integral in the representation formula is convergent for any $\rho \in [0,1]$. Moreover, from our assumptions we have that, for $\rho < 1$,

$$\begin{aligned} g_\rho(x) &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} m_k \rho^k L_k^\alpha(x) e^{-\frac{x}{2}} \\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \left(\int_0^\infty \rho^k e^{-kt} d\psi(t) \right) L_k^\alpha(x) e^{-\frac{x}{2}} \\ &= \lim_{N\to+\infty} \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{N} \left(\int_0^\infty \rho^k e^{-kt} d\psi(t) \right) L_k^\alpha(x) e^{-\frac{x}{2}} \\ &= \lim_{N\to+\infty} \frac{1}{\Gamma(\alpha+1)} e^{-\frac{x}{2}} \int_0^\infty Z_{\alpha,x}^{(N)}(\rho e^{-t})\, d\psi(t) \end{aligned} \tag{4.27}$$

where

$$Z_{\alpha,x}^{(N)}(w) = \sum_{k=0}^{N} L_k^\alpha(x) w^k$$

denotes a partial sum of the series for $Z_{\alpha,x}(w)$. Now, since $\rho < 1$, that series converges uniformly in the interval $[0,\rho]$, so that given $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that

$$|Z_{\alpha,x}(w) - Z_{\alpha,x}^{(N)}(w)| < \varepsilon \text{ if } N \geq N_0.$$

Using this estimate and hypothesis $(H1)$, we obtain

$$\begin{aligned} &\left| \int_0^\infty Z_{\alpha,x}(\rho e^{-t})\, d\psi(t) - \int_0^\infty Z_{\alpha,x}^{(N)}(\rho e^{-t})\, d\psi(t) \right| \\ &\leq \int_0^\infty |Z_{\alpha,x}(\rho e^{-t}) - Z_{\alpha,x}^{(N)}(\rho e^{-t})|\, |d\psi|(t) \\ &\leq C\varepsilon \end{aligned}$$

from which we conclude that

$$\lim_{N\to+\infty} \int_0^\infty Z_{\alpha,x}^{(N)}(\rho e^{-t})\, d\psi(t) = \int_0^\infty Z_{\alpha,x}(\rho e^{-t})\, d\psi(t) \tag{4.28}$$

and, replacing (4.28) into (4.27) we obtain (4.26).

(3) We have already observed that the integral in (4.26) is convergent for $\rho = 1$. Moreover, the bound we have proved in (4.2.1) for $Z_{\alpha,x}$, and $(H1)$ imply that we can apply the Lebesgue bounded convergence theorem to this integral (with a constant majorant function, which is integrable with respect to $|d\psi|(t)$ by $(H1)$), to conclude that (4.25) holds.

(4) Let δ be as in $(H2)$ and observe that

$$\begin{aligned}
\Gamma(\alpha+1)g_\rho(x) &= e^{-\frac{x}{2}}\int_0^\infty Z_{\alpha,x}(\rho e^{-t})d\psi(t)\\
&= e^{-\frac{x}{2}}\int_0^\delta Z_{\alpha,x}(\rho e^{-t})d\psi(t) + e^{-\frac{x}{2}}\int_\delta^\infty Z_{\alpha,x}(\rho e^{-t})d\psi(t)\\
&= \underbrace{e^{-\frac{x}{2}}\int_0^\delta Z'_{\alpha,x}(\rho e^{-t})\rho e^{-t}\psi(t)\,dt}_{(i)} + \underbrace{e^{-\frac{x}{2}}Z_{\alpha,x}(\rho e^{-\delta})\psi(\delta)}_{(ii)}\\
&\quad - \underbrace{e^{-\frac{x}{2}}Z_{\alpha,x}(\rho)\psi(0)}_{(iii)} + \underbrace{e^{-\frac{x}{2}}\int_\delta^\infty Z_{\alpha,x}(\rho e^{-t})d\psi(t)}_{(iv)}
\end{aligned}$$

Since $|Z_{\alpha,x}(\rho e^{-\delta})| \le (1-\rho e^{-\delta})^{-\alpha-1} \le C_\delta$, $\psi(0)=0$, and $\sigma-\alpha-1<0$, clearly $(ii) \le Cx^{\sigma-\alpha-1}$ and (iii)*vanishes*.

To bound (iv), notice that if $\omega = \rho e^{-t}$ and $t>\delta$, $0 \le Z_{\alpha,x}(\omega) \le M_\delta$. Therefore, using $(H1)$ and the fact that $\sigma-\alpha-1<0$ we obtain

$$(iv) \le e^{-\frac{x}{2}}M_\delta\int_\delta^\infty |d\psi|(t) \le Cx^{\sigma-\alpha-1}.$$

Now, observing that

$$Z'_{\alpha,x}(\omega) = (\alpha+1)Z_{\alpha+1,x}(\omega) - xZ_{\alpha+2,x}(\omega).$$

and using $(H2)$, we obtain

$$\begin{aligned}
(i) \le Ce^{-\frac{x}{2}}&\int_0^\delta Z_{\alpha+1,x}(\rho e^{-t})\rho e^{-t}t^\sigma\,dt\\
&+ e^{-\frac{x}{2}}\int_0^\delta xZ_{\alpha+2,x}(\rho e^{-t})\rho e^{-t}t^\sigma\,dt
\end{aligned}$$

and the wanted estimates in this case follow by a direct application of the following lemma. □

Lemma 4.4. *In the conditions of Lemma 4.3(4), if*

$$I(x) = e^{-\frac{x}{2}}\int_0^\delta Z_{\beta,x}(\rho e^{-t})\rho e^{-t}t^\sigma\,dt,$$

and $\beta=\alpha+1$ or $\beta=\alpha+2$ then, $|I(x)| \le Cx^{\sigma-\beta}$ with $C=C(\beta,\sigma,\delta,\rho_0)$.

Proof. Making the change of variables $w = \rho e^{-t}$, and recalling the definition of $Z_{\beta,x}(w)$ given by (4.23), we see that

$$I(x) = e^{-\frac{x}{2}} \int_{\rho e^{-\delta}}^{\rho} (1-w)^{-\beta-1} e^{-\frac{xw}{1-w}} \log^{\sigma}\left(\frac{\rho}{w}\right) dw$$

Making the change of variables $u = \frac{1}{2} + \frac{w}{1-w}$ and setting $c_\delta = e^{-\delta}$ this is

$$\begin{aligned} I(x) &= \int_{\frac{1}{2}+\frac{c_\delta \rho}{1-c_\delta \rho}}^{\frac{1}{2}+\frac{\rho}{1-\rho}} \left(u+\frac{1}{2}\right)^{\beta+1} e^{-ux} \left[\log\left(\rho\frac{u+\frac{1}{2}}{u-\frac{1}{2}}\right)\right]^{\sigma} \frac{1}{\left(u+\frac{1}{2}\right)^2} du \\ &\leq C \int_{\frac{1}{2}+\frac{c_\delta \rho}{1-c_\delta \rho}}^{\frac{1}{2}+\frac{\rho}{1-\rho}} u^{\beta-1} e^{-ux} \left(u-\frac{1}{2}\right)^{-\sigma} \underbrace{\left[u(\rho-1)+\frac{1}{2}(\rho+1)\right]^{\sigma}}_{:=\tilde{u}(\rho)} du \qquad (4.29) \end{aligned}$$

where in (4.29) we have used that, since

$$\rho\frac{u+\frac{1}{2}}{u-\frac{1}{2}} = 1 + \frac{u(\rho-1)+\frac{1}{2}(\rho+1)}{u-\frac{1}{2}},$$

then

$$\log\left(\rho\frac{u+\frac{1}{2}}{u-\frac{1}{2}}\right) \leq \frac{u(\rho-1)+\frac{1}{2}(\rho+1)}{u-\frac{1}{2}}.$$

Since $\frac{1}{2} < u \leq \frac{1}{2} + \frac{\rho}{1-\rho}$, it is immediate that

$$0 \leq u(\rho-1) + \frac{1}{2}(\rho+1) \leq \rho,$$

which, using that $\sigma \geq 0$, implies $\tilde{u}(\rho) \leq 1$.

