Cuerdas y D-branas en espacio-tiempos curvos

Baron, Walter Helmut

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Walter Helmut Baron

Director de tesis: Dra. Carmen Alicia Nuñez

Consejero de estudios: Dr. Gustavo Sergio Lozano

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En esta tesis estudiamos el modelo AdS_3 Wess-Zumino-Novikov-Witten. Calculamos la Expansión en Producto de Operadores de campos primarios y de sus imágenes bajo el automorfismo de flujo espectral en todos los sectores del modelo considerado como una rotación de Wick del modelo coset H_3^+. Argumentamos que las simetrías afines del álgebra requieren un truncado que determina la clausura de las reglas de fusión del espacio de Hilbert. Estos resultados son luego utilizados para discutir la factorización de las funciones de cuatro puntos con la ayuda del formalismo conocido como bootstrap.

También realizamos un estudio de las propiedades modulares del modelo. Los caracteres sobre el toro Euclídeo divergen de una manera poco controlable. La regularización propuesta en la literatura es poco satisfactoria pues elimina información del espectro y se pierde así la relación uno a uno entre caracteres y representaciones del álgebra de simetría que forman el espectro. Proponemos estudiar entonces los caracteres definidos sobre el toro Lorentziano los cuales están perfectamente definidos sobre el espacio de funcionales lineales, recuperando así la biyección entre caracteres y representaciones. Luego obtenemos las transformaciones modulares generalizadas y las utilizamos para estudiar la conexión con los correladores que determinan los acoplamientos a las branas simétricas en tal espacio de fondo, obteniendo que en los casos particulares de branas puntuales o dS_2 branas se recuperan resultados típicos de Teorías de Campos Conformes Racionales como soluciones tipo Cardy o fórmulas tipo Verlinde.

Palabras claves: *teoría de cuerdas, teorías conformes no racionales, D-branas, AdS_3, reglas*

de fusión, transformaciones modulares.

Strings and D-branes in curved space-time

In this thesis we study the AdS_3 Wess-Zumino-Novikov-Witten model. We compute the Operator Product Expansion of primary fields as well as their images under the spectral flow automorphism in all sectors of the model by considering it as a Wick rotation of the H_3^+ coset model. We argue that the symmetries of the affine algebra require a truncation which establishes the closure of the fusion rules on the Hilbert space of the theory. These results are then used to discuss the factorization of four point functions by applying the bootstrap approach.

We also study the modular properties of the model. Although the Euclidean partition function is modular invariant, the characters on the Euclidean torus diverge and the regularization proposed in the literature removes information on the spectrum, so that the usual one to one map between characters and representations of rational models is lost. Reconsidering the characters defined on the Lorentzian torus and focusing on their structure as distributions, we obtain expressions that recover those properties. We then study their generalized modular properties and use them to discuss the relation between modular data and one point functions associated to symmetric D-branes, generalizing some results from Rational Conformal Field Theories in the particular cases of point like and dS_2 branes, such as Cardy type solutions or Verlinde like formulas.

Keywords: *string theory, non rational conformal field theories, D-branes, AdS_3, fusion rules, modular transformations.*

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Contents

Chapter 1

Introduction

String theory is one of the most ambitious projects in the field of high energy physics. Even though it was originally conceived to explain quark confinement, its development forced the opening of different roads and the extension of its original aims. Shortly after its beginning, people realized every consistent string theory had a massless spin two particle in the spectrum, a graviton. This opened a new possibility to reach a consistent theory of quantum gravity, with extended objects to quantize instead of point-like particles, and so to reach the dream of every theoretical physicist, *i.e.* a unified theory for all known interactions or a theory of everything.

This theory, yet under construction, is the best candidate to give a unified description of all interactions, but also has the beauty of having offered important results going beyond the theory itself. It gave rise to an amazing development of Conformal Field Theories (CFTs) providing several tools to deal with two dimensional statistical models. The consistency of the theory led to the idea of supersymmetry. The gauge/gravity duality, whose validity is not restricted to that of string theory, has offered the first realization of the holographic principle, leading not only to a theoretical frame relating gravitational theories with gauge theories in flat space, but a powerful tool allowing to study non perturbative regions of superconformal, but also of non conformal field theories or models with less supersymmetry or in different space-time dimensions [1] than those proposed in the first example of this duality [2, 3], including holographic renormalization group flows [4, 5, 6], rotating strings [7, 8, 9], applications to the study of baryonic symmetry breaking [10, 11], applications to cosmology [12, 13], applications to holographic QCD [14, 15, 16, 17], to electroweak symmetry breaking [18], and non relativistic quantum-mechanical systems [19, 20, 21, 22], among others.

Many years have passed since its birth and it appears frustrating not having found yet a solid confirmation of the theory. There is not known way to reduce the theory to one which exactly reproduces all aspects of the Standard Model, or which solves the cosmological constant problem, among other drawbacks. But these puzzles must not be seen as a failure of the theory, but as a consequence of the complexity of the problems to be solved. The knowledge of the theory has increased considerably. There is a good understanding of the perturbative regime, and the discovery of $D-$branes, extended dynamical objects on which the ends of open strings live, has been crucial in the development of dualities that allow to explore non perturbative regions of the theory.

One of the major challenges of the theory is to find a mechanism univocally leading to an effective reduced theory with a solution phenomenologically contrastable with the real world. Instead, one finds a large number of vacua, roughly 10^{500}, a problem frequently named the *landscape* of string theory [23].

There is no doubt that D-branes play a fundamental role in the resolution of these problems and so a fundamental step consists in the study of branes in non trivial curved backgrounds.

A very powerful approach to explore string theories on non trivial curved backgrounds is to consider vacua with a lot of symmetry. This was successfully reached by considering the space-time to be given by the group manifold of a continuous compact group, G. In such a case the worldsheet theory is a Rational Conformal Field Theory (RCFT), *i.e.* it contains a finite number of primary states. To date these theories can be solved exactly by using only algebraic tools [24, 25, 26]. The situation is very different in the non compact case, as a consequence of the presence of a continuous spectrum of states (see for instance [27] for a review). This thesis is devoted to the study of the worldsheet of string theory on a three dimensional anti de Sitter (AdS_3) space time. This theory is very interesting because it is one of the simplest models to test string theory in curved non compact backgrounds and with a non trivial timelike direction, it can be used to learn more on the not well known non RCFT, but also because of its relevance in the AdS/CFT correspondence.

Most of the studies regarding the AdS/CFT duality were explored only within the supergravity approximation. An example where the correspondence was successfully explored beyond the supergravity approximation is string theory in PP waves backgrounds [28, 29] with RR fields, obtained by taking the Penrose limit of AdS. Another accessible background is AdS_3 with Neveu Schwarz (NS) antisymmetric field, appearing within backgrounds like $\mathrm{AdS}_3 \times S^3 \times M^4$, with $M^4 = T^4$ or K^3 (obtained as the near horizon limit of the $D1 - D5$ brane setup in the background $R^6 \times M^4$) [30, 31, 32, 33, 34, 35].

The worldsheet of the bosonic string propagating on AdS_3 is described by the Wess Zumino Novikov Witten (WZNW) model associated to the universal cover of the $SL(2,\mathbb{R})$ group, $\widetilde{SL(2,\mathbb{R})}$ but for short we will refer to this as the AdS_3 WZNW model in order to avoid confusion with the WZNW model on the single cover of $SL(2,\mathbb{R})$. So it is expected to be exactly solvable. The studies on this model started within the seminal work of O'Raifeartaigh, Balog, Forgacs and Wipf [36] on the $SU(1,1)$ WZNW model more than twenty years ago, but it was necessary to wait for more than ten years until the work of Maldacena and Ooguri [37] correctly defined the spectrum, which is considerably more involved that those of RCFT. It consists of long strings with continuous energy spectrum arising from the principal continuous representation of $sl(2)$ and its spectral flow images, and short strings with discrete physical spectrum resulting from the highest-weight discrete representations and their spectral flow images.

By extending the result of Evans, Gaberdiel and Perry [38] to nontrivial spectral flow sectors, a no ghost theorem for this spectrum was proved in [37] and verified in [39] through the computation of the one-loop partition function on a Euclidean AdS_3 background at finite temperature. Amplitudes of string theory in AdS_3 on the sphere were computed in [40], analytically continuing the expressions obtained for the Euclidean $\mathrm{H}_3^+ = \frac{SL(2,\mathbb{C})}{SU(2)}$ (gauged) WZNW model in [41, 42]. Some subtleties of the analytic continuation relating the H_3^+ and AdS_3 models were clarified in [40] and this allowed to construct, in particular, the four-point functions of unflowed short strings. Integrating over the moduli space of the worldsheet, it was shown that the string amplitude can be expressed as a sum of products of three-point functions with intermediate physical states, *i.e.* the structure of the factorization agrees with the Hilbert space of the theory.

A step up towards a proof of consistency and unitarity of the theory involves the construction of four-point functions including states in different representations and the verification that only unitary states corresponding to long and short strings in agreement with the spectral flow selection rules are produced in the intermediate channels. To achieve this goal, the analytic and algebraic structure of the AdS_3 WZNW model should be explored further.

Most of the important progress achieved is based on the better understood Euclidean H_3^+ model. The absence of singular vectors and the lack of chiral factorization in the relevant current algebra representations obstruct the use of the powerful techniques from rational conformal field theories. Nevertheless, a generalized conformal bootstrap approach was successfully applied in [41, 42] to the H_3^+ model on the punctured sphere, allowing to discuss the factorization of four-point functions. In principle, this method offers the possibility to unambiguously determine any $n > 3-$point function in terms of two- and three-point functions once the operator product expansions of two operators and the structure constants are known.

We carried out some initial steps along the development of this thesis [43], by examining the role of the spectral flow symmetry on the analytic continuation of the operator product expansion from H_3^+ to the relevant representations of $SL(2,\mathbb{R})$ and on the factorization properties of four-point functions. These results give fusion rules establishing the closure of the Hilbert space and the unitarity of the full interacting string theory.

In RCFT, a practical derivation of the fusion rules (*i.e.* of the representations contained in the Operator algebra) can be performed through the Verlinde theorem [44], often formulated as the statement that the S matrix of modular transformations diagonalizes the fusion rules. Moreover, besides leading to a Verlinde formula, the S matrix allows a classification of modular invariants and a systematic study of boundary states for symmetric branes. It is interesting to explore whether analogous of these properties can be found in the AdS_3 WZNW model. However, the relations among fusion algebra, boundary states and modular transformations are difficult to identify and have not been very convenient in non compact models [45]. In general, the characters have an intricate behavior under the modular group [46]-[48] and, as is often the case in theories with discrete and continuous representations, these mix under S transformations. In the forthcoming chapters we will discuss these subjects based on our previous results [49].

This thesis is organized as follows, in Chapter 2 we review the geometry, symmetries, and give some basis functions of AdS_3 and related spaces which are the basic objects in the minisuperspace limit. Chapter 3 is devoted to introduce the WZNW models. We begin with a short introduction on general WZNW models and then present the AdS_3 WZNW model, and related ones like H_3^+ and the $\frac{SL(2,\mathbb{R})_k/U(1)\times U(1)_{-k}}{\mathbb{Z}_{N_k}}$ models. In Chapter 4 we discuss interactions in the model. We consider two and three point functions of the Euclidean H_3^+ model, and assuming they are related to those of the AdS_3 model [40], we compute the Operator Product Expansion (OPE) for primary fields as well as for their images under spectral flows in all the sectors of the theory. After discussing the extension to descendant fields, we show that the spectral flow symmetry requires a truncation of the fusion rules determining the closure of the operator algebra on the Hilbert space of the theory. In Chapter 5 we consider the factorization of four-point functions and study some of its properties. Chapter 6 is devoted to the characters of the relevant representations of the AdS_3 WZNW model. Since the standard Euclidean characters diverge and lack good modular properties, extended characters were originally introduced in [50] (see also [51])[1]. A different approach was followed in [37] where the standard characters were computed on the Lorentzian torus and it was shown that the modular invariant partition function of the H_3^+ model obtained in [55] is recovered after performing analytic continuation and discarding contact terms. However, this trivial regularization removes information on the spectrum and the usual one to one map between characters and representations of rational models is lost. With the aim of overcoming these problems, we review (and redefine) the characters on the Lorentzian torus, focusing on their structure as distributions and compute the full set of generalized modular transformations in Chapter 7.

In order to explore the properties of the modular S matrix, in Chapter 8 we consider the maximally symmetric D-branes of the model. We explicitly construct the Ishibashi states and show that the coefficients of the boundary states turn out to be determined from the generalized

[1]Similar problems in non-compact coset models have also been considered in [52]-[54]

S matrix, suggesting that a Verlinde-like formula could give some information on the spectrum of open strings attached to certain D-branes. Furthermore, we show that a generalized Verlinde formula reproduces the fusion rules of the finite dimensional degenerate representations of $sl(2,\mathbb{R})$ appearing in the boundary spectrum of the point-like D-branes.

In chapter 9 we give a summary of the thesis and discuss the actual status and future challenges and perspectives regarding open problems. We also list the original contributions to the subject presented along the thesis.

Some basic facts about CFTs are reviewed in appendix A, some technical details of the calculations are included in appendices B, D and E and a discussion of the moduli space of the Lorentzian torus is found in appendix C.

Chapter 2

Geometric aspects of maximally symmetric spaces

Before introducing the AdS_3 WZNW model, it is instructive to spend some pages reviewing some aspects of the geometry of maximally symmetric spaces and the minisuperspace limit in AdS_3 and related models.

2.1 Geometry and symmetries

Along the bulk of the thesis the reader will find discussions concerning different types of geometries such as hyperbolic, Anti de Sitter or de Sitter spaces, so a good point to begin with is by defining all these geometries.

2.1.1 Maximally symmetric spaces

Maximally symmetric spaces are defined as those metric spaces with maximal number of isometries in a given spacetime dimension[1]. Due to this important property such geometries were extensively studied in the literature. Here we will give a short *breviatum*. For a deeper study of the subject we redirect the reader to [56].

[1]In the concrete case of D spacetime dimensions these spaces admit $D(D+1)/2$ linear independent Killing vectors.

Every D dimensional maximally symmetric space has constant curvature and can be realized as a pseudosphere embedded in a D+1 dimensional flat space.

To be more precise, let $\left(X^0, X^1, \ldots, X^D\right)$ represent the Cartesian coordinates of a particular point in such a flat space, the symmetric spaces can be realized as the hypersurfaces constrained by

$$\epsilon R^2 = X^\mu X_\mu, \tag{2.1.1}$$

where the indices are lowered with the background metric

$$\eta_{\mu\nu} = diag\left(\epsilon_0, \epsilon_1, \ldots, \epsilon_D\right) \tag{2.1.2}$$

and the ϵ's are signs. R is frecuently called the radius of the space because of the similarity with the radius of a sphere, but it must not be confused with the Ricci curvature scalar, $\mathcal{R}$, which is given by

$$\mathcal{R} = \frac{\epsilon D(D-1)}{R^2}. \tag{2.1.3}$$

We are specially interested in Euclidean or Lorentzian (one timelike direction) D dimensional spaces, so we fix $\epsilon_1 = \cdots = \epsilon_{D-1} = 1$.

The cases of interest for us are anti de Sitter, Hyperbolic and de Sitter spaces.

Anti de Sitter space: $\mathbf{AdS}_D$

This geometry corresponds to the case where $\epsilon_0 = \epsilon_D = \epsilon = -1$. The space has Lorentzian signature and $SO(D-1,2)$ isometry group. The time like direction is compact.

The particular case $D = 3$ will be of special interest for us.

Notice that the topology of AdS_3 coincides with that of the $SL(2,\mathbb{R})$ group manifold as can be checked from the following parametrization of this group

$$g = R^{-1}\begin{pmatrix} X^0+X^1 & X^2+X^3 \\ X^2-X^3 & X^0-X^1 \end{pmatrix}, \qquad X^\mu \in \mathbb{R}. \tag{2.1.4}$$

This relation is intimately linked to the fact that the $SO(2,2)$ isometry group is locally isomorphic to $SL(2)\otimes SL(2)$.

For obvious reasons we will decompactify the time-like direction when discussing physical applications. And from now on we will denote this space as AdS_3, which is nothing but the group manifold of the universal covering of $SL(2,\mathbb{R})$, *i.e.* $\widetilde{SL(2,\mathbb{R})}$.

Another useful coordinate system, frequently used in the literature, is the so called global or cylindrical coordinate system (ρ,θ,τ), related to the previous one via

$$\begin{aligned} X^0+iX^3 &= e^{i\tau}\cosh\rho, \\ X^1+iX^2 &= e^{i\theta}\sinh\rho. \end{aligned} \tag{2.1.5}$$

Hyperbolic space: $\mathbf{H}_D$

This geometry is realized when $\epsilon=\epsilon_0=-\epsilon_D=-1$. It has Euclidean signature, $SO(D,1)$ isometry and decomposes in two disconnected branches, the upper sheet ($X^0\geq R$) denoted by H_D^+ and the lower one ($X^0\leq -R$) denoted by H_D^-.

For the special case of $D=3$, this space has the topology of the group coset manifold of the subset of hermitian matrices of unit determinant in $SL(2,\mathbb{C})$, *i.e.* $SL(2,\mathbb{C})/SU(2)$. The subspace H_3^+ can be interpreted as a Euclidean Wick rotation of AdS_3, in fact it can be parametrized with the analytic continuation $\tau\to i\tau$ of the cylindrical coordinate system of AdS_3 so breaking the periodicity of the time-like direction.

Another useful coordinate system is the one defined by the complex variables $(\phi,\gamma,\bar{\gamma})$, where

$$\begin{aligned} \gamma &= e^{\tau+i\theta}\tanh\rho, \\ \bar{\gamma} &= e^{\tau-i\theta}\tanh\rho, \end{aligned}$$

$$e^{\phi} \quad = \quad e^{-\tau} \cosh \rho. \tag{2.1.6}$$

In terms of these coordinate, the elements of the coset are parametrized by

$$h = \begin{pmatrix} e^{\phi} & e^{\phi}\,\bar{\gamma} \\ e^{\phi}\,\gamma & e^{\phi}\gamma\bar{\gamma} + e^{-\phi} \end{pmatrix} \tag{2.1.7}$$

and the coset structure is manifest when h is written as $h = v\,v^{\dagger}$, with

$$v = \begin{pmatrix} e^{\phi/2} & 0 \\ e^{\phi/2}\,\gamma & e^{-\phi/2} \end{pmatrix}. \tag{2.1.8}$$

Clearly h is invariant under $v \to v\,u$, with $u \in SU(2)$ and this explicit realization of the coset structure is the reason why these coordinates are implemented when constructing the H_3^+ model by gauging the $SU(2)$ subgroup of the $SL(2,\mathbb{C})$ WZNW model.

de Sitter space: dS$_D$

In this case only $\epsilon_0 = -1$. The space has Lorentzian signature, $SO(D,1)$ isometry group and its topology coincides with the group manifold of unimodular antihermitian matrices

$$g = R^{-1} \begin{pmatrix} i\left(X^0 + X^1\right) & \left(X^2 + iX^3\right) \\ -\left(X^2 - iX^3\right) & i\left(X^0 - X^1\right) \end{pmatrix}, \qquad X^{\mu} \in \mathbb{R}. \tag{2.1.9}$$

Other geometries (not discussed in the thesis) are the (Euclidean) D-dimensional sphere with SO(D+1) isometry group ($\epsilon_{\mu} = 1, \forall \mu$) and the "two time" pseudosphere with $SO(D-1,2)$ isometry corresponding to $\epsilon_0 = \epsilon_D = -\epsilon = -1$.

2.2 Basis functions

In this section we describe some of the basis functions for $H_3^+, SL(2,\mathbb{R})$ and AdS_3, where by abuse of notation we refer to $SL(2,\mathbb{R})$ as the group manifold of the single cover of $SL(2,\mathbb{R})$. The fact that both H_3^+ and $SL(2,\mathbb{R})$ admit matricial representations makes the study of functions over these spaces much easier than in the AdS_3 case where such representations are not present.

Let us begin with $SL(2,\mathbb{R})$, and the parametrization (2.1.5), then the elements of the group, g are written as

$$g(\rho,\theta,\tau) = \begin{pmatrix} \cosh\rho\cos\tau + \sinh\rho\cos\theta & \cosh\rho\sin\tau + \sinh\rho\sin\theta \\ \sinh\rho\sin\theta - \cosh\rho\sin\theta & \cosh\rho\cos\tau - \sinh\rho\cos\theta \end{pmatrix} \qquad (2.2.1)$$

As we commented in the previous section, AdS_3 is obtained by decompactifying τ. If we write $\tau = 2\pi(q+\lambda)$, where $q\in\mathbb{Z},\ \lambda\in[0,1)$, the elements $G(\rho,\theta,\tau)$ of AdS_3 admit a parametrization in terms of $SL(2,\mathbb{R})$

$$G = (g,q), \qquad g\in SL(2,\mathbb{R}),\ q\in\mathbb{Z} \qquad (2.2.2)$$

The AdS_3 product is built from the one in the single cover as

$$GG' = (g,q)(g',q') = (gg', q+q'+F(g)+F(g')-F(gg')) \qquad (2.2.3)$$

Notice that both $SL(2,\mathbb{R})$ and the universal cover carry a natural left and right multiplication by themselves. For instance, for AdS_3 we have $(G_L,G_R)\cdot G = G_L G G_R$ and this will be the symmetry of the model under consideration (see section 3.1).

The geometric symmetry group of $SL(2,\mathbb{R})$ is $\frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ is the center of the group. In the case of AdS_3 the symmetry group is $\frac{AdS_3\times AdS_3}{\mathbb{Z}}$, where the center is isomorphic to $\mathbb{Z}$. Indeed, it can be easily checked, from (2.2.3), that it is given by the subgroup $\{(\pm id,q),\ q\in\mathbb{Z}\}$, which is the subgroup freely generated by the element $(-id,0)$.

A useful parametrization for H_3^+, different from the ones considered in the previous section is given by

$$h(\rho,\theta,\tilde{\tau}) = \begin{pmatrix} e^{\tilde{\tau}}\cosh\rho & e^{i\theta}\sinh\rho \\ e^{-i\theta}\sinh\rho & e^{-\tilde{\tau}}\cosh\rho \end{pmatrix} \qquad (2.2.4)$$

Notice that the product of two matrices in H_3^+, h_1 and h_2 with $\theta_1 \neq \theta_2$ is outside of the hyperbolic space. This is a not surprising because H_3^+ is not a group, but the group coset $SL(2,\mathbb{C})/SU(2)$. In this case the left and right action take the element out of the space and the geometric symmetry group acts as $k \cdot h = khk^\dagger$, with $k \in SL(2,\mathbb{C})$. So it is given by $\frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$.

2.2.1 Continuous basis

As we commented at the beginning of this section, the fact that H_3^+ and $SL(2,\mathbb{R})$ admit matrix representations simplifies matters.

A useful basis for H_3^+ is the well known *x-basis*,

$$\phi^j(x,\bar{x}|h) = \frac{2j+1}{\pi}\left|(\bar{x}\ 1)h\begin{pmatrix} x \\ 1 \end{pmatrix}\right|^{2j}, \tag{2.2.5}$$

whit $x, \bar{x} \in \mathbb{C}$. These functions have simple behavior under symmetry transformations. If k is parametrized by $k^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\phi^j(x,\bar{x}|khk^\dagger) = |cx+d|^{4j}\phi^j(k\cdot x, \bar{x}\cdot k^\dagger|h), \tag{2.2.6}$$

where

$$k \cdot x = \frac{ax+b}{cx+d} \tag{2.2.7}$$

It was shown [57] that a complete basis of functions in the Hyperbolic space is generated by $\phi^j(x,\bar{x}|h)$, with $j \in -\frac{1}{2} + i\mathbb{R}^+$.

Functions with j and $j' = -j-1$ are related by the reflection symmetry

$$\phi^{-j-1}(x,\bar{x}|h) = \frac{R(-1-j)}{\pi}\int d^2y|x-y|^{4j}\phi^j(y,\bar{y}|h) \tag{2.2.8}$$

where the reflection $R(j)$ is such that

$$R(j)R(-j-1) = -(2j+1)^2. \tag{2.2.9}$$

A continuous basis for $SL(2,\mathbb{R})$ is given by the so called *t-basis*

$$\phi^{j,\eta}(t_L, t_R|g) = \frac{2j+1}{\pi} \left|(1 \ \ -t_L)g \begin{pmatrix} t_R \\ 1 \end{pmatrix}\right|^{2j} sgn^{2\eta}\left[(1 \ \ -t_L)g \begin{pmatrix} t_R \\ 1 \end{pmatrix}\right] \tag{2.2.10}$$

The parity $\eta \in \{0, \frac{1}{2}\}$, and the symmetry group acts as

$$\begin{aligned} \phi^{j,\eta}(t_L, t_R|g_L^{-1} g g_R) &= |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} \, sgn^{2\eta}\left[(c_R t_R + d_R)(c_L t_L + d_L)\right] \\ &\times \ \phi^{j,\eta}(g_L \cdot t_L, g_R \cdot t_R|g), \end{aligned} \tag{2.2.11}$$

where

$$g \cdot t = \frac{at+b}{ct+d}. \tag{2.2.12}$$

A complete basis of functions is known to be $\{\phi^{j,\eta}(t_L, t_R|g); j \in -\frac{1}{2} + i\mathbb{R}^+, t_L, t_R \in \mathbb{R}$, $\eta \in \{0, \frac{1}{2}\}\ \} \cup \{\phi^{j,\eta}(t_L, t_R|g); j \in -1 - \frac{1}{2}\mathbb{N}, t_L, t_R \in \mathbb{R}, \eta = j \text{ mod } 1\}$

The situation is subtler in AdS_3 where the correct behavior under symmetry transformations was found in [58]

$$\begin{aligned} \phi^{j,\alpha}(t_L, t_R|G_L^{-1} G G_R) &= |(c_R t_R + d_R)(c_L t_L + d_L)|^{2j} \, e^{2\pi i \alpha[N(G_L|t_L) - N(G_R|t_R)]} \\ &\times \ \phi^{j,\alpha}(G_L \cdot t_L, G_R \cdot t_R|G), \end{aligned} \tag{2.2.13}$$

with $\alpha \in [0, 1)$, $G \cdot t \equiv g \cdot t$, g being the $SL(2,\mathbb{R})$ projection of $G \in AdS_3$. $N(G|t)$ is a function with the following properties

$$N(G'G|t) = N(G'|Gt) + N(G|t)$$

$$N(\,(id,q)\,|t) \;\;=\;\; q. \tag{2.2.14}$$

For instance $N(G|t)$ can be taken as the number of times $G' \cdot t$ crosses infinity when G' moves from $(id,0)$ to G. A function satisfying (2.2.13) was found in [59] and is given by

$$\phi^{j,\alpha}(t_L,t_R|G) = \frac{2j+1}{\pi} e^{2\pi i\alpha\, n(G|t_L,t_R)} \left| (1,-t_L) g \begin{pmatrix} t_R \\ 1 \end{pmatrix} \right|^{2j}, \tag{2.2.15}$$

where n is the function defined such that $n(G|t_L,t_R) - \frac{1}{2} sgn(t_L,t_R)$ gives the number of times $G' \cdot t$ crosses t_R as G' moves from $(id,0)$ to G.

2.2.2 Discrete basis

The continuous x and t basis have simple properties under symmetry transformations. But as g in (2.2.1) and h in (2.2.4) are not related by Wick rotation, neither the continuous functions defined above are related by Wick rotation.

So it is useful to introduce another type of basis, the so called "*m-basis*". These are sets of functions parametrized by discrete parameters $m, \bar{m}$, which even though not having a simple behavior under symmetry transformations have a simple connection via Wick rotation.

The *m-basis* of H_3^+ can be defined as a kind of Fourier transformation of the *x-basis*

$$\phi^j_{m\bar{m}}(h) \equiv \int d^2x \; x^{j+m} \bar{x}^{j+\bar{m}} \phi^j(x,\bar{x}|h). \tag{2.2.16}$$

The combinations $m - \bar{m}$ and $m + \bar{m}$ are proportional to the momentum numbers along the compact θ-direction and the non-compact τ direction respectively, which implies the decomposition

$$\begin{aligned} m &= \frac{n+p}{2} \\ \bar{m} &= \frac{-n+p}{2}, \end{aligned} \tag{2.2.17}$$

with $n \in \mathbb{Z},\; p \in i\mathbb{R}$. Notice that $m - \bar{m} \in \mathbb{Z}$ ensures the monodromy in (2.2.16).

The explicit computation of (2.2.16) gives

$$\phi^j_{m\bar{m}} = -4\,\frac{\Gamma(1+j+\frac{|n|+p}{2})\Gamma(1+j+\frac{|n|-p}{2})}{\Gamma(|n|+1)\Gamma(1+2j)}e^{p\tilde{\tau}-in\theta}\sinh^{|n|}\rho\cosh^{-p}\rho \times F(1+j+\frac{|n|-p}{2},\ -j+\frac{|n|-p}{2},|n|+1;-\sinh^2\rho) \qquad (2.2.18)$$

The reflection property (2.2.8) translates in the *m-basis* to

$$\phi^j_{m\bar{m}} = R^j_{m\bar{m}}\ \phi^{-j-1}_{m\bar{m}}, \qquad (2.2.19)$$

where

$$R^j_{m,\bar{m}} = \frac{\Gamma(-2j-1)\Gamma(j+1+m)\Gamma(j+1-\bar{m})}{\Gamma(2j+1)\Gamma(-j+m)\Gamma(-j-\bar{m})} \qquad (2.2.20)$$

We take the *m-basis* of the Lorentzian models as the Wick rotation $\tilde{\tau} \to i\tau$ of the basis above and by abuse of notation we use the same name, *i.e.* $\phi^j_{m,\bar{m}}$. The first difference of the Wick rotated functions is that p must be real in order to ensure the (delta function) normalization.

If we introduce the parameter $\alpha \in [0,1)$ such that $m,\ \bar{m} \in \alpha+\mathbb{Z}$, the change of basis between m and t-basis is

$$\phi^j_{m,\bar{m}}(G) = c^{j,\alpha}\int_{-\infty}^{\infty} dt_L(1+t_L^2)^j\left(\frac{1+it_L}{1-it_L}\right)^m \times \int_{-\infty}^{\infty} dt_R(1+t_R^2)^j\left(\frac{1-it_R}{1+it_R}\right)^{\bar{m}}\phi^{-1-j,\alpha}(t_L,t_R|G), \qquad (2.2.21)$$

where

$$c^{j,\alpha} = -\frac{4^{-2-2j}\sin 2\pi j}{\sin\pi(j-\alpha)\sin\pi(j+\alpha)}. \qquad (2.2.22)$$

All the functions introduced in this section have a relevant role in the field theory description as they describe what is known as the minisuperspace limit of the WZNW models associated with $SL(2,\mathbb{R})$, AdS_3 or their Euclidean rotation, the H_3^+ space. In such a limit these functions represent the zero mode contribution of the primary fields. A thorough study of the minisuperspace limit of the H_3^+ coset model and the AdS_3 WZNW model was presented in [57] and [59] respectively. The basis functions discussed in this section were proved to be a complete set of functions in each space. These functions not necesarily belong to the squared integrable set but form an orthogonal basis in the same sense that $\{e^{ik\,x}|k \in \mathbb{R}\}$ is a complete basis over the space of real functions. The completeness of the continuous and discrete basis of functions on H_3^+ was proved in [57], and as commented in [59] the proof of the completeness of the discrete basis of AdS_3 follows from the results of [60] where a plancherel formula for AdS_3 was proved. The completeness of the continuous basis is a consequence of the integral relation (2.2.21). And finally the completeness of the basis functions for $SL(2,\mathbb{R})$ follows from the observation that these are nothing but the functions of AdS_3 with 2π periodicity on τ.

Chapter 3

AdS_3 WZNW Model

In this chapter we present the AdS_3 WZNW model and other coset models related to the former by Wick rotations. In next section we present a brief introduction on nongauged and gauged WZNW models, with the aim of introducing some basics tools, setting the notation and to present the formulae we will be using along the thesis. Then we turn to a description of the AdS_3 WZNW model, the H_3^+ or $SL(2,\mathbb{C})/SU(2)$ Coset model and the Coset $\frac{SL(2,\mathbb{R})_k/U(1)\times U(1)_{-k}}{\mathbb{Z}_{Nk}}$ model.

3.1 WZNW models

WZNW models are CFTs with a Lie group symmetry, where the spectrum is built over representations of the affine algebra. These theories have the peculiarity of being defined with an action, a feature that usually does not occur in CFT's.

Sigma models defined with semisimple group manifolds as target space constitute a natural starting point to construct a theory with the properties mentioned above. Nevertheless, even though these theories are classically scale invariant, the β function of the renormalization group is nonzero and so the effective theory becomes massive and a scale anomaly emerges at the quantum level.