Also, since

$$u \geq \frac{1}{2} + \frac{c_\delta \rho_0}{1 - c_\delta \rho_0} > \frac{1}{2}$$

we have that

$$\left(u - \frac{1}{2}\right)^{-\sigma} \leq C u^{-\sigma}$$

where the constant depends only on ρ_0 and δ. Therefore,

$$\begin{aligned} I(x) &\leq C \int_0^\infty u^{\beta-\sigma-1} e^{-ux} du \\ &= C x^{-\beta+\sigma} \int_0^\infty v^{\beta-\sigma-1} e^{-v} dv \qquad (4.30) \\ &\leq C x^{-\beta+\sigma} \qquad (4.31) \end{aligned}$$

where in (4.30) we have made the change of variables $v = ux$, and in (4.31) we have used that $\beta - \sigma - 1 > -1$ because $\beta = \alpha + 1$ or $\beta = \alpha + 2$.

$\square$

4.2.2 Step 2: weighted estimates for the generalized Euclidean convolution

Following the idea of the previous section, we define a regularized multiplier operator $M_{\alpha,m,\rho}$ by:

$$M_{\alpha,m,\rho} f(x) := \sum_{k=0}^{\infty} m_k \rho^k a_{k,\alpha}(f) l_k^{\alpha}(x) \tag{4.32}$$

In this section we will obtain the estimate

$$\left(\int_0^{\infty} |M_{\alpha,m,\rho}(f)|^q x^{\alpha - bq} \, dx \right)^{\frac{1}{q}} \leq C \left(\int_0^{\infty} |f|^p x^{\alpha + ap} \, dx \right)^{\frac{1}{p}}$$

for $f \in L^p(\mathbb{R}_+, x^{\alpha+ap})$ with a constant C independent of the regularization parameter ρ and appropriate a, b (see Theorem 4.5).

Indeed, the operator can be expressed as before as a twisted generalized convolution with kernel $G_\rho(y) = g_\rho(y^2)$ (in place of G), and by Lemma 4.3, if $F(y) = f(y^2)$, we have the pointwise bound

$$|M_{\alpha,m,\rho} f(x^2)| \leq (|F| \star |G_\rho|)(x) \leq C(|F| \star |x^{2(\sigma-\alpha-1)}|)(x).$$

Therefore, (4.2.2) will follow from a weighted inequality for the generalized Euclidean convolution with kernel $K_\sigma := x^{2(\sigma-\alpha-1)}$ (Theorem 4.5).

Once we have (4.2.2), Theorem 4.1 will follow by a standard density argument. Indeed, if we consider the space

$$E = \{f(x) = p(x)e^{-\frac{x}{2}} : 0 \leq x, \, p(x) \text{ a polynomial}\},$$

any $f \in E$ has only a finite number of non-vanishing Laguerre coefficients. In that case, it is straightforward that $M_{\alpha,m} f(x)$ is well-defined and:

$$M_{\alpha,m} f(x) = \lim_{\rho \to 1} M_{\alpha,m,\rho} f(x)$$

Then, by Fatou's lemma,

$$\int_0^{\infty} |M_{\alpha,m}(f)|^q x^{\alpha - bq} \, dx \leq \lim_{\rho \to 1} \int_0^{\infty} |M_{\alpha,m,\rho}(f)|^q x^{\alpha - bq} \, dx$$

and, therefore, we obtain

$$\left(\int_0^\infty |M_{\alpha,m,\rho}(f)|^q x^{\alpha-bq}\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f|^p x^{\alpha+ap}\,dx\right)^{\frac{1}{p}} \ \forall f \in E$$

Since E is dense in $L^p(\mathbb{R}_+, x^{\alpha+ap})$, we deduce that $M_{\alpha,m}$ can be extended to a bounded operator from $L^p(\mathbb{R}_+, x^{\alpha+ap})$ to $L^q(\mathbb{R}_+, x^{\alpha-bq})$. Moreover, the extended operator satisfies:

$$M_{\alpha,m} f = \lim_{\rho\to 1} M_{\alpha,m,\rho} f$$

This means, that the formula (4.3) is valid for $f \in L^p(\mathbb{R}_+, x^{\alpha+ap})$ if the summation is interpreted in the Abel sense with convergence in $L^q(\mathbb{R}_+, x^{\alpha-bq})$.

Now we can conclude the proof of Theorem 4.1 in the case $\alpha \ge 0$.

Theorem 4.5. *Let $\alpha \ge 0, 0 < \sigma < \alpha+1$ and $M_{\alpha,m,\rho}$ be given by* (4.32) *such that it satisfies* $(H1)$ *and* $(H2)$*. Then, for all $f \in L^p(\mathbb{R}_+, x^{\alpha+ap})$, the following estimate holds*

$$\|M_{\alpha,m,\rho} f(x^2) x^{-2b}\|_{L^q(\mathbb{R}_+, x^{2\alpha+1})} \le \|f(x^2) x^{2a}\|_{L^p(\mathbb{R}_+, x^{2\alpha+1})} \tag{4.33}$$

provided that conditions (4.7), (4.8), (4.9) *hold, and that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha+1}. \tag{4.34}$$

Proof. Let $K_\sigma(x) := x^{2(\sigma-\alpha-1)}$, $F(y) = f(y^2)$ and recall that

$$|M_{\alpha,m,\rho} f(x^2)| \le C(|F| \star |K_\sigma|)(x)$$

where $\star$ denotes the generalized euclidean convolution defined by (4.21).

We begin by computing the generalized Euclidean translation of K_σ given by (4.22). Making the change of variables

$$t = \cos\theta \Rightarrow dt = -\sin\theta\, d\theta = -\sqrt{1-t^2}\, d\theta$$

we see that

$$\tau_x^E K_\sigma(y) = C(\alpha)\int_{-1}^1 (x^2+y^2-2xyt)^{\sigma-\alpha-1}(1-t^2)^{\alpha-\frac{1}{2}}\,dt.$$

Recalling the notation of Chapter 1, if we let

$$I_{\gamma,k}(r) := \int_{-1}^1 \frac{(1-t^2)^k}{(1-2rt+r^2)^{\frac{\gamma}{2}}}\,dt,$$

then

$$\tau_x^E K_\sigma(y) = C(\alpha) y^{2(\sigma-\alpha-1)} I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}\left(\frac{x}{y}\right)$$

and, therefore,

$$\begin{aligned}
K_\sigma \star F(x) &= C\int_0^\infty y^{2(\sigma-\alpha-1)} I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}\left(\frac{x}{y}\right) F(y) y^{2\alpha+1}dy \\
&= C\int_0^\infty y^{2\sigma} I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}\left(\frac{x}{y}\right) F(y)\frac{dy}{y}
\end{aligned} \tag{4.35}$$