This observation is sufficient to realize that this is not the theory we are looking for. Another indication follows from the fact that the conserved currents do not satisfy the factorization property of CFTs, *i.e.* they do not factorize in a holomorphic current and an antiholomorphic one.

The requested theory is obtained when the sigma model action is supplemented with a Wess-Zumino term. The WZNW action is [61, 62, 63]

$$\begin{aligned} S^{WZNW} &= S^{sigma} + S^{WZ} \\ &= \frac{-k}{16\pi} \int d^2x \; Tr' \left(\partial^\mu g^{-1} \partial_\mu g\right) \\ &+ \frac{ik}{24\pi} \int_{B^3} d^3y \; \epsilon_{\alpha\beta\gamma} \; Tr' \left(\tilde{g}^{-1}\partial^\alpha \tilde{g} \; \tilde{g}^{-1}\partial^\beta \tilde{g} \; \tilde{g}^{-1}\partial^\gamma \tilde{g} \right) \end{aligned} \tag{3.1.1}$$

where B^3 is the space whose boundary is the compactification of the space on which we defined the sigma model. The prime in the trace means a normalization in the trace such that in any representation the generators of the Lie algebra satisfy

$$Tr' \left(t^a \; t^b\right) = 2\delta_{ab}. \tag{3.1.2}$$

The field $g(x)$ lives in a unitary representation of the semisimple group G in order to ensure the sigma model action be real. $\tilde{g}(y)$ is the extension to the three-dimensional space B^3. The coupling constant k, usually referred to as the level, must be quantized because B^3 has the topology of a sphere.

Even though the Wess-Zumino term is an integral over a three dimensional space, its variation being a divergence can be written as an integral over the two dimensional boundary and the solution to the Euler-Lagrange equation of the WZNW action is, after the change of variable $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$, $g(z, \bar{z}) = f(z)\bar{f}(\bar{z})$ with independent holomorphic and antiholomorphic functions. The conserved currents are

$$\begin{aligned} J(z) &= k \; \partial g \; g^{-1}, \\ \bar{J}(\bar{z}) &= -k \; g^{-1}\bar{\partial} g, \end{aligned} \tag{3.1.3}$$

where the notation $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$ was used. They are associated with the following invariance of the action

$$g(z,\bar{z}) \to \Omega(z)\ g(z,\bar{z})\ \bar{\Omega}^{-1}(\bar{z}), \tag{3.1.4}$$

with $\Omega(z),\ \bar{\Omega}(\bar{z})$ two arbitrary functions living on G so that the global symmetry $G \times G$ of the sigma model was lifted to a local one by adding the Wess-Zumino term. The transformation law of the currents is easily read out from (3.1.4) and (3.1.3) and the *current algebra* can be determined using the Ward identities. It is found to be

$$J^a(z_1)J^b(z_2) \sim \frac{-k\delta_{ab}}{(z_1-z_2)^2} + \sum_c if_{abc}\frac{J^c(z_2)}{z_1-z_2}, \tag{3.1.5}$$

where $\sim$ means equal up to regular terms and J^a are the components of J in the t^a basis. So defining the Laurent modes as

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n, \tag{3.1.6}$$

the *current algebra* leads to the desired affine Lie algebra $\hat{g}$

$$[J^a_m, J^b_n] = \sum_c if_{abc}J^c_{m+n} - kn\delta_{ab}\delta_{m+n,0} \tag{3.1.7}$$

and similarly for the antiholomorphic sector. The OPE of holomorphic and antiholomorphic currents has no singular terms implying that the modes commute with each other.

The classical energy momentum tensor is obtained from varying the action with respect to the metric[1].

$$T(z)_{classic} = -\frac{1}{2k}\sum_a J^a(z)J^a(z). \tag{3.1.8}$$

Fields are not free, so that this expression will be corrected at quantum level. If the product

[1]From now we will work only with the holomorphic sector, the antiholomorphic sector being analogous.

of currents in (3.1.8) is replaced by a normal ordered product, namely

$$: A(z_1)B(z_2) := \frac{1}{2\pi i}\oint_{z_2} dz_1 \frac{A(z_1)B(z_2)}{z-w}, \tag{3.1.9}$$

and the coefficient is left as a free parameter to be fixed by the requirement that the OPE between two energy momentum tensors be as required by a CFT, *i.e*

$$T(z_1)T(z_2) \sim \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{z_1-z_2}, \tag{3.1.10}$$

one finds the quantum corrected energy momentum tensor

$$T(z) = \frac{-1}{2(k-\mathfrak{g}_c)} \sum_a : J^a J^a : (z), \tag{3.1.11}$$

where $\mathfrak{g}_c$ is the dual coxeter number and the central charge is found to be

$$c = \frac{k\ dim\, g}{k-\mathfrak{g}_c}. \tag{3.1.12}$$

It is bounded from below

$$c \geq Rank\ g, \tag{3.1.13}$$

so that $c \geq 1$.

This realization of the energy momentum tensor in terms of the currents is usually named in the literature as the *Sugawara construction*.

After expanding T according to

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \tag{3.1.14}$$

the Virasoro algebra is realized, namely

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}. \tag{3.1.15}$$

The commutator between Virasoro and current modes is

$$[L_m, J_n^a] = -nJ^a_{m+n}. \tag{3.1.16}$$

Of course holomorphic and antiholomorphic modes commute with each other.

There is much more to say about WZNW models, like discussing the Knizhnik-Zamolodchikov (KZ) equation, free field representations, the modular data, fusion rules and many other matters. We will only discuss the properties we need and at the appropriate time. We end this short review with a few comments on primary fields.

In conformal field theory one defines primary fields, $\Phi(z,\ \bar{z})$, as those transforming covariantly with respect to scale transformations and satisfying

$$T(z_1)\phi(z_2,\bar{z}_2) \sim \frac{\Delta}{(z_1-z_2)^2}\Phi(z_2,\bar{z}_2) + \frac{1}{z_1-z_2}\partial_{z_2}\Phi(z_2,\bar{z}_2), \tag{3.1.17}$$

where Δ is the conformal dimension of Φ. So that

$$\begin{aligned}[L_n, \Phi(z,\bar{z})] &= \frac{1}{2\pi i}\oint_z dw\ w^{n+1}T(w)\Phi(z,\bar{z}) \\ &= \Delta(n+1)z^n\ \Phi(z,\bar{z}) + z^{n+1}\partial\Phi(z,\bar{z}), \qquad n \geq -1\end{aligned} \tag{3.1.18}$$

On the other hand, in the case of WZNW models primary fields are those transforming covariantly under $G(z)\times G(\bar{z})$ and so satisfying the following OPE with the current

$$J^a(z_1)\ \Phi_{\mu,\bar{\mu}}(z_2,\bar{z}_2) \sim \frac{-t^a_\mu\ \Phi_{\mu,\bar{\mu}}(z_2,\bar{z}_2)}{z_1-z_2}, \tag{3.1.19}$$

where $\mu,\bar{\mu}$ denote the holomorphic and antiholomorphic representations of the field, and t^a_μ is the realization of the generator t^a in such representation. These conditions translate into

$$\begin{aligned}J^a_0|\Phi_{\mu,\bar{\mu}}> &= -t^a_\mu|\Phi_{\mu,\bar{\mu}}>, \\ J^a_n|\Phi_{\mu,\bar{\mu}}> &= 0\ , \qquad n>0,\end{aligned} \tag{3.1.20}$$

where $|\Phi_{\mu,\bar{\mu}}>$ represents the primary state $\Phi_{\mu,\bar{\mu}}(0)|0>$.

As a consequence of the realization of the conformal symmetry via the *Sugawara construction*, WZNW primaries are also conformal primaries. They satisfy

$$\begin{aligned} L_n|\Phi_{\mu,\bar{\mu}}> &= 0\ , \qquad n>0, \\ L_0|\Phi_{\mu,\bar{\mu}}> &= \Delta_\mu|\Phi_{\mu,\bar{\mu}}>, \end{aligned} \tag{3.1.21}$$

where the conformal weight Δ_μ is

$$\Delta_\mu = \frac{-\,t^a_\mu t^a_\mu}{2(k-\mathfrak{g}_c)}, \tag{3.1.22}$$

and $t^a_\mu t^a_\mu$ is the quadratic Casimir.

But the reader has to bear in mind that the inverse is not always true. A Virasoro primary can be a WZNW descendant.

3.1.1 Gauged WZNW models

We now consider the construction of Coset or Gauged WZNW models. These are constructed from two WZNW models where the first group is a subgroup of the second one. Contrary to what happens in WZNW models (see (3.1.13)), these theories are less restrictive as there are no bounds for the central charge[2]. Moreover it is expected that this framework provides a full classification of all RCFT [24].

As we saw above a WZNW model with group G is invariant under $G(z)\times G(\bar{z})$, thus it has a global symmetry $G\times G$. Given two subgroups $H_\pm \in G$ it is sometimes possible to obtain a theory with local $H_-(z,\bar{z})\times H_+(z,\bar{z})$ symmetry.

The gauged action was obtained in [64]. This is given by

$$S(g,\mathcal{A}) = S^{sigma}(g,\mathcal{A}) + S^{WZ}(g,\mathcal{A}), \tag{3.1.23}$$

[2]The central charge of the Coset is the difference of the central charges of both WZNW models.

where

$$
\begin{aligned}
S^{sigma}(g,\mathcal{A}) &= \frac{-k}{16\pi}\int Tr'\left(g^{-1}Dg\wedge *g^{-1}Dg\right),\\
S^{WZ}(g,\mathcal{A}) &= S^{WZ}(g)+\frac{ik}{16\pi}\int_{\Sigma} Tr'\left(\mathcal{A}_{-}\wedge dgg^{-1}+\mathcal{A}_{+}\wedge g^{-1}dg+\mathcal{A}_{+}g^{-1}\wedge\mathcal{A}-Lg\right), \quad (3.1.24)
\end{aligned}
$$

D denotes the covariant derivative $Dg = dg + \mathcal{A}_{-}g - g\mathcal{A}_{+}$ and $\mathcal{A}_{\pm}$ are the gauge fields associated to $H_{\pm}$ respectively. It was found that (3.1.23) is invariant under local transformation ($\xi_{L/R} = \xi^a(z,\bar{z})t_{a,L/R}$)

$$
\delta g = \xi_{-}g - g\xi_{+} \quad (3.1.25)
$$

when the gauge fields transform as

$$
\delta\mathcal{A}_{\pm} = -d\xi_{\pm} + [\xi_{\pm},\mathcal{A}_{\pm}], \quad (3.1.26)
$$

and $H_{\pm}$ are anomaly free subgroups, *i.e.* their Lie algebra generators ($t_{a,\pm}$) satisfy [64]

$$
Tr\,(t_{a,-}t_{b,-} - t_{a,+}t_{b,+}) = 0 \quad (3.1.27)
$$

The origin of this unusual restriction can be traced back to the fact that the S^{WZ} is a three dimensional term which defines a two dimensional theory and the standard machinery implemented to gauge a field theory fails.[3]

3.2 AdS_3 WZNW and related models

3.2.1 The spectrum

The spectrum of the AdS_3 WZNW model is built with representations of the affine $\hat{sl}(2)$ algebra, but which are the representations to consider is a subtle question.

The representations of the affine algebra are generated from those of the global $sl(2)$ alge-

[3]Of course the standard approach perfectly works for S^{sigma}, *i.e.* replacing a derivative with a covariant derivative gives a gauge invariant expression.

bra by freely acting with the current modes $J^a_n,\ \bar{J}^a_n,\ a = 3, \pm,\ n \in \mathbb{Z}$, obeying the following commutation relations

$$
\begin{aligned}
[J^3_n, J^3_m] &= -\frac{k}{2} n \delta_{n+m,0} \,, \\
[J^3_n, J^\pm_m] &= \pm J^\pm_{n+m} \,, \\
[J^+_n, J^-_m] &= -2J^3_{n+m} + kn\delta_{n+m,0} \,,
\end{aligned}
\tag{3.2.1}
$$

with level $k \in \mathbb{R}_{>2}$.

In the first attempts to define a consistent spectrum for the worldsheet theory of string theory on $SL(2,\mathbb{R})$ and AdS_3 [36, 38][65]-[69] only representations with L_0 bounded from below were considered. These decompose into direct products of the normalizable continuous, highest and lowest weight discrete representations.

The principal continuous representations $\mathcal{C}^\alpha_j \times \mathcal{C}^\alpha_j$ contain the states $|j, \alpha, m, \bar{m} >$ with, $\alpha \in [0, 1)$, $m, \bar{m} \in \alpha + \mathbb{Z}$ and they are unitary as long as $j \in -\frac{1}{2} + i\mathbb{R}^+$. The lowest weight principal discrete representations $\mathcal{D}^+_j \times \mathcal{D}^+_j$ contain the states $|j, m, \bar{m} >$ with $j \in \mathbb{R}$, $m, \bar{m} \in -j + \mathbb{Z}_{\geq 0}$. The highest weight principal discrete representations $\mathcal{D}^-_j \times \mathcal{D}^-_j$ contain the states $|j, m, \bar{m} >$ with $j \in \mathbb{R}$, $m, \bar{m} \in j - \mathbb{Z}_{\geq 0}$. Both highest and lowest weight representations are unitary when the spin j is constrained as $j < 0$.

The spectrum is supplemented by adding the affine descendants of the global representations defined above, *i.e.* $\hat{\mathcal{D}}^\pm_j \times \hat{\mathcal{D}}^\pm_j$ and $\hat{\mathcal{C}}^\alpha_j \times \hat{\mathcal{C}}^\alpha_j$. Even though these representations are not unitary for $j < 0$, a no ghost theorem has been proved in [38] for $-\frac{k}{2} < j < 0$ guaranteeing the spectrum is unitary when the theory is supplemented with a unitary CFT such that the full central charge be $c = 26$ and the Virasoro constraint is imposed.

This spectrum raised the problem that it leads to a non-modular invariant partition function. And it also raises two puzzles: an upper limit in the string mass spectrum and the absence of states corresponding to the long strings, which were expected to exist from the results of [70, 71].

These problems were all solved when it was realized in [37] that the spectrum must be extended to include representations with L_0 not bounded from below. These representations are nothing but the spectral flow images of the representations considered above. The no ghost theorem for the new representations requires the new bounds $-\frac{k-1}{2} < j < -\frac{1}{2}$ to hold [37].[4]

The spectral flow transformation is generated by the operators $U_w, \bar{U}_{\bar{w}}$, defined by their action on the SL(2,$\mathbb{R}$) currents $J^3, J^{\pm}$ as

$$\begin{cases} \tilde{J}^3(z) = U_w J^3(z) U_{-w} &= J^3(z) - \frac{k}{2}\frac{w}{z} \ , \\ \\ \tilde{J}^{\pm}(z) = U_w J^{\pm}(z) U_{-w} &= z^{\pm w} J^{\pm}(z) \ , \end{cases} \tag{3.2.2}$$

where $U_{-w} = U_w^{-1}$ and similarly for the antiholomorphic sector.

It is not hard to see that such a transformation leaves the algebra (3.2.1) invariant. The right and left spectral flow numbers $w, \bar{w}$ are independent in the single cover of SL(2,$\mathbb{R}$) where $\bar{w} - w$ is the winding number around the compact closed timelike direction. But in the universal covering, they are forced to be equal, $w = \bar{w} \in \mathbb{Z}$. Using the *Sugawara construction*, the action of $U_w, \bar{U}_{\bar{w}}$ on the Virasoro generators is found to be

$$\tilde{L}_n = U_{-w} L_n U_w = L_n + w J_n^3 - \frac{k}{4} w^2 \delta_{n,0} \ . \tag{3.2.3}$$

The conformal weights are easily read as

$$\Delta_j = \tilde{\Delta}_j + wm - \frac{k}{4} w^2 \delta_{n,0} \ , \qquad \tilde{\Delta}_j = -\frac{j(j+1)}{k-2}. \tag{3.2.4}$$

We define the nontrivial spectral flow representations as the discrete and continuous representations considered above but with respect to $\tilde{J}^a$. So that even though $\tilde{L}_0$ is bounded from below, this is clearly not the case for L_0 (see (3.2.3)). This implies that unlike in the compact $SU(2)$ case, different amounts of spectral flow give inequivalent representations of the current algebra of $SL(2,\mathbb{R})$.

We will use the notation $|j, w, m, \bar{m}>= U_w \bar{U}_w |j, m, \bar{m}>$. The action of the currents on the

[4]The bound $j < -\frac{1}{2}$ already appears in the semiclassical limit. Indeed it is not hard to see that the wave functions in AdS_3 are squared integrable iff $j < -\frac{1}{2}$.

spectral flowed states is

$$\begin{aligned} J_0^3|j,w,m,\bar{m}> &= (m+\frac{k}{2}w)|j,w,m,\bar{m}> \\ J_n^3|j,w,m,\bar{m}> &= 0\ , \qquad n=1,2,\ldots \\ J_n^+|j,w,m=-j,\bar{m}> &= 0\ , \qquad n=w,w+1,\ldots \\ J_n^-|j,w,m=j,\bar{m}> &= 0\ , \qquad n=-w,-w+1,\ldots \end{aligned} \tag{3.2.5}$$

Let us end this presentation about the spectrum by noting that there is an overcounting when considering the spectral flow images of the representations above. In fact the spectral flow image of highest and lowest weight representations are related via the identification $\hat{\mathcal{D}}_j^{+,w} \equiv \hat{\mathcal{D}}^{-,w+1}_{-\frac{k}{2}-j}$. The reader can easily check from (3.2.2) and (3.2.3) that they have the same spectrum. So, in what follows we will only consider continuous and lowest weight discrete representations and their images under spectral flow.

The spectrum of the H_3^+ model is simpler. It was determined in [55] and it is built from the principal continuous representations, but contrary to the AdS_3 case, it does not factorize between holomorphic and antiholomorphic representations[5]. The Virasoro primary states are parametrized by $|j,m,\bar{m}>$, with $j \in -\frac{1}{2}+i\mathbb{R}^+$ and $m,\bar{m}$ are given by (2.2.17).

3.2.2 Primary fields

The interpretation of the Lorentzian field theory is subtle and it is instructive to begin with a discussion on the sigma model whose target space is a Euclidean rotation of AdS_3, with a non-zero NS-NS 2 form field $B_{\mu\nu}$. This is the H_3^+ model, constructed by gauging the $SL(2,\mathbb{C})$ WZNW model with an $SU(2)$ right action [55]. A thorough study of the model was presented in [41, 42]. The Lagrangian formulation was developed in [55] and it follows from

$$\mathcal{L} = \frac{k}{\pi}(\partial\phi\overline{\partial}\phi + e^{2\phi}\overline{\partial}\gamma\partial\overline{\gamma})\,. \tag{3.2.6}$$

[5]It is important to note that even though the representations of the spectrum, and so the fields, do not factorize in holomorphic-antiholomorphic sectors, the correlation functions do.

The relevance of this model relies in that it is used to compute the correlation functions of the Lorentzian AdS_3 model. The Euclidean action (3.2.6) is real valued and positive definite and normalizable fields have positive conformal weights, so that Euclidean path integrals are expected to be well defined. As a consequence of the B field, the Euclidean action is not invariant under Euclidean time inversion and so the Lorentzian action is not real and the theory is not unitary.

Normalizable operators in the H_3^+ model, $\Phi_j(x,\overline{x};z,\overline{z})$, $x,z\in\mathbb{C}$, are labeled by the spin j of a principal continuous representation of $SL(2,\mathbb{C})$. These are primary operators with respect to the Virasoro generators (but not with respect to the $sl(2)$ ones) and can be semiclassically identified with the expression

$$\Phi_j(x,\overline{x};z,\overline{z}) = \frac{2j+1}{\pi}\left((\gamma-x)(\overline{\gamma}-\overline{x})e^{\phi}+e^{-\phi}\right)^{2j}. \tag{3.2.7}$$

The variables $x,\bar{x}$ allow to build a continuous representation where the generators of the $sl(2)$ algebra are differential operators. But as is discussed in [72], these variables have a very important interpretation in string theory as the coordinates of the operators in the dual CFT living in the boundary of H_3^+.

These fields satisfy the following OPE with the holomorphic $SL(2,\mathbb{C})$ currents

$$J^a(z)\Phi_j(x,\overline{x};z',\overline{z}') \sim \frac{D^a\Phi_j(x,\overline{x};z',\overline{z}')}{z-z'}\,, \qquad a=\pm,3\,, \tag{3.2.8}$$

where $D^-=\partial_x\,, D^3=x\partial_x - j\,, D^+=x^2\partial_x-2jx$ and similarly with the antiholomorphic modes, and they have conformal weight $\widetilde{\Delta}_j$ as defined in (3.2.4). The asymptotic $\phi\to\infty$ expansion, given by

$$\Phi_j(x,\overline{x}|z,\overline{z}) \sim: e^{2(-1-j)\phi(z)}:\delta^2\left(\gamma(z)-x\right)+B(j):e^{2j\phi(z)}:|\gamma(z)-x|^{4j}\,, \tag{3.2.9}$$

fixes a normalization and determines the relation between Φ_j and Φ_{-1-j} as

$$\Phi_j(x,\overline{x}|z,\overline{z}) = B(j)\int_{\mathbb{C}} d^2x'|x-x'|^{4j}\Phi_{-1-j}(x',\overline{x}';z,\overline{z})\,, \tag{3.2.10}$$

which explicitly realizes the equivalence between representations with "spin" j and $j' = -1 - j$. The reflection coefficient $B(j)$ is given by

$$B(j) = \frac{k-2}{\pi} \frac{\nu^{1+2j}}{\gamma\left(-\frac{1+2j}{k-2}\right)}\,, \quad \nu = \pi \frac{\Gamma\left(1 - \frac{1}{k-2}\right)}{\Gamma\left(1 + \frac{1}{k-2}\right)}\,, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{3.2.11}$$

As the path integral of the Lorentzian theory is ill defined, the correlation functions are defined via analytic continuation from those of the H_3^+ model. Even those involving fields in non trivial spectral flow sectors, absent in the Euclidean model, can be obtained from the latter by certain formal manipulations [40]. Then, as it is clear from section 2.2 we will need to Fourier transform these expressions to the *m-basis* in order to obtain fields which can be interpreted as fields of the AdS_3 model after analytic continuation. This transformation is exactly the same carried out for the basis functions of section 2.2.2, *i.e.*

$$\Phi^j_{m,\overline{m}}(z,\overline{z}) = \int d^2x\ x^{j+m}\ \overline{x}^{j+\overline{m}}\ \Phi_{-1-j}(x,\overline{x};z,\overline{z})\,, \tag{3.2.12}$$

where $m = \frac{n+is}{2}$, $\overline{m} = \frac{-n+is}{2}$, $n \in \mathbb{Z}, s \in \mathbb{R}$. The fields $\Phi^j_{m,\overline{m}}$ have the following OPE with the chiral currents

$$\begin{aligned} J^{\pm}(z)\Phi^j_{m,\overline{m}}(z',\overline{z}') &\sim \frac{\mp j + m}{z - z'}\Phi^j_{m\pm 1,\overline{m}}(z',\overline{z}')\,, \\ J^3(z)\Phi^j_{m,\overline{m}}(z',\overline{z}') &\sim \frac{m}{z - z'}\Phi^j_{m,\overline{m}}(z',\overline{z}')\,, \end{aligned} \tag{3.2.13}$$

and the relation between $\Phi^j_{m,\overline{m}}$ and $\Phi^{-1-j}_{m,\overline{m}}$ is given by

$$\Phi^j_{m,\overline{m}}(z,\overline{z}) = B(-1-j)c^{-1-j}_{m,\overline{m}}\Phi^{-1-j}_{m,\overline{m}}(z,\overline{z}) \tag{3.2.14}$$

which generalized the reflection relation of the superspace limit (2.2.19). The reflection coefficient

$c^j_{m,\bar{m}}$ is defined as

$$c^j_{m,\bar{m}} = \frac{-\pi}{1+2j}\frac{\Gamma(1+2j)\ \Gamma(-j+m)\ \Gamma(-j-\overline{m})}{\Gamma(-1-2j)\Gamma(1+j+m)\Gamma(1+j-\overline{m})}\,. \tag{3.2.15}$$

We take the fields of the AdS_3 WZNW model as the Wick rotated from those of the H_3^+ coset but we will use the same notation, with the difference that now the quantum numbers are in agreement with the representations appearing in the former model.

An affine primary state in the unflowed sector is mapped by the automorphism (3.2.2) to a highest/lowest-weight state of the global $sl(2)$ algebra. We denote these fields in the spectral flow sector w as $\Phi^{j,w}_{m,\overline{m}}$. Their explicit expressions will not be needed below. It is only necessary to know that they verify the following OPE with the currents:

$$\begin{aligned}
J^3(z)\Phi^{j,w}_{m,\overline{m}}(z',\overline{z}') &\sim \frac{m+\frac{k}{2}w}{z-z'}\Phi^{j,w}_{m,\overline{m}}(z',\overline{z}')\,,\\
J^{\pm}(z)\Phi^{j,w}_{m,\overline{m}}(z',\overline{z}') &\sim \frac{\mp j+m}{(z-z')^{\pm w}}\Phi^{j,w}_{m\pm 1,\overline{m}}(z',\overline{z}')+\cdots
\end{aligned} \tag{3.2.16}$$

$m-\overline{m}\in\mathbb{Z}$, $m+\overline{m}\in\mathbb{R}$ and dots denote we only write down the leading singular term in the OPE. Its conformal weight is given by Δ_j as defined in (3.2.4).

These fields satisfy the same reflection relation as those of the $w=0$ sector (3.2.14),

$$\Phi^{j,w}_{m,\overline{m}}(z,\overline{z}) = B(-1-j)c^{-1-j}_{m,\overline{m}}\Phi^{-1-j,w}_{m,\overline{m}}(z,\overline{z})\,. \tag{3.2.17}$$

Let us end this section presenting a model constructed from the axial coset $SL(2,\mathbb{R})/U(1)_A$, which also admits an analytic continuation to the AdS_3 model. It was shown that such a model has some advantages with respect to the H_3^+ model in that the spectrum is similar to that of AdS_3 and it was shown that the correlation functions [40] and the partition function [37] obtained from the hyperbolic model can be exactly reproduced with this new construction [73] using path integral techniques. It was also used to compute one point functions associated with symmetric and symmetry breaking D-branes (see [74] and section 8.3 for the case of maximally symmetric D-branes).

The construction rests on the observation that, after doing a T duality in the timelike direction, the N-th cover of SL(2,$\mathbb{R}$), *i.e.* SL(2,$\mathbb{R}$)$_k^N$, is given by the orbifold

$$\frac{SL(2,\mathbb{R})_k/U(1)\times U(1)_{-k}}{\mathbb{Z}_{Nk}}\,. \tag{3.2.18}$$

Because now the timelike direction is a free compact boson, the analytic continuation to Euclidean space is simply obtained by replacing $U(1)_{-k}\to U(1)_{R^2k}$. Thus, one can construct arbitrary correlation functions in AdS_3 from those in the cigar and the free compact boson theories, after taking the limits $N\to\infty$, $R^2\to -1$. The effect of the orbifold is to produce new (twisted) sectors. These can be read in the following modification of the left and right momentum modes in the coset and the free boson models, respectively,

$$\begin{aligned}\frac{(n+k\omega,n-k\omega)}{\sqrt{2k}} &\longrightarrow \frac{\left(n+k\omega-\frac{\gamma}{N},n-k\omega+\frac{\gamma}{N}\right)}{\sqrt{2k}},\quad \gamma\in\mathbb{Z}_{kN}\,,\\ \frac{(\tilde{n}+R^2k\tilde{\omega},\tilde{n}-R^2k\tilde{\omega})}{R\sqrt{2k}} &\longrightarrow \frac{\left(n+kNp+R^2k\tilde{\omega}+\frac{R^2\gamma}{N},n+kNp-R^2k\tilde{\omega}-\frac{R^2\gamma}{N}\right)}{R\sqrt{2k}}\,,\end{aligned} \tag{3.2.19}$$

with $p\in\mathbb{Z}$ and $\omega,\tilde{\omega}$ being the winding numbers in the cigar and U(1) respectively. In the N-th cover, k has to be an integer, but in the universal covering, the theory can be defined for arbitrary real level $k>2$ [74].

The vertex operators for the orbifold theory are the product of the vertices in each space, namely

$$V^j_{n\omega\gamma p\tilde{\omega}}(z,\bar{z})=\Phi^{sl(2)/u(1)}_{j,n,\omega-\frac{\gamma}{kN}}(z,\bar{z})\;\;\Phi^{u(1)}_{n+kNp,\tilde{\omega}+\frac{\gamma}{kN}}(z,\bar{z})\,. \tag{3.2.20}$$

In the universal covering, the discrete momentum $\frac{\gamma}{kN}$ becomes a continuous parameter $\lambda\in[0,1)$, the $J_0^3, \bar{J}_0^3$ quantum numbers read

$$M = -\frac{n}{2} + \frac{k}{2}(\tilde{\omega} + \lambda), \qquad \bar{M} = \frac{n}{2} + \frac{k}{2}(\tilde{\omega} + \lambda), \tag{3.2.21}$$

and the spectral flow number is given by

$$w = \omega + \tilde{\omega}. \tag{3.2.22}$$

Chapter 4

Operator algebra

The OPE in non RCFTs, as for instance in the case of WZNW (or gauged WZNW) models with non compact groups, is much subtler than those of RCFT. The structure constants can be determined by the usage of null states as commented in section A.3 and generalizing the bootstrap approach of [75] complementing this with the KZ equation (in the case of WZNW) models. So that these new approaches are analytic rather than algebraic as in the RCFT case.

We begin this chapter presenting a brief review of the OPE in the H_3^+ Coset model. Then we turn to the construction of the Operator Algebra in AdS_3 WZNW model by an appropriate Wick rotation of that of the H_3^+ model and by adding the non preserving spectral flow structure constants.

4.1 Operator algebra in H_3^+

In the case of H_3^+ the degenerate (reducible) representations of $sl(2)$ are not included in the spectrum and this is the reason why the conjecture that the correlation functions are analytic in their parameters is so important. Two and three point functions of the hyperbolic model was determined by Teschner in [41, 42].

The following operator product expansion for any product $\Phi_{j_1}\Phi_{j_2}$ in the H_3^+ model was determined in *loc. cit.*:

$$\Phi_{j_2}(x_2|z_2)\Phi_{j_1}(x_1|z_1) = \int_{\mathcal{P}^+} dj_3\, C(-j_1,-j_2,-j_3)\; |z_2-z_1|^{-\widetilde{\Delta}_{12}} \int_{\mathbb{C}} d^2x_3 |x_1-x_2|^{2j_{12}} \times |x_1-x_3|^{2j_{13}}|x_2-x_3|^{2j_{23}}\Phi_{-1-j_3}(x_3|z_1) + \text{descendants}. \quad (4.1.1)$$

Here, the integration contour is $\mathcal{P}^+ = -\frac{1}{2}+i\mathbb{R}_+$, the structure constants $C(j_1,j_2,j_3)$ are given by

$$C(j_1,j_2,j_3) = -\frac{G(1-j_1-j_2-j_3)G(-j_{12})G(-j_{13})G(-j_{23})}{2\pi^2\nu^{j_1+j_2+j_3-1}\gamma\left(\frac{k-1}{k-2}\right)G(-1)G(1-2j_1)G(1-2j_2)G(1-2j_3)}\,, \quad (4.1.2)$$

with $G(j) = (k-2)^{\frac{j(1-j-k)}{2(k-2)}}\,\Gamma_2(-j|1,k-2)\,\Gamma_2(k-1+j|1,k-2)$, $\Gamma_2(x|1,w)$ being the Barnes double Gamma function, $\widetilde{\Delta}_{12} = \widetilde{\Delta}(j_1)+\widetilde{\Delta}(j_2)-\widetilde{\Delta}(j_3)$ and $j_{12} = j_1+j_2-j_3$, etc.

The OPE (4.1.1) holds for a range of values of j_1, j_2 given by

$$|\mathrm{Re}(j_{21}^{\pm})| < \frac{1}{2}\,, \qquad j_{21}^{+} = j_2+j_1+1\,, \qquad j_{21}^{-} = j_2-j_1\,. \quad (4.1.3)$$

This is the maximal region in which j_1, j_2 may vary such that none of the poles of the integrand hits the contour of integration over j_3. However, as long as the imaginary parts of $j_{21}^{\pm}$ do not vanish, J. Teschner [42] showed that (4.1.1) admits an analytic continuation to generic complex values of j_1, j_2, defined by deforming the contour $\mathcal{P}^+$. The deformed contour is given by the sum of the original one plus a finite number of circles around the poles leading to a finite sum of residue contributions to the OPE. When $j_{21}^{\pm}$ are real one can give them a small imaginary part which is sent to zero after deforming the contour.

4.2 Operator algebra in AdS_3

4.2.1 Correlation functions in AdS_3

Following [40],[76]-[79] we assume that correlation functions of primary fields in the AdS_3 WZNW model are those of H_3^+ with $j_i, m_i, \overline{m}_i$ taking values in representations of $\widetilde{SL(2,\mathbb{R})}$. The spectral flow operation is straightforwardly performed in the $m-$basis where the only change in the $w-$conserving expectation values of fields $\Phi^{j,w}_{m,\overline{m}}$ in different w sectors is in the powers of the coordinates $z_i, \overline{z}_i$. Correlation functions may violate $w-$conservation according to the following spectral flow selection rules established in [40]

$$-N_t + 2 \ \leq \ \sum_{i=1}^{N_t} \ w_i \ \ \leq \ N_c - 2\,, \qquad \text{at least one state in } \widehat{\mathcal{C}}^{\alpha,w}_j \otimes \widehat{\mathcal{C}}^{\alpha,w}_j, \tag{4.2.1}$$

$$-N_d + 1 \ \leq \ \sum_{i=1}^{N_t} \ w_i \ \ \leq \ -1\,, \qquad \text{all states in } \widehat{\mathcal{D}}^{+,w}_j \otimes \widehat{\mathcal{D}}^{+,w}_j, \tag{4.2.2}$$

with $N_t = N_c + N_d$ and N_c, N_d are the total numbers of operators in $\widehat{\mathcal{C}}^{\alpha,w}_j \otimes \widehat{\mathcal{C}}^{\alpha,w}_j$ and $\widehat{\mathcal{D}}^{+,w}_j \otimes \widehat{\mathcal{D}}^{+,w}_j$, respectively.

The spectral flow preserving two-point function is given by

$$\begin{aligned} \langle \Phi^{j,w}_{m,\overline{m}}(z,\overline{z}) \Phi^{j',-w}_{m',\overline{m}'}(z',\overline{z}') \rangle \ \ &= \ \ \delta^2(m+m')\,(z-z')^{-2\Delta(j)}(\overline{z}-\overline{z}')^{-2\overline{\Delta}(j)} \\ &\times \ \left[\delta(j+j'+1) + B(-1-j) c^{-1-j}_{m,\overline{m}}\, \delta(j-j') \right], \end{aligned} \tag{4.2.3}$$

where $\Delta(j) = \widetilde{\Delta}(j) - wm - \frac{k}{4}w^2 = -\frac{j(j+1)}{k-2} - wm - \frac{k}{4}w^2$, $B(j)$ and $c^j_{m,\bar{m}}$ were defined in (3.2.11) and (3.2.15) respectively

For states in discrete series it is convenient to work with spectral flow images of both lowest- and highest-weight representations related by the identification $\widehat{\mathcal{D}}^{+,w}_j \equiv \widehat{\mathcal{D}}^{-,w+1}_{-\frac{k}{2}-j}$, which determines the range of values for the spin

$$-\frac{k-1}{2} < j < -\frac{1}{2}\,, \tag{4.2.4}$$

and allows to obtain the (± 1) unit spectral flow two-point functions from (4.2.3).

Spectral flow conserving three-point functions are the following:

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i,w_i}_{m_i,\overline{m}_i}(z_i,\overline{z}_i) \right\rangle = \delta^2(\sum m_i)\, C(1+j_i) W \begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} \prod_{i<j} z_{ij}^{-\Delta_{ij}} \overline{z}_{ij}^{-\overline{\Delta}_{ij}}, \quad (4.2.5)$$

where $z_{ij} = z_i - z_j$ and $C(j_i)$ is given by (4.1.2). The function W is

$$W \begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \int d^2x_1\, d^2x_2\, x_1^{j_1+m_1} \overline{x}_1^{j_1+\overline{m}_1} x_2^{j_2+m_2} \overline{x}_2^{j_2+\overline{m}_2} \times\ |1-x_1|^{-2j_{13}-2} |1-x_2|^{-2j_{23}-2} |x_1-x_2|^{-2j_{12}-2}, \quad (4.2.6)$$

and we omit the obvious $\overline{m}-$dependence in the arguments to lighten the notation. This integral was computed in [80].

The one unit spectral flow three-point function [40] is given by [1]

$$\left\langle \prod_{i=1}^{3} \Phi^{j_i,w_i}_{m_i,\overline{m}_i}(z_i,\overline{z}_i) \right\rangle = \delta^2(\sum m_i \pm \frac{k}{2}) \frac{\widetilde{C}(1+j_i)\widetilde{W} \begin{bmatrix} j_1 \;\;,\;\; j_2 \;\;,\;\; j_3 \\ \pm m_1, \pm m_2, \pm m_3 \end{bmatrix}}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \prod_{i<j} z_{ij}^{-\Delta_{ij}} \overline{z}_{ij}^{-\overline{\Delta}_{ij}}, \quad (4.2.7)$$

where $\sum_i w_i = \pm 1$, the $\pm$ signs corresponding to the $\pm$ signs in the r.h.s.,

$$\widetilde{C}(j_i) \sim B(-j_1) C\left(\frac{k}{2} - j_1, j_2, j_3\right), \quad (4.2.8)$$

up to $k-$dependent, $j-$independent factors and

$$\widetilde{W} \begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \frac{\Gamma(1+j_1+m_1)}{\Gamma(-j_1-\overline{m}_1)} \frac{\Gamma(1+j_2+\overline{m}_2)}{\Gamma(-j_2-m_2)} \frac{\Gamma(1+j_3+\overline{m}_3)}{\Gamma(-j_3-m_3)}. \quad (4.2.9)$$

For discrete states, this expression is related to the $\sum_i w_i = \pm 2$ three-point function through $\widehat{\mathcal{D}}^{+,w}_j \equiv \widehat{\mathcal{D}}^{-,w+1}_{-\frac{k}{2}-j}$.

[1] For an independent calculation of three-point functions using the free field approach see [81]

4.2.2 Operator Product Expansion in AdS_3

A non-trivial check on the OPE (4.1.1) and structure constants (4.1.2) of the H_3^+ WZNW model is that the well-known fusion rules of degenerate representations [82] are exactly recovered by analytically continuing $j_i, i = 1, 2$ [41]. On the other hand, it was argued in [40]-[42], [76]-[79] that correlation functions in the H_3^+ and AdS_3 WZNW models are related by analytic continuation and moreover, the $k \to \infty$ limit of the OPE of unflowed fields computed along these lines in [76, 77] exhibits complete agreement with the classical tensor products of representations of $SL(2, \mathbb{R})$ [83]. It seems then natural to conjecture that the OPE of all fields in the spectrum of the AdS_3 WZNW model can be obtained from (4.1.1) analytically continuing j_1, j_2 from the range (4.1.3).

However, the spectral flowed fields do not belong to the spectrum of the H_3^+ model and moreover, the spectral flow symmetry transforms primaries into descendants. Thus, a better knowledge of these representations seems necessary in order to obtain the fusion rules in the AdS_3 model and these cannot be simply obtained by a straight forward analytic continuation. Nevertheless, we will show that the OPE and fusion rules obtained from the H_3^+ model by analytic continuation and by taking into account the $w-$violating structure constants in addition to (4.1.2) require a truncation imposed by the spectral flow symmetry and once it is carried out the OPE and fusion rules close in the spectrum of the theory and satisfy many consistency checks. For instance, the selection rules of arbitrary correlation functions with fields in any representation of the model and the appropriate semiclassical limit are reproduced. In this section we explore this possibility in order to get the OPE of primary fields and their spectral flow images in the AdS_3 WZNW model.

To deal with highest/lowest-weight and spectral flow representations it is convenient to work in the $m-$basis. We have to keep in mind that when j is real, new divergences appear in the transformation from the $x-$basis and this must be performed for certain values of $m_i, \overline{m}_i$, $i = 1, 2$. Indeed, to transform the OPE (4.1.1) to the $m-$basis using (3.2.12), the integrals over x_1, x_2 in the r.h.s. must be interchanged with the integral over j_3 and this process does not commute in general if there are divergences. However, restricting j_1, j_2 to the range (4.1.3), one can check that the integrals commute and are regular when $|m_i| < \frac{1}{2}$ and $|\overline{m}_i| < \frac{1}{2}$, $i = 1, 2, 3$, where $m_3 = m_1 + m_2, \overline{m}_3 = \overline{m}_1 + \overline{m}_2$. For other values of $m_i, \overline{m}_i$ the OPE must be defined, as usual, by analytic continuation of the parameters. Therefore, after performing the x_1, x_2 integrals, the OPE (4.1.1) in the $m-$basis is found to be

$$\Phi^{j_1}_{m_1,\overline{m}_1}(z_1,\overline{z}_1)\Phi^{j_2}_{m_2,\overline{m}_2}(z_2,\overline{z}_2)\Big|_{w=0} = \int_{\mathcal{P}} dj_3\ |z_{12}|^{-2\widetilde{\Delta}_{12}}\ Q^{w=0}\begin{bmatrix} j_1\ , j_2,\ j_3 \\ m_1, m_2, m_3 \end{bmatrix} \Phi^{j_3}_{m_3,\overline{m}_3}(z_1,\overline{z}_2) + \text{ descendants}, \tag{4.2.10}$$

where we have defined

$$Q^{w=0}\begin{bmatrix} j_1\ , j_2,\ j_3 \\ m_1, m_2, m_3 \end{bmatrix} = C(1+j_1, 1+j_2, -j_3) W \begin{bmatrix} j_1, j_2, -1-j_3 \\ m_1, m_2, -m_3 \end{bmatrix} . \tag{4.2.11}$$

It is easy to see that the integrand is symmetric under $j_3 \to -1 - j_3$ using the identity [77]

$$\frac{W\begin{bmatrix} j_1\ , j_2\ , j_3 \\ m_1, m_2, m_3 \end{bmatrix}}{W\begin{bmatrix} j_1, j_2, -1-j_3 \\ m_1\ , m_2\ , m_3 \end{bmatrix}} = \frac{C(1+j_1, 1+j_2, -j_3)}{C(1+j_1, 1+j_2, 1+j_3)} B(-1-j_3) c^{-1-j_3}_{m_3,\overline{m}_3}\, , \tag{4.2.12}$$

and as a consequence of (3.2.14). In the $x-$basis, every pole in (4.1.1) appears duplicated, one over the real axis and another one below, and the $j_3 \to -1 - j_3$ symmetry implies that the integral may be equivalently performed either over Im $j_3 > 0$ or over Im $j_3 < 0$ [42]. In the $m-$basis, the $(j_1, j_2)-$dependent poles are also duplicated but the $m-$dependent poles are not. The $j_3 \to -1 - j_3$ symmetry is still present, as we discussed above, because of poles and zeros in the normalization of $\Phi^j_{m,\overline{m}}$. The integral must be extended to the full axis $\mathcal{P} = -\frac{1}{2} + i\mathbb{R}$ before performing the analytic continuation in m_1, m_2 because the $m-$dependent poles fall on the real axis.

Since the $w-$conserving structure constants of operators $\Phi^{j,w}_{m,\overline{m}} \in \mathcal{C}^{\alpha,w}_j$ or $\mathcal{D}^{+,w}_j$ in different

w sectors do not change in the $m-$basis [2], the OPE (4.2.10) should also hold for fields obtained by spectral flowing primaries to arbitrary w sectors, as long as they satisfy $w_1 + w_2 = w_3$. But this OPE only reproduces the w-preserving three-point functions, which are the ones directly obtained from H_3^+. The full OPE requires to additionally take into account the spectral flow non-preserving structure constants and to consider the following OPE [3]

$$\Phi^{j_1,w_1}_{m_1,\overline{m}_1}(z_1,\overline{z}_1)\Phi^{j_2,w_2}_{m_2,\overline{m}_2}(z_2,\overline{z}_2) = \sum_{w=-1}^{1}\int_{\mathcal{P}} dj_3\; Q^w z_{12}^{-\Delta_{12}}\overline{z}_{12}^{-\overline{\Delta}_{12}}\Phi^{j_3,w_3}_{m_3,\overline{m}_3}(z_2,\overline{z}_2) + \cdots\,, \tag{4.2.13}$$

with $w = w_3 - w_1 - w_2$, $m_3 = m_1 + m_2 - \frac{k}{2}w$, $\overline{m}_3 = \overline{m}_1 + \overline{m}_2 - \frac{k}{2}w$, and

$$\begin{aligned} Q^{w=\pm1}(j_i;m_i,\overline{m}_i) &= \widetilde{W}\begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ \mp m_1, \mp m_2, \pm m_3 \end{bmatrix} \frac{\widetilde{C}(j_i+1)}{B(-1-j_3)c^{-j_3-1}_{m_3,\overline{m}_3}\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \\ &\sim \frac{\Gamma(\pm\overline{m}_3 - j_3)}{\Gamma(1+j_3\mp m_3)}\prod_{a=1}^{2}\frac{\Gamma(1+j_a\mp m_a)}{\Gamma(-j_a\pm\overline{m}_a)}\frac{C(\frac{k}{2}-1-j_1,1+j_2,1+j_3)}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})}\,. \end{aligned} \tag{4.2.14}$$

For completeness, according to the spectral flow selection rules (4.2.2), we should also include terms with $w = \pm 2$ in the sum. However, we shall show in the next section that these do not affect the results of the OPE. The integrand is also symmetric under $j_3 \to -1 - j_3$. This follows from (4.2.12) and the analogous identity

$$\frac{\widetilde{W}\begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix}}{\widetilde{W}\begin{bmatrix} j_1, j_2, -1-j_3 \\ m_1\,,\, m_2\,,\, m_3 \end{bmatrix}} = \frac{\widetilde{C}(1+j_1,1+j_2,-j_3)\gamma(j_1+j_2+j_3+3-\frac{k}{2})}{\widetilde{C}(1+j_1,1+j_2,1+j_3)\gamma(j_1+j_2-j_3+2-\frac{k}{2})}B(-1-j_3)c^{-1-j_3}_{m_3,\overline{m}_3}\,, \tag{4.2.15}$$

together with the reflection relation (3.2.17). The dots in (4.2.13) stand for spectral flow images

[2] We denote the series containing the highest/lowest-weight states obtained by spectral flowing primaries as $\mathcal{C}^{\alpha,w}_j, \mathcal{D}^{+,w}_j$.

[3] A similar expression was proposed in [84] and some supporting evidence was presented from the relation between the H_3^+ model and Liouville theory.

of current algebra descendants with the same J_0^3 eigenvalues $m_3, \overline{m}_3$. This expression is valid for j_1, j_2 in the range (4.1.3) and the restrictions on m_1, m_2 depend on Q^w. The maximal regions in which they may vary such that none of the poles hit the contour of integration are, $min\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} < \frac{1}{2}$ and $max\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} > -\frac{1}{2}$ for $Q^{w=0}$, $min\,\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} < -\frac{k-1}{2}$ for $Q^{w=-1}$ and $max\,\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} > \frac{k-1}{2}$ for $Q^{w=+1}$. So that the bounds for $Q^{w=1}$ and $Q^{w=-1}$ ensure the convergence for the contributions from $Q^{w=0}$. For other values of j_1, j_2 and m_1, m_2 the OPE must be defined by analytic continuation. In the rest of this section we perform this continuation.

To specifically display the contributions to (4.2.13) we have to study the analytic structure of Q^w. We first consider the simpler case $w = \pm 1$ and we refer to the terms proportional to $Q^{w=\pm 1}$ as *spectral flow non-preserving contributions* to the OPE. Then, we investigate $Q^{w=0}$ and obtain the *spectral flow preserving contributions.*

4.2.3 Spectral flow non-preserving contributions

Let us study the analytic structure of $Q^{w=\pm 1}$ in (4.2.14). The m-independent poles arising from the last factor are the same for both $w = \pm 1$ sectors and are explicitly given by

$$\begin{aligned} j_3 &= \pm j_{21}^- + \frac{k}{2} - 1 + p + q(k-2)\,, \qquad & j_3 &= \pm j_{21}^- - \frac{k}{2} - p - q(k-2)\,, \\ j_3 &= \pm j_{21}^+ + \frac{k}{2} - 1 + p + q(k-2)\,, \qquad & j_3 &= \pm j_{21}^+ - \frac{k}{2} - p - q(k-2)\,, \end{aligned} \tag{4.2.16}$$

with $p, q = 0, 1, 2, \ldots$ The m-dependent poles, instead, vary according to the spectral flow sector. However they are connected through $(m, \overline{m}) \leftrightarrow (-m, -\overline{m})$ and thus going from $w = -1$ to $w = +1$ involves the change $\mathcal{D}_{j_i}^{-,\,w_i} \otimes \mathcal{D}_{j_i}^{-,\,w_i} \leftrightarrow \mathcal{D}_{j_i}^{+,\,w_i} \otimes \mathcal{D}_{j_i}^{+,\,w_i}$. Therefore we concentrate on the contributions from $w = -1$.

By abuse of notation, from now on we denote the states by the representations they belong to and we write only the holomorphic sector for short, *e.g.* when $\Phi^{j_i,w_i}_{m_i,\overline{m}_i} \in \mathcal{D}_{j_i}^{+,w_i} \otimes \mathcal{D}_{j_i}^{+,w_i}$, $i = 1, 2$, we write the set of all possible operator products $\Phi^{j_1,w_1}_{m_1,\overline{m}_1} \Phi^{j_2,w_2}_{m_2,\overline{m}_2}$ for generic quantum numbers within these representations as $\mathcal{D}_{j_1}^{+,w_1} \times \mathcal{D}_{j_2}^{+,w_2}$.

Let us study the OPE of fields in all different combinations of representations. First consider the case $\Phi^{w_i,j_i}_{m_i,\overline{m}_i} \in \mathcal{C}^{\alpha_i,w_i}_{j_i} \otimes \mathcal{C}^{\alpha_i,w_i}_{j_i}$, $i=1,2$, *i.e.*

- $\mathcal{C}^{\alpha_1,\, w_1}_{j_1} \times \mathcal{C}^{\alpha_2,\, w_2}_{j_2}$

The pole structure of $Q^{w=-1}$ is represented in Figure 1.a) for $min\,\{m_1+m_2,\, \overline{m}_1+\overline{m}_2\} < -\frac{k-1}{2}$. Recalling that $m_3 = m_1+m_2+\frac{k}{2}$, then $min\,\{m_3,\overline{m}_3\} < \frac{1}{2}$, and therefore the poles from the factor $\frac{\Gamma(-j_3-\overline{m}_3)}{\Gamma(1+j_3+m_3)}$ are to the right of the integration contour. Moreover, given that all $m-$independent poles are to the right of the axis $\frac{k}{2}-1$ or to the left of $-\frac{k}{2}$, we conclude that the OPE $\mathcal{C}^{\alpha_1,w_1}_{j_1} \times \mathcal{C}^{\alpha_2,w_2}_{j_2}$ receives no spectral flow violating contributions from discrete representations when $min\{m_1+m_2,\overline{m}_1+\overline{m}_2\} < -\frac{k-1}{2}$.

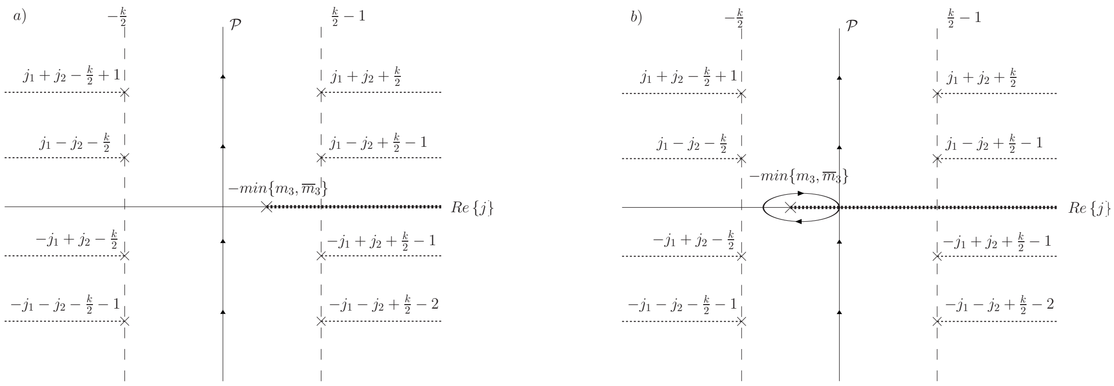

Figure 1: Case $\mathcal{C}^{\alpha_1,w_1}_{j_1} \times \mathcal{C}^{\alpha_2,w_2}_{j_2}$. The solid line indicates the integration contour $\mathcal{P} = -\frac{1}{2} + i\mathbb{R}$ in the j_3 complex plane. The dots above or below the real axis represent the (j_1,j_2)-dependent poles and those on the real axis correspond to the $m-$dependent poles. The crosses are the positions of the first poles in the series. a) When $m_1+m_2 < -\frac{k-1}{2}$ or $\overline{m}_1+\overline{m}_2 < -\frac{k-1}{2}$, there are no poles crossing the contour of integration. b) When $m_1+m_2 > -\frac{k-1}{2}$ and $\overline{m}_1+\overline{m}_2 > -\frac{k-1}{2}$, poles from the factor $\frac{\Gamma(-j_3-\overline{m}_3)}{\Gamma(1+j_3+m_3)}$ cross the contour, indicating the contribution to the OPE from states in discrete representations.

Some poles cross the integration contour when $min\,\{m_1+m_2,\, \overline{m}_1+\overline{m}_2\} > -\frac{k-1}{2}$. They are sketched in Figure 1.b) and indicate contributions from the discrete series $\mathcal{D}^{+,w_3=w_1+w_2-1}_{j_3}$ with $j_3 = -min\,\{m_3,\overline{m}_3\}+n$, $n=0,1,2,\ldots$, and such that $j_3 < -\frac{1}{2}$. Since $Q^{w=\pm1}$ does not vanish for $j_3 = -\frac{1}{2}+i\mathbb{R}$ and m_3 not correlated with j_3, there are terms from $\mathcal{C}^{\alpha_3,w_3=w_1+w_2-1}_{j_3}$ in this OPE as well. Therefore we get

$$\mathcal{C}_{j_1}^{\alpha_1,\,w_1} \times \mathcal{C}_{j_2}^{\alpha_2,\,w_2}\Big|_{|w|=1} \;=\; \sum_{j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{+,\,w_3=w_1+w_2-1} + \sum_{j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{-,\,w_3=w_1+w_2+1}$$
$$+ \sum_{w=-1,1} \int_{\mathcal{P}} dj_3\, \mathcal{C}_{j_3}^{\alpha_3,\,w_3=w_1+w_2+w} + \cdots\,, \qquad (4.2.17)$$

where $|_{|w|=1}$ denotes that only spectral flow non-preserving contributions are displayed in the right-hand side.

- $\mathcal{C}_{j_1}^{\alpha_1,\,w_1} \times\ \mathcal{D}_{j_2}^{\pm,\,w_2}$

To analyze this case, we need to perform the analytic continuation for j_2 away from $-\frac{1}{2}+is_2$. When is_2 is continued to the real interval $(-\frac{k-2}{2},\ 0)$, the series of $m-$independent poles changes as shown in Figure 2. It is easy to see that these poles do not cross the contour of integration. For instance, $\mathrm{Re}\left\{j_1+j_2+\frac{k}{2}\right\}>0$, $\mathrm{Re}\left\{j_1-j_2+\frac{k}{2}-1\right\}>\frac{k}{2}-1$, etc. Similarly as in the previous case, only poles from $\frac{\Gamma(-j_3-\overline{m}_3)}{\Gamma(1+j_3+m_3)}$ can cross the contour, but due to the factor $\frac{\Gamma(1+j_2+\overline{m}_2)}{\Gamma(-j_2-m_2)}$ there are contributions from the discrete series just for $\Phi^{j_2,w_2}_{m_2,\overline{m}_2}\in\mathcal{D}^{-,w_2}_{j_2}\otimes\mathcal{D}^{-,w_2}_{j_2}$. Therefore we get

$$\mathcal{C}_{j_1}^{\alpha_1,\,w_1} \times\ \mathcal{D}_{j_2}^{\pm,\,w_2}\Big|_{|w|=1} = \int_{\mathcal{P}} dj_3\, \mathcal{C}_{j_3}^{\alpha_3,\,w_3=w_1+w_2\pm1} + \sum_{j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{\mp,\,w_3=w_1+w_2\pm1} + \cdots\,. \qquad (4.2.18)$$

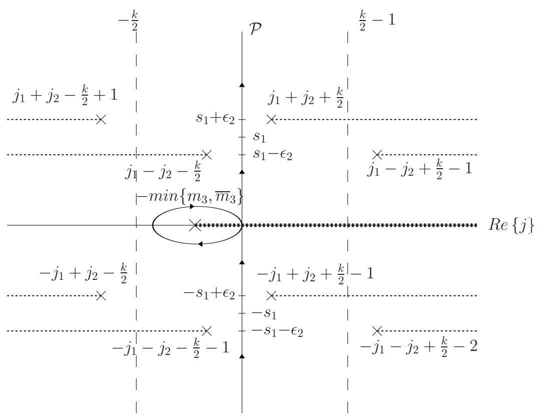

Figure 2: Case $\mathcal{C}_{j_1}^{\alpha_1,w_1}\times\mathcal{D}_{j_2}^{\pm,w_2}$. Only $m-$dependent poles can cross the contour of integration. This occurs when both m_1+m_2 and $\overline{m}_1+\overline{m}_2$ are larger than $-\frac{k-1}{2}$. We have given j_2 an infinitesimal imaginary part, ϵ_2, to better display the (j_1,j_2)-dependent series of poles.

- $\mathcal{D}^{\pm,\,w_1}_{j_1} \times \mathcal{D}^{\pm,\,w_2}_{j_2}$ and $\mathcal{D}^{\pm,\,w_1}_{j_1} \times \mathcal{D}^{\mp,\,w_2}_{j_2}$

Let us first analytically continue both j_1 and j_2 to the interval $(-\frac{k-1}{2},\ -\frac{1}{2})$, which is shown in Figure 3. The correct way to do this is to consider that both j_1 and j_2 have an infinitesimal imaginary part, ϵ_1 and ϵ_2 respectively, which is sent to zero after computing the integral.

The $m-$independent poles cross the contour of integration only when $j_1 + j_2 < -\frac{k+1}{2}$. However, due to the factors $\frac{\Gamma(1+j_1+m_1)}{\Gamma(-j_1-\overline{m}_1)}\frac{\Gamma(1+j_2+\overline{m}_2)}{\Gamma(-j_2-m_2)}$ in $Q^{w=-1}$, the contributions from these poles only survive when the quantum numbers of both $\Phi^{j_1,w_1}_{m_1,\overline{m}_1}$ and $\Phi^{j_2,w_2}_{m_2,\overline{m}_2}$ are in $\mathcal{D}^{-,w_i}_{j_i} \otimes \mathcal{D}^{-,w_i}_{j_i}, i = 1, 2$. In this case, the poles at $j_3 = j_1 + j_2 + \frac{k}{2} + n$ give contributions from $\mathcal{D}^{-,w_3=w_1+w_2-1}_{j_3}$. This may be seen noticing that $j_3 = m_1 + m_2 + \frac{k}{2} + n_3 = \overline{m}_1 + \overline{m}_2 + \frac{k}{2} + \overline{n}_3$, with $n_3 = n + n_1 + n_2$ and $\overline{n}_3 = n + \overline{n}_1 + \overline{n}_2$, or using $m_3 = m_1 + m_2 + \frac{k}{2}$, $\overline{m}_3 = \overline{m}_1 + \overline{m}_2 + \frac{k}{2}$, so that $j_3 = m_3 + n_3 = \overline{m}_3 + \overline{n}_3$. Instead, the contributions from the poles at $j_3 = -j_1 - j_2 - \frac{k}{2} - 1 - n$ seem to cancel due to the factor $\frac{\Gamma(-j_3-\overline{m}_3)}{\Gamma(1+j_3+m_3)}$. However, these zeros are canceled because the operator diverges. In fact, using (3.2.17) and relabeling $j_3 \to -1 - j_3$, it is straightforward to recover exactly the same contribution from the poles at $j_3 = j_1 + j_2 + \frac{k}{2} + n$. Obviously, this was expected as a consequence of the symmetry $j_3 \leftrightarrow -1 - j_3$ of the integrand in (4.2.13).

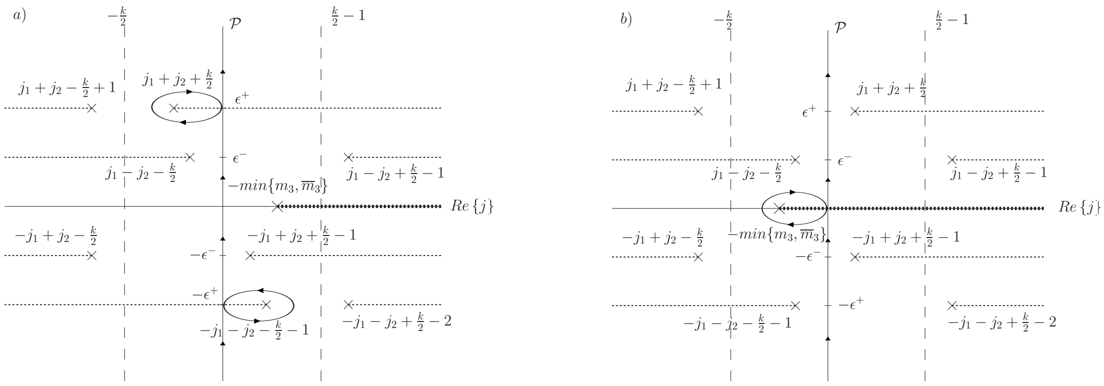

Figure 3: Case $\mathcal{D}^{w_1}_{j_1} \times \mathcal{D}^{w_2}_{j_2}$. Both $m-$dependent and $m-$independent poles can cross the contour of integration. There are two possibilities: 1) $\mathcal{D}^{-,w_1}_{j_1} \times \mathcal{D}^{-,w_2}_{j_2}$. When $j_1 + j_2 < -\frac{k+1}{2}$, only $m-$independent poles can cross the contour, as shown in Figure 3.a) and when $j_1 + j_2 > -\frac{k-1}{2}$, only $m-$dependent poles can cross as shown in Figure 3.b). 2) $\mathcal{D}^{\mp,w_1}_{j_1} \times \mathcal{D}^{\pm,w_2}_{j_2}$. Both $m-$dependent and $m-$independent poles can cross the contour but only the former survive after taking the limit $\epsilon^+, \epsilon^- \to 0$, where $\epsilon^{\pm} = \epsilon_1 \pm \epsilon_2$.

Finally, the $m-$dependent poles give contributions from $\mathcal{D}^{+,w_3=w_1+w_2-1}_{j_3}$. Actually, when

$min\{m_3, \overline{m}_3\} > \frac{1}{2}$ some of the $m-$dependent poles cross the contour. Using $m-$conservation it is not difficult to check that these contributions fall inside the range (4.2.4).

Let us continue the analysis, considering the OPE $\mathcal{D}^{\mp,w_1}_{j_1} \times \mathcal{D}^{\pm,w_2}_{j_2}$. For instance, take the limiting case $j_1 = m_1 + n_1 + i\epsilon_1$ and $j_2 = -m_2 + n_2 + i\epsilon_2$ with $\epsilon_1,\ \epsilon_2 \to 0$. The factor $\frac{\Gamma(1+j_2+\overline{m}_2)}{\Gamma(-j_2-m_2)}$ vanishes as a simple zero. However, some poles from the series $j_3 = j_2 - j_1 - \frac{k}{2} - n$ will overlap with the $m-$dependent poles. But because the $m-$independent simple poles are outside the contour of integration, in the limit $\epsilon_i \to 0$ they may cancel the simple zeros. The way to compute this limit is determined by the definition of the three-point function. We assume that a finite and nonzero term remains in the limit [4].