Now,

$$\begin{aligned}
\|M_{\alpha,m,\rho} f(x^2)x^{-2b}\|_{L^q(\mathbb{R}_+,x^{2\alpha+1})} &\le C\|[K_\sigma \star F(x)]x^{-2b}\|_{L^q(\mathbb{R}_+,x^{2\alpha+1})} \\
&= C\left(\int_0^\infty |K_\sigma \star F(x)x^{-2b}|^q x^{2\alpha+1}\,dx\right)^{\frac{1}{q}} \\
&= C\left(\int_0^\infty \left|K_\sigma \star F(x)x^{\frac{2\alpha+2}{q}-2b}\right|^q \,\frac{dx}{x}\right)^{\frac{1}{q}}
\end{aligned}$$

but, by (4.35),

$$\begin{aligned}
&[K_\sigma \star F(x)]x^{\frac{2\alpha+2}{q}-2b} \\
&\quad = C\int_0^\infty y^{2\sigma} x^{\frac{2\alpha+2}{q}-2b} I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}\left(\frac{x}{y}\right) F(y)\frac{dy}{y} \\
&\quad = C\int_0^\infty \left(\frac{y}{x}\right)^{-[\frac{2\alpha+2}{q}-2b]} I_{2(1-\alpha-\sigma),\alpha-\frac{1}{2}}\left(\frac{x}{y}\right) F(y) y^{2\sigma+\frac{2\alpha+2}{q}-2b}\frac{dy}{y} \\
&\quad = [y^{\frac{2\alpha+2}{q}-2b} I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}(y) * F(y) y^{2\sigma+\frac{2\alpha+2}{q}-2b}](x)
\end{aligned}$$

where $*$ denotes the convolution in $\mathbb{R}_+$ with respect to the Haar measure $\frac{dx}{x}$.

Then, by Young's inequality:

$$\begin{aligned}
&\|M_{\alpha,m,\rho} f(x^2)x^{-2b}\|_{L^q(\mathbb{R}_+,x^{2\alpha+1})} \\
&\qquad \le \|F(x)x^{2\sigma+\frac{2\alpha+2}{q}-2b}\|_{L^p\left(\frac{dx}{x}\right)} \|x^{\frac{2\alpha+2}{q}-2b} I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}(x)\|_{L^{s,\infty}\left(\frac{dx}{x}\right)}
\end{aligned}$$

provided that:

$$\frac{1}{p}+\frac{1}{s} = 1+\frac{1}{q}. \tag{4.36}$$

Since by hypothesis (4.34),

$$\begin{aligned}\|F(x)x^{2\sigma+\frac{2\alpha+2}{q}-2b}\|_{L^p(\frac{dx}{x})} &= \left(\int_0^\infty |F(x)x^{2\sigma+\frac{2\alpha+2}{q}-2b}|^p\,\frac{dx}{x}\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty |F(x)x^{2a+\frac{2\alpha+2}{p}}|^p\,\frac{dx}{x}\right)^{\frac{1}{p}} \\ &= \|F(x)x^{2a}\|_{L^p(\mathbb{R}_+,x^{2\alpha+1})} \\ &= \|f(x^2)x^{2a}\|_{L^p(\mathbb{R}_+,x^{2\alpha+1})}\end{aligned}$$

to conclude the proof of the theorem it suffices to see that

$$\|x^{\frac{2\alpha+2}{q}-2b}I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}(x)\|_{L^{s,\infty}(\frac{dx}{x})} < +\infty. \tag{4.37}$$

Using Lemma 1.7, it is clear that when $x \to 1$ and $2(\alpha+1-\sigma) \le 2(\alpha-\frac{1}{2})$ the norm (4.37) is bounded.

In the case $2(\alpha+1-\sigma) > 2(\alpha-\frac{1}{2})$ (that is, $\sigma < 3$), the integrability condition is

$$-s\left[2(\alpha+1-\sigma)-2\left(\alpha-\frac{1}{2}\right)-2\right] \ge -1.$$

But, using (4.36) and (4.34), we see that this is equivalent to hypothesis (4.9).

When $x=0$, the integrability condition is

$$\frac{2\alpha+2}{q}-2b > 0$$

which holds because of hypothesis (4.8).

Finally, when $x \to \infty$, since $I_{\alpha-\frac{1}{2},2(\alpha+1-\sigma)}(x) \sim x^{-2(\alpha+1-\sigma)}$, the condition we need to fulfill is

$$\frac{2\alpha+2}{q}-2b-2(\alpha+1-\sigma) < 0$$

which, by (4.34), is equivalent to (4.7). $\square$

4.2.3 Extension to the case $-1 < \alpha < 0$ and end of proof of Theorem 4.1

First, notice that if condition $(H2)$ holds for a certain $0 < \sigma_0 < \alpha+1$, then it also holds for any $0 < \sigma < \sigma_0$. Therefore, it suffices to prove Theorem 4.1 in the case $\frac{1}{q} = \frac{1}{p} - \frac{\sigma-a-b}{\alpha+1}$.

When $\alpha \ge 0$ this is exactly Theorem 4.5 above. To extend this result to the case $-1 < \alpha < 0$ let us consider $-1 < \alpha < \beta$, where $\beta \ge 0$, and use a transplantation result form [15, Corollary 6.19 (ii)], that we recall here as a lemma for the sake of completeness:

Lemma 4.6 ([15])**.** *Let $1 < q < \infty$. Given $\alpha, \beta > -1$, we define the transplantation operator*

$$\mathbb{T}_\beta^\alpha f = \sum_{k=0}^{\infty} \left(\int_0^\infty f(y) l_k^\alpha(y) y^\alpha \, dy \right) l_k^\beta.$$

Then, if $\sigma_0 \in \mathbb{R}$ and $\sigma_1 = \sigma_0 + (\alpha - \beta)(\frac{1}{p} - \frac{1}{2})$, $\mathbb{T}_\beta^\alpha : L^q_{\sigma_0}(\mathbb{R}_+, x^\alpha \, dx) \to L^q_{\sigma_1}(\mathbb{R}_+, x^\beta \, dx)$ and $\mathbb{T}_\alpha^\beta : L^q_{\sigma_1}(\mathbb{R}_+, x^\beta \, dx) \to L^q_{\sigma_0}(\mathbb{R}_+, x^\alpha \, dx)$ are bounded operators if and only if

$$-\frac{1+\alpha}{q} < \sigma_0 < \frac{1+\alpha}{q'}.$$

Using this lemma, we can write

$$\begin{aligned}\|M_{\alpha,m} f |x|^{-b}\|_{L^q(\mathbb{R}_+, x^\alpha \, dx)} &= \|\mathbb{T}_\alpha^\beta (M_{\beta,m}(\mathbb{T}_\beta^\alpha f))|x|^{-b}\|_{L^q(\mathbb{R}_+, x^\alpha \, dx)} \\ &\le C \|M_{\alpha,m,\beta}(\mathbb{T}_\beta^\alpha f)|x|^{-\tilde{b}}\|_{L^q(\mathbb{R}_+, x^\beta \, dx)}\end{aligned}$$

provided that

$$-1 < \alpha < \beta \tag{4.38}$$

$$-\tilde{b} = -b + (\alpha - \beta)\left(\frac{1}{p} - \frac{1}{2}\right) \tag{4.39}$$

and

$$-\frac{1+\alpha}{q} < -b < \frac{1+\alpha}{q'} \tag{4.40}$$

and, using Theorem 4.5 for $M_{\beta,m}$ with $\beta \ge 0$,

$$\|M_{\alpha,m,\beta}(\mathbb{T}_\beta^\alpha f)|x|^{-\tilde{b}}\|_{L^q(\mathbb{R}_+, x^\beta \, dx)} \le C \|\mathbb{T}_\beta^\alpha f |x|^{\tilde{a}}\|_{L^p(\mathbb{R}_+, x^\beta \, dx)}$$

provided that

$$0 < \sigma < \beta + 1 \tag{4.41}$$

$$\tilde{a} < \frac{\beta+1}{p'} \tag{4.42}$$

$$\tilde{b} < \frac{\beta+1}{q} \tag{4.43}$$

$$2(\tilde{a} + \tilde{b}) \ge \left(\frac{1}{q} - \frac{1}{p}\right)(2\beta + 1) \tag{4.44}$$

and

$$\frac{1}{q} = \frac{1}{p} - \frac{\sigma - \tilde{a} - \tilde{b}}{\beta + 1} \tag{4.45}$$