Including the contributions from continuous representations, we get the following results:

$$\mathcal{D}^{\pm,\,w_1}_{j_1} \times \mathcal{D}^{\pm,\,w_2}_{j_2}\Big|_{|w|=1} = \int_{\mathcal{P}+} dj_3\ \mathcal{C}^{\alpha_3,\,w_3=w_1+w_2\pm1}_{j_3} + \sum_{-j_1-j_2-\frac{k}{2}\le j_3<-\frac{1}{2}} \mathcal{D}^{\mp,\,w_3=w_1+w_2\pm1}_{j_3} + \sum_{j_1+j_2+\frac{k}{2}\le j_3<-\frac{1}{2}} \mathcal{D}^{\pm,\,w_3=w_1+w_2\pm1}_{j_3} + \cdots . \qquad (4.2.19)$$

$$\mathcal{D}^{+,\,w_1}_{j_1} \times \mathcal{D}^{-,\,w_2}_{j_2}\Big|_{|w|=1} = \sum_{j_3<j_2-j_1-\frac{k}{2}} \mathcal{D}^{-,w_3=w_1+w_2+1}_{j_3} + \sum_{j_3<j_1-j_2-\frac{k}{2}} \mathcal{D}^{+,w_3=w_1+w_2-1}_{j_3} + \cdots \qquad (4.2.20)$$

4.2.4 Spectral flow preserving contributions

The analytic structure of $Q^{w=0}(j_i; m_i, \overline{m}_i)$ in (4.2.11) was studied in [77]. Here we present the analysis mainly to discuss some subtleties which are crucial to perform the analytic continuation

[4] In the limit $\epsilon_1, \epsilon_2 \to 0$, $Res(Q^{w=-1}) \sim \frac{\epsilon_2}{\epsilon_2-\epsilon_1}$. The same ambiguity appears in the three-point function including $\Phi^{j_1,w_1}_{m_1,\overline{m}_1} \in \mathcal{D}^{-,w_1}_{j_1} \otimes \mathcal{D}^{-,w_1}_{j_1}$, $\Phi^{j_2,w_2}_{m_2,\overline{m}_2} \in \mathcal{D}^{+,w_2}_{j_2} \otimes \mathcal{D}^{+,w_2}_{j_2}$, with $n_1 \le n_2$ such that $j_3 = j_1 - j_2 - \frac{k}{2} - \mathbb{Z}_{n\ge0}$. The resolution of this ambiguity requires an interpretation of the divergences. The $w-$selection rules allow to assume that a finite term survives in the limit. For instance, consider a generic three-point function $\langle \mathcal{D}^{-,w_1}_{j_1} \mathcal{D}^{+,w_2}_{j_2} \mathcal{D}^{+,w_3}_{j_3} \rangle$ with $w_1 + w_2 + w_3 = -1$. According to (4.2.2) this is non-vanishing (for certain values of j_i, not determined from the $w-$selection rules). Indeed, the divergence from the $\delta^2(\sum_i m_i - \frac{k}{2})$ in (4.2.7) cancels the zero from $\Gamma(-j_3 - m_3)$ and then the pole in $\widetilde{C}(1 + j_i) \sim \frac{1}{\epsilon_2-\epsilon_1}$ must cancel the zero from $\frac{\Gamma(1+j_2+\overline{m}_2)}{\Gamma(-j_2-\overline{m}_2)} \sim \epsilon_2$, leaving a finite and non vanishing contribution.

of $m_i, \overline{m}_i, i = 1, 2$. Although our treatment of the $m-$dependent poles differs from that followed in [77], we show in this section that the results coincide.

The function $C(1 + j_i)$ has zeros at $j_i = \frac{j-1}{2}$, $i = 1, 2, 3$ and poles at $j = -j_1 - j_2 - j_3 - 2$, $-1-j_1-j_2+j_3$, $-1-j_1-j_3+j_2$, or $-1-j_2-j_3+j_1$ where $j := p+q(k-2), -(p+1)-(q+1)(k-2)$, $p, q = 0, 1, 2, \cdots$. To explore the behavior of the function W, we use the expression [77]

$$W \begin{bmatrix} j_1\, ,\, j_2\, ,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = (i/2)^2 \left[C^{12}\overline{P}^{12} + C^{21}\overline{P}^{21}\right] \,, \tag{4.2.21}$$

with $\quad (i/2)^2 P^{12} = s(j_1 + m_1)s(j_2 + m_2)C^{31} - s(j_2 + m_2)s(m_1 - j_2 + j_3)C^{13}\,,$

$$\begin{aligned} C^{12} &= \frac{\Gamma(-N)\Gamma(1+j_3-m_3)}{\Gamma(-j_3-m_3)} G \begin{bmatrix} -m_3 - j_3,\ -j_{13},\ 1 + m_2 + j_2 \\ -m_3 - j_1 + j_2 + 1,\ m_2 - j_1 - j_3 \end{bmatrix}, \\ C^{31} &= \frac{\Gamma(1+j_3+m_3)\Gamma(1+j_3-m_3)}{\Gamma(1+N)} G \begin{bmatrix} 1 + N,\ 1 + j_1 + m_1,\ 1 - m_2 + j_2 \\ j_3 + j_2 + m_1 + 2,\ j_1 + j_3 - m_2 + 2 \end{bmatrix}, \\ G \begin{bmatrix} a, b, c \\ e, f \end{bmatrix} &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)} F \begin{bmatrix} a, b, c \\ e, f \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)\Gamma(n+1)}\,, \end{aligned} \tag{4.2.22}$$

and $N = 1 + j_1 + j_2 + j_3$, $s(x) = \sin(\pi x)$. $\overline{P}^{ab}\,(\overline{C}^{ab})$ is obtained from $P^{ab}\,(C^{ab})$ by replacing $(m_i \to \overline{m}_i)$ and $P^{ba}\,(C^{ba})$ from $P^{ab}\,(C^{ab})$ by changing $(j_1, m_1 \leftrightarrow j_2, m_2)$ and $F \begin{bmatrix} a, b, c \\ e, f \end{bmatrix} =$ ${}_3F_2(a, b, c; e, f; 1)$. An equivalent expression for W which will be useful below is the following [77]

$$W \begin{bmatrix} j_1\, ,\, j_2\, ,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = D_1 C^{12}\overline{C}^{12} + D_2 C^{21}\overline{C}^{21} + D_3[C^{12}\overline{C}^{21} + C^{21}\overline{C}^{12}]\,, \tag{4.2.23}$$

where

$$\begin{aligned}
D_1 &= \frac{s(j_2+m_2)s(j_{13})}{s(j_1-m_1)s(j_2-m_2)s(j_3+m_3)}[s(j_1+m_1)s(j_1-m_1)s(j_2+m_2) \\
&\quad -s(j_2-m_2)s(j_2-j_3-m_1)s(j_2+j_3-m_1)]\,, \\
D_2 &= D_1(j_1,m_1 \leftrightarrow j_2,m_2)\,, \\
D_3 &= -\frac{s(j_{13})s(j_{23})s(j_1+m_1)s(j_2+m_2)s(j_1+j_2+m_3)}{s(j_1-m_1)s(j_2-m_2)s(j_3+m_3)}\,. \qquad (4.2.24)
\end{aligned}$$

Studying the analytic structure of $Q^{w=0}$ is a difficult task as a consequence of the complicated form of W. The analysis greatly simplifies when analytically continuing the quantum numbers of one operator to those of a discrete representation. Indeed, when $j_1 = -m_1 + n_1 = -\overline{m}_1 + \overline{n}_1$, and $n_1, \overline{n}_1 = 0, 1, 2, \cdots$, $W\begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix}$ reduces to $W_1 = D_1 C^{12}\overline{C}^{12}$ [77], *i.e.*

$$W_1\begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \frac{(-)^{m_3-\overline{m}_3+\overline{n}_1}\pi^2\gamma(-N)}{\gamma(-2j_1)\gamma(1+j_{12})\gamma(1+j_{13})}\frac{\Gamma(1+j_3-m_3)\Gamma(1+j_3-\overline{m}_3)}{\Gamma(1+j_3-m_3-n_1)\Gamma(1+j_3-\overline{m}_3-\overline{n}_1)}$$

$$\times\prod_{i=2,3}\frac{\Gamma(1+j_i+m_i)}{\Gamma(-j_i-\overline{m}_i)}F\begin{bmatrix} -n_1, -j_{12}, 1+j_{23} \\ -2j_1, 1+j_3-m_3-n_1 \end{bmatrix}F\begin{bmatrix} -\overline{n}_1, -j_{12}, 1+j_{23} \\ -2j_1, 1+j_3-\overline{m}_3-\overline{n}_1 \end{bmatrix}. \qquad (4.2.25)$$

It is easy to see that

$$\frac{\Gamma(1+j_3-m_3)}{\Gamma(1+j_3-m_3-n_1)}F\begin{bmatrix} -n_1, -j_{12}, 1+j_{23} \\ -2j_1, 1+j_3-m_3-n_1 \end{bmatrix} =$$

$$\sum_{n=0}^{n_1}\frac{(-)^n n_1!}{n!(n_1-n)!}\frac{\Gamma(n-j_{12})}{\Gamma(-j_{12})}\frac{\Gamma(n+1+j_{23})}{\Gamma(1+j_{23})}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j_3-m_3)}{\Gamma(n+1+j_3-m_3-n_1)} \qquad (4.2.26)$$

Recall that the OPE involves the function $W\begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, -m_3 \end{bmatrix}$ and then the change $(m_3, \overline{m}_3) \to (-m_3, -\overline{m}_3)$ is required in the above expressions to analyze $Q^{w=0}$. Thus, for generic $2j_i \notin \mathbb{Z}$, the poles and zeros of $Q^{w=0}(j_i; m_i, \overline{m}_i)$ are contained in

$$C(1+j_i)\frac{\gamma(-1-j_1-j_2-j_3)}{\gamma(1+j_{12})\gamma(1+j_{13})}\frac{\Gamma(1+m_2+j_2)\Gamma(-m_3-j_3)}{\Gamma(-\overline{m}_2-j_2)\Gamma(1+\overline{m}_3+j_3)}\,, \tag{4.2.27}$$

plus possible additional zeros in (4.2.26) and its antiholomorphic equivalent expression (see appendix B). The $(j_1,j_2)-$dependent poles in (4.2.27) are at $j_3=\pm j_{21}^{\pm}+p+(q+1)(k-2)$, $\pm j_{21}^{\pm}-(p+1)-q(k-2)$, $\mp j_{21}^{\pm}+p+q(k-2)$, $\mp j_{21}^{\pm}-(p+1)-(q+1)(k-2)$. There are also zeros at $1+2j_i=p+q(k-2)$, $-(p+1)-(q+1)(k-2)$, $i=1,2,3$.

Let us first consider $\Phi^{w_1,j_1}_{m_1,\overline{m}_1}\in\mathcal{D}^{+,w_1}_{j_1}\otimes\mathcal{D}^{+,w_1}_{j_1}$ and note that when $\Phi^{w_1,j_1}_{m_1,\overline{m}_1}\in\mathcal{D}^{-,w_1}_{j_1}\otimes\mathcal{D}^{-,w_1}_{j_1}$ the OPE follows directly using the symmetry of the spectral flow conserving two- and three-point functions under $(m_i,\overline{m}_i)\leftrightarrow(-m_i,-\overline{m}_i),\forall\, i=1,2,3.$[5]

- $\mathcal{D}^{\pm,\,w_1}_{j_1}\times\,\mathcal{C}^{\alpha_2,\,w_2}_{j_2}$

Consider $j_1=-m_1+n_1+i\epsilon_1$ with $n_i\in\mathbb{Z}_{\geq 0}$ and ϵ_1 an infinitesimal positive number, and $j_2=-\frac{1}{2}+is_2$ not correlated with m_2. In this case, $W\begin{bmatrix} j_1\,,\,j_2\,,\,j_3\\ m_1,m_2,m_3\end{bmatrix}\approx W_1\begin{bmatrix} j_1\,,\,j_2\,,\,j_3\\ m_1,m_2,m_3\end{bmatrix}$.

The $m-$independent poles are to the right or to the left of the contour of integration as sketched in Figure 4.a). If $min\,\{m_3,\overline{m}_3\}<\frac{1}{2}$, none of the $m-$dependent poles cross the contour, implying that only continuous series contribute to the spectral flow conserving terms of the OPE $\mathcal{D}^{+,w_1}_{j_1}\times\mathcal{C}^{\alpha_2,w_2}_{j_2}$. On the other hand if $min\,\{m_3,\overline{m}_3\}>\frac{1}{2}$, this OPE also receives contributions from $\mathcal{D}^{+,w_3=w_1+w_2}_{j_3}$. Note that when $j_1\approx m_1+n_1$, $W\begin{bmatrix} j_1\,,\,j_2\,,\,j_3\\ m_1,m_2,m_3\end{bmatrix}\approx W_1\begin{bmatrix} j_1\,,\,j_2\,,\,j_3\\ -m_1,-m_2,-m_3\end{bmatrix}$, which implies that the spectral flow conserving terms in the OPE $\mathcal{D}^{-,w_1}_{j_1}\times\mathcal{C}^{\alpha_2,w_2}_{j_2}$ contain contributions from the continuous representations as well as from $\mathcal{D}^{-,w_3}_{j_3}$ when $max\,\{m_3,\overline{m}_3\}<-\frac{1}{2}$. So we find

$$\mathcal{D}^{\pm,\,w_1}_{j_1}\times\,\mathcal{C}^{\alpha_2,\,w_2}_{j_2}\Big|_{w=0}=\int_{\mathcal{P}}dj_3\,\mathcal{C}^{\alpha_3,\,w_3=w_1+w_2}_{j_3}+\sum_{j_3<-1/2}\mathcal{D}^{\pm,\,w_3=w_1+w_2}_{j_3}+\cdots\,. \tag{4.2.28}$$

- $\mathcal{D}^{\pm,\,w_1}_{j_1}\otimes\,\mathcal{D}^{\mp,\,w_2}_{j_2}$ and $\mathcal{D}^{\mp,\,w_1}_{j_1}\otimes\,\mathcal{D}^{\mp,\,w_2}_{j_2}$

When j_2 is continued to $(-\frac{k-1}{2}+i\epsilon_2,-\frac{1}{2}+i\epsilon_2)$, ϵ_2 being an infinitesimal positive number,

[5] This symmetry follows directly from the integral expression for $W\begin{bmatrix} j_1\,,\,j_2\,,\,j_3\\ m_1,m_2,m_3\end{bmatrix}$ performing the change of variables $(x_i,\overline{x}_i)\to(x_i^{-1},\overline{x}_i^{-1})$ in (4.2.6).

W is again well approximated by W_1 as long as $j_2 \neq -m_2+n_2+i\epsilon_2, -\overline{m}_2+\overline{n}_2+i\epsilon_2$. Otherwise, one also has to consider $W_2 \equiv D_2 C^{21}\overline{C}^{21}$, but the result coincides exactly with the one obtained using W_1, so we restrict to this. Two $m-$independent series of poles may cross the contour of integration: $j_3 = j_1 - j_2 - 1 - p - q(k-2)$ and $j_3 = j_2 - j_1 + p + q(k-2)$, both with $q = 0$. The former has $j_3 > -\frac{1}{2}$ and the latter, $j_3 < -\frac{1}{2}$. The $m-$dependent poles in $Q^{w=0}$ arise from $\frac{\Gamma(-j_3-\overline{m}_3)}{\Gamma(1+j_3+m_3)}$. When $j_2 = -m_2+n_2+i\epsilon_2$, because of the factor $\Gamma(-j_2-m_2)^{-1}$, only $m-$dependent poles give contributions from discrete series. To see this, consider the $m-$independent poles at $j_3 = j_1+j_2-p-q(k-2)$. These are outside the contour of integration and in the limit $\epsilon_1, \epsilon_2 \to 0$ some of them may overlap with the $m-$dependent ones. Again, one may argue that this limit leaves a finite and non-vanishing factor.

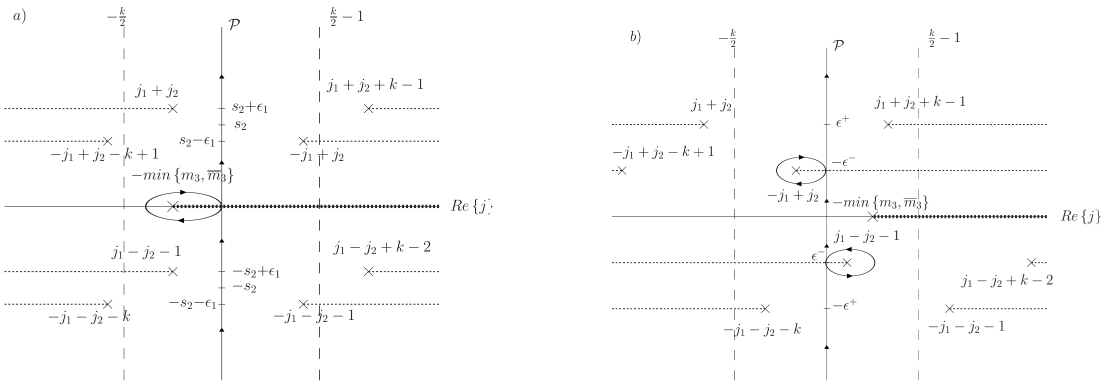

Figure 4: Analytic continuation of $Q^{w=0}$ for (j_1, j_2)-values away from the axis $-\frac{1}{2}+i\mathbb{R}$, using W_1 instead of W. In 4.a) $j_2 = -\frac{1}{2}+is_2$ and only $m-$dependent poles can cross the contour of integration. In 4.b) $-\frac{k-1}{2} < j_2 < -\frac{1}{2}$ was considered. While $m-$independent poles only cross the contour when $j_2 < j_1$, $m-$dependent poles can cross independently of the values of j_1, j_2, but they are annihilated unless $j_2 > j_1$.

When $j_2 = m_2 + n_2 + i\epsilon_2$, at first sight there are no zeros. If $j_2 - j_1 < -\frac{1}{2}$, some poles with $q = 0$ in the series $j_3 = j_2 - j_1 + p + q(k-2)$ and $j_3 = j_1 - j_2 - 1 - p - q(k-2)$ cross the contour, as shown in Figure 4.b). Using the relation between j_i and m_i and $m-$conservation, it follows that the former poles can be rewritten as $j_3 = m_3 + n_3 = \overline{m}_3 + \overline{n}_3$, where $n_3 = n_2 - n_1 + p$ and $\overline{n}_3 = \overline{n}_2 - \overline{n}_1 + p$. Obviously, if $n_2 \geq n_1$ and $\overline{n}_2 \geq \overline{n}_1$ all the residues picked up by the contour deformation imply contributions to the OPE from $\mathcal{D}_{j_3}^{-,w_3=w_1+w_2}$. When $n_2 < n_1$ or $\overline{n}_2 < \overline{n}_1$, only those values of p for which both n_3 and $\overline{n}_3$ are non-negative integers remain after taking the limit $\epsilon_1, \epsilon_2 \to 0$. This is because of extra zeros appearing in W_1 which are not explicit in (4.2.25) (see appendix B). Using the results in the Appendix and the identity (3.2.17) it is straightforward to see that the latter series of poles give the same contributions.

The poles at $j_3 = -min\,\{m_3, \overline{m}_3\} + n_3$ may cross the contour. If this happens they overlap with the $m-$independent poles. But there are double zeros canceling these contributions.

If $j_2 - j_1 > -\frac{1}{2}$, only $m-$dependent poles may cross the contour. But they give contributions only if they do not overlap with the poles at $j_3 = j_1 - j_2 - 1 - n$, again because of the presence of double zeros. Therefore, these contributions remain only for $j_3 \geq j_1 - j_2$.

Putting all together we get

$$\begin{aligned} \mathcal{D}^{+,\,w_1}_{j_1} \times \mathcal{D}^{-,\,w_2}_{j_2}\Big|_{w=0} &= \int_{\mathcal{P}} dj_3\, \mathcal{C}^{\alpha_3,\,w_3=w_1+w_2}_{j_3} + \sum_{j_2-j_1\leq j_3<-\frac{1}{2}} \mathcal{D}^{-,\,w_3=w_1+w_2}_{j_3} \\ &\quad + \sum_{j_1-j_2\leq j_3<-\frac{1}{2}} \mathcal{D}^{+,\,w_3=w_1+w_2}_{j_3} + \cdots , \end{aligned} \tag{4.2.29}$$

$$\mathcal{D}^{\pm,\,w_1}_{j_1} \times \mathcal{D}^{\pm,\,w_2}_{j_2}\Big|_{w=0} = \sum_{j_3\leq j_1+j_2} \mathcal{D}^{\pm,\,w_3=w_1+w_2}_{j_3} + \cdots . \tag{4.2.30}$$

- $\mathcal{C}^{\alpha_1,\,w_1}_{j_1} \times \mathcal{C}^{\alpha_2,\,w_2}_{j_2}$

The zero and pole structure of $Q^{w=0}$ is given by

$$\begin{aligned} Q^{w=0}(j_i; m_i, \overline{m}_i) \;\sim\; & C(1+j_i)\frac{\gamma(-N)}{s(\overline{m}_3+j_3)} G\begin{bmatrix} m_3-j_3,\ -j_{13},\ 1+m_2+j_2 \\ m_3-j_1+j_2+1,\ m_2-j_1-j_3 \end{bmatrix} \\ & \times \Bigg\{ s(\overline{m}_1+j_1) G\begin{bmatrix} 1+N,\ 1+\overline{m}_1+j_1,\ 1-\overline{m}_2+j_2 \\ 2+\overline{m}_1+j_2+j_3,\ 2-\overline{m}_2+j_1+j_3 \end{bmatrix} \\ & \quad - \ s(\overline{m}_1-j_2+j_3) G\begin{bmatrix} 1+N,\ 1+\overline{m}_2+j_2,\ 1-\overline{m}_1+j_1 \\ 2+\overline{m}_2+j_1+j_3,\ 2-\overline{m}_1+j_2+j_3 \end{bmatrix} \Bigg\} \\ & + (j_1, m_1, \overline{m}_1) \leftrightarrow (j_2, m_2, \overline{m}_2) . \end{aligned}$$

$G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix}$ has simple poles at $a, b, c = 0, -1, -2, \ldots$ as well as at $u = e + f - a - b - c = 0, -1, -2, \ldots,$ if $a, b, c \neq 0, -1, -2, \cdots$. The pole structure of $Q^{w=0}$ is much subtler when

$j_i = -\frac{1}{2} + is_i, i = 1, 2$ as the naive poles cancel because of the presence of hidden zeros. Actually, the correct behavior of $Q^{w=0}$, must be of the form [43]

$$Q^{w=0} \sim \frac{\Gamma(-j_3 - m_3)\Gamma(-j_3 + \overline{m}_3)}{\Gamma(1 + j_3 - m_3)\Gamma(1 + j_3 + \overline{m}_3)}, \tag{4.2.31}$$

for generic j_1, j_2 and for m_1, m_2 not correlated with them, up to regular and non-vanishing contributions for $j_3 = \pm m_3 + n_3 = \pm\overline{m}_3 + \overline{n}_3$, with $n_3, \overline{n}_3 \in \mathbb{Z}$. No other m-dependent pole series appears and the m-independent pole series do not cross the integration contour

We may now analyze the OPE $\mathcal{C}_{j_1}^{\alpha_1, w_1} \times \mathcal{C}_{j_2}^{\alpha_2, w_2}$. A sum over continuous representations appears because $Q^{w=0}$ does not vanish for $j_3 \in -\frac{1}{2} + i\mathbb{R}$. On the other hand, the expression (4.2.31) shows that there are no contributions from discrete representations provided $min\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} < \frac{1}{2}$ and $max\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} > -\frac{1}{2}$. Obviously both bounds cannot be violated at the same time. When the first one is violated, operators belonging to spectral flow images of lowest-weight representations contribute to the OPE. On the contrary, when the second bound is not satisfied, operators in spectral flow images of highest-weight representations appear in the OPE.

Therefore, we conclude that the $w-$conserving contributions to the OPE of two continuous representations are the following:

$$\mathcal{C}_{j_1}^{\alpha_1, w_1} \times \mathcal{C}_{j_2}^{\alpha_2, w_2}\Big|_{w=0} \sim \int_{\mathcal{P}} dj_3\, \mathcal{C}_{j_3}^{\alpha_3, w_3=w_1+w_2} + \sum_{j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{+, w_3=w_1+w_2} + \sum_{j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{-, w_3=w_1+w_2}, \tag{4.2.32}$$

up to descendants. Note that, in a particular OPE with $m_i, \overline{m}_i$ fixed, only one of the discrete series contributes, depending on the signs of $m_3, \overline{m}_3$.

Collecting all the results, the OPE for primary fields and their spectral flow images in the spectrum of the AdS_3 WZNW model are the following:

$$\mathcal{D}_{j_1}^{\pm, w_1} \times \mathcal{D}_{j_2}^{\pm, w_2} = \sum_{j_3 \leq j_1+j_2} \mathcal{D}_{j_3}^{\pm, w_3=w_1+w_2} + \sum_{-j_1-j_2-\frac{k}{2} \leq j_3 < -\frac{1}{2}} \mathcal{D}_{j_3}^{\mp, w_3=w_1+w_2\pm 1}$$

$$+ \sum_{j_1+j_2+\frac{k}{2}\leq j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{\pm,\, w_3=w_1+w_2\pm 1} + \int_{\mathcal{P}} dj_3\ \mathcal{C}_{j_3}^{\alpha_3,\, w_3=w_1+w_2\pm 1} + \cdots . \tag{4.2.33}$$

$$\begin{aligned} \mathcal{D}_{j_1}^{+,\, w_1} \times \mathcal{D}_{j_2}^{-,\, w_2} &= \sum_{j_1-j_2\leq j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{+,\, w_3=w_1+w_2} + \sum_{j_2-j_1\leq j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{-,\, w_3=w_1+w_2} + \\ &+ \sum_{j_3\leq j_2-j_1-\frac{k}{2}} \mathcal{D}_{j_3}^{-,\, w_3=w_1+w_2+1} + \sum_{j_3\leq j_1-j_2-\frac{k}{2}} \mathcal{D}_{j_3}^{+,\, w_3=w_1+w_2-1} \\ &+ \int_{\mathcal{P}} dj_3\ \mathcal{C}_{j_3}^{\alpha_3,\, w_3=w_1+w_2} + \cdots , \end{aligned} \tag{4.2.34}$$

$$\begin{aligned} \mathcal{D}_{j_1}^{\pm,\, w_1} \times \mathcal{C}_{j_2}^{\alpha_2,\, w_2} &= \sum_{w=0}^{1} \int_{\mathcal{P}} dj_3\ \mathcal{C}_{j_3}^{\alpha_3,\, w_3=w_1+w_2\pm w} + \sum_{j_3<-\frac{1}{2}} \mathcal{D}_{j_3}^{\pm,\, w_3=w_1+w_2} \\ &+ \sum_{j_3<-\frac{1}{2}} D_{j_3}^{\mp, w_3=w_1=w_2\pm 1} + \cdots , \end{aligned} \tag{4.2.35}$$

$$\begin{aligned} \mathcal{C}_{j_1}^{\alpha_1,\, w_1} \times \mathcal{C}_{j_2}^{\alpha_2,\, w_2} &= \sum_{w=0}^{1} \sum_{j_3<-\frac{1}{2}} \left(\mathcal{D}_{j_3}^{+,\, w_3=w_1+w_2-w} + \mathcal{D}_{j_3}^{-,\, w_3=w_1+w_2+w} \right) \\ &+ \sum_{w=-1}^{1} \int_{\mathcal{P}} dj_3\ \mathcal{C}_{j_3}^{\alpha_3,\, w_3=w_1+w_2+w} + \cdots . \end{aligned} \tag{4.2.36}$$

Satoh's prescription

In order to analyze these results, let us first restrict to the spectral flow conserving contributions and for the particular case of $w_i = 0,\ i = 1, 2$. In this case, exactly the same results were obtained in [77] using the following prescription for the OPE of $w = 0$ primary fields $\Phi^{j_1}_{m_1,\overline{m}_1}\ \Phi^{j_2}_{m_2,\overline{m}_2}$ [6]:

$$\Phi^{j_1}_{m_1,\overline{m}_1}(z_1,\overline{z}_1)\Phi^{j_2}_{m_2,\overline{m}_2}(z_2,\overline{z}_2) \underset{z_1\to z_2}{\sim} \sum_{j_3} |z_{12}|^{-2\widetilde{\Delta}_{12}} Q^{w=0}(j_i; m_i, \overline{m}_i) \Phi^{j_3}_{m_1+m_2,\overline{m}_1+\overline{m}_2}(z_2,\overline{z}_2), \tag{4.2.37}$$

[6] See [76] for previous work involving highest-weight representations.

where $Q^{w=0}$ was obtained using the standard procedure, *i.e.* multiplying both sides of (4.2.37) by a fourth field in the $w = 0$ sector and taking expectation values. The formal symbol $\sum_{j_3}$ denotes integration over $\mathcal{D}^{\pm}_{j_3}$ and $\mathcal{C}^{\alpha_3}_{j_3}$, namely

$$\sum_{j_3} = \int_{\mathcal{P}^+} dj_3 + \delta_{\mathcal{D}^{\pm}_{j_3}} \oint_{\mathcal{C}} dj_3\,. \tag{4.2.38}$$

The integration over $\mathcal{P}^+$ stands for summation over C^{α}_j. The contour integral along $\mathcal{C}$ encloses the poles from $\mathcal{D}^{\pm}_{j_3}$ and $\delta_{\mathcal{D}^{\pm}_{j_3}}$ means that j_3 is picked up from the poles in $Q^{w=0}$ by the contour $\mathcal{C}$ only when it belongs to a discrete representation. The range of j_3 is Re $j_3 \leq -\frac{1}{2}$ and Im $j_3 \geq 0$, consistently with the argument which determined $Q^{w=0}$ because $\sum_{j_3}$ picks up only one term in (4.2.3). This prescription to deal with the $j-$dependent $m-$independent poles was shown to be compatible with the one suggested in [42] for the H_3^+ model. The strategy designed in (4.2.38) for the treatment of $m-$dependent poles, which were absent in [42], aimed to reproducing the classical tensor product of representations of $SL(2,\mathbb{R})$ in the limit $k \to \infty$. This proposal for the OPE includes in addition the requirement that poles with divergent residues should not be picked up.

In this section, we have followed a different path. We have treated the $j-$ and $m-$dependent poles alike. However, although the equivalence between both prescriptions is not obvious a priori, we obtained the same results for the OPE of unflowed primary fields [7]. Indeed, notice that poles in $Q^{w=0}$ at values of quantum numbers in C^{α}_j or $\mathcal{D}^{\pm}_{j_3}$ would not contribute to the OPE determined by (4.2.13) if they do not cross the contour $\mathcal{P}$, unlike to (4.2.37). On the other hand, contributions from operators in other representations, *i.e.* neither in C^{α}_j nor in $\mathcal{D}^{\pm}_{j_3}$, could have appeared in (4.2.33)$-$(4.2.36), but they did not. Moreover, by a careful analysis of the analytic structure of $Q^{w=0}$ we have shown that there are no double poles, so that the regularization proposed in [77] is not really necessary. [8].

[7]More generally, it can be shown that a generalization of the *ansatz* (4.2.37) for fields $\Phi^{j_1,w_1}_{m_1,\overline{m}_1}\ \Phi^{j_2,w_2}_{m_2,\overline{m}_2}$, by adding the contributions from terms proportional to $Q^{w=\pm1}$ and replacing $\delta_{D^{\pm}_{j_3}}$ by $\delta_{D^{\pm,w_3}_{j_3}}$, leads to the same results (4.2.33)$-$(4.2.36).

[8]This is very important because the double poles discussed in [77] would lead to inconsistencies in the analytic continuation of the OPE from H_3^+ that we have performed in this chapter. In particular, they would give divergent contributions to the OPE $\mathcal{D}^+_j \times \mathcal{D}^-_j$ and, in addition, this OPE would be incompatible with $\mathcal{D}^-_j \times \mathcal{D}^+_j$,

In the case $w_1 = w_2 = 0$, $k \to \infty$, the $w-$conserving contributions to the OPE of representations of the zero modes in (4.2.33)−(4.2.36) reproduce the classical tensor products of representations of $SL(2,\mathbb{R})$ obtained in [83]. Continuous series appear twice in the product of two continuous representations due to the existence of two linearly independent Clebsh-Gordan coefficients. As noted in [77], this is in agreement with the fact that both terms C^{12} and C^{21} in (4.2.21) contribute to $Q^{w=0}$ in the fusion of two continuous series. Moreover, it was also observed that the analysis can be applied for finite k without modifications. The results are given by replacing $\mathcal{D}_j^{\pm}, \mathcal{C}_j^{\alpha}$ in (4.2.33)−(4.2.36) by the corresponding affine representations $\widehat{\mathcal{D}}_j^{\pm}, \widehat{\mathcal{C}}_j^{\alpha}$. It is easy to see that this OPE of unflowed fields in the spectrum of the AdS_3 WZNW model is not closed, *i.e.* it gets contributions from discrete representations with $j_3 < -\frac{k-1}{2}$. When spectral flow is turned on, incorporating all the relevant representations of the theory and the complete set of structure constants as we have done in this section, the OPE still does not close, namely there are contributions from discrete representations outside the range (4.2.4). In particular, this feature of the OPE of fields in discrete representations differs from the results in [40] where the factorization limit of the four-point function of $w = 0$ short strings was shown to be in accord with the Hilbert space of the string dual theory.

In the following section we will show how the spectral flow symmetry imposes a truncation of the OPE (4.2.33)−(4.2.36) which guaranties the closure in the physical spectrum of the model.

4.2.5 Truncation of the operator algebra and fusion rules

The analysis of the previous section involved primary operators and their spectral flow images. Then, the OPE (4.2.33)-(4.2.36) explicitly includes some descendant fields. Assuming the appearance of spectral flow images of primary states in the fusion rules indicates that there are also contributions from descendants not obtained by spectral flowing primaries but descendants with the same J_0^3 eigenvalue. As we commented in section A.2, the presence of a descendant in the OPE requires the presence of the Virasoro primary associated to it, so the inclusion of descendants implies the replacement of $\mathcal{D}_j^{\pm,w}, \mathcal{C}_j^{\alpha,w}$ by $\hat{\mathcal{D}}_j^{\pm,w}, \hat{\mathcal{C}}_j^{\alpha,w}$ in (4.2.33)-(4.2.36), we will show that this observation implies some interesting conclusions.

in contradiction with expectations from the symmetries of the function W.

For instance, consider the spectral flow non-preserving terms in the OPE $\mathcal{D}_{j_1}^{-,w_1} \times \mathcal{D}_{j_2}^{-,w_2}$, (4.2.33). If they are extended to the affine series, using the spectral flow symmetry they may be identified as

$$\sum_{-\frac{k-1}{2}<\tilde{j}_3\leq j_1+j_2} \widehat{\mathcal{D}}^{+,\, w_3=w_1+w_2-1}_{-\frac{k}{2}-\tilde{j}_3} \equiv \sum_{-\frac{k-1}{2}<j_3\leq j_1+j_2} \widehat{\mathcal{D}}^{-,\, w_3=w_1+w_2}_{j_3} . \qquad (4.2.39)$$

This reproduces the spectral flow conserving terms in the first sum in (4.2.33). However, there is an important difference: here j_3 is automatically restricted to the region (4.2.4).

Analogously, applying the spectral flow symmetry to the discrete series contributing to the OPE $\mathcal{D}_{j_1}^{+,w_1} \times \mathcal{D}_{j_2}^{-,w_2}\Big|_{|w|=1}$ in (4.2.20) leads to contributions from $\sum_{j_2-j_1\leq j_3} \widehat{\mathcal{D}}^{-,\, w_3=w_1+w_2}_{j_3}$ as well as from $\sum_{j_1-j_2\leq j_3} \widehat{\mathcal{D}}^{+,\, w_3=w_1+w_2}_{j_3}$, which were found among the spectral flow conserving terms with the extra condition $j_3 < -\frac{1}{2}$.

In order to see further implications of the spectral flow symmetry on the OPE (4.2.33)-(4.2.36), let us now consider operator products of descendants. Take the OPE $\widehat{D}_{j_1}^{+,w_1=0} \otimes \widehat{D}_{j_2}^{-,w_2=1}$ [9]. Equation (4.2.34) gives spectral flow conserving contributions from $\widehat{\mathcal{D}}_{j_3}^{-,w_3=1}$, for certain $m_i, \overline{m}_i, i = 1, 2$, with j_3 verifying (4.2.4). Using the spectral flow symmetry, one might infer that the contributions from $\widehat{\mathcal{D}}_{j_3}^{+,w_3=0}$ to the OPE $\widehat{\mathcal{D}}_{j_1}^{+,w_1=0} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2=0}$ in (4.2.33) would also be within the region (4.2.4). On the contrary, we found terms in $\widehat{\mathcal{D}}_{j_3}^{+,w_3=0}$ with $j_3 < -\frac{k-1}{2}$. Moreover, using the spectral flow symmetry again, these terms can be identified with contributions from $\widehat{\mathcal{D}}_{j_3}^{-,w_3=1}$ with $j_3 > -\frac{1}{2}$ to the OPE $\widehat{\mathcal{D}}_{j_1}^{+,w_1=0} \otimes \widehat{\mathcal{D}}_{j_2}^{-,w_2=1}$, in contradiction with our previous result.

Similar puzzles are found identifying $\sum_{j_3<-\frac{1}{2}} \widehat{\mathcal{D}}^{+,\, w_3=w_1+w_2-1}_{j_3} = \sum_{-\frac{k-1}{2}<j_3} \widehat{\mathcal{D}}^{-,\, w_3=w_1+w_2}_{j_3}$ in (4.2.35), which gives some of the spectral flow conserving contributions. It is interesting to note that only the states within the region (4.2.4) contribute in both cases, explicitly $j_3 = j_1 + \alpha_2 + n$, with $n \in \mathbb{Z}$ such that $-\frac{k-1}{2} < j_3 < -\frac{1}{2}$. It is also important to stress the following observation. For given j_1, m_1 and j_2, m_2 the spectral flow conserving part of the OPE (4.2.35) receives contributions from states with $\tilde{j}_3, \tilde{m}_3$ verifying $\tilde{j}_3 = \tilde{m}_3 + \tilde{n}_3$ with $\tilde{n}_3 = 0, 1, \cdots, \tilde{n}_3^{max}$, $\tilde{n}_3^{max}$ being

[9] We use the tensor product symbol $\otimes$ to denote the OPE of fields in representations of the current algebra, to distinguish it from that of highest/lowest-weight fields.

the maximum integer such that $\tilde{j}_3 < -\frac{1}{2}$. On the other hand, the spectral flow non-conserving terms get contributions from $j_3 = -m_3 + n_3$ with $n_3 = 0, 1, 2, \dots, n_3^{max}$ and here n_3^{max} is the maximal non-negative integer such that $j_3 < -\frac{1}{2}$. So, identifying both series implies considering $\tilde{j}_3 = -\frac{k}{2} - j_3$ and now n_3^{max} (which is the same as before) has to be the maximal non-negative integer for which $\tilde{j}_3 > -\frac{k-1}{2}$. There is just one operator appearing in both contributions to the OPE. It has $\tilde{n}_3 = 0$ in the former and $n_3 = 0$ in the latter. This is a consequence of the relation $\Phi^{j,w=0}_{m=\overline{m}=-j} = \frac{\nu^{\frac{k}{2}-1}}{(k-2)} \frac{1}{B(-1-j')} \Phi^{j',w'=1}_{m'=\overline{m}'=j'}$ with $j' = -\frac{k}{2} - j$ [40]. One can check that the $w-$conserving three-point functions containing $\Phi^{j,w=0}_{m=\overline{m}=-j}$ reduce to the $w-$non-conserving ones involving $\Phi^{j',w'=1}_{m'=\overline{m}'=j'}$. This result can be generalized for arbitrary w sectors in the $m-$basis, *i.e.* $\Phi^{j,w}_{m=\overline{m}=-j} \sim \Phi^{j',w'=w+1}_{m'=\overline{m}'=j'}$ up to a regular normalization for j in the region (4.2.4). For instance, one can reduce a spectral flow conserving three-point function including $\Phi^{j,w}_{m=\overline{m}=-j}$ to a one unit violating amplitude containing $\Phi^{j',w+1}_{m'=\overline{m}'=j'}$ using the identity

$$C(1+j_1, 1+j_2, 1+j_3) = \frac{\nu^{k-2}\gamma(k-2-j_{23})\gamma(2-k-2j_1)C(k+j_1-1, 1+j_2, 1+j_3)}{(k-2)\gamma(1+2j_1)\gamma(-N)\gamma(-j_{12})\gamma(-j_{13})} \quad (4.2.40)$$

which is a consequence of the relation $G(j) = (k-2)^{1+2j}\gamma(-j)G(j-k+2)$.

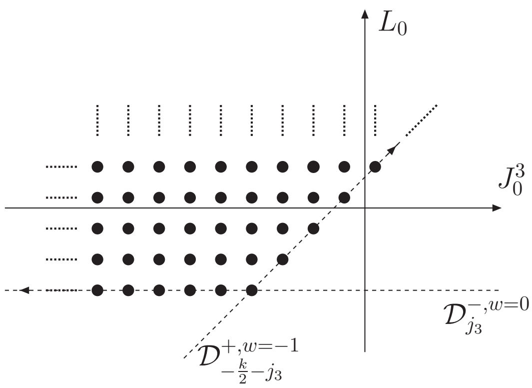

Figure 5: Weight diagram of $\widehat{\mathcal{D}}^{-,w=0}_{j_3}$. The lines with arrows indicate the states in $\mathcal{D}^{-,w=0}_{j_3}$ and $\mathcal{D}^{+,w=-1}_{-\frac{k}{2}-j_3}$. Consider a state in $\widehat{\mathcal{D}}^{+,w=0}_{\tilde{j}}$, at level $\tilde{N}$ and weight $\tilde{m} = -\tilde{j} + \tilde{n}$. It follows from (3.2.2), (3.2.3) that after spectral flowing by (-1) unit, this state maps to a state in $\widehat{\mathcal{D}}^{-,w=0}_{j}$, with $j = -\frac{k}{2} - \tilde{j}$, level $N = \tilde{n}$ and weight $m = j - n$, with $n = \tilde{N}$. For instance primary states in $\widehat{\mathcal{D}}^{+,w=0}_{-\frac{k}{2}-j_3}$, denoted simply by $\mathcal{D}^{+,w=0}_{-\frac{k}{2}-j_3}$, map to highest-weight states in $\widehat{\mathcal{D}}^{-,w=0}_{j_3}$. So, only one state in $\mathcal{D}^{+,w=-1}_{-\frac{k}{2}-j_3}$ coincides with one in $\mathcal{D}^{-,w=0}_{j_3}$, namely that with $\tilde{n} = 0$.

The OPE (4.2.35) was obtained for states in $\mathcal{C}_j^{\alpha,w}$ and $\mathcal{D}_j^{\pm,w}$. When replacing operators in, say $\mathcal{D}_j^{-,w}$ by those in $\widehat{\mathcal{D}}_j^{-,w}$, the latter can be interpreted as having been obtained by performing w units of spectral flow on primaries of $\widehat{\mathcal{D}}_j^{-,w=0}$ or $w-1$ units of spectral flow on primaries of $\mathcal{D}^+_{-\frac{k}{2}-j}$, that is w units of spectral flow from $\mathcal{D}^{+,w=-1}_{-\frac{k}{2}-j}$, which in turn may be thought of as the highest-weight field in $\widehat{\mathcal{D}}_j^{-,w=0}$ (see figure 5). Only the spectral flowed primary of highest-weight appears in both sets of contributions, *i.e.* the one with $n_3 = \tilde{n}_3 = 0$. This behavior was observed in all other cases, namely, the same discrete series arising in the OPE from $Q^{w=0}$ can be also seen to arise from $Q^{w=1}$ or $Q^{w=-1}$, but only one operator appears in both simultaneously.

Thus, even if the calculations involved operators in the series $\mathcal{D}_j^{\pm,w}$ and $\mathcal{C}_j^{\alpha,w}$, we collect here the results for the fusion rules [10] assuming $\Phi^{j_i,w_i}_{m_i,\overline{m}_i}(z_i,\overline{z}_i) \in \widehat{\mathcal{D}}^{\pm,w_i}_{j_i}$ or $\widehat{\mathcal{C}}^{\alpha_i,w_i}_{j_i}$, $i=1,2,3$. Using the spectral flow symmetry to identify $\widehat{\mathcal{D}}_j^{-,w} = \widehat{\mathcal{D}}^{+,w-1}_{-\frac{k}{2}-j}$, we obtain:

1. $$\widehat{\mathcal{D}}^{+,\,w_1}_{j_1} \otimes \widehat{\mathcal{D}}^{+,\,w_2}_{j_2} = \int_{\mathcal{P}} dj_3\, \widehat{\mathcal{C}}^{\alpha_3,\,w_3=w_1+w_2+1}_{j_3} \oplus \sum_{-\frac{k-1}{2}<j_3\leq j_1+j_2} \widehat{\mathcal{D}}^{+,\,w_3=w_1+w_2}_{j_3} \oplus \sum_{j_1+j_2+\frac{k}{2}\leq j_3<-\frac{1}{2}} \widehat{\mathcal{D}}^{+,\,w_3=w_1+w_2+1}_{j_3},$$

2. $$\widehat{\mathcal{D}}^{+,\,w_1}_{j_1} \otimes \widehat{\mathcal{C}}^{\alpha_2,\,w_2}_{j_2} = \sum_{-\frac{k-1}{2}<j_3<-\frac{1}{2}} \widehat{\mathcal{D}}^{+,\,w_3=w_1+w_2}_{j_3} \oplus \sum_{w=0}^{1} \int_{\mathcal{P}} dj_3\, \widehat{\mathcal{C}}^{\alpha_3,\,w_3=w_1+w_2+w}_{j_3},$$

3. $$\widehat{\mathcal{C}}^{\alpha_1,\,w_1}_{j_1} \otimes \widehat{\mathcal{C}}^{\alpha_2,\,w_2}_{j_2} = \sum_{w=-1}^{0} \sum_{-\frac{k-1}{2}<j_3<-\frac{1}{2}} \widehat{\mathcal{D}}^{+,\,w_3=w_1+w_2+w}_{j_3} \oplus \sum_{w=-1}^{1} \int_{\mathcal{P}} dj_3\, \widehat{\mathcal{C}}^{\alpha_3,\,w_3=w_1+w_2+w}_{j_3}.$$

We have truncated the spin of the contributions from discrete representations following the criterion that processes related through the identity $\widehat{\mathcal{D}}_j^{+,w} \equiv \widehat{\mathcal{D}}^{-,w+1}_{-\frac{k}{2}-j}$ must be equal, *i.e.* equivalent operator products should get the same contributions. Indeed, one finds contradictions unless the OPE is truncated to keep j_3 within the region (4.2.4). As we have seen through some examples, extending the OPE (4.2.33)-(4.2.36) to representations of the current algebra, discrepancies occur both when comparing $w-$conserving with non-conserving contributions as well as when comparing $w-$conserving terms among themselves. So the truncation is imposed by self-consistency.

[10] Actually, the fusion rules for two representations determine the exact decomposition of their tensor products. These not only contain information on the conformal families appearing in the r.h.s of the OPE, but also on their multiplicities. We shall not attempt to determine the latter here.

A strong argument in support of the fusion rules 1.−3. is that only operators violating the bound (4.2.4) must be discarded. Indeed, the cut amounts to keeping just contributions from states in the spectrum[11], *i.e.* it implies that the operator algebra is closed on the Hilbert space of the theory. However, the spectrum involves irreducible representations and there are no singular vectors to decouple states like in $SU(2)$ [85] [12]. So there is a yet to be discovered physical mechanism decoupling states.

Nevertheless, the results listed in items 1.−3. above are supported by several consistency checks. First, the limit $k \to \infty$ contains the classical tensor products of representations of $SL(2,\mathbb{R})$ [83] when restricted to $w = 0$ fields. Second, as mentioned in the previous paragraph, once the OPE is truncated to keep only contributions from the spectrum, one can verify full consistency. In particular, the OPE $\widehat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2}$ is consistent with the results in [40] (see the discussion in appendix 5.2.1). Finally, based on the spectral flow selection rules (4.2.1) and (4.2.2), the following alternative analysis can be performed. Let us consider, for instance, the operator product $\widehat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2}$. Applying equation (4.2.2) to correlators involving three discrete states in $\widehat{\mathcal{D}}_{j}^{+,w}$ requires either *i*) $w_3 = -w_1 - w_2 - 1$ or *ii*) $w_3 = -w_1 - w_2 - 2$. Therefore, together with m conservation, *i*) implies that the three-point function $< \widehat{\mathcal{D}}_{j_1}^{+,w_1} \widehat{\mathcal{D}}_{j_2}^{+,w_2} \widehat{\mathcal{D}}_{j_3}^{+,w_3=-w_1-w_2-1} >$ will not vanish as long as the OPE $\widehat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2}$ contains a state in $\widehat{\mathcal{D}}_{j_3}^{-,w=w_1+w_2+1}$, which is equivalent to $\widehat{\mathcal{D}}_{\tilde{j}_3}^{+,w=w_1+w_2}$. Indeed, this contribution appeared above. Similarly, *ii*) implies that in order for $< \widehat{\mathcal{D}}_{j_1}^{+,w_1} \widehat{\mathcal{D}}_{j_2}^{+,w_2} \widehat{\mathcal{D}}_{j_3}^{+,w_3=-w_1-w_2-2} >$ to be non-vanishing, the OPE $\widehat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2}$ must have contributions from $\widehat{\mathcal{D}}_{j_3}^{-,w_3=w_1+w_2+2} \equiv \widehat{\mathcal{D}}_{\tilde{j}_3}^{+,w_3=w_1+w_2+1}$, which in fact were found. Finally, when the third state involved in the three-point function is in the series $\widehat{\mathcal{C}}_{j_3}^{\alpha_3,w_3}$, equation (4.2.1) leaves only one possibility, namely $w_3 = -w_1 - w_2 - 1$, and thus the OPE must include terms in $\widehat{\mathcal{C}}_{j_3}^{\alpha_3,w_3=w_1+w_2+1}$, which actually appear in the list above. Although this analysis based on the spectral flow selection rules does not allow to determine either the range of j_3-values or the OPE coefficients, it is easy to check that the series content in 1.−3. is indeed completely

[11] It is important to stress that the truncation is not discarding contributions from the *microstates* associated to the $(j_1, j_2)-$dependent poles that were found in [42]. Only $m-$dependent poles which are absent in the $x-$basis present inconsistences with the spectral flow symmetry.

[12] The spectral flow operators $\Phi^{-\frac{k}{2}}_{\pm\frac{k}{2},\pm\frac{k}{2}}$ have null descendants. Even though they are excluded from the range (4.2.4) they are necessary auxiliary fields to construct the states in spectral flow representations. Although the physical mechanism is not clear to us, these operators might play a role in the decoupling.

reproduced in this way.

As mentioned in the previous section, in principle $w = \pm 2$ three-point functions should have been considered. However, the contributions from these terms are already contained in our results. If they gave contributions from discrete representations outside the spectrum, they should be truncated since the equivalent terms listed above do not include them. Contributions from operators in $\widehat{\mathcal{D}}_{j_3}^{-,w_3=w_1+w_2+2}$ can only appear in case 1., namely $\widehat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2}$, for $j_3 = -k - j_1 - j_2 - n$. These correspond to the terms denoted as Poles_2 in [40], where they could not be interpreted in terms of physical string states and were then truncated. See section 5.2.1 for a detailed discussion.

In conclusion, the results presented in this section are in agreement with the spectral flow selection pattern (4.2.1)-(4.2.2), they are consistent with the results in [40] and determine the closure of the operator algebra when properly treating the spectral flow symmetry. The full consistency of the OPE should follow from a proof of factorization and crossing symmetry of the four-point functions, but closed expressions for these amplitudes are not known, even in the simpler H_3^+ model. In order to make some preliminary progress in this direction, in the next chapter we discuss certain properties of the factorization of four-point amplitudes involving states in different representations of the AdS_3 WZNW model, constructed along the lines in [42].

Chapter 5

Factorization of four-point functions

In this chapter we discuss the issue of the factorization of four point functions in AdS_3 and its consistency with the OPE found in previous chapter. After discussing the factorization in the Euclidean rotation of AdS_3, the H_3^+ model we turn to the Lorentzian model.

Although a complete description of the contributions of descendant operators is not available to complete the bootstrap program, in section 5.2 we display some interesting properties of the amplitudes that can be useful to achieve a resolution of the theory. We will display a very odd property of the factorization in the AdS_3 WZNW model which follows from the assumption that correlators are related through analytic continuation to those in H_3^+: Both, spectral flow conserving and spectral flow non conserving channels seem to give the same numerical result. This observation was later explicitly confirmed in some particular examples in the Coulomb-Gas formalism [87]. Then, we perform a qualitative study of the contributions of primaries and flowed primaries in the intermediate channels of the amplitudes and finally, we discuss the consistency of the factorization with the spectral flow selection rules.

5.1 Factorization in H_3^+

A decomposition of the four-point function in the Euclidean H_3^+ model was worked out in [41, 42] using the OPE (4.1.1) for pairs of primary operators $\Phi_{j_1}\Phi_{j_2}$ and $\Phi_{j_3}\Phi_{j_4}$. The $s-$channel factorization was written as follows

$$
\langle \Phi_{j_1}(x_1|z_1)\Phi_{j_2}(x_2|z_2)\Phi_{j_3}(x_3|z_3)\Phi_{j_4}(x_4|z_4)\rangle = |z_{34}|^{2(\tilde\Delta_2+\tilde\Delta_1-\tilde\Delta_4-\tilde\Delta_3)}|z_{14}|^{2(\tilde\Delta_2+\tilde\Delta_3-\tilde\Delta_4-\tilde\Delta_1)}
$$
$$
\times\ |z_{24}|^{-4\tilde\Delta_2}\ |z_{13}|^{2(\tilde\Delta_4-\tilde\Delta_1-\tilde\Delta_2-\tilde\Delta_3)} \int_{\mathcal{P}^+} dj\ \mathcal{A}(j_i,j)\ \mathcal{G}_j(j_i,z,\overline{z},x_i,\overline{x}_i)\ |z|^{2(\Delta_j-\Delta_1-\Delta_2)}. \tag{5.1.1}
$$

Here

$$
\mathcal{A}(j_i,j) = C(-j_1,-j_2,-j)B(-j-1)C(-j,-j_3,-j_4) \tag{5.1.2}
$$

and

$$
\mathcal{G}_j(j_i,z,\overline{z},x_i,\overline{x}_i) = \sum_{n,\overline{n}=0}^{\infty} z^n \overline{z}^{\overline{n}} D^{(n)}_{x,j}(j_i,x_i)\overline{D}^{\overline{n}}_{\overline{x},j}(j_i,\overline{x}_i)G_j(j_i,x_i,\overline{x}_i)\,, \tag{5.1.3}
$$

where $D^{(n)}_{x,j}(j_i,x_i)$ are differential operators containing the contributions from intermediate descendant states and

$$
\begin{aligned}
G_j(j_i,x_i,\overline{x}_i) \quad &= \quad |x_{12}|^{2(j_1+j_2-j)}|x_{34}|^{2(j_3+j_4-j)}\int d^2x d^2x' |x_1-x|^{2(j_1+j-j_2)}|x_2-x|^{2(j_2+j-j_1)} \\
&\times\ |x_3-x'|^{2(j_3+j-j_4)}|x_4-x'|^{2(j_4+j-j_3)}|x-x'|^{-4j-4}\,,
\end{aligned} \tag{5.1.4}
$$

which may be rewritten as

$$
\begin{aligned}
G_j(j_i,x_i,\overline{x}_i) \quad &= \quad \frac{\pi^2}{(2j+1)^2}|x_{34}|^{2(j_4+j_3-j_2-j_1)}|x_{24}|^{4j_2}|x_{14}|^{2(j_4+j_1-j_2-j_3)}|x_{13}|^{2(j_3+j_2+j_1-j_4)} \\
&\times\ \left\{|F_j(j_i,x)|^2 + \frac{\gamma(1+j+j_4-j_3)\gamma(1+j+j_3-j_4)}{\gamma(2j+1)\gamma(j_1-j_2-j)\gamma(j_2-j_1-j)}|F_{-1-j}(j_i,x)|^2\right\}\,,
\end{aligned}
$$

with $F_j(j_i,x) \equiv x^{j_1+j_2-j}{}_2F_1(j_1-j_2-j,j_4-j_3-j;-2j;x)$ and $x = \frac{x_{12}x_{34}}{x_{13}x_{24}}$.

The properties of (5.1.1) under $j \to -1-j$ allow to extend the integration contour from $\mathcal{P}^+$ to the full axis $\mathcal{P} = -\frac{1}{2} + i\mathbb{R}$ and rewrite it in a holomorphically factorized form. Crossing symmetry follows from similar properties of a five-point function in Liouville theory to which this model is closely relates and it amounts to establishing the consistency of the H_3^+ WZNW model [88].

5.2 Analytic continuation to AdS_3

Expression (5.1.1) is valid for external states Φ_{j_1}, Φ_{j_2} in the range (4.1.3) and similarly for Φ_{j_3}, Φ_{j_4}. In particular, it holds for operators in continuous representations of the AdS_3 WZNW model. The analytic continuation to other values of j_i was performed in [40]. In this process, some poles in the integrand cross the integration contour and the four-point function is defined as (5.1.1) plus the contributions of all these poles. This procedure allowed to analyze the factorization of four-point functions of $w = 0$ short strings in the boundary conformal field theory, obtained from primary states in discrete representations $\mathcal{D}_j^{w=0} \otimes \mathcal{D}_j^{w=0}$, by integrating over the world-sheet moduli. It is important to stress that the aim in [40] was to study the factorization in the boundary conformal field theory with coordinates $x_i, \overline{x}_i$, so the $x-$basis was found convenient. The conformal blocks were expanded in powers of the cross ratios $x, \overline{x}$ and then integrated over the worldsheet coordinates $z, \overline{z}$. To study the factorization in the AdS_3 WZNW model instead, we expand the conformal blocks in powers of $z, \overline{z}$, and in order to consider the various sectors, we find convenient to translate (5.1.1) to the $m-$basis.

To this purpose, one can verify that the integral over j commutes with the integrals over $x_i, \overline{x}_i$, $i = 1, \dots, 4$ and that it is regular for $j_{21}^{\pm}$ and $j_{43}^{\pm}$ in the range (4.1.3) and for all of $|m|, |\overline{m}|, |m_i|, |\overline{m}_i| < \frac{1}{2}$, where we have introduced $m = m_1 + m_2 = -m_3 - m_4$, $\overline{m} = \overline{m}_1 + \overline{m}_2 = -\overline{m}_3 - \overline{m}_4$. Integrating in addition over x and x' in (5.1.4), we get

$$\left\langle \Phi^{j_1}_{m_1,\overline{m}_1} \Phi^{j_2}_{m_2,\overline{m}_2} \Phi^{j_3}_{m_3,\overline{m}_3} \Phi^{j_4}_{m_4,\overline{m}_4} \right\rangle = |z_{34}|^{2(\tilde{\Delta}_2+\tilde{\Delta}_1-\tilde{\Delta}_4-\tilde{\Delta}_3)} |z_{14}|^{2(\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\Delta}_4-\tilde{\Delta}_1)} |z_{24}|^{-4\tilde{\Delta}_2}$$
$$\times \ |z_{13}|^{2(\tilde{\Delta}_4-\tilde{\Delta}_1-\tilde{\Delta}_2-\tilde{\Delta}_3)} \int_{\mathcal{P}^+} dj \ \mathbb{A}_j^{w=0}(j_i; m_i, \overline{m}_i) \ |z|^{2(\tilde{\Delta}_j-\tilde{\Delta}_1-\tilde{\Delta}_2)} + \cdots \ , \quad (5.2.1)$$

where

$$\mathbb{A}_j^{w=0}(j_i; m_i, \overline{m}_i) \ = \ \delta^{(2)}(m_1 + \cdots + m_4) \ C(1+j_1, 1+j_2, 1+j) \ W \begin{bmatrix} j_1 \, , j_2 , \ j \\ m_1, m_2, -m \end{bmatrix}$$
$$\times \ \frac{1}{B(-1-j) \ c_{m,\overline{m}}^{-1-j}} \ C(1+j_3, 1+j_4, 1+j) \ W \begin{bmatrix} j_3 \, , j_4 , \ j \\ m_3, m_4, m \end{bmatrix} . \quad (5.2.2)$$

An alternative representation of (5.2.2) was found in [86] in terms of higher generalized hypergeometric functions ${}_4F_3$. This new identity among hypergeometric functions is an interesting by-product of the present result.

The dots in (5.2.1) refer to higher powers of $z, \overline{z}$ corresponding to the integration of terms of the form $\mathbb{A}_j^{N,w=0}|z|^{2(\Delta_j^{(N)}-\tilde{\Delta}_1-\tilde{\Delta}_2)}$, where $\mathbb{A}_j^{N,w=0}$, $N = 1, 2, 3, \dots$ stand for contributions from descendant operators at level N with conformal weights $\Delta_j^{(N)} = \tilde{\Delta}_j + N$.

Notice that the symmetry under $j \leftrightarrow -1-j$ in (5.2.2), which can be easily checked by using the identity (4.2.12), allows to extend the integral to the full axis $\mathcal{P} = -\frac{1}{2} + i\mathbb{R}$.

Given that correlation functions in the AdS_3 WZNW model in the $m-$basis depend on the sum of w_i numbers, except for the powers of the coordinates $z_i, \overline{z}_i$, if the Lorentzian and Euclidean theories are simply related by analytic continuation, this result should hold, in particular, for states in continuous representations in arbitrary spectral flow sectors (with $|m_i|, |\overline{m}_i|, |m| < \frac{1}{2}$), as long as $\sum_i w_i = 0$, *i.e.*

$$\left\langle \Phi^{j_1,w_1}_{m_1,\overline{m}_1} \Phi^{j_2,w_2}_{m_2,\overline{m}_2} \Phi^{j_3,w_3}_{m_3,\overline{m}_3} \Phi^{j_4,w_4}_{m_4,\overline{m}_4} \right\rangle_{\sum_{i=1}^4 w_i=0} = z_{34}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3} z_{14}^{\Delta_2+\Delta_3-\Delta_4-\Delta_1} z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3}$$
$$\times\ z_{24}^{-2\Delta_2} \times\ c.c.\ \times \int_{\mathcal{P}} dj\ \mathbb{A}_j^{w=0}(j_i; m_i, \overline{m}_i)\ z^{\Delta_j-\Delta_1-\Delta_2} \overline{z}^{\overline{\Delta}_j-\overline{\Delta}_1-\overline{\Delta}_2} + \cdots , \quad (5.2.3)$$

where $\Delta_j = -\frac{j(j+1)}{k-2} - m(w_1+w_2) - \frac{k}{4}(w_1+w_2)^2$ and *c.c.* stands for the obvious antiholomorphic $\overline{z}_i-$dependence. For other values of $j_1, \cdots, j_4$, $m_1, \dots, \overline{m}_4$ the integral may diverge and must be defined by analytic continuation.

That a generic $w-$conserving four-point function involving primaries or highest/lowest-weight states in $\mathcal{C}_j^{\alpha,w}$ or $\mathcal{D}_j^{\pm,w}$ should factorize as in (5.2.3), if the amplitude with four $w = 0$ states is given by (5.2.1), can be deduced from the relation [84]:

$$\left\langle \prod_{i=1}^n \Phi^{j_i,w_i}_{m_i,\overline{m}_i}(z_i, \overline{z}_i) \right\rangle_{\sum_{i=1}^n w_i=0} = \kappa\overline{\kappa} \left\langle \prod_{i=1}^n \Phi^{j_i,\widetilde{w}_i=0}_{m_i,\overline{m}_i}(z_i, \overline{z}_i) \right\rangle , \quad (5.2.4)$$

where $\kappa = \displaystyle\prod_{i<j} z_{ij}^{-w_i m_j - w_j m_i - \frac{k}{2} w_i w_j}$, $\overline{\kappa} = \displaystyle\prod_{i<j} z_{ij}^{-w_i \overline{m}_j - w_j \overline{m}_i - \frac{k}{2} w_i w_j}$, after Taylor expanding around

$z=0$ the r.h.s. of the following identity:

$$\kappa\ \ z_{34}^{\tilde{\Delta}_2+\tilde{\Delta}_1-\tilde{\Delta}_4-\tilde{\Delta}_3} z_{14}^{\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\Delta}_4-\tilde{\Delta}_1} z_{24}^{-2\tilde{\Delta}_2} z_{13}^{\tilde{\Delta}_4-\tilde{\Delta}_1-\tilde{\Delta}_2-\tilde{\Delta}_3} z^{\tilde{\Delta}_j-\tilde{\Delta}_1-\tilde{\Delta}_2} = \\ z_{34}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3} z_{14}^{\Delta_2+\Delta_3-\Delta_4-\Delta_1} z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3} z_{24}^{-2\Delta_2} z^{\Delta_j-\Delta_1-\Delta_2} (1-z)^{-m_2w_3-m_3w_2-\frac{k}{2}w_2w_3} . \tag{5.2.5}$$

The conclusion is that, if the H_3^+ and AdS_3 models are simply related by analytic continuation, then (5.2.3) and its analytic continuation should hold for generic $w-$conserving four-point functions of fields in $\mathcal{C}_j^{\alpha,w}$ or $\mathcal{D}_j^{\pm,w}$ [1]. However, expression (5.2.3) appears to be in contradiction with the factorization *ansatz* and the OPE found in section 4.2.2 for the AdS_3 WZNW model, because it seems to contain just $w-$conserving channels. Actually, directly applying the factorization *ansatz* based on the OPE (4.2.13) would give the following expression for both $w-$conserving and violating four-point functions:

$$\left\langle \Phi^{j_1,w_1}_{m_1,\overline{m}_1}\Phi^{j_2,w_2}_{m_2,\overline{m}_2}\Phi^{j_3,w_3}_{m_3,\overline{m}_3}\Phi^{j_4,w_4}_{m_4,\overline{m}_4}\right\rangle \sim z_{34}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3} z_{14}^{\Delta_2+\Delta_3-\Delta_4-\Delta_1} z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3} z_{24}^{-2\Delta_2} \times c.c. \\ \times\ \delta^2\Big(\sum_{i=1}^4 m_i+\frac{k}{2}w_i\Big)\sum_{w=-1}^{1}\int_{\mathcal{P}} dj\ Q^wQ^{-w-\sum_{i=1}^4 w_i}B(-1-j)c_{m,\overline{m}}^{-1-j} z^{\Delta_j-\Delta_1-\Delta_2}\overline{z}^{\overline{\Delta}_j-\overline{\Delta}_1-\overline{\Delta}_2}+\cdots \tag{5.2.6}$$

with $m=m_1+m_2-\frac{k}{2}w=-m_3-m_4-\frac{k}{2}w$, $\overline{m}=\overline{m}_1+\overline{m}_2-\frac{k}{2}w=-\overline{m}_3-\overline{m}_4-\frac{k}{2}w$ and $\Delta_j=-\frac{j(j+1)}{k-2}-m(w_1+w_2+w)-\frac{k}{4}(w_1+w_2+w)^2$ (similarly for $\overline{\Delta}_j$). Actually, in the $m-$basis, the starting point for the $w-$conserving four-point function would have been (5.2.3) plus an analogous contribution involving one unit spectral flow three-point functions, *i.e.* (5.2.3) rewritten in terms of $\mathbb{A}_j^{w=1}$ or $\mathbb{A}_j^{w=-1}$ instead of $\mathbb{A}_j^{w=0}$, where

$$\mathbb{A}_j^{w=\pm1}(j_i;m_i,\overline{m}_i) = \delta^{(2)}\Big(\sum_{i=1}^4 m_i\Big)\frac{\widetilde{C}(1+j_1,1+j_2,1+j)}{\gamma(j_1+j_2+j+3-\frac{k}{2})}\widetilde{W}\begin{bmatrix} j_1\ , & j_2\ , & j \\ \mp m_1, & \mp m_2, & \pm m\end{bmatrix} \\ \times\ \frac{1}{B(-1-j)c_{m,\overline{m}}^{-1-j}}\frac{\widetilde{C}(1+j_3,1+j_4,1+j)}{\gamma(j_3+j_4+j+3-\frac{k}{2})}\widetilde{W}\begin{bmatrix} j_3\ , & j_4\ , & j \\ \pm m_3, & \pm m_4, & \pm m\end{bmatrix} \tag{5.2.7}$$

[1]See section 5.3 for an alternative discussion directly in the $m-$basis, independent of the $x-$basis.