Finally, using Lemma 4.6 again, we obtain

$$\|M_{\alpha,m} f|x|^{-b}\|_{L^q(\mathbb{R}_+, x^\alpha\, dx)} C \|f|x|^a\|_{L^p(\mathbb{R}_+, x^\alpha\, dx)} \tag{4.46}$$

provided that

$$\tilde{a} = a + (\alpha - \beta)\left(\frac{1}{p} - \frac{1}{2}\right) \tag{4.47}$$

$$-\frac{1+\alpha}{p} < a < \frac{1+\alpha}{p'} \tag{4.48}$$

Now, replacing (4.47) and (4.39) into (4.45) we obtain

$$\frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1} \tag{4.49}$$

and replacing (4.47) and (4.39) into (4.44), we obtain

$$2a + 2b \geq \left(\frac{1}{q} - \frac{1}{p}\right)(2\alpha + 1).$$

To conclude the proof of the theorem we need to see that the restrictions $a > -\frac{1+\alpha}{p}$ in (4.48) and $b > -\frac{1+\alpha}{q'}$ in (4.40) are redundant. Indeed, the first one follows from (4.49) and $b < \frac{\alpha+1}{q}$, while the second one follows from (4.49) and $a < \frac{\alpha+1}{p'}$.

4.3 Proof of Theorem 4.2

In this section we exploit the well-known relation between Hermite and Laguerre poynomials to obtain an analogous result to that of the previous section in the Hermite case. Indeed, recalling that

$$H_{2k}(x) = (-1)^k 2^{2k} k! L_k^{-\frac{1}{2}}(x^2)$$
$$H_{2k+1}(x) = (-1)^k 2^{2k} k! x L_k^{\frac{1}{2}}(x^2)$$

it is immediate that

$$h_{2k}(x) = l_k^{-1/2}(x^2)$$
$$h_{2k+1}(x) = x l_k^{\frac{1}{2}}(x^2)$$

It is then natural to decompose $f = f_0 + f_1$ where

$$f_0(x) = \frac{f(x) + f(-x)}{2} \quad , \quad f_1(x) = \frac{f(x) - f(-x)}{2}$$

and, clearly, when $k = 2j$, if we let $g_0(y) = f_0(\sqrt{y})$ we obtain:

$$c_k(f) = \langle f_0, h_k\rangle = 2\int_0^\infty f_0(x) l_j^{-\frac{1}{2}}(x^2)\, dx = a_{-\frac{1}{2},j}(g_0)$$

while if $k = 2j+1$, and we let $g_1(y) = \frac{1}{\sqrt{y}} f_1(\sqrt{y})$ we have:

$$c_k(f) = \langle f_1, h_k\rangle = 2\int_0^\infty f_1(x) x l_j^{\frac{1}{2}}(x^2)\, dx = a_{\frac{1}{2},j}(g_1)$$

Then,

$$\begin{aligned} M_{H,m}f(x) &= \sum_{j=0}^{\infty} m_{2j} a_{-\frac{1}{2},j}(g_0) l_j^{-\frac{1}{2}}(x^2) + \sum_{j=0}^{\infty} m_{2j+1} a_{\frac{1}{2},j}(g_1) x l_j^{\frac{1}{2}}(x^2) \\ &= M_{-\frac{1}{2},m_0} g_0(x^2) + x M_{\frac{1}{2},m_1} g_1(x^2) \end{aligned}$$

where $(m_0)_k = m_{2k}$ and $(m_1)_k = m_{2k+1}$.

To apply Theorem 4.1 to this decomposition, we need to check first that m_0 and m_1 are Laplace-Stiltjes functions of certain functions ψ_0 and ψ_1. Indeed, notice that $m_{2k} = \mathfrak{L}\psi_0(k)$ where

$$\psi_0(u) = \frac{1}{2}\psi(\frac{u}{2})$$

and $m_{2k+1} = \mathfrak{L}\psi_1(k)$ where

$$\psi_1(u) = \frac{1}{2}\int_0^{\frac{u}{2}} e^{-\tau} d\psi(\tau).$$

It is also easy to see that ψ_0 satisfies the hypotheses of Theorem 4.1 for $\alpha = -\frac{1}{2}$ whereas ψ_1 satisfies the hypotheses for $\alpha = \frac{1}{2}$ (in this case condition $(H2)$ follows after an integration by parts).

Then,

$$\begin{aligned} \|M_{H,m}f|x|^{-b}\|_{L^q(\mathbb{R})} &= \left(\int_{\mathbb{R}} |M_{H,m}f(x)|^q |x|^{-bq}\, dx\right)^{\frac{1}{q}} \\ &= C\left(\int_{\mathbb{R}} \left|M_{-\frac{1}{2},m_0} g_0(x^2) + x M_{\frac{1}{2},m_1} g_1(x^2)\right|^q |x|^{-bq}\, dx\right)^{\frac{1}{q}} \qquad (4.50) \end{aligned}$$

Using Minkowski's inequality and making the change of variables $y = x^2, dx =$

$\frac{1}{2}y^{-\frac{1}{2}}\,dy$, we see that

$$
\begin{aligned}
(4.50) \sim &\left(\int \left|M_{-\frac{1}{2},m_0}g_0(y)\right|^q |y|^{-\frac{bq}{2}-\frac{1}{2}}\,dy\right)^{\frac{1}{q}} + \left(\int \left|M_{\frac{1}{2},m_1}g_1(y)\right|^q |y|^{\frac{(-b+1)q}{2}-\frac{1}{2}}\,dy\right)^{\frac{1}{q}}\\
&= \|M_{-\frac{1}{2},m_0}g_0(y)|y|^{-\frac{b}{2}}\|_{L^q(\mathbb{R},x^{-\frac{1}{2}}\,dx)} + \|M_{\frac{1}{2},m_1}g_1(y)|y|^{\frac{-b+1}{2}-\frac{1}{q}}\|_{L^q(\mathbb{R},x^{\frac{1}{2}}\,dx)}\\
&\le C\|g_0(y)|y|^{\tilde{a}}\|_{L^p(\mathbb{R},x^{-\frac{1}{2}}\,dx)} + C\|g_1(y)|y|^{\hat{a}}\|_{L^p(\mathbb{R},x^{\frac{1}{2}}\,dx)}
\end{aligned}
$$

where the last inequality follows from Theorem 4.1 provided that:

$$\tilde{a} < \frac{1}{2p'} \tag{4.51}$$

$$b < \frac{1}{q} \tag{4.52}$$

$$\frac{1}{q} \ge \frac{1}{p} - \frac{s-\tilde{a}-\frac{b}{2}}{\frac{1}{2}} \tag{4.53}$$

$$\tilde{a} + \frac{b}{2} \ge 0 \tag{4.54}$$

$$\hat{a} < \frac{3}{2p'} \tag{4.55}$$

$$\frac{1}{q} \ge \frac{1}{p} - \frac{s-\hat{a}+\frac{1-b}{2}-\frac{1}{q}}{\frac{3}{2}} \tag{4.56}$$

$$\hat{a} + \hat{b} \ge \left(\frac{1}{q} - \frac{1}{p}\right) \tag{4.57}$$

Therefore,

$$
\begin{aligned}
\|M_{H,m}f|x|^{-b}\|_{L^q(\mathbb{R})} &\le C\left(\int |g_0(x)|^p|x|^{\tilde{a}p-\frac{1}{2}}\,dx\right)^{\frac{1}{p}} + C\left(\int |g_1(x)|^p|x|^{\hat{a}p+\frac{1}{2}}\,dx\right)^{\frac{1}{p}}\\
&= C\left(\int |f_0(\sqrt{x})|^p|x|^{\tilde{a}p-\frac{1}{2}}\,dx\right)^{\frac{1}{p}} + C\left(\int |f_1(\sqrt{x})|^p|x|^{\hat{a}p+\frac{1}{2}-\frac{p}{2}}\,dx\right)^{\frac{1}{p}}\\
&= C\left(\int |f_0(x)|^p|x|^{2\tilde{a}p}\,dx\right)^{\frac{1}{p}} + C\left(\int |f_1(x)|^p|x|^{2\hat{a}p+2-p}\,dx\right)^{\frac{1}{p}}\\
&\le C\|f(x)|x|^a\|_{L^p(\mathbb{R})}
\end{aligned}
$$

provided that

$$a = 2\tilde{a} = 2\hat{a} + \frac{2}{p} - 1. \tag{4.58}$$

Therefore, by (4.58), (4.51) and (4.55) there must hold

$$a < \frac{1}{p'}$$

while, also by (4.58), (4.53) and (4.56) are equivalent to

$$\frac{1}{q} \geq \frac{1}{p} - (2s - a - b)$$

and (4.54) and (4.57) are equivalent to

$$a + b \geq 0.$$

Remark 4.7. *It follows from the proof of Theorem 4.2 that a better result holds if the function f is odd.*

4.4 Examples and further remarks

First, we should point out that it is clear that, since a Stieltjes integral of a continuous function with respect to a function of bounded variation can be thought as an integral with respect to the corresponding Lebesgue-Stieltjes measure, we could equivalently have formulated all our results in terms of integrals with respect to signed Borel measures in $\mathbb{R}_+$. However, we have found convenient to use the framework of Stieltjes integrals since many of the classical references on Laplace transforms are written in that framework (for instance [49]), and leave the details of a possible restatement of the theorems in the case of regular Borel measure to the reader.

We also recall that the Laplace-Stieltjes transform contains as particular cases both the ordinary Laplace transform of (locally integrable) functions (when $\psi(t)$ is absolutely continuous), and Dirichlet series (see below). In particular, if ψ is absolutely continuous and $\phi(t) = \psi'(t)$ (defined almost everywhere), the assumptions $(H1)$ and $(H2)$ of Theorem 4.1 can be replaced by:

(H1ac)

$$\int_0^\infty |\phi(x)|\, dx < +\infty \quad \text{i.e. } \phi \in L^1(\mathbb{R}_+)$$

(H2ac) there exist $\delta > 0$, $0 < \sigma < \alpha + 1$, and $C > 0$ such that

$$\left| \int_0^t \phi(x)\, dx \right| \leq C t^\sigma \quad \text{for } 0 < t \leq \delta.$$

In particular, assumption $(H2ac)$ holds if $\phi(t) = O(t^{\sigma-1})$ when $t \to 0$.

As we have already mentioned in the introduction, B. Wróbel [50, Corollary 2.6] has recently proved that Laplace type multipliers (with the definition given in [40]) are bounded on $L^p(\mathbb{R}^d, \omega)$, $1 < p < \infty$, for all $\omega \in A_p$ and $\alpha \in (\{-\frac{1}{2}\} \cup [\frac{1}{2}, \infty))^d$. In the case of power weights in one dimension this means that $\omega(x) = |x|^\beta$ must satisfy $-1 < \beta < p-1$, while taking $p = q$ and letting the weight be $|x|^\beta$ on both sides, Theorem 4.1 can easily be seen to imply $-1 < \beta < p - 1 + \alpha p$.

Also, weighted estimates had been obtained before for the case of some particular operators covered by our definition. Indeed, recall that one of the main examples of the kind of multipliers we are considering is the Laguerre fractional integral introduced in [16], which corresponds to the choice $m_k = (k+1)^{-\sigma}$.

In [30, Theorem 4.2], A. Nowak and K. Stempak considered multi-dimensional Laguerre expansions and used a slightly different definition of the fractional integral operator, given by the negative powers of the differential operator (4.1).

As they point out, their theorem contains as a special case the result of [16] (in the one dimensional case). To see that both operators are indeed equivalent, they rely on a deep multiplier theorem [44, Theorem 1.1].

Instead, we can see that Theorem 4.1 is applicable to both definitions by choosing:

$$m_k = (k+c)^{-\sigma}, \quad \phi(t) = \frac{1}{\Gamma(\sigma)} t^{\sigma-1} e^{-ct} \quad (c > 0)$$

The case $c = 1$ corresponds to the definition in [16], whereas the choice $c = \frac{\alpha+1}{2}$ corresponds to the definition in [30]. Therefore, Theorem 4.1 applied to these choices, coincides in the first case with the result of [17, Theorem 1] (which is an improvement of [16, Theorem 3.1]) and improves in the second case the one-dimensional result of [30, Theorem 4.2].

The same choice of m_k and ϕ in Theorem 4.2 gives a two-weight estimate for the Hermite fractional integral, which corresponds to the one-dimensional version of [30, Theorem 2.5].

Another interesting example is the operator $(L^2 + I)^{-\frac{\alpha}{2}}$, where L is given by (4.1). In this case, Theorem 4.1 with hypotheses $(H1ac)$ and $(H2ac)$ instead of $(H1)$ and $(H2)$ applies with $\alpha = \sigma$ and

$$\phi(t) = \frac{1}{C_\alpha} e^{-\frac{\alpha+1}{2}t} J_{\frac{\alpha-1}{2}}(t) t^{\frac{\alpha-1}{2}}$$

since, by [48, formula 5, p. 386],

$$\int_0^\infty e^{-st} J_{\frac{\alpha-1}{2}}(t) t^{\frac{\alpha-1}{2}}\, dt = C_\alpha (s^2+1)^{-\frac{\alpha}{2}}$$

and, when $t \to 0$, $J_{\frac{\alpha-1}{2}}(t) t^{\frac{\alpha-1}{2}} \sim t^{\alpha-1}$.