But if correlation functions in this model are to be obtained from those in the H_3^+ model [40]-[42], [76]-[79], spectral flow conserving and non-conserving channels should give the same result for the $w-$conserving four-point functions. This does not imply that $\mathbb{A}_j^{w=0}$ and $\mathbb{A}_j^{w=\pm1}$ carry the same amount of information [2]. In general, if both expressions for the four-point functions were equivalent, one would expect that part of the information in $\mathbb{A}_j^{w=0}$ were contained in $\mathbb{A}_j^{w=\pm1}$ and the rest in the contributions from descendants in $\mathbb{A}_j^{N,w=\pm1}$.

A proof of this statement would require making explicit the higher order terms and possibly some contour manipulations, which we shall not attempt. Nevertheless there are several indications supporting this claim. A similar proposition was advanced in [84] for the H_3^+ model and some evidence was given that these possibilities might not be exclusive, depending on which correlator the OPE is inserted in. Furthermore, $w = 1$ long strings were found in the $s-$channel factorization of the four-point amplitude of $w = 0$ short strings in [40] starting from the holomorphically factorized expression for (5.1.1), rewriting the integrand and moving the integration contour. Moreover, in the $m-$basis, spectral flow non-conserving channels can be seen to appear naturally from (5.2.3) in certain special cases, as we now show.

Identities among different expansions of four-point functions containing at least one field in discrete representations can be generated using the spectral flow symmetry. In particular, $w-$conserving four-point functions involving the fields $\Phi^{j_1,w_1}_{m_1=\overline{m}_1=-j_1}$ and $\Phi^{j_3,w_3}_{m_3=\overline{m}_3=j_3}$ coincide (up to $B(j_1), B(j_3)$ factors) with the $w-$conserving amplitudes involving $\Phi^{j_1'=-\frac{k}{2}-j_1,w_1'=w_1+1}_{m_1'=\overline{m}_1'=j_1'}$ and $\Phi^{j_3'=-\frac{k}{2}-j_3,w_3'=w_3-1}_{m_3'=\overline{m}_3'=-j_3'}$ [3]. This allows to expand the four-point amplitude in two alternative ways, namely

$$\int_{\mathcal{P}} dj\ \mathbb{A}_j^{w=0}(j_1,j_2,j_3,j_4;m_1,\ldots,\overline{m}_3,\overline{m}_4)\ z^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}\overline{z}^{\overline{\Delta}(j)-\overline{\Delta}(j_1)-\overline{\Delta}(j_2)}+\cdots \qquad (5.2.8)$$

[2]In other words, both expressions seem to give the same contribution in $w-$conserving four-point functions. However one cannot always use either one of them. In particular, this is not expected to hold for $w-$violating amplitudes.

[3]This is a consequence of the identities discussed in the paragraph containing equation (4.2.40) in the previous section.

or

$$\beta_{1,3}\int_{\mathcal{P}} dj\ \mathbb{A}_j^{w=0}(j_1',j_2,j_3',j_4;m_1',\dots,\overline{m}_3',\overline{m}_4)\ z^{\Delta'(j)-\Delta(j_1')-\Delta(j_2)}\overline{z}^{\overline{\Delta}'(j)-\overline{\Delta}(j_1')-\overline{\Delta}(j_2)}+\cdots, \qquad (5.2.9)$$

where $\beta_{1,3}\equiv\frac{B(-1-j_3)}{B(-1-j_1')}$ and the dots refer to contributions from descendants and, in addition, to residues at poles in $\mathbb{A}_j^{w=0}$ crossing $\mathcal{P}$ after analytic continuation of j_i ($i=1,3$ and eventually $2,4$) to the region (4.2.4). Explicitly, $\mathbb{A}_j^{w=0}(j_1',j_2,j_3',j_4;m_1',\dots,\overline{m}_3',\overline{m}_4)$ is given by

$$C(1+j_1',1+j_2,1+j)C(1+j_3',1+j_4,1+j)\frac{\pi^3\gamma(2+2j)}{B(-1-j)}\frac{\gamma(j-j_1'-j_2)\gamma(j_2-j_1'-j)}{\gamma(2+j_1'+j_2+j)\gamma(-2j_1')}$$

$$\times\frac{\gamma(j-j_3'-j_4)\gamma(j_4-j_3'-j)}{\gamma(2+j_3'+j_4+j)\gamma(-2j_3')}\frac{\Gamma(1+j_2-m_2)\Gamma(1+j_4+\overline{m}_4)}{\Gamma(-j_2+\overline{m}_2)\Gamma(-j_4-m_4)}\frac{\Gamma(-j-\overline{m})\Gamma(1+j-m)}{\Gamma(1+j+m)\Gamma(-j+\overline{m})}.$$

Using (4.2.40) and rewriting this expression in terms of j_i, m_i, the following equivalence can be shown

$$(5.2.9)=\int_{\mathcal{P}} dj\ \mathbb{A}_j^{w=1}(j_1,j_2,j_3,j_4;m_1,\dots,\overline{m}_3,\overline{m}_4)\ z^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}\overline{z}^{\overline{\Delta}(j)-\overline{\Delta}(j_1)-\overline{\Delta}(j_2)}+\cdots \qquad (5.2.10)$$

Notice that not only the coefficient $\mathbb{A}_j^{w=1}$ but also the $z_i,\overline{z}_i$ dependence are as expected. In fact, $\Delta(j_1')=\tilde{\Delta}(j_1')-m_1'w_1'-\frac{k}{4}{w_1'}^2=\tilde{\Delta}(j_1)-m_1w_1-\frac{k}{4}w_1^2=\Delta(j_1)$ and $\Delta'(j)=\tilde{\Delta}(j)-(m_1'+m_2)(w_1'+w_2)-\frac{k}{4}(w_1'+w_2)^2=\tilde{\Delta}(j_1)-mw-\frac{k}{4}w^2=\Delta(j_1)$, where $m=m_1+m_2-\frac{k}{2}$ and $w=w_1+w_2+1$. Therefore, we have seen in a particular example that spectral flow conserving and violating channels can give the same result for four-point functions. This is a nontrivial result showing that the spectral flow symmetry allows to exhibit $w-$non-conserving channels that are not equivalent to other $w-$conserving ones in expressions constructed as sums over $w-$conserving exchanges.

In section 5.3 we show that the terms explicitly displayed in both (5.2.3) and (5.2.10) are solutions of the Knizhnik-Zamolodchikov (KZ) equations. However, these equations do not give enough information to confirm that the full expressions (5.2.3) and (5.2.10) are equivalent.

The factorization of four-point functions reproduces the field content of the OPE. Therefore, the truncation imposed on the operator algebra by the spectral flow symmetry must be realized in physical amplitudes. Again, to confirm this would require more information on the contributions from descendant fields and studying crossing symmetry. Here, we just illustrate this point with one example. Take for instance the following four-point function [4]:

$$\left\langle \mathcal{D}_{j_1}^{+,w_1=0} \mathcal{D}_{j_2}^{+,w_2=-1} \mathcal{D}_{j_3}^{-,w_3=0} \mathcal{D}_{j_4}^{-,w_4=-1} \right\rangle , \tag{5.2.11}$$

in the particular case with $n_i = 0, \forall i$ (where $m_i = \pm j_i \mp n_i$) and $j_1 + j_2 = j_3 + j_4 < -\frac{k-1}{2}$. The OPE (4.2.33) implies one intermediate state in the $s-$channel in $\mathcal{D}_j^{+,w=-1}$, with $j = j_1 + j_2 = -m$ as well as exchanges of states in $\mathcal{D}_j^{+,w=0}$ if $j_1 + j_2 = j_3 + j_4 < -\frac{k+1}{2}$ with $j = j_1 + j_2 + \frac{k}{2} + n$, $n = 0, 1, 2, \ldots$ such that $j < -\frac{1}{2}$, and also of continuous states in $\mathcal{C}_j^{\alpha,w=0}$. The unique state found in $\mathcal{D}_j^{+,w=-1}$ is equivalent to the highest-weight state in $\mathcal{D}_{\tilde{j}}^{-,w=0}$ with $\tilde{j} = -\frac{k}{2} - j > -\frac{1}{2}$.

This four-point function must coincide with the following one:

$$\left\langle \mathcal{D}_{j_1}^{+,w_1=0} \mathcal{D}_{\tilde{j}_2}^{-,w_2=0} \mathcal{D}_{j_3}^{-,w_3=0} \mathcal{D}_{\tilde{j}_4}^{+,w_4=0} \right\rangle , \tag{5.2.12}$$

where as usual $\tilde{j}_i = -\frac{k}{2} - j_i$ (notice that this holds without "hats" because $n_i = 0, \forall i$). Now $\tilde{j}_2 - j_1 = \tilde{j}_4 - j_3 > -\frac{1}{2}$. Therefore, (4.2.34) implies that only states from $\mathcal{C}_j^{\alpha,w=0}$ as well as from $\mathcal{D}_j^{+,w=0}$ with $j = j_1 - \tilde{j}_2 + n = j_1 + j_2 + \frac{k}{2} + n$ propagate in the intermediate $s-$channel, the latter requiring the extra condition $\tilde{j}_2 - j_1 = \tilde{j}_4 - j_3 > \frac{1}{2}$, *i.e.* $j_1 + j_2 = j_3 + j_4 < -\frac{k+1}{2}$. The important remark is that no intermediate states from $\mathcal{D}_{\tilde{j}}^{-,w=0}$ appear in the factorization. This behavior was discussed in the previous chapter when studying the consequences of the spectral flow symmetry on the OPE. However, we have considered this case carefully here because it explicitly displays the fact that the same four-point function factorizes in two different ways and the unique difference is an extra state violating the bounds (4.2.4). Recall that we are only considering primaries and their spectral flow images. We expect that some consistency requirements, such as crossing symmetry, will automatically realize the OPE displayed in the previous chapter in physical amplitudes.

An indication in favor of the factorization of this non-rational CFT is that the expressions

[4]Here, as in the previous chapter, we denote the states by the representations they belong to and we omit the antiholomorphic part for short.

reproduce the spectral flow selection rules (4.2.1) and (4.2.2) for four-point functions in different sectors. Indeed, let us analyze this feature in a four-point function involving only external discrete states or their spectral flow images. The bounds (4.2.2) require $-3 \leq \sum_{i=1}^{4} w_i \leq -1$, in agreement with the factorization of this amplitude in any channel. Indeed, consider for instance

$$\left\langle \widehat{\mathcal{D}}_{j_1}^{+,w_1} \widehat{\mathcal{D}}_{j_2}^{+,w_2} \widehat{\mathcal{D}}_{j_3}^{+,w_3} \widehat{\mathcal{D}}_{j_4}^{+,w_4} \right\rangle . \tag{5.2.13}$$

The OPE $\widehat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \widehat{\mathcal{D}}_{j_2}^{+,w_2}$ computed in the previous chapter (and similarly for j_3, j_4) requires either $w_1 + w_2 = -w_3 - w_4 - 1$ or $w_1 + w_2 = -w_3 - w_4 - 2$ or $w_1 + w_2 = -w_3 - w_4 - 3$ for discrete intermediate states and $w_1 + w_2 = -w_3 - w_4 - 2$ for continuous intermediate states. And similarly in the other channels.

Repeating this analysis for four-point functions involving fields in different representations, it is straightforward to conclude that the spectral flow selection rules for four-point functions in different sectors can be obtained from those for two- and three-point functions, or equivalently from the OPE found in chapter 4.

5.2.1 Relation to [40]

This section contains some comments about the relation between our work and [40]. For simplicity, we use the conventions of the latter, related to ours by $j \to -j$ in the $x-$basis, up to normalizations. The range of j for discrete representations is now $\frac{1}{2} < j < \frac{k-1}{2}$ and for continuous representations, $j = \frac{1}{2} + i\mathbb{R}$.

One of the aims of [40] was to study the factorization of four-point functions involving $w = 0$ short strings in the boundary conformal field theory. The $x-$basis seems appropriate for this purpose since $x_i, \overline{x}_i$ can be interpreted as the coordinates of the boundary. Naturally, both the OPE and the factorization look very different in the $m-$ and $x-$basis. For instance, it is not obvious how discrete series would appear in the OPE or factorization of fields in continuous representations if they are to be obtained from the analogous expressions in the H_3^+ model in the $x-$basis. However, when discrete representations are involved, there are certain similarities. Actually, in agreement with the fusion rules $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$ obtained in the previous chapter,

$w = 1$ long strings and $w = 0$ short strings were found in the factorization studied in [40]. Conversely, it was interpreted that $w = 1$ short strings do not propagate in the intermediate channels, while we found spectral flow non-preserving contributions of discrete representations in the OPE. In this section we analyze this issue. We reexamine the three-point functions involving two $w = 0$ strings and one $w = 1$ short string and certain divergences in the four-point functions of $w = 0$ short strings, namely the so-called Poles$_2$, which seem to break the factorization.

- *Three-point functions involving one $w = 1$ short string and two $w = 0$ strings*

The $w-$conserving two-point functions of short strings in the target space ($w \geq 0$) are given by

$$\langle \Phi^{\omega,j}_{J,\bar{J}}(x_1,\overline{x}_1)\Phi^{\omega,j}_{J,\bar{J}}(x_2,\overline{x}_2)\rangle \sim |2j-1\pm(k-2)\omega|\frac{\Gamma(2j+p)\Gamma(2j+\bar{p})}{\Gamma(2j)^2 p!\bar{p}!}\frac{\mathcal{B}(j)}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}}\,, \qquad (5.2.14)$$

where $\mathcal{B}(j) = B(-j)$ and the upper (lower) sign holds for $J = j + p + \frac{k}{2}w$ ($J = -j - p + \frac{k}{2}w$), $p, \overline{p}$ being non-negative integers. Three-point functions of $w = 0$ string states are

$$\langle \Phi_{j_1}(x_1,\overline{x}_1)\Phi_{j_2}(x_2,\overline{x}_2)\Phi_{j_3}(x_3,\overline{x}_3)\rangle = C(j_1,j_2,j_3)\prod_{i>j}|x_{ij}|^{-2j_{ij}}\,, \qquad (5.2.15)$$

and for one $w = 1$ short string and two $w = 0$ strings they are given by (we omit the $x, \overline{x}-$dependence)

$$\begin{aligned}\langle \Phi^{j_1,\omega=1}_{J_1,\bar{J}_1}(x_1,\overline{x}_1)\Phi_{j_2}(x_2,\overline{x}_2)\Phi_{j_3}(x_3,\overline{x}_3)\rangle &\sim \frac{1}{\Gamma(0)}\mathcal{B}(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)\times\\ &\frac{\Gamma(j_2+j_3-J_1)}{\Gamma(1-j_2-j_3+\bar{J}_1)}\frac{\Gamma(j_1+J_1-\frac{k}{2})}{\Gamma(1-j_1-\bar{J}_1+\frac{k}{2})}\frac{1}{\gamma(j_1+j_2+j_3-\frac{k}{2})}\,. \qquad (5.2.16)\end{aligned}$$

The $\Gamma(0)^{-1}$ factor is absent when the $w = 1$ operator is a long string state. This three-point function was obtained in [40] from an equivalent expression in the $m-$basis. $J_1, \overline{J}_1$ label the global AdS_3 representations and can be written in terms of parameters $m_1, \overline{m}_1$ as $J_1 = \mp m_1 + \frac{k}{2}$, $\overline{J}_1 = \mp\overline{m}_1 + \frac{k}{2}$, depending if the correlator involved the field $\Phi^{j_1,w_1=\mp 1}_{m_1,\overline{m}_1}$.

As observed in [40], when $J_1 = \frac{k}{2} - j_1 - p$, $\overline{J}_1 = \frac{k}{2} - j_1 - \overline{p}$, the factor $\frac{\Gamma(j_1+J_1-\frac{k}{2})}{\Gamma(1-j_1-\overline{J}_1+\frac{k}{2})}$ cancels the $\Gamma(0)$ and the three-point function is finite and can be interpreted as a $w-$conserving amplitude.

To see this, recall that if it was obtained from a $w=-1$ three-point function in the $m-$basis and $m_1 = j_1 + p$, then

$$\langle \Phi^{j_1,w=1}_{J_1,\bar{J}_1}(x_1,\overline{x}_1)\Phi_{j_2}(x_2,\overline{x}_2)\Phi_{j_3}(x_3,\overline{x}_3)\rangle \sim (-)^{p+\bar{p}}\mathcal{B}(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)\times \frac{\Gamma(j_2+j_3+j_1-\frac{k}{2}+p)}{p!\Gamma(j_2+j_3+j_1-\frac{k}{2})}\frac{\Gamma(j_2+j_3+j_1-\frac{k}{2}+\bar{p})}{\bar{p}!\Gamma(j_2+j_3+j_1-\frac{k}{2})} \qquad (5.2.17)$$

reduces to (5.2.15) when $p=\overline{p}=0$ and $j_1 \to \frac{k}{2}-j_1$, as expected from spectral flow symmetry. Similarly, if $w=+1$ and $m_1=-j_1-p$, the same interpretation holds.

On the contrary, for $w=-1$ ($w=+1$) and $m_1=-j_1-p$ ($m_1=j_1+p$), the $\Gamma(j_1+J_1-\frac{k}{2})$ does not cancel the factor $\Gamma(0)^{-1}$ and then, it was concluded in [40] that the three-point function vanishes in this case.

However, notice that if $J_1=\frac{k}{2}+j_1+n=j_2+j_3+p$, $\overline{J}_1=\frac{k}{2}+j_1+\overline{n}=j_2+j_3+\overline{p}$, $n,\overline{n}\in\mathbb{Z}_{\geq 0}$, the r.h.s. of (5.2.16) can also be rewritten as the r.h.s. of (5.2.17), but now this non-vanishing amplitude corresponds to a $w=1$ three-point function which is not equivalent to a $w-$conserving one. Indeed, (5.2.17) is regular as long as $n<p$ ($\overline{n}<\overline{p}$) and when $n\geq p$ ($\overline{n}\geq\overline{p}$) there are divergences in $C(\frac{k}{2}-j_1,j_2,j_3)$ at $j_1=j_2+j_3-\frac{k}{2}-q$ with $q=0,1,2,\cdots$. Using the spectral flow symmetry, the $w=1$ short string can be identified with a $w=2$ short string with $\tilde{j}_1=\frac{k}{2}-j_1=k-j_2-j_3+q$, which correspond to the Poles$_2$ in [40].

- *Factorization of four-point functions of $w=0$ short strings*

The four-point amplitude of $w=0$ short strings was extensively studied in [40]. The conformal blocks were rearranged as sums of products of positive powers of x times functions of $u=z/x$. In order to perform the integral over the worldsheet before the $j-$integral, it was necessary to change the $j-$integration contour from $\frac{1}{2}+i\mathbb{R}$ to $\frac{k-1}{2}+i\mathbb{R}$, and in this process two types of sequences of poles were picked up, namely

$$\begin{aligned} \text{Poles}_1: &\quad j_3=j_1+j_2+n,\\ \text{Poles}_2: &\quad j_3=k-j_1-j_2+n, \end{aligned}$$

where $n = 0, 1, 2, \ldots$. Only values of n for which $j_3 < \frac{k-1}{2}$ contribute to the factorization, so Poles$_1$ appear when $j_1 + j_2 < \frac{k-1}{2}$ and Poles$_2$ when $j_1 + j_2 > \frac{k+1}{2}$. The contributions from Poles$_1$ were identified as two particle states of short strings in the boundary conformal field theory, but no interpretation was found for Poles$_2$ as $s-$channel exchange.

Recall that we found Poles$_1$ among the $w-$conserving discrete contributions to the OPE $\mathcal{D}_{j_i}^{+,w_i} \times \mathcal{D}_{j_i}^{+,w_i}$ (see (4.2.33)) and Poles$_2$ in the $w-$violating terms with $\tilde{j}_3 = \frac{k}{2} - j_3 = j_1 + j_2 - \frac{k}{2} - n$. Therefore, it seems tempting to consider Poles$_2$ as two particle states of $w = 1$ short strings in the boundary conformal field theory. However, neither the powers of $x, \overline{x}$ nor the residues of the poles in the four-point function studied in [40] allow this interpretation and thus the Poles$_2$ had to be truncated. Clearly, more work is necessary to determine the four-point function and understand the factorization.

5.3 Knizhnik-Zamolodchikov equation

5.3.1 KZ equations in the $m-$basis and the factorization *ansatz*

In this section we show some consistency conditions of the expressions used in the previous section.

Let us start by considering the KZ equation for $w-$conserving $n-$point functions in the $m-$basis, namely [84]

$$\mathcal{E}_i \ \kappa^{-1} \left\langle \prod_{\ell=1}^{n} \Phi_{m_\ell, \overline{m}_\ell}^{j_\ell, w_\ell}(z_\ell, \overline{z}_\ell) \right\rangle = 0, \tag{5.3.1}$$

where

$$\mathcal{E}_i \equiv (k-2)\frac{\partial}{\partial z_i} + \sum_{j \neq i} \frac{Q_{ij}}{z_{ji}}, \qquad Q_{ij} = -2t_i^3 t_j^3 + t_i^- t_j^+ + t_i^+ t_j^-, \tag{5.3.2}$$

t^a are defined by $\tilde{J}_0^a|j, m, \overline{m}, w >= -t^a|j, m, \overline{m}, w >$, $|j, m, \overline{m}, w >$ being the state corresponding to the field $\Phi_{m,\overline{m}}^{j,w}$ and κ was introduced in (5.2.4).

Since a generic $w-$conserving four-point function can be obtained from the expression involving four $w=0$ fields, we concentrate on

$$\left\langle \prod_{i=1}^{4} \Phi^{j_i, w_i=0}_{m_i,\overline{m}_i}(z_i,\overline{z}_i) \right\rangle = |z_{34}|^{2(\tilde{\Delta}_2+\tilde{\Delta}_1-\tilde{\Delta}_4-\tilde{\Delta}_3)}|z_{14}|^{2(\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\Delta}_4-\tilde{\Delta}_1)}|z_{13}|^{2(\tilde{\Delta}_4-\tilde{\Delta}_1-\tilde{\Delta}_2-\tilde{\Delta}_3)} \times |z_{24}|^{-4\tilde{\Delta}_2}\, \mathcal{F}_j(z,\overline{z}) \,,$$

$\mathcal{F}_j(z,\overline{z})$ being a function of the cross ratios $z,\overline{z}$, not determined by conformal symmetry. The KZ equation (5.3.1) implies the following constraint

$$\frac{\partial \mathcal{F}_j(z,\overline{z})}{\partial z} = \frac{1}{k-2}\left[\frac{Q_{21}}{z} + \frac{Q_{23}}{z-1}\right]\mathcal{F}_j(z,\overline{z})\,. \tag{5.3.3}$$

Assuming that $\mathcal{F}_j(z,\overline{z})$ has the following form

$$\mathcal{F}_j(z,\overline{z}) = \sum_{N,\overline{N}=0}^{\infty} \int dj \left\{ A_j^{(N,\overline{N})} \begin{bmatrix} j_1\,,\, j_2\,,\, j_3\,,\, j_4 \\ m_1, m_2, \dots, \overline{m}_4 \end{bmatrix} z^{\Delta_j-\tilde{\Delta}_1-\tilde{\Delta}_2+N}\overline{z}^{\Delta_j-\tilde{\Delta}_1-\tilde{\Delta}_2+\overline{N}} \right\}, \tag{5.3.4}$$

inserting it into (5.3.3) with $\Delta_j = \tilde{\Delta}_j \equiv -\frac{j(1+j)}{k-2}$, then $A_j^{(0,0)} \begin{bmatrix} j_1\,,\, j_2\,,\, j_3\,,\, j_4 \\ m_1, m_2, \dots, \overline{m}_4 \end{bmatrix}$ satisfies

$$\begin{aligned} \{2m_1m_2 - j(1+j) + j_1(1+j_1) + j_2(1+j_2)\}\, A_j^{(0,0)} \begin{bmatrix} j_1\,,\, j_2\,,\, j_3\,,\, j_4 \\ m_1, m_2, \dots, \overline{m}_4 \end{bmatrix} &= \\ = (m_1-j_1)(m_2+j_2) A_j^{(0,0)} \begin{bmatrix} j_1\,,\, j_2\,,\, j_3\,,\, j_4 \\ m_1+1, m_2-1, \dots, \overline{m}_4 \end{bmatrix} & \\ + (m_1+j_1)(m_2-j_2) A_j^{(0,0)} \begin{bmatrix} j_1\,,\, j_2\,,\, j_3\,,\, j_4 \\ m_1-1, m_2+1, \dots, \overline{m}_4 \end{bmatrix} & . \end{aligned} \tag{5.3.5}$$

The equations relating coefficients $A_j^{(N,\overline{N})}$ with $N,\overline{N}\neq 0$, are much more complicated because they mix terms with different values of $m_i,\overline{m}_i$ with terms at different levels $N,\overline{N}$.

This equation does not have enough information to determine $A_j^{(0,0)}$ completely. So we just check that the expression found in (5.2.1) is consistent with an analysis performed directly in the $m-$basis. Inserting $A_j^{(0,0)}\begin{bmatrix} j_1\,,\, j_2\,,\, j_3\,,\, j_4 \\ m_1, m_2, \ldots, \overline{m}_4 \end{bmatrix} = \mathbb{A}_j^{w=0}(j_1,\ldots,j_4;m_1,\ldots,\overline{m}_4)$ into (5.3.5) reproduces the same equation with $A_j^{(0,0)}$ replaced by $W(j_1,j_2,j;m_1,m_2,m)$. Because of the complicated expressions known for W, we focus on the case in which one of the fields in the four-point function is a discrete primary, namely $\Phi^{j_1,w_1=0}_{m_1,\overline{m}_1} \in \mathcal{D}^{+,w=0}_{j_1}$. In this case, using (4.2.25) one can show that (5.3.5) is equivalent to

$$
\begin{aligned}
0 \;=\; & \sum_{n=0}^{n_1-1}(-)^n \binom{n_1}{n}\left[j-m+\frac{(m_1-j_1)(1+j_1+m_1)}{n_1+1-n}+\frac{(m_2-j_2)(1+j_2+m_2)(n_1-n)}{n+1+j+m-n_1}\right] \\
& \times \frac{\Gamma(n-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n+1+j+j_2-j_1)}{\Gamma(1+j+j_2-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j+m)}{\Gamma(n-n_1+1+j+m)} \\
- \; & (-)^{n_1}\left[m_1(1-m_1)+j_1(1+j_1)\right]\frac{\Gamma(n_1-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n_1+1+j+j_2-j_1)}{\Gamma(1+j+j_2-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n_1-2j_1)}
\end{aligned}
$$

where $n_1 = m_1 + j_1$ and $m = m_1 + m_2$. Using $m-$conservation this can be rewritten as

$$
\begin{aligned}
0 \;=\; & \sum_{n=0}^{n_1-1}(-)^n \binom{n_1}{n}\left[-n\frac{1-n+2j_1}{n_1+1-n}+\frac{(n-j_1-j_2+j)(n+1+j_2+j-j_1)}{n+1+j+m-n_1}\right] \\
& \times \frac{\Gamma(n-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j+m)}{\Gamma(n-n_1+1+j+m)} \\
- \; & (-)^{n_1}\left[m_1(1-m_1)+j_1(1+j_1)\right]\frac{\Gamma(n_1-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n_1+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n_1-2j_1)}\,.
\end{aligned}
$$

To see that this vanishes, it is sufficient to note that

$$\sum_{n=0}^{n_1-1}(-)^n\begin{pmatrix}n_1\\ n\end{pmatrix}\left[-n\frac{1-n+2j_1}{n_1+1-n}\right]\frac{\Gamma(n-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}$$

$$\times\ \frac{\Gamma(1+j+m)}{\Gamma(n-n_1+1+j+m)}$$

$$-(-)^{n_1}\left[m_1(1-m_1)+j_1(1+j_1)\right]\frac{\Gamma(n_1-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n_1+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n_1-2j_1)}$$

$$=-\sum_{\tilde{n}=0}^{n_1-1}(-)^{\tilde{n}}\begin{pmatrix}n_1\\ \tilde{n}\end{pmatrix}\left[\frac{(\tilde{n}-j_1-j_2+j)(\tilde{n}+1+j_2+j-j_1)}{\tilde{n}+1+j+m-n_1}\right]\frac{\Gamma(\tilde{n}-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}$$

$$\times\ \frac{\Gamma(\tilde{n}+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(\tilde{n}-2j_1)}\frac{\Gamma(1+j+m)}{\Gamma(\tilde{n}-n_1+1+j+m)}\,,$$

where $\tilde{n}=n-1$.

Let us now discuss the other possible *ansatz*, namely (5.2.7). To see that $\mathbb{A}_j^{w=1}$ also verifies the KZ equation, consider $\Delta_j=-\frac{j(1+j)}{k-2}-m-\frac{k}{4}$ and $m=m_1+m_2-\frac{k}{2}$ in (5.3.3). In this case, the equation to be satisfied by $A_j^{(0,0)}$, obtained by replacing (5.3.4) into (5.3.3), is the following:

$$\left\{2m_1m_2-j(1+j)+j_1(1+j_1)+j_2(1+j_2)-(k-2)(m_1+m_2-\frac{k}{4})\right\}A_j^{(0,0)}\begin{bmatrix}j_1\,,\,j_2\,,\,j_3\,,\,j_4\\ m_1,m_2,\ldots,\overline{m}_4\end{bmatrix}$$

$$\begin{aligned}=\ &(m_1-j_1)(m_2+j_2)A_j^{(0,0)}\begin{bmatrix}j_1\,,\,j_2\,,\,j_3\,,\,j_4\\ m_1+1,m_2-1,\ldots,\overline{m}_4\end{bmatrix}\\ &+(m_1+j_1)(m_2-j_2)A_j^{(0,0)}\begin{bmatrix}j_1\,,\,j_2\,,\,j_3\,,\,j_4\\ m_1-1,m_2+1,\ldots,\overline{m}_4\end{bmatrix}\\ &-(m_2-j_2)(m_3+j_3)A_j^{(0,0)}\begin{bmatrix}j_1\,,\,j_2\,,\,j_3\,,\,j_4\\ m_1,m_2+1,m_3-1,\ldots,\overline{m}_4\end{bmatrix}.\end{aligned}\qquad(5.3.6)$$

It is not difficult to check that $A_j^{(0,0)}\begin{bmatrix}j_1\,,\,j_2\,,\,j_3\,,\,j_4\\ m_1,m_2,\ldots,\overline{m}_4\end{bmatrix}=\mathbb{A}_j^{w=1}(j_1,\ldots,j_4;m_1,\ldots,\overline{m}_4)$ is a solution of this equation.

Obviously, $A_j^{w=-1}$ is also a solution of (5.3.3) when $\Delta_j = -\frac{j(1+j)}{k-2} + m - \frac{k}{4}$ and $m = m_1 + m_2 + \frac{k}{2}$.

Here, we have considered the simple case of four $w = 0$ fields. However, these results can be generalized for arbitrary $w-$conserving correlators using the identity (5.2.5).

Chapter 6

Characters on the Lorentzian torus

The spectrum of a quantum model is built up by appropriately choosing a subset of the representation spaces of the symmetry group. There are two quantities storing this information, the characters associated to each representation and the partition function storing the information of the full spectrum. Such a subset of representations is chosen by constraining the spectrum with physical conditions, *e.g.* unitarity.

A consistent CFT must be well defined independently of the boundary conditions imposed on the fields, which means that the theory can be defined in any Riemannian surface. Consistency is guarantied by imposing modular invariance on the surface. The particular case with periodic boundary conditions in the two directions gives place to a CFT defined on a torus topology and a well defined CFT on this surface requires the partition function to be invariant under $SL(2,\mathbb{Z})$, the modular group of the torus. This is the reason why a standard method, used in CFTs theories to determine the appropriate representations contained in the spectrum of the model is looking for modular invariant quantities, which are then interpreted as the partition function. In this chapter we will concentrate on the definition of characters of the Lorentzian AdS_3 WZNW model. The next chapter is devoted to discuss the issue of modular trnasformations.

The partition function of the AdS_3 WZNW model was computed on the Lorentzian torus in [37] because it diverges on the Euclidean signature torus, and it was shown that a modular invariant expression is obtained after analytic continuation of the modular parameters. In this chapter we rederive the characters of the relevant representations and stress some important issues related to the regions of convergence of the expressions involved, focussing on their structure as distributions.

The characters on the Lorentzian signature torus are defined from the standard expressions as

$$\chi_{\mathcal{V}_L}(\theta_-,\tau_-,u_-) = \mathrm{Tr}_{\mathcal{V}_L} e^{2\pi i\tau_-(L_0-\frac{c}{24})} e^{2\pi i\theta_- J_0^3} e^{\pi i u_- K}\,,$$
$$\chi_{\mathcal{V}_R}(\theta_+,\tau_+,u_+) = \mathrm{Tr}_{\mathcal{V}_R} e^{2\pi i\tau_+(\bar{L}_0-\frac{\bar{c}}{24})} e^{2\pi i\theta_+ \bar{J}_0^3} e^{\pi i u_+ K}\,, \qquad (6.0.1)$$

where $\tau_\pm, \theta_\pm, u_\pm$ are independent real parameters, $c = \bar{c}$ are the left- and right-moving central charges and K is the central element of the affine algebra. The traces are taken over the left and right representation modules of the Hilbert space of the theory, $\mathcal{V}_L$ and $\mathcal{V}_R$, respectively. The Euclidean version of (6.0.1) is obtained replacing the real parameters by complex ones. For completeness, a description of the moduli space of the Lorentzian torus is presented in appendix C.

In the remaining of this chapter we compute the complete set of characters of the relevant representations making up the spectrum of the bulk AdS_3 conformal field theory and of the finite dimensional representations appearing in the open string spectrum of some brane solutions.

To lighten notation, from now on τ, θ, u will denote the real parameters τ_-, θ_-, u_- and the following compact notation will be used: $\chi_j^{\pm,w} := \chi_{\hat{\mathcal{D}}_j^{\pm,w}}$, $\chi_j^{\alpha,w} := \chi_{\hat{\mathcal{C}}_j^{\alpha,w}}$.

6.1 Discrete representations

The naive computation of the characters (6.0.1) for the discrete representations leads to θ and τ dependent divergences. This is not a problem because the characters are typically not functions but distributions. Indeed, similarly as the characters of the continuous representations, which contain a series of delta functions [37], those of the discrete representations need also be interpreted as distributions.

Let us consider the distributions constructed from the series defining the characters of the discrete representations. Shifting $\tau \to \tau + i\xi_1$ and $\theta \to \theta + i\xi_2^w$ in (6.0.1), where ξ_1, ξ_2^w are two real non vanishing parameters, a regular distribution can be defined. Indeed, the deformed characters of discrete representations in an arbitrary spectral flow sector w can be written in terms of those of unflowed representations as

$$\chi^{+,w}_{j,\xi_2^w,\xi_1}(\theta,\tau,u) = e^{i\pi k u}\sum_{\rm n} \epsilon_{\rm n} < {\rm n}|U_{-w} e^{2\pi i(\tau+i\xi_1)(L_0-\frac{c}{24})} e^{2\pi i(\theta+i\xi_2^w)J_0^3} U_w|{\rm n}> ,$$

where $|{\rm n}>$ is a complete orthonormal basis in $\hat{\mathcal{D}}^{+,0}_j$, with norm $\epsilon_{\rm n} = \pm 1$ (remember that this model is not unitary unless the Virasoro constraint is imposed). Since U_w is unitary, $U_w|{\rm n}>$ defines an orthonormal basis in $\hat{\mathcal{D}}^{+,w}_j$ and from (3.2.2) one can rewrite

$$\chi^{+,w}_{j,\xi_2^w,\xi_1} = e^{i\pi k u} e^{-2\pi i \tau \frac{k}{4} w^2} e^{2\pi i \theta \frac{k}{2} w} \sum_{\rm n} \epsilon_{\rm n} < {\rm n}| e^{2\pi i(\tau+i\xi_1)(L_0-\frac{c}{24})} e^{2\pi i(\theta - w\tau + i(\xi_2^w - w\xi_1))J_0^3}|{\rm n}> . \quad (6.1.1)$$

Choosing an orthonormal basis of eigenvectors of L_0 and J_0^3, the following behavior of the sum is easy to see

$$\chi^{+,w}_{j,\xi_2^w,\xi_1} \sim \sum_{N,n=0}^{\infty} \rho(n,N)\; e^{2\pi i[(1+w)\tau-\theta+i\left((1+w)\xi_1-\xi_2^w\right)]N} e^{2\pi i[\theta - w\tau + i(\xi_2^w - w\xi_1)]n} ,$$

where $\rho(n,N)$ gives the degeneracy of states. This expression requires the necessary condition

$$\begin{cases} \xi_1 > 0 , \\ (1+w)\xi_1 > \xi_2^w > w\xi_1 , \end{cases} \quad (6.1.2)$$

and it gives

$$\chi^{+,w}_{j,\xi_2^w,\xi_1} = e^{i\pi k u} e^{-2\pi i(\tau+i\xi_1)\frac{k}{4}w^2} e^{2\pi i(\theta+i\xi_2^w)\frac{k}{2}w} \frac{e^{-\frac{2\pi i(\tau+i\xi_1)}{k-2}(j+\frac{1}{2})^2} e^{-2\pi i(\theta+i\xi_2^w - w(\tau+i\xi_1))(j+\frac{1}{2})}}{i\vartheta_{11}(\theta+i\xi_2^w - w(\tau+i\xi_1), \tau+i\xi_1)} . \quad (6.1.3)$$

Analyzing expression (6.1.3) it is found that region (6.1.2) is free of poles and that the nearest poles are located when the inequalities saturate. So that (6.1.2) are not only necessary but also sufficient conditions.

This character defines a regular distribution and, given that the series of regular distributions are continuous with respect to the weak limit, this implies

$$\chi_j^{+,w}(\theta,\tau,u) = e^{i\pi k u}\,\frac{e^{-\frac{2\pi i\tau}{k-2}(j+\frac{1}{2}-w\frac{k-2}{2})^2}e^{-2\pi i\theta(j+\frac{1}{2}-w\frac{k-2}{2})}}{i\vartheta_{11}(\theta+i\epsilon_2^w,\tau+i\epsilon_1)}, \tag{6.1.4}$$

where we have used the identity

$$\vartheta_{11}\left(\theta+i\epsilon_2^w-w(\tau+i\epsilon_1),\tau+i\epsilon_1\right)=(-)^w e^{-\pi i\tau w^2+2\pi i\theta w}\vartheta_{11}(\theta+i\epsilon_2^w,\tau+i\epsilon_1) \tag{6.1.5}$$

and the $i\epsilon$'s denote the usual $i0$ prescriptions, but constrained as the corresponding finite parameters in (6.1.2), which dictate how to avoid the poles of ϑ_{11}^{-1} at $n\tau\in\mathbb{Z},\ m\tau+\theta\in\mathbb{Z}$, for $n\in\mathbb{N}, m\in\mathbb{Z}$. These poles are easily seen in the following alternative expression for the elliptic theta function

$$\begin{aligned}\frac{1}{\vartheta_{11}(\theta+i\epsilon_2^w,\tau+i\epsilon_1)} &= \frac{-e^{-i\frac{\pi}{4}\tau}}{\sin\left[\pi\left(\theta+i\epsilon_2^w\right)\right]\prod_{n=1}^{\infty}\left[1-e^{2\pi i n(\tau+i\epsilon_1)}\right]}\\ &\times\frac{1}{\prod_{n=1}^{\infty}\left[1-e^{2\pi i(n\tau-\theta+i\epsilon_3^{n,w})}\right]\left[1-e^{2\pi i(n\tau+\theta+i\epsilon_4^{n,w})}\right]},\end{aligned} \tag{6.1.6}$$

with

$$\begin{cases}\epsilon_3^{n,w}=n\epsilon_1-\epsilon_2^w\\ \epsilon_4^{n,w}=n\epsilon_1+\epsilon_2^w\end{cases}, \tag{6.1.7}$$

i.e., $\epsilon_3^{n,w}>0\ (<0)$ for $n\geq 1+w\ (n\leq w)$ and $\epsilon_4^{n,w}>0\ (<0)$ for $n\geq -w\ (n\leq -1-w)$.

Notice that, in the weak limit, one can take ϵ_1, $\epsilon_2^w=0$ in the arguments of the exponential terms in (6.1.4) because they are perfectly regular.

It is useful to rewrite (6.1.4) using the identity (D.0.1), which allows to change the signs of ϵ_2^w, $\epsilon_3^{n,w}$ and $\epsilon_4^{n,w}$, in order to get the following expressions in terms of only one parameter, say $\epsilon_2^{w'}$, with arbitrary w':

$$\chi_j^{+,w<w'}(\theta,\tau,u) = (-)^w e^{i\pi k u}\frac{e^{-\frac{2\pi i\tau}{k-2}(j+\frac{1}{2}-w\frac{k-2}{2})^2}e^{-2\pi i\theta(j+\frac{1}{2}-w\frac{k-2}{2})}}{i\vartheta_{11}(\theta+i\epsilon_2^{w'},\tau+i\epsilon_1)}$$

$$
\begin{aligned}
& - (-)^w e^{i\pi k u} \frac{e^{-\frac{2\pi i \tau}{k-2}(j+\frac{1}{2}-w\frac{k-2}{2})^2} e^{-2\pi i\theta(j+\frac{1}{2}-w\frac{k-2}{2})}}{\eta^3(\tau+i\epsilon_1)} \\
& \times \sum_{n=1+w}^{w'} (-)^n e^{2i\pi\tau\frac{n^2}{2}} \sum_{m=-\infty}^{\infty} (-)^m \delta(\theta - n\tau + m)
\end{aligned}
\tag{6.1.8}
$$

and

$$
\begin{aligned}
\chi_j^{+,w>w'}(\theta,\tau,0) \; = \; & (-)^w e^{i\pi k u} \frac{e^{-\frac{2\pi i \tau}{k-2}(j+\frac{1}{2}-w\frac{k-2}{2})^2} e^{-2\pi i\theta(j+\frac{1}{2}-w\frac{k-2}{2})}}{i\vartheta_{11}(\theta+i\epsilon_2^{w'},\tau+i\epsilon_1)} \\
& + (-)^w e^{i\pi k u} \frac{e^{-\frac{2\pi i \tau}{k-2}(j+\frac{1}{2}-w\frac{k-2}{2})^2} e^{-2\pi i\theta(j+\frac{1}{2}-w\frac{k-2}{2})}}{\eta^3(\tau+i\epsilon_1)} \\
& \times \sum_{n=1+w'}^{w} (-)^n e^{2i\pi\tau\frac{n^2}{2}} \sum_{m=-\infty}^{\infty} (-)^m \delta(\theta - n\tau + m)\,.
\end{aligned}
\tag{6.1.9}
$$

These expressions are in perfect agreement with the spectral flow symmetry, which implies $\chi_j^{+,w}(-\theta,\tau,u) \;=\; \chi_{-\frac{k}{2}-j}^{+,-w-1}(\theta,\tau,u)$. They lead to the following contribution to the partition function

$$
Z_{\mathcal{D}}^{AdS_3} = \sqrt{\frac{k-2}{2i(\tau_- - \tau_+)}} \frac{e^{i\pi k(u_- - u_+)} e^{2\pi i \frac{k-2}{4}\frac{(\theta_- - \theta_+)^2}{\tau_- - \tau_+}}}{\vartheta_{11}(\theta_- + i\epsilon_2^0, \tau_- + i\epsilon_1)\vartheta_{11}^*(\theta_+ - i\epsilon_2^0, \tau_+ - i\epsilon_1)} \;+\; \ldots\,, \tag{6.1.10}
$$

where the ellipses stand for the contributions of the contact terms. This expression differs formally from the equivalent one in [37], where no ϵ prescription or contact terms were considered. Nevertheless, the ultimate goal in [37] was to reproduce the Euclidean partition function continuing the modular parameters away from the real axes and discarding contact terms such as those of the characters of the continuous representations.

6.2 Continuous representations

A similar analysis can be performed for the characters of the continuous representations. Using (6.1.1), one can compute these characters in terms of those of the unflowed continuous representations. The result is

$$\begin{aligned}\chi_j^{\alpha,w} &= e^{i\pi k u}\frac{-2\sin[\pi(\theta-w\tau)]e^{-2\pi i\tau\frac{k}{4}w^2}e^{2\pi i\theta\frac{k}{2}w}e^{-\frac{2\pi i\tau}{k-2}(j+\frac{1}{2})^2}e^{2\pi i(\theta-w\tau)\alpha}}{\vartheta_{11}(\theta-w\tau,\tau+i\epsilon_1)}\sum_{n=-\infty}^{\infty}e^{2\pi i(\theta-w\tau)n}\\ &= e^{i\pi k u}\frac{e^{2\pi i\tau\left(\frac{s^2}{k-2}+\frac{k}{4}w^2\right)}}{\eta^3(\tau+i\epsilon_1)}\sum_{m=-\infty}^{\infty}e^{-2\pi i m\left(\alpha+\frac{k}{2}w\right)}\delta\left(\theta-w\tau+m\right)\,, \end{aligned} \tag{6.2.1}$$

where the following identity was used

$$\sum_{n=-\infty}^{\infty}e^{2\pi i x n}=\sum_{m=-\infty}^{\infty}\delta\left(x+m\right)\,. \tag{6.2.2}$$

In previous attempts to find the modular S-transformation of the principal continuous series (see for instance [45, 89]) the θ variable was turned off and so the factor $\sum_m e^{-2\pi i m\alpha}\delta(\theta+m)\sim\delta(\theta)$ was interpreted as the infinite volume of the target space and was factorized out in the transformation. It is clear that the situation is much more involved because such a term has a non trivial behavior under $SL(2,\mathbb{Z})$. After a modular S transformation one finds $\delta\left(\frac{\theta}{\tau}\right)=|\tau|\delta(\theta)$, which prevents one from simply taking the limit $\theta=0$ discarding the δ function. The modular transformation will differ from the $\theta\neq 0$ case, and so it will not give the correct modular S matrix (which must not depend on θ).

In this case, the characters are defined as the weak limit ϵ_1, $\epsilon_2^w\to 0$, with the constraints

$$\begin{cases}\epsilon_1>0\,,\\ \epsilon_2^w-w\epsilon_1=0\,,\end{cases} \tag{6.2.3}$$

and they give the following contribution to the partition function:

$$\begin{aligned}Z_{\mathcal{C}}^{AdS_3} &= \sqrt{\frac{2-k}{8i(\tau_--\tau_+)}}\frac{e^{i\pi k(u_--u_+)}}{\eta^3(\tau_-+i\epsilon_1)\eta^{*3}(\tau_+-i\epsilon_1)}\\ &\times\sum_{m,w=-\infty}^{\infty}e^{-2\pi i\frac{k}{4}w(\theta_--\theta_+)}\delta(\theta_--w\tau_-+m)\delta(\theta_+-w\tau_++m)\,. \end{aligned} \tag{6.2.4}$$

6.3 Degenerate representations

Degenerate representations are not contained in the spectrum of the AdS_3 WZNW model but they play an important role in the description of the boundary CFT. Indeed, using worldsheet duality, it was argued that they make up the Hilbert space of open string excitations of S^2 branes in the H_3^+ model [89, 93]. For the analysis that we shall perform in the forthcoming chapters, it is useful to note the relation among their characters and those of discrete and continuous representations of the universal cover of SL(2,$\mathbb{R}$) discussed above.

The finite dimensional degenerate representations are labeled by the spin $j_{rs}^{\pm}$ defined by $1+2j_{rs}^{\pm} = \pm\,(r + s(k-2))$, with $r, s+1 = 1, 2, 3, \ldots$ for the upper sign and $r, s = 1, 2, 3, \ldots$ for the lower one. Here we consider $J = j_{r0}^{+}$, with characters given by

$$\chi_J(\theta,\tau,u) = -\frac{2e^{i\pi k u}e^{-2\pi i\tau\frac{(2J+1)^2}{4(k-2)}}\sin\left[\pi\theta(2J+1)\right]}{\vartheta_{11}(\theta+i\epsilon_2,\tau+i\epsilon_1)}\,, \tag{6.3.1}$$

where the ϵ's are restricted to

$$\begin{cases} \epsilon_1 > 0\,, \\ |\epsilon_2| < \epsilon_1\,. \end{cases} \tag{6.3.2}$$

Extrapolating the values of the spins in the expressions obtained in the previous sections, (6.3.1) can be rewritten as

$$\chi_J(\theta,\tau,u) \;=\; \chi_J^{+,w=0}(\theta,\tau,u) + \chi_{-\frac{k}{2}-J}^{+,w=-1}(\theta,\tau,u) - \chi_J^{\alpha=\{J\},w=0}(\theta,\tau,u)\,, \tag{6.3.3}$$

where $\{J\}$ is the sawtooth function. Actually, this relation could have been guessed from a simple inspection of the spectrum (see Figure 6). This can be seen as a non trivial check of the characters defined above and, simultaneously, it shows the important role played by the $i0$ prescription in the definition of the characters of discrete representations. A naive computation of these characters, ignoring the $i0'$s, would yield the (wrong) conclusion $\chi_J = \chi_J^{+,w=0} + \chi_{-\frac{k}{2}-J}^{+,w=-1}$.

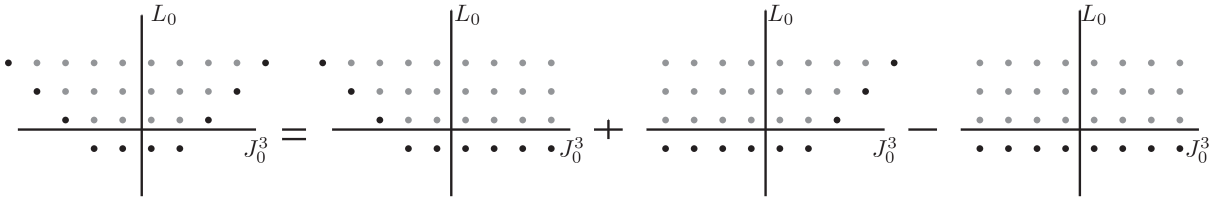

Figure 6: The weight diagram of the degenerate representations with spin $J = j^+_{r0} = \frac{r-1}{2}$, $r = 1, 2, 3...$ can be decomposed as the sum of the weight diagrams of the lowest and highest weight unflowed discrete representations minus that of the continuous representation of spin J.

Chapter 7

Modular properties

In this chapter we discuss the modular properties of the Lorentzian characters defined in the previous chapter. Even though the Lorentzian torus is not modular invariant, the characters transform as pseudovectors and the full modular S and T matrices can be rigorously defined. It is shown that these satisfy the expected relations $S^2 = (ST)^3 = C$, C being the charge conjugation matrix. In the next chapter we will explore if these modular matrices have a similar role as the modular S matrices in the microscopic description of boundary CFTs.

7.1 Modular group of the torus

The modular transformation $\tau \to \frac{a\tau+b}{c\tau+d}$, with integer parameters a, b, c, d such that $ad - bc = 1$, can be easily extended to include θ, u. Characters generating a representation space of the modular group transform as [24]

$$\chi_\mu\left(\frac{\theta}{c\,\tau+d}, \frac{a\,\tau+b}{c\,\tau+d}, u + \frac{c\,\theta^2}{2(c\,\tau+d)}\right) = \sum_\nu M_\mu{}^\nu \chi_\nu(\theta, \tau, u), \tag{7.1.1}$$

M being the matrix associated to the group element. Insofar as τ and u are concerned, the sign of all the parameters a, b, c, d may be simultaneously changed without affecting the transformation. In models where the representations are self conjugate (*e.g.* the $SU(2)$ WZNW model), the invariance $\theta \leftrightarrow -\theta$ allows to put $\theta = 0$ and so the modular group is simply PSL(2, $\mathbb{Z}$) $= \frac{SL(2,\mathbb{Z})}{\mathbb{Z}_2}$.

When this is not the case, the characters form a representation of $SL(2,\mathbb{Z})$ which is freely generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where S^2 is no longer the identity (as in $SU(2)$) but the charge conjugation matrix. In fact, S^2 produces time and parity inversion on the torus geometry and, by CPT invariance, it transforms a character into its conjugate.

7.2 The S matrix

Below we will find explicit expressions for generalized S transformations of the characters introduced in the previous chapter, setting $u = 0$ for short, as[1]

$$\chi_\mu(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0) = e^{-2\pi i \frac{k}{4} \frac{\theta^2}{\tau}} \sum_\nu S_\mu{}^\nu \chi_\nu(\theta, \tau, 0)\,, \tag{7.2.1}$$

and we will show that, unlike standard expressions, they contain a sign of τ factor. This result can already be inferred from the S modular transformation of the partition function. Indeed, ignoring the ϵ's and the contact terms, one finds for the contributions from discrete representations[2]

$$\tilde{Z}^{AdS_3}_{\mathcal{D}}(\tau'_-, \theta'_-, u'_-; \tau'_+, \theta'_+, u'_+) \;=\; sgn(\tau_-\,\tau_+)\;\; \tilde{Z}^{AdS_3}_{\mathcal{D}}(\tau_-, \theta_-, u_-; \tau_+, \theta_+, u_+)\,, \tag{7.2.2}$$

while the contributions from the continuous series verify

$$Z^{AdS_3}_{\mathcal{C}}(\tau'_-, \theta'_-, u'_-; \tau'_+, \theta'_+, u'_+) \;=\; sgn(\tau_-\,\tau_+)\;\; Z^{AdS_3}_{\mathcal{C}}(\tau_-, \theta_-, u_-; \tau_+, \theta_+, u_+)\,, \tag{7.2.3}$$

where the primes denote the S modular transformed parameters. This suggests that the block $S_{d_i}{}^{d_j}$, d_i labeling discrete representations, is given by $sgn(\tau)\ \mathcal{S}_{d_i}{}^{d_j}$ with $\mathcal{S}_{d_i}{}^{d_j}$ being unitary. Moreover, since the characters of the continuous representations contain purely contact terms, one expects that they close among themselves. This together with (7.2.3) suggest that the

[1]Some authors use the $\tilde{S}$ matrix generating $\chi_\mu\left(-\frac{\theta}{\tau}, -\frac{1}{\tau}, u + \frac{\theta^2}{2\tau}\right)$. This is given by $\tilde{S}_\mu{}^\nu = S_\mu{}^{\nu^+}$, where ν^+ labels the conjugate ν-representation.

[2]$\tilde{Z}^{AdS_3}_{\mathcal{D}}$ is the contribution to the partition function for θ and τ far from $\theta + n\tau \in \mathbb{Z}$, $\forall n \in \mathbb{Z}$.

block ${S_{c_i}}^{c_j}$, c_i labeling continuous representations, is given by $sgn(\tau)\ {\mathcal{S}_{c_i}}^{c_j}$ with ${\mathcal{S}_{c_i}}^{c_j}$ being unitary. We will explicitly show these features of the generalized modular transformations in the next section. In this sense, the characters of the AdS_3 model on the Lorentzian torus are pseudovectors with respect to the standard modular S transformations.

A naive treatment of the Lorentzian partition function as a Wick rotation of the Euclidean path integral, would suggest the appearance of this sign after an S transformation from the measure, when one takes into account the change in the metric (see appendix C). However, it will be clear from the results of section 7.2.3, that the failure in the modular invariance of $Z_{\mathcal{D}}^{AdS_3}$ is subtler than just the sign appearing in (7.2.2). Of course this is not a problem, the Lorentzian partition functions are not modular invariant, the S transformation exchanges time and space directions and this is generically not a symmetry. The relevant question is how to perform the Wick rotation to a finite modular invariant quantity without lost on the information of the spectrum.

7.2.1 The strategy

The modular S matrix is an interesting object in itself and proving its existence is a fundamental step to ensure the consistency of a given CFT, but it also plays an important role in the microscopic description of a string theory, *e.g.* in RCFT worldsheet it was proved the S matrix defines the coupling (one point functions) to maximally symmetric D-branes and determines the fusion rules of the theory. We will come back to these issues in the next chapter.

In order to present and to motivate the relevance of the modular transformation of the characters defined in the previous chapter let us concentrate for a moment in the simplest Lorentzian model, where the target space is the D-dimensional Minkowski spacetime.

Contrary to the AdS_3 case, string theory in flat space can be consistently defined with a Lorentzian target space and a Euclidean worldsheet. The representations building up the worldsheet spectrum are labeled by the D-dimensional momentum k^μ and the characters on the

Euclidean torus are defined as (we set $\alpha' = 1$)

$$\chi_{\mathbf{k}}(\tau) = V \prod_{\mu=0}^{D-1} \chi_\mu(\tau), \tag{7.2.4}$$

where V is the volume of the space time and the normalized characters, χ_μ are defined by

$$\begin{aligned} \chi_0(\tau) &= \frac{1}{\eta(\tau)} e^{-2\pi i \tau \frac{k_0^2}{2}} , \\ \chi_j(\tau) &= \frac{1}{\eta(\tau)} e^{2\pi i \tau \frac{k_j^2}{2}} , \; j = 1, \ldots, D-1. \end{aligned} \tag{7.2.5}$$

The partition function and the modular transformations are ill defined, so one makes a Wick rotation to the Euclidean target space, where $k_0 \to ik_0$. The rotated character, $\chi^E_{\mathbf{k}}$ has a well defined modular S transformation given by

$$\chi^E_{\mathbf{k}}(-\frac{1}{\tau}) = \int d^D k' \; S_{\mathbf{k},\mathbf{k}'} \; \chi^E_{\mathbf{k}'}(\tau) \; , \qquad S_{\mathbf{k},\mathbf{k}'} = e^{2\pi i \mathbf{k}\cdot\mathbf{k}'}. \tag{7.2.6}$$

The partition function is modular invariant and the indices in the modular S matrix are in a one to one relation with the representations of the original Lorentzian model.

The situation is completely different in the AdS_3 model. Here the representations in the Lorentzian and Euclidean models are completely different. The characters of H_3^+ do not even factorize in holomorphic and antiholomorphic factors and their Wick rotation has no information on the AdS_3 spectrum so that there is no reason to expect that the H_3^+ modular S matrix has some relation with the microscopic description of AdS_3.

An interesting observation is that the S matrix of the Minkowski space can be obtained without invoking the Euclidean rotation. In fact the characters on the Lorentzian torus are given by (7.2.5) where now τ is a real parameter and the Dedekind function is now interpreted as the distribution $\frac{1}{\eta(\tau+i0^+)}$. Even though these characters are not in a vector representation of the modular group, their transformations are perfectly well defined (in a distributional sense) and do not require Wick rotation or other regularization. They transform as pseudovectors as their transformations introduces a $sgn(\tau)$ factor. In fact by using

$$\frac{1}{\eta(-\frac{1}{\tau}+i0^+)} = \frac{1}{\eta(-\frac{1}{\tau+i0^+})} = \frac{e^{sgn(\tau)\ \frac{i\pi}{4}}}{\sqrt{|\tau|}\ \eta(\tau+i0^+)}, \tag{7.2.7}$$

$$e^{-2\pi i\frac{1}{\tau}\frac{\lambda^2}{2}} = e^{-sgn(\tau)\frac{i\pi}{4}}\sqrt{|\tau|}\int d\lambda'\ e^{2\pi i\lambda\lambda'}e^{2\pi i\tau\frac{\lambda'^2}{2}}, \tag{7.2.8}$$

it is found that

$$\chi_{\mathbf{k}}(-\frac{1}{\tau}) = sgn(\tau)\int d^Dk'\ \mathcal{S}_{\mathbf{k},\mathbf{k}'}\ \chi_{\mathbf{k}'}(\tau)\ , \qquad \mathcal{S}_{\mathbf{k},\mathbf{k}'} = i\,e^{2\pi i\mathbf{k}\cdot\mathbf{k}'}. \tag{7.2.9}$$

So, up to the $sign(\tau)$ factor and the i phase which can be interpreted as coming from the Euclidean rotation $dk^0 \to i\,dk^0$, we have obtained the modular S matrix of the model without any reference to the Euclidean theory.

In the rest of the chapter we will generalize this procedure to the more involved AdS_3 model and we will find that the characters found in the previous chapter transform as pseudovectors with respect to the modular group.

7.2.2 Continuous representations

The S transformed characters of continuous representations can be written as:

$$\chi_j^{\alpha,w}(\frac{\theta}{\tau},-\frac{1}{\tau},0) = \frac{e^{-2\pi i\left(\frac{s^2}{k-2}+\frac{k}{4}w^2\right)\frac{1}{\tau}}}{(-i\tau)^{\frac{3}{2}}\eta^3(\tau+i\epsilon_1)}\sum_{m=-\infty}^{\infty} e^{2\pi i m(\alpha+\frac{k}{2}w)}\delta\left(\frac{\theta}{\tau}+\frac{w}{\tau}-m\right), \tag{7.2.10}$$

where (7.2.7) was used. After inserting equation (7.2.8) with the appropriate relabeling we find

$$\begin{aligned}\chi_j^{\alpha,w}(\frac{\theta}{\tau},-\frac{1}{\tau},0) &= \frac{e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}}}{\tau}\int_{-\infty}^{+\infty} ds'\ \tilde{\mathcal{S}}_s{}^{s'}\,\frac{e^{\frac{2\pi i}{k-2}\tau s'^2}}{\eta^3(\tau+i\epsilon_1)} \\ &\times \sum_{m=-\infty}^{\infty} e^{2\pi i\frac{k}{4}\tau m^2}e^{2\pi i m\alpha}\ \delta\left(\frac{\theta}{\tau}+\frac{w}{\tau}-m\right)\ ,\end{aligned} \tag{7.2.11}$$

with $\tilde{\mathcal{S}}_s{}^{s'} = i\sqrt{\frac{2}{k-2}}e^{-4\pi i\frac{ss'}{k-2}}$.