A further example is obtained by choosing $\psi(t) = e^{-s_0 t} H(t-\tau)$ with $s_0 = \frac{\alpha+1}{2}$, where H is the Heaviside unit step function:

$$H(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

and we see that Theorem 4.1 is applicable to the Heat diffusion semigroup (considered for instance in [43] and [26])

$$M_\tau = e^{-\tau L}$$

associated to the operator L for any $\sigma > 0$. More generally, the same conclusion holds for

$$\psi(t) = \sum_{n=1}^{\infty} a_n e^{-s_0 t} H(t - \tau_n)$$

provided that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{-\tau_n s}, \quad 0 < \tau_1 < \tau_2 < \dots$$

conveges absolutely for $s = s_0$ (which corresponds to hypothesis $(H1)$).

As a final comment, we remark that finding a function ψ of bounded variation such that $m_k = \mathfrak{L}\psi(k)$ holds (see (4.5)) is equivalent to solving the clasical Hausdorff moment problem (see [49, Chapter III]).

4.5 Weighted $A_{p,q}$-type estimates for certain values of α and some remarks on open problems

As we have pointed out in the introduction to this chapter, both the fractional integral of radially symmetric functions and the multipliers of Laplace transform type for Laguerre expansions can be bounded by a convolution in the multiplicative group $(\mathbb{R}_+, \cdot)$ with kernels of the same kind. This suggest that known results on the suitable weights for fractional integrals could be transplanted to analogous results for Laguerre multipliers with little effort, at least in some particular cases. More precisely, the following theorem holds:

Theorem 4.8. *Let $\alpha \geq 0$, $0 < \sigma < \alpha + 1$, $1 \leq p \leq q < \infty$and assume that $M_{\alpha,m}$ is multiplier of Laplace transform type for Laguerre expansions given by* (4.3) *and* (4.5) *which satisfies hypotheses* $(H1)$ *and* $(H2)$. *Then, if $2\alpha + 2 \in \mathbb{N}$ and $V(\mathbf{x}) = v(x)$, $W(\mathbf{x}) = w(x)$ are radially symmetric weights which for which the fractional integral of radial functions is continuous from $L^p(\mathbb{R}^n, W(\mathbf{x}))$ into $L^q(\mathbb{R}^n, V(\mathbf{x}))$, then*

$$\|M_{\alpha,m} f(x)\|_{L^q(\mathbb{R}_+, w(x^{1/2})\, x^\alpha\, dx)} \leq C \|f(x)\|_{L^p(\mathbb{R}_+, v(x^{1/2})\, x^\alpha\, dx)}$$

Proof. As we have seen in the proof of Theorem 4.5,

$$\begin{aligned}|M_{\alpha,m,\rho}f(x^2)|x^{\frac{2\alpha+2}{q}} &\leq C(|F|\star x^{2(\sigma-\alpha-1)})x^{\frac{2\alpha+2}{q}}\\ &\leq C\left[y^{2\sigma+\frac{2\alpha+2}{q}}F(y) * y^{\frac{2\alpha+2}{q}}I_{2(1+\alpha-\sigma),\alpha-\frac{1}{2}}(y)\right](x)\end{aligned}$$

with $F(y)=f(y^2)$ where, $\star$ denotes the generalized euclidean convolution and $*$ denotes the convolution in the multiplicative group $(\mathbb{R}_+,\cdot)$.

But, by the results in Chapter 1, Section 1.3, if $\mathbf{x}\in\mathbb{R}^n$, $|\mathbf{x}|=x$ and $g\in L^p(\mathbb{R}^n)$ is a radially symmetric function, $g(\mathbf{x})=g_0(x)$, then

$$T_\gamma g(\mathbf{x})\,x^{\frac{n}{q}}=\omega_{n-2}\left[y^{n-\gamma+\frac{n}{q}}g_0(y) * y^{\frac{n}{q}}I_{\gamma,\frac{n-3}{2}}(y)\right](x).$$

Therefore, if

$$\frac{n-3}{2}=\alpha+\frac{1}{2}\quad\text{ and }\quad\gamma=2(1+\alpha-\sigma)$$

and $g_0(x)=f(x^2)$ we see that

$$|M_{\alpha,m,\rho}f(x^2)|x^{\frac{2\alpha+2}{q}}\leq C|T_\gamma g(\mathbf{x})|\,x^{\frac{n}{q}}$$

where the fact that $0<\gamma<n$ follows from the assumption $0<\sigma<\alpha+1$. Informally, this says that the multiplier operator can be thought of as a fractional integral of the radial function $g(\mathbf{x})=f(x^2)$ in $\mathbb{R}^{2\alpha+2}$. Then, by our assumptions on the weights,

$$\begin{aligned}\|M_{\alpha,m,\rho}f(x^2)\|_{L^q(\mathbb{R}_+,w(x)x^{2\alpha+1}\,dx)} &= \left(\int_0^\infty |M_{\alpha,m,\rho}f(x^2)|^q w(x)x^{2\alpha+1}\,dx\right)^{\frac{1}{q}}\\ &\leq C\left(\int_0^\infty |T_\gamma g(x)|^q w(x)x^{n-1}\,dx\right)^{\frac{1}{q}}\\ &\leq C\left(\int_{\mathbb{R}^n} |T_\gamma g(\mathbf{x})|^q W(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{q}}\\ &\leq C\left(\int_{\mathbb{R}^n} |g(\mathbf{x})|^p V(\mathbf{x})\,d\mathbf{x}\right)^{\frac{1}{p}}\\ &\leq C\left(\int_0^\infty |g_0(x)|^p v(x)x^{n-1}\,dx\right)^{\frac{1}{p}}\\ &= C\left(\int_0^\infty |f(x^2)|^p v(x)x^{2\alpha+1}\,dx\right)^{\frac{1}{p}}\\ &= C\|f(x^2)\|_{L^p(\mathbb{R}_+,v(x)x^{2\alpha+1}\,dx)}\end{aligned}$$

This is clearly equivalent to

$$\left(\int_0^\infty |M_{\alpha,m,\rho}f(x)|^q w(x^{\frac{1}{2}})x^\alpha\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)|^p v(x^{\frac{1}{2}})x^\alpha\,dx\right)^{\frac{1}{p}}$$

and, as in the case of the power weights, one can see that the same bound holds for $M_{\alpha,m}$.

□

Remark 4.9. *In particular, the previous theorem holds if $p < q$, and $V(\mathbf{x}), W(\mathbf{x})$ are radially symmetric reverse doubling weights that satisfy the $A_{p,q}$ condition:*

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q W\right)^{\frac{1}{q}}\left(\int_Q V^{1-p'}\right)^{\frac{1}{p'}} \le C$$

for all cubes $Q \subset \mathbb{R}^n$. Indeed, this corresponds to a special case of the characterization given by E. Sawyer and R. L. Wheeden of the weights for the $L^p - L^q$ continuity of the fractional integral [37, Theorem 1(B)].

However, it follows from our results in Chapter 1 that if $p \neq q$ the radially symmetric weights appropriate for the $L^p - L^q$ continuity of fractional integrals of radially symmetric functions are a strictly larger class than that of the radially symmetric weights which satisfy the $A_{p,q}$ condition (because the class of admissible power weights is strictly larger than that of the power weights in $A_{p,q}$).

Remark 4.10. *Besides the lack of a complete characterization of which are the appropriate radial weights for the fractional integral of radial functions, the main drawback of Theorem 4.8 is the fact that it is only applicable to $\alpha \ge 0$ and $2\alpha + 2 \in \mathbb{N}$. The first restriction is due to the fact that representation of the multipliers as a generalized twisted convolution used holds in this case only, and therefore a transplantation theorem for more general weights would be needed to overcome this restriction.*

The restriction $2\alpha + 2 \in \mathbb{N}$, in contrast, is due to the fact that $2\alpha + 2$ plays the role of the dimension. But the convolution with kernel $I_{\gamma,k}$ makes sense for any γ and k. In fact, the "higher dimensional" interpretation of the parameter α played no role in the proof of Theorem 4.1.

A different approach to the same problem is to consider the following formula proved by B. S. Rubin [33], that he used in his proof of the admissible power weights mentioned in Chapter 1 (it was indeed after conjecturing such a formula that we became aware of his result):

$$(T_{n-\alpha}f)(\sqrt{t}) = 2^{-\alpha}t^{1-\frac{n}{2}}R_{\frac{\alpha}{2}}(s^{\frac{n-\alpha}{2}-1}W_{\frac{\alpha}{2}}f_0(\sqrt{\tau}))(t) \tag{4.59}$$

whenever f is a radially symmetric function in $\mathbb{R}^n$ with trace f_0 and R, W are the Riemann-Liouville and Weyl fractional integrals given by

$$R_\lambda f(x) = \frac{1}{\Gamma(\lambda)}\int_0^x \frac{f(t)\,dt}{(x-t)^{1-\lambda}} \quad (0 < \lambda < 1)$$

$$W_\lambda f(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty \frac{f(t)\,dt}{(t-x)^{1-\lambda}} \quad (0<\lambda<1)$$

Since the weights for one-sided fractional integrals as R_λ *and* W_λ *have been extensively studied, it is reasonable to expect that formula* (4.59) *will provide information on the class of weights we are trying to characterize, at least for* $0<\alpha<2$*. Also, since the right-hand side of formula* (4.59) *makes sense for any* $\alpha \in (0,2)$*, rewriting the multiplier operator* $M_{\alpha,m}$ *in this fashion would give a result similar to Theorem 4.8 for a larger class of* α*'s.*

Appendix A

A.1 Alternative proof or Stein's theorem for singular integrals with power weights

Here we will show how Lemma 1.7 can be used to obtain an alternative proof of Stein's result [39] in the case of radially symmetric functions, namely,

Theorem A.1. *Let $f(x) = f_0(|x|)$ be a radially symmetric function in $\mathbb{R}^n$ and*

$$(Tf)(x) = P.V. \int_{\mathbb{R}^n} \frac{H(x, x-y)}{|x-y|^n} f(y)\, dy$$

and assume that $\|Tf(x)\|_p \leq C\|f(x)\|_p, 1 < p < \infty$. Assume further that $|H(x, x-y)| \leq A$. Then

$$\|(Tf)(x)|x|^\beta\|_p \leq C\|f(x)|x|^\beta\|_p$$

if $1 < p < \infty$ and $-\frac{n}{p} < \beta < \frac{n}{p'}$.

Remark A.2. *Although the power weights of the above theorem are exactly the power weights belonging to Muckenhoupt's A_p class, the advantage of Stein's result is that it only requires boundedness of the singular integral in $L^p(\mathbb{R}^n)$ and no additional regularity on the kernel $H(x, x-y)$ and, therefore, it can be applied even in the case of rough kernels.*

Proof. Since $\|T(|x|^\beta f(x))\|_p \leq C\||x|^\beta f(x)\|_p$, it suffices to see that $\|T(|x|^\beta f(x)) - |x|^\beta Tf(x)\|_p \leq C\||x|^\beta f(x)\|_p$.

But

$$\begin{aligned}
|T(|x|^\beta f(x)) - |x|^\beta Tf(x)| &= \left| \int_{\mathbb{R}^n} \frac{H(x, x-y)}{|x-y|^n} (|y|^\beta - |x|^\beta) f(y)\, dy \right| \\
&= \left| \int_{\mathbb{R}^n} \frac{H(x, x-y)}{|x-y|^n} \left(1 - \frac{|x|^\beta}{|y|^\beta}\right) |y|^\beta f(y)\, dy \right| \\
&\leq A \int_{\mathbb{R}^n} K(x, y) |y|^\beta |f(y)|\, dy
\end{aligned}$$

where

$$K(x,y) = \frac{1 - \frac{|x|^\beta}{|y|^\beta}}{|x-y|^n} \tag{4.1}$$

The theorem will be then a consequence of the following lemma:

Lemma A.3. *Let K be as in* (4.1) *and*

$$Uf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

Then, $\|Uf\|_p \le C\|f\|_p$ *if* $1 < p < \infty$ *and* $-\frac{n}{p} < \beta < \frac{n}{p'}$.

Let

$$y = ry', |y| = r, y' \in S^{n-1}$$

and

$$x = \rho x', |x| = \rho, x' \in S^{n-1}.$$

Then, since $f(x) = f_0(|x|)$,

$$\begin{aligned}
Uf(x) &= \int_0^\infty \int_{S^{n-1}} \frac{1 - \frac{\rho^\beta}{r^\beta}}{(\rho^2 - 2\rho r x'\cdot y' + r^2)^{\frac{n}{2}}}\, dy' f_0(r) r^{n-1}\, dr \\
&= \omega_{n-2} \int_0^\infty f_0(r) \left(1 - \frac{\rho^\beta}{r^\beta}\right) r^{n-1} \left\{ \int_{-1}^1 \frac{(1-t^2)^{(n-3)/2}}{(\rho^2 - 2\rho r t + r^2)^{n/2}}\, dt \right\} dr \\
&= \omega_{n-2} \int_0^\infty f_0(r) \left(1 - \frac{\rho^\beta}{r^\beta}\right) \left\{ \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{((\frac{\rho}{r})^2 - 2(\frac{\rho}{r})t + 1)^{\frac{n}{2}}}\, dt \right\} \frac{dr}{r}
\end{aligned}$$

Therefore, in the notation of Lemma 1.7 we have that

$$Uf(x) = \omega_{n-2} \int_0^\infty f_0(r) \left(1 - \frac{\rho^\beta}{r^\beta}\right) I_{\gamma,k}\left(\frac{\rho}{r}\right) \frac{dr}{r}$$

with $\gamma = n, k = \frac{n-3}{2}$.

Now, remark that

$$\|Uf\|_{L^p(\mathbb{R}^n)} = w_{n-1} \|Uf\, r^{\frac{n}{p}}\|_{L^p(\frac{dr}{r})}$$

and

$$\begin{aligned}
\rho^{\frac{n}{p}} Uf &= \omega_{n-2} \int_0^\infty f_0(r) r^{\frac{n}{p}} \left(\frac{\rho}{r}\right)^{\frac{n}{p}} \left(1 - \frac{\rho^\beta}{r^\beta}\right) I_{\gamma,k}\left(\frac{\rho}{r}\right) \frac{dr}{r} \\
&= \omega_{n-2} (f_0(r) r^{\frac{n}{p}}) * (r^{\frac{n}{p}} (1 - r^\beta) I_{\gamma,k}(r)).
\end{aligned}$$

Therefore, by Young's inequality,

$$\begin{aligned}\|Uf\,r^{\frac{n}{p}}\|_{L^p(\frac{dr}{r})} &\le C\|f_0(r)r^{\frac{n}{p}}\|_{L^p(\frac{dr}{r})}\|r^{\frac{n}{p}}(1-r^\beta)I_{\gamma,k}(r)\|_{L^1(\frac{dr}{r})}\\ &= C\|f\|_{L^p(\mathbb{R}^n)}\|r^{\frac{n}{p}}(1-r^\beta)I_{\gamma,k}(r)\|_{L^1(\frac{dr}{r})}\end{aligned}$$

and to conclude the proof of the lemma it suffices to see that

$$\|r^{\frac{n}{p}}(1-r^\beta)I_{\gamma,k}(r)\|_{L^1(\frac{dr}{r})} < +\infty$$

Indeed, when $r \to \infty$, we know that $I_{\gamma,k} \sim C_k r^{-\gamma}$, which implies that the integrability condition at ∞ is $\frac{n}{p} + \beta - n < 0$, that is, $\beta < \frac{n}{p'}$.

When $r \to 0$, since $I_{\gamma,k}(0) < C$, the integrability condition is $\frac{n}{p} + \beta > 0$, that is, $\beta > -\frac{n}{p}$.

When $r \to 1$, since we are in the case $\gamma > 2k+2$, by Lemma 1.7 $|I_{\gamma,k}(r)| \le C_{\gamma,k}|1-r|^{-\gamma+2k+2}$, and it follows that the resulting integral is finite.

□

Appendix B

B.1 Alternative proof of the weighted imbedding theorem in Chapter 2

Here we present the original proof of the weighted imbedding theorem that appeared in [11]. This proof of the theorem is based on an $L^p - L^q$ estimate for the so-called Fourier-Bessel (or Hankel) transform due to L. De Carli [8]. In this section, we shall therefore follow the notations in that paper, that we recall here for sake of completeness:

For given parameters α, ν, μ De Carli introduced the operator

$$L^{\alpha}_{\nu,\mu} f(y) = y^{\mu} \int_0^{\infty} (xy)^{\nu} f(x) J_{\alpha}(xy)\, dx$$

where J_α denotes the Bessel function of order α. A particular case of this operator is the Fourier-Bessel transform:

$$\tilde{\mathcal{H}}_{\alpha} f(x) = L^{\alpha}_{\alpha+1,-2\alpha-1} f(x) \tag{4.1}$$

The importance of this operator for our purposes is due to the fact that it provides an expresion for the Fourier transform of a radial function $u(x) = u_0(|x|)$:

$$\hat{u}(|\omega|) = (2\pi)^{\frac{n}{2}} \tilde{H}_{\frac{n}{2}-1}(u_0)(|\omega|). \tag{4.2}$$

Morover, we recall that we have the inversion formula:

$$\tilde{\mathcal{H}}_{\alpha}(\tilde{\mathcal{H}}_{\alpha}(u))(x) = u(x) \ \text{ (equation (2.4) from [8])} \tag{4.3}$$

Now, we state De Carli's theorem (Theorem 1.1 in [8]):

Theorem B.1. *$L^{\alpha}_{\nu,\mu}$ is a bounded operator from $L^p(0,\infty)$ to $L^q(0,\infty)$ whenever $\alpha \geq -\frac{1}{2}$, $1 \leq p \leq q \leq \infty$, if and only if*

$$\mu = \frac{1}{p'} - \frac{1}{q} \;\; and \;\; -\alpha - \frac{1}{p'} < \nu \leq \frac{1}{2} - max\left(\frac{1}{p'} - \frac{1}{q}, 0\right).$$

Finally, we observe that the De Carli operators $L^{\alpha}_{\nu,\mu}$ enjoy two invariance properties that will be useful in obtaining weighted estimates (and that are immediate from their definition):

$$y^e L^{\alpha}_{\nu,\mu}(f)(y) = L^{\alpha}_{\nu,\mu+e}(f)(y) \tag{4.4}$$

$$L^{\alpha}_{\nu,\mu}(f) = L^{\alpha}_{\nu-\sigma,\mu+\sigma}(x^{\sigma} f). \tag{4.5}$$

Now we are ready to give or proof:

Proof. Let $u(x) = u_0(|x|) \in H^s_{rad}(\mathbb{R}^n)$. Using polar coordinates we have that:

$$\left(\int_{\mathbb{R}^n} |x|^c |u|^q \, dx\right)^{\frac{1}{q}} = C\left(\int_0^{\infty} r^{c+n-1} |u_0(r)|^q \, dr\right)^{\frac{1}{q}}.$$

Thanks to the inversion formula (4.3) for the Fourier-Bessel transform of order $\alpha = \frac{n}{2} - 1$ (which is just the usual Fourier inversion formula for radial functions) we obtain:

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |x|^c |u|^q \, dx\right)^{\frac{1}{q}} &= C\left(\int_0^{\infty} r^{c+n-1} |\tilde{\mathcal{H}}_{\alpha}(\tilde{\mathcal{H}}_{\alpha}(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \\
&= C\left(\int_0^{\infty} r^{c+n-1} |L^{\alpha}_{\alpha+1,-2\alpha-1}(\tilde{\mathcal{H}}_{\alpha}(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \text{ (using (4.1))} \\
&= C\left(\int_0^{\infty} |L^{\alpha}_{\alpha+1,-2\alpha-1+\frac{c+n-1}{q}}(\tilde{\mathcal{H}}_{\alpha}(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \text{ (using (4.4))} \\
&= C\left(\int_0^{\infty} |L^{\alpha}_{\alpha+1-\sigma,-2\alpha-1+\frac{c+n-1}{q}+\sigma}(r^{\sigma}\tilde{\mathcal{H}}_{\alpha}(u_0))(r)|^q \, dr\right)^{\frac{1}{q}} \text{ (using (4.5))}
\end{aligned}$$

where the value of the parameter σ can be chosen to fulfill our needs.

Indeed, now we apply Theorem B.1 with the following choice of parametes

$$p = \frac{nq}{nq - n - c}, \quad \sigma = \frac{n-1}{p}, \quad \alpha = \frac{n}{2} - 1, \quad \nu = \alpha + 1 - \sigma, \quad \mu = -2\alpha - 1 + \frac{c+n-1}{q} + \sigma$$

Since it easy to see that, under the hypotheses of our theorem, all the restrictions of Theorem B.1 are fulfilled, we get the bound:

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |x|^c |u|^q \, dx\right)^{\frac{1}{q}} &\le C \left(\int_0^\infty |r^\sigma \tilde{\mathcal{H}}_\alpha(u_0)(r)|^p \, dr\right)^{\frac{1}{p}} \\
&= C \left(\int_0^\infty (1+r^2)^{\frac{sp}{2}} (1+r^2)^{-\frac{sp}{2}} |\tilde{\mathcal{H}}_\alpha(u_0)(r)|^p \, r^{n-1} \, dr\right)^{\frac{1}{p}} \\
&\le C \left(\int_0^\infty (1+r^2)^s |\tilde{\mathcal{H}}_\alpha(u_0)(r)|^2 r^{n-1} \, dr\right)^{\frac{1}{2}} \left(\int_0^\infty (1+r^2)^{-\frac{sp}{2-p}} r^{n-1} \, dr\right)^{\frac{2-p}{2p}} \\
&\le C \|u\|_{H^s},
\end{aligned}$$

where in the last inequality we have used (4.2) and the fact that, under the restrictions of our theorem,

$$\int_0^\infty (1+r^2)^{-\frac{sp}{2-p}} \, r^{n-1} \, dr < +\infty \ \left(\text{recall that } \frac{2n}{2s+n} < p\right).$$

It remains to prove that the imbedding $H^s_{rad}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^c)$ is compact, which can be done in exactly the same way as in Chapter 2, Section 2.

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