From $\delta\left(\frac{\theta}{\tau}+\frac{w}{\tau}-m\right)=|\tau|\,\delta\left(\theta+w-m\tau\right)$ and renaming variables, one gets

$$
\begin{aligned}
\chi_j^{\alpha,w}(\frac{\theta}{\tau},-\frac{1}{\tau},0) &= e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}}sgn(\tau) \\
&\times \sum_{w'=-\infty}^{\infty}\int_{-\infty}^{+\infty} ds'\tilde{\mathcal{S}}_s{}^{s'}\frac{e^{2\pi i\tau\left(\frac{s'^2}{k-2}+\frac{k}{4}w'^2\right)}}{\eta^3(\tau+i\epsilon_1)}e^{2\pi i w'\alpha}\delta\left(\theta-w'\tau+w\right). \quad (7.2.12)
\end{aligned}
$$

In order to reconstruct the character $\chi_{j'}^{\alpha',w'}$ in the *r.h.s.*, we use the identity

$$
\delta\left(\theta-w'\tau+w\right)=\sum_{m'=-\infty}^{\infty}\int_0^1 d\alpha' e^{2\pi i\left(w\alpha'+\frac{k}{2}ww'\right)}e^{-2\pi i m'\left(\alpha'+\frac{k}{2}w'\right)}\delta\left(\theta-w'\tau+m'\right)\,, \quad (7.2.13)
$$

and exchanging summation and integration[3], (7.2.12) can be rewritten as

$$
\chi_j^{\alpha,w}(\frac{\theta}{\tau},-\frac{1}{\tau},0)=e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}}sgn(\tau)\sum_{w'=-\infty}^{\infty}\int_0^{+\infty}ds'\int_0^1 d\alpha'\;\;\mathcal{S}_{s,\alpha,w}{}^{s',\alpha',w'}\chi_{j'=-\frac{1}{2}+is'}^{\alpha',w'}(\theta,\tau,0)\,,
$$

with

$$
\mathcal{S}_{s,\alpha,w}{}^{s',\alpha',w'}=2i\sqrt{\frac{2}{k-2}}\;\;\cos\left(4\pi\frac{ss'}{k-2}\right)\;\;e^{2\pi i\left(w\alpha'+w'\alpha+\frac{k}{2}ww'\right)}\,, \quad (7.2.14)
$$

which is symmetric and, as expected from (7.2.3), unitary, *i.e.*

$$
\sum_{w'=-\infty}^{\infty}\int_0^{\infty}ds'\int_0^1 d\alpha'\mathcal{S}_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'}\mathcal{S}^{\dagger}_{s',\alpha',w'}{}^{s_2,\alpha_2,w_2}=\delta(s_1-s_2)\delta(\alpha_1-\alpha_2)\delta_{w_1,w_2}\,. \quad (7.2.15)
$$

7.2.3 Discrete representations

The structure of the characters of the discrete representations is more involved than that of the continuous ones. A priori, we expect that characters of both discrete and continuous representations appear in the generalized modular transformations. So, generically we can assume

$$
\chi_j^{+,w}(\frac{\theta}{\tau},-\frac{1}{\tau},0) \;=\; e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}}sgn(\tau)\sum_{w'=-\infty}^{\infty}\left\{\int_{-\frac{k-1}{2}}^{-\frac{1}{2}}\mathcal{S}_{j,w}{}^{j',w'}\;\chi_{j'}^{+,w'}(\theta,\tau,0)\right.
$$

[3]Here, summation and integration can be exchanged because, for a fixed w', the series always reduces to a finite sum when it is considered as a distribution acting on a test function.

$$+ \quad \int_0^1 d\alpha' \int_0^\infty ds' \; \mathcal{S}_{j,w}{}^{s',\alpha',w'} \; \chi^{\alpha',w'}_{j'=-\frac{1}{2}+is'}(\theta,\tau,0) \Big\} \; .$$

Fortunately, it is easy to separate the contributions from discrete and continuous representations. If one considers generic values of θ and τ far from $\theta + n\tau \in \mathbb{Z}$ for $n \in \mathbb{Z}$, the contributions of the continuous series in the $r.h.s.$ can be neglected as well as all contact terms and ϵ's. On the other hand, if $\theta + n\tau \notin \mathbb{Z}, \forall n \in \mathbb{Z}$ then $\frac{\theta}{\tau} - p\frac{1}{\tau} \notin \mathbb{Z}, \forall p \in \mathbb{Z}$ and all contact terms and ϵ's of the $l.h.s.$ can be neglected too. Thus, we obtain

$$\begin{aligned} \chi^{+,w}_j(\frac{\theta}{\tau},-\frac{1}{\tau},0) &= \frac{(-)^w e^{\frac{2\pi i}{k-2}\frac{1}{\tau}\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)^2} e^{-2\pi i\frac{\theta}{\tau}\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)}}{i\vartheta_{11}(\frac{\theta}{\tau},-\frac{1}{\tau})} \\ &= (-)^{w+1}\frac{e^{\frac{2\pi i}{k-2}\frac{1}{\tau}\left(j+\frac{1}{2}-(w+\theta)\frac{k-2}{2}\right)^2} e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}}}{i\sqrt{i\tau}\vartheta_{11}(\theta,\tau)} \,, \end{aligned} \tag{7.2.16}$$

where the following identity was used for $\tau \in \mathbb{R}$:

$$\vartheta_{11}(\frac{\theta}{\tau},-\frac{1}{\tau}) = sgn(\tau)\; e^{\pi i\frac{\theta^2}{\tau}} e^{sgn(\tau)\; i\frac{\pi}{4}}\sqrt{|\tau|}\;\vartheta_{11}(\theta,\tau)\,. \tag{7.2.17}$$

Inserting

$$e^{\frac{2\pi i}{k-2}\frac{1}{\tau}\left(j+\frac{1}{2}-(w+\theta)\frac{k-2}{2}\right)^2} = e^{sgn(\tau)\; i\frac{\pi}{4}}\sqrt{\frac{2|\tau|}{k-2}}\int_{-\infty}^{+\infty} d\lambda' e^{\frac{4\pi i}{k-2}\lambda'\left(j+\frac{1}{2}-(w+\theta)\frac{k-2}{2}\right)} e^{-\frac{2\pi i}{k-2}\tau\lambda'^2} \tag{7.2.18}$$

into (7.2.16), changing the integration variable to $j' + \frac{1}{2} - w'\frac{k-2}{2}$ and using (7.2.17), we get

$$\chi^{+,w}_j(\frac{\theta}{\tau},-\frac{1}{\tau},0) = e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}} sgn(\tau) \sum_{w'=-\infty}^{\infty}\int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj' \mathcal{S}_{j,w}{}^{j',w'}\chi^{+,w'}_{j'}(\theta,\tau,0), \tag{7.2.19}$$

with

$$\mathcal{S}_{j,w}{}^{j',w'} = (-)^{w+w'+1}\sqrt{\frac{2}{k-2}}\; e^{\frac{4\pi i}{k-2}\left(j'+\frac{1}{2}-w'\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)}\,. \tag{7.2.20}$$

Notice that this block of the $\mathcal{S}$ matrix is symmetric and, again as expected from (7.2.2), unitary[4].

[4]Changing $e^{\pm i\frac{\pi}{4}}\sqrt{\tau}$ by $\sqrt{i\tau}$, the validity of (7.2.18) can be extended to the full lower half plane and that of

While the identity (7.2.18), which is essential to reconstruct the discrete characters in the *r.h.s.* of (7.2.19), only makes sense for Im $\tau \leq 0$, the characters are only well defined for Im $\tau \geq 0$. Therefore, to determine the generalized S transformation, it is crucial that $\tau \in \mathbb{R}$.

Finding the block ${\mathcal{S}_{j,w}}^{s',\alpha',w'}$ mixing discrete with continuous representations is a much more technical issue, which we discuss in appendix D. Here we simply display the result, namely

$$ {\mathcal{S}_{j,w}}^{s',\alpha',w'} = -i\sqrt{\frac{2}{k-2}} e^{-2\pi i\left(w'j - w\alpha' - ww'\frac{k}{2}\right)} \left[\frac{e^{\frac{4\pi}{k-2}s'\left(j+\frac{1}{2}\right)}}{1+e^{-2\pi i(\alpha'-is')}} + \frac{e^{-\frac{4\pi}{k-2}s'\left(j+\frac{1}{2}\right)}}{1+e^{-2\pi i(\alpha'+is')}} \right] . \quad (7.2.22) $$

This block prevents the full $\mathcal{S}$ matrix from being unitary. Instead, we find $\mathcal{S}^*\mathcal{S} = id$. This implies that the full partition function defined from the product of characters is not modular invariant, not only due to the sign of the modular parameters. Actually, after a modular transformation, the mixing block introduces terms where the left modes are in discrete representations and the right ones in continuous series, and vice versa, as well as new terms containing left and right continuous representations.

In section 7.4, we explicitly check that the blocks of the S matrix determined here have the correct properties.

7.2.4 Degenerate representations

The modular properties discussed above can be used to write the S transformation of the characters of the degenerate representations with $1+2J \in \mathbb{N}$ as:

$$ \begin{aligned} \chi_J(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0) \;=\; & e^{-2\pi i \frac{k}{4}\frac{\theta^2}{\tau}} sgn(\tau) \sum_{w=-\infty}^{\infty} \left\{ \int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj \; {\mathcal{S}_J}^{j,w} \chi_j^{+,w}(\theta,\tau,0) \right. \\ & \left. + \int_0^1 d\alpha \int_{-\frac{k-1}{2}}^{-\frac{1}{2}} ds \; {\mathcal{S}_J}^{s,\alpha,w} \chi^{\alpha,w}_{j=-\frac{1}{2}+is}(\theta,\tau,0) \right\} , \end{aligned} $$

(7.2.17) can be extended to the upper half plane, giving

$$ \vartheta_{11}(\frac{\theta}{\tau}, -\frac{1}{\tau}) = -e^{\pi i \frac{\theta^2}{\tau}} \sqrt{i\tau}\, \vartheta_{11}(\theta,\tau) . \quad (7.2.21) $$

Nevertheless, one cannot cancel the $\sqrt{i\tau}$ terms and ignore the sign factor due to the different branches.

where

$$\mathcal{S}_J{}^{j,w} = 2i\sqrt{\frac{2}{k-2}}(-)^{w+1}\sin\left[\frac{\pi}{k-2}\left(1+2j-w(k-2)\right)(2J+1)\right], \tag{7.2.23}$$

and

$$\begin{aligned}\mathcal{S}_J{}^{s,\alpha,w} &= -i(-)^{2Jw}\sqrt{\frac{2}{k-2}}e^{\frac{4\pi}{k-2}s(J+\frac{1}{2})}\left(1+\frac{1}{1+e^{-2\pi i(\alpha-is)}}+\frac{1}{1+e^{2\pi i(\alpha+is)}}\right)\\ &+ \ (s\leftrightarrow -s)\,.\end{aligned} \tag{7.2.24}$$

7.3 The T matrix

Together with the S matrix, the T matrix defines a basis over the space of modular transformations. Using

$$\vartheta_{11}(\theta,\tau+1) = e^{\frac{\pi i}{4}}\vartheta_{11}(\theta,\tau), \qquad \eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau)\,, \tag{7.3.1}$$

the characters of the discrete and continuous representations transform respectively with

$$T_{j,w}{}^{j',w'} = \delta_{w,w'}\delta(j-j')e^{-\frac{2\pi i}{k-2}\left(j'+\frac{1}{2}-w'\frac{k-2}{2}\right)^2-\frac{\pi i}{4}} \tag{7.3.2}$$

and

$$T_{s,\alpha,w}{}^{s',\alpha',w'} = \delta_{w,w'}\delta(\alpha-\alpha')\delta(s-s')e^{2\pi i\left(\frac{s^2}{k-2}-\frac{k}{4}w^2-w\alpha-\frac{1}{8}\right)}, \tag{7.3.3}$$

while the T transformation of the characters of the degenerate representations is given by

$$\chi_J(\theta,\tau+1,0) = e^{-\frac{2\pi i}{k-2}\left(J+\frac{1}{2}\right)^2}e^{-\frac{\pi i}{4}}\chi_J(\theta,\tau,0)\,. \tag{7.3.4}$$

7.4 Properties of the S and T matrices

The expressions $(ST)^3$ and S^2 must give the conjugation matrix, C. We have found above that the characters of the AdS_3 model do not expand a representation space for the modular group since the generators depend on the sign of τ. Nevertheless, in terms of the τ independent part of S, that we have denoted $\mathcal{S}$, these identities read $C = (ST)^3 = sgn(\tau+1)$ $sgn(\frac{\tau}{\tau+1})sgn(-\frac{1}{\tau})(\mathcal{S}T)^3 = -(\mathcal{S}T)^3$ and $C = S^2 = sgn(\tau)sgn(-\frac{1}{\tau})\mathcal{S}^2 = -\mathcal{S}^2$.

As a consistency check on the expressions found above for S and T, an explicit computation gives

$$-(\mathcal{S}T)^3{}_{j_1,w_1}{}^{j_2,w_2} = -\mathcal{S}^2{}_{j_1,w_1}{}^{j_2,w_2} = \delta_{w_1+w_2+1,0}\,\delta\left(j_1+j_2+\frac{k}{2}\right), \tag{7.4.1}$$

which corresponds to the conjugation matrix restricted to the discrete sector, since $\hat{\mathcal{D}}_j^{+,w}$ is the conjugate representation of $\hat{\mathcal{D}}_j^{-,-w}$, which in turn can be identified with $\hat{\mathcal{D}}^{+,-w-1}_{-\frac{k}{2}-j}$ using the spectral flow symmetry. Similarly, for the block of continuous representations we get

$$-(\mathcal{S}T)^3_{s_1,\alpha_1,w_1}{}^{s_2,\alpha_2,w_2} = -\mathcal{S}^2{}_{s_1,\alpha_1,w_1}{}^{s_2,\alpha_2,w_2} = \delta_{w_1,-w_2}\delta(s_1-s_2)\delta(\alpha_1+\alpha_2-1), \tag{7.4.2}$$

which is again the charge conjugation matrix, since $\hat{\mathcal{C}}_j^{1-\alpha,-w}$ is the conjugate representation of $\hat{\mathcal{C}}_j^{\alpha,w}$.

Of course, one also needs to show that the non diagonal terms vanish. The equalities $(\mathcal{S}T)^3{}_{s_1,\alpha_1,w_1}{}^{j_2,w_2} = \mathcal{S}^2{}_{s_1,\alpha_1,w_1}{}^{j_2,w_2} = 0$ are trivially satisfied as a consequence of $\mathcal{S}_{s_1,\alpha_1,w_1}{}^{j_2,w_2} = 0$. One can also show that $(\mathcal{S}T)^3{}_{j_1,w_1}{}^{s_2,\alpha_2,w_2} = \mathcal{S}^2{}_{j_1,w_1}{}^{s_2,\alpha_2,w_2}=0$, but this computation is more involved, so the details are left to appendix D.

Chapter 8

D-branes in AdS_3

D-branes can be characterized by the one-point functions of the states in the bulk, living on the upper half plane. In RCFT, these one-point functions can be determined from the entries of the S matrix, a property that we will call a *Cardy structure*. This property is closely related to the Verlinde formula and, *a priori*, there is no reason for it to hold in non RCFT. In this chapter we explore the scope of this connexion for the AdS_3 model.

D-branes in AdS_3 and related models have been studied in several works (see for instance [89]-[103] and references therein). Here, we shall restrict to the maximally symmetric D-branes discussed in [90]. Because the Lorentzian AdS_3 geometry is obtained by sewing an infinite number of SL(2,$\mathbb{R}$) group manifolds, these D-brane solutions can be trivially obtained from those of SL(2,$\mathbb{R}$). Their geometry was considered semiclassically in [90], where it was found that solutions of the Dirac Born Infeld action stand for regular and twined conjugacy classes of SL(2,$\mathbb{R}$). The model also has symmetry breaking D-brane solutions, but in this case, the open string spectrum is not a sum of $sl(2)$ representations and then the one-point functions cannot be determined by the S matrix.

We begin this chapter with a short introduction to Boundary Conformal Field Theories (BCFT) and the geometry of D-branes in AdS_3. A very comprehensive study about the (twined) conjugacy classes of SL(2,$\mathbb{R}$) and a semiclassical analysis of branes can be found in [90] and [91]. Both can be easily extended to the universal covering. Here, we review the analysis of the conjugacy classes in order to make the discussion self contained and discuss the extension to the universal covering with the aim of obtaining the key relation (8.2.8) obtained in [49].

Third section is devoted to give a brief review of the one point functions computed in [74] and to translate them to our conventions. Then we turn to the explicit construction of the Ishibashi states for regular and twisted boundary gluing conditions which give rise to the maximally symmetric D-branes. These conditions were solved in the past for the single cover of SL(2,$\mathbb{R}$) (see [92] for twisted gluing conditions) with different amounts of spectral flow in the left and right sectors, namely $w_L = -w_R$, and therefore, these solutions are not contained in the spectrum of the AdS_3 model (with the obvious exception of $w = 0$ discrete and $w = 0$, $\alpha = 0, \frac{1}{2}$ continuous representations).

We will show that the one-point functions of states in discrete representations coupled to point-like and H_2 branes exhibit a *Cardy structure* and we present a generalized Verlinde formula giving the fusion rules of the degenerate representations with $1 + 2J \in \mathbb{N}$.

8.1 Boundary CFT

In this section we present a brief review on CFTs with boundaries in order to develop the machinery to deal with D-branes on Conformal theories.

8.1.1 Closed String Sector

Bulk fields constitute the field content of the worldsheet theory describing closed strings. These can be defined in CFT with and without boundaries. In both cases there is a correspondence between states of the Hilbert space and fields of the CFT. Even though the fields differ in each case the Hilbert space ($\mathcal{H}_C = \oplus_{m\overline{m}}\mathcal{V}_m \otimes \overline{\mathcal{V}}_{\overline{m}}$ [1]) is the same in both, the theory over the full plane (P) and the one over the upper half plane (H). So that for a given state $|m\,\overline{m}>\in \mathcal{H}_C$ the state operator correspondence is

$$\varphi_{m,\overline{m}}(z,\bar{z}) \quad = \quad \Phi^{(P)}(|m\,\overline{m}>;z,\bar{z}),$$

[1]We assume the spectrum factorizes as a sum of representations, $\mathcal{V}_m$ and $\overline{\mathcal{V}}_{\overline{m}}$ of certain symmetry algebra generated by chiral currents $J(z)$ and $\bar{J}(\bar{z})$ respectively.

$$\phi_{m,\overline{m}}(z,\bar{z}) \;=\; \Phi^{(H)}\left(|m\,\overline{m}>; z, \bar{z}\right), \tag{8.1.1}$$

where $\varphi(z,\bar{z})$ are well defined over the full complex plane and $\phi(z,\bar{z})$ is only required to be well defined for Im $z > 0$. The condition of introducing a D-brane in the background defined by the Bulk theory is to require that Bulk fields of the BCFT and those of the CFT without boundaries (Bulk CFT) are equivalent, where equivalence means that both spectra coincide but also that the OPE is the same in both theories.

The conformal symmetry requires

$$T(z) = \overline{T}(\bar{z}), \qquad z = \bar{z}. \tag{8.1.2}$$

It is important to stress that a given Bulk CFT may be connected with several BCFTs and that there is no systematic approach to dealing with all possibilities.

In this short review we will restrict to the case of maximally symmetric D-branes, that is we will consider boundary conditions preserving the maximal amount of symmetry of the parent Bulk CFT. Assuming that the chiral fields $J(z), \bar{J}(\bar{z})$ can be analytically continued to the real line we will look for an automorphism Ω, that we will denote as *gluing map*, such that

$$J(z) = \Omega\bar{J}(\bar{z}), \qquad z = \bar{z} \tag{8.1.3}$$

Notice that this map, Ω, induces a map over the different sectors, $j \to w(j)$.[2]

One point functions

Ward identities of the chiral currents and the OPE of the bulk fields can be exploited to reduce the computation of correlation functions with arbitrary number of points of bulk fields on the disk to the calculus of one point functions.

[2]For instance in a theory with $U(1)$ symmetry and Dirichlet gluing map $\Omega(J) = -J, \quad \Rightarrow \Omega(\alpha_n) = -\alpha_n$. Then, taking into account that $\alpha_0|k>= \sqrt{\alpha'}k|k>$

$$\Omega(\alpha_0)|k>= -\alpha_0|k>= -\sqrt{\alpha'}k|k>= \alpha_0|-k>,$$

which induces the map $w(k) = -k$.

The transformation properties of the bulk fields in the BCFT, $\phi_{m\bar{m}}$ (the indices label left and right representations) with the modes $L_n,\ n = \pm 1, 0$ and the zero modes of the currents, J_0

$$\begin{aligned}
&[J_0, \phi_{m\overline{m}}(z,\bar{z})] = X_J^m \phi_{m\overline{m}}(z,\bar{z}) - \phi_{m\overline{m}}(z,\bar{z}) X_{\Omega\bar{J}}^{\overline{m}} \ , \\
&[L_n, \phi_{m\overline{m}}(z,\bar{z})] = z^n[z\partial + \Delta_m(n+1)]\ \phi_{m\overline{m}}(z,\bar{z}) + \bar{z}^n[\bar{z}\bar{\partial} + \bar{\Delta}_{\overline{m}}(n+1)]\ \phi_{m\overline{m}}(z,\bar{z}) \ ,
\end{aligned} \quad (8.1.4)$$

determine the structure of the one point functions on the disk to be

$$\langle \phi_{i\bar{i}}(z,\bar{z}) \rangle_\alpha = \frac{\mathcal{A}^\alpha_{i\bar{i}}}{|z-\bar{z}|^{\Delta_i + \bar{\Delta}_{\bar{i}}}}, \quad (8.1.5)$$

where $\mathcal{A}^\alpha_{m\overline{m}} : \mathcal{V}^0_{\overline{m}} \to \mathcal{V}^0_m$, obey $X^m_J \mathcal{A}^\alpha_{m\overline{m}} = \mathcal{A}^\alpha_{m\overline{m}} X^{\overline{m}}_{\Omega\bar{J}}$, which implies $\overline{m} = w(m^+) \equiv m^w$ is a necessary condition for the one point function to be non vanishing. m^+ is the conjugate representation of m and X^m_J is the action of the zero mode J_0 of the affine algebra over the primaries *i.e.*

$$X^m_J := J_0|_{\mathcal{V}^0_m} : \mathcal{V}^0_m \to \mathcal{V}^0_m, \quad (8.1.6)$$

The zero modes of the currents are irreducible in the subspaces $\mathcal{V}^0_m$, thus Schur's lemma implies

$$\mathcal{A}^\alpha_{m\overline{m}} = A^\alpha_m\ \delta_{\overline{m},m^w}\ \mathcal{U}_{m\overline{m}}, \quad (8.1.7)$$

where $\mathcal{U}_{m\overline{m}}$ intertwines between the zero modes of the algebra X^m_J and $X^{\overline{m}}_{\Omega\bar{J}}$ and is normalized such that $\mathcal{U}^*_{m\overline{m}}\mathcal{U}_{m\overline{m}} = 1$. α indexes different Boundary Theories appearing for a given *gluing map* Ω.

We have found that one point functions for different boundary conditions associated to a given Bulk Theory and the same *gluing map* Ω may differ just in a set of scalars A^α_m. Once this set is known we will have completely solved the Boundary Theory (including the open string sector that we have not considered yet).

Boundary states

It is possible to store all the information of the couplings A_m^α in a unique object, the so called *boundary state*.

The boundary state is a linear combination of coherent states known as Ishibashi states [105] and the coefficients of this expansion are essentially the couplings A_m^α.

A way to introduce *boundary states* is by equaling Bulk fields in the upper half plane of the Boundary theory and in the exterior of the unit disk in the Bulk theory.

Let $z, \bar{z}$ be coordinates in the upper half and $\zeta, \bar{\zeta}$ those of the exterior unit disk. They are related through

$$\zeta = \frac{1 - iz}{1 + iz} \quad ; \qquad \bar{\zeta} = \frac{1 + i\bar{z}}{1 - i\bar{z}}. \tag{8.1.8}$$

Thus if $|0>$ is the vacuum of the Bulk CFT, the *boundary state* $|\alpha>$ is defined via the identity

$$\left\langle \Phi^{(H)}(|\varphi>; z, \bar{z}) \right\rangle_\alpha = \left(\frac{d\zeta}{dz}\right)^h \left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{h}} < 0|\Phi^{(P)}(|\varphi>; \zeta, \bar{\zeta})|\alpha> \tag{8.1.9}$$

The boundary condition (8.1.3) of the BCFT at $z = \bar{z}$ translates in the Bulk CFT as the condition

$$\left[J(\zeta) - \bar{\zeta}^{2\Delta_J}(-)^{\Delta_J}\Omega\bar{J}(\bar{\zeta})\right]|\alpha>= 0, \qquad \zeta\bar{\zeta} = 1, \tag{8.1.10}$$

where Δ_J is the conformal weight of J. The constraint above can be rewritten in terms of the Laurent modes as

$$\left[J_n - (-)^{\Delta_J}\Omega\bar{J}_{-n}\right]|\alpha>_\Omega= 0, \tag{8.1.11}$$

where the label Ω was written to make the *gluing map* considered explicit.

These constraints are linear and leave each representation invariant. Given an automorphism Ω there exists a unique solution in each representation of the form $(m, w(m^+))$ [105] and this is given by a coherent state (or Ishibashi state) $|m \gg_\Omega$ univocally defined up to a normalization which can be fixed such that

$$_{\Omega} \ll m|\ \tilde{q}^{L_0^{(P)}-\frac{c}{24}}\ |n \gg_{\Omega} = \delta_{mn}\ \chi(\tilde{q}). \tag{8.1.12}$$

From (8.1.13), (8.1.9) and (8.1.12) one can prove that the *boundary states*, $|\alpha >_{\Omega}$ are

$$|\alpha >_{\Omega} = \sum_m A^{\alpha}_{m^+}\ |m \gg_{\Omega} \tag{8.1.13}$$

Open String Sector

World sheet duality changes open string channels and closed string channels when time and space coordinates are exchanged. Thus the boundary partition function, $Z_{\alpha\beta}(q),\ q = e^{2\pi i\tau}$, associated with the Boundary spectrum with α and β boundary conditions[3] ($\equiv$ One-loop vacuum open string amplitude) must agree with the tree level amplitude of a boundary state emitted in the brane α and reabsorbed in the brane β, *i.e.* it is interpreted as the probability that the boundary state be emitted and reabsorbed in a time interval $\tilde{\tau} = -1/\tau$ [4]

$$Z_{\alpha\beta}(q) = <\Theta\beta|\tilde{q}^{H^{(p)}}|\alpha>, \tag{8.1.14}$$

where Θ denotes the worldsheet CPT operator, defined such that

$$\Theta A^{\beta}_{m^+}|m \gg = (A^{\beta}_{m^+})^*|m^+ \gg . \tag{8.1.15}$$

Using that the Hamiltonian in the Bulk CFT is $H^{(P)} = \frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}$ and the gluing condition of the energy momentum tensor (8.1.2) imposes in the Bulk CFT the equation $[L_n - \bar{L}_{-n}]|\alpha >= 0$ it is found that

$$< \Theta\beta|\tilde{q}^{H^{(p)}}|\alpha > = \sum_{m,n} \left(\Theta A^{\beta}_{m^+}\right)^* \ll m|\tilde{q}^{L_0-\frac{c}{24}}|n \gg A^{\alpha}_{n^+}$$

[3]This is interpreted in string theory as the spectrum of open strings with one end in the brane α and the other one in the brane β.

[4]Remember that the modular S matrix exchanges cycles a and b in the torus and so exchanges time and space coordinates

$$= \sum_{m,n} A^{\beta}_{m^+} A^{\alpha}_{n^+} \ll m^+ | \tilde{q}^{L_0 - \frac{c}{24}} | n \gg = \sum_m A^{\beta}_{m^+} A^{\alpha}_{m^+} \chi_{m^+}(\tilde{q}). \qquad (8.1.16)$$

Then

$$Z_{\alpha\beta}(q) = \sum_{m,n} A^{\beta}_{m^+} A^{\alpha}_{m^+} S_{m^+}{}^{n} \chi_n(q), \qquad (8.1.17)$$

with S_{m^+n} being the modular S matrix.

Boundary conditions discussed here preserve part of the chiral symmetry, thus the boundary spectrum must be decomposed as a sum over certain representations $\mathcal{V}_m$. Let $N_{\alpha\beta}{}^{m}$ be its spectrum degeneracy, thus the *Cardy condition* [106]

$$N_{\alpha\beta}{}^{n} = \sum_m A^{\beta}_{m^+}\ A^{\alpha}_{m}\ S_{m^+}{}^{n} \in \mathbb{Z}_{\geq 0}, \qquad (8.1.18)$$

must be satisfied.

Cardy Solution

Let us consider a Bulk RCFT with modular invariant partition function

$$Z(q, \bar{q}) = \sum_m \chi_m(q) \chi_{\overline{m}}(\bar{q}), \qquad (8.1.19)$$

with m taking values in some given set $\mathcal{M}$ and $\overline{m}$ being paired with some given m, assuming that[5]

$$m^w \equiv m(j)^+ = \overline{m}. \qquad (8.1.20)$$

Cardy claimed [106] that the number of boundary theories coincide with the number of representations in the boundary spectrum. In other words that there is a one to one map between the labels α and the indices $m \in \mathcal{M}$.

Then Cardy proposed that the one point function coefficients A^n_m (before A^{α}_m) are determined

[5]Remember that the condition $m^w = \overline{m}$ was required in order to get non vanishing one point functions.

by the entries of the modular S matrix

$$A^n_m = \frac{S_n{}^m}{\sqrt{S_0{}^m}}, \tag{8.1.21}$$

with 0 indexing the representation containing the identity field. The disk one point functions of primary fields are

$$\langle \phi_{m,m^w}(z\bar{z}) \rangle_n = \frac{S_n{}^m}{\sqrt{S_0{}^m}} \frac{\mathcal{U}_{mm^w}}{|z-\bar{z}|^{2\Delta_m}}, \tag{8.1.22}$$

with $\mathcal{U}_{jj^w}$ the unitary intertwiner operator previously defined.

The *Cardy ansatz* follows from the observation that such a solution naturally solves the *Cardy condition*. Indeed, after inserting (8.1.21) in (8.1.18) one obtains

$$\sum_q A^n_{q^+}\ A^m_q\ S_{q^+}{}^k = \sum_q \frac{S_n{}^{q^+}\ S_m{}^q\ S_{q^+}{}^k}{S_0}{}_q = N_{mn^+}{}^k \in \mathbb{Z}_{\geq 0} \tag{8.1.23}$$

where the last equality is nothing but the famous *Verline formula* determining the fusion rules ($N^k_{mn^+}$) in RCFTs. It guaranties the ansatz satisfies the *Cardy condition* but does not prove that it is a solution. It may happen that fusion rules do not reproduce the spectral decomposition of the Boundary CFT or it may occur that Verlinde formula does not works, as in most Non RCFTs. Nevertheless it has shown to be a powerful framework to solve Rational BCFTs.

8.2 Conjugacy classes in AdS_3

As is well known [103], the world-volume of a symmetric D-brane on the G group manifold is given by the (twined) conjugacy classes

$$\mathcal{W}^\omega_g = \left\{\omega(h)gh^{-1}, \forall h \in G\right\}\ , \tag{8.2.1}$$

where ω determines the gluing condition connecting left and right moving currents, $\omega(g) = \omega^{-1} g\, \omega$. When ω is an inner automorphism, $\mathcal{W}^\omega_g$ can be seen as left group translations of the regular conjugacy class (of the element ωg). So, one can restrict attention to the case $\omega = id.$, and in the case of $SL(2,\mathbb{R})$ the conjugacy classes are simply given by the solution to (see (2.1.4))

$$tr\ g = 2\frac{X_0}{\ell} = 2\tilde{C}\,. \tag{8.2.2}$$

The geometry of the world-volume is then parametrized by the constant $\tilde{C}$ as

$$-{X_1}^2 - {X_2}^2 + {X_3}^2 = \ell^2\left(1 - \tilde{C}^2\right)\,. \tag{8.2.3}$$

Different geometries can be distinguished for $\tilde{C}^2$ bigger, equal or smaller than one. The former gives rise to a two dimensional de Sitter space, dS_2, the latter to a two dimensional hyperbolic space, H_2, and the case $|\tilde{C}| = 1$ splits into three different geometries: the apex, the future and the past of a light-cone.

A more convenient way to parametrize these solutions is given by the redefinition

$$\tilde{C} = \cos\sigma\,. \tag{8.2.4}$$

For $|\tilde{C}| > 1$, $\sigma = ir + \pi v,\ r \in \mathbb{R}^+,\ v \in \mathbb{Z}_2$. The world-volumes are given by

$$\cosh\rho\cos t = \pm\cosh r\,. \tag{8.2.5}$$

Each circular D-string is emitted and absorbed at the boundary in a time interval of width π but does not reach the origin unless $r = 0$. Their lifetime is determined by v.

For $|\tilde{C}| < 1$, σ is real and

$$\cosh\rho\cos t = \cos\sigma\,. \tag{8.2.6}$$

If one restricts $\sigma \in (0,\pi)$, there are two different solutions for each σ, for instance one with $t \in (-\frac{\pi}{2}, -\sigma]$ and another one with $t \in [\sigma, \frac{\pi}{2})$. To distinguish between these two solutions we can take $\sigma = \lambda + \pi v,\ \lambda \in (-\pi, 0)\,,\ v \in \mathbb{Z}_2$, such that $t = \arccos(\cos\sigma/\cosh\rho)$, taking the branch where $t = \sigma$ when it crosses over the origin. Because these solutions have Euclidean signature, they are identified as instantons in AdS_3. In fact, they represent constant time slices in hyperbolic coordinates.

For $|\tilde{C}| = 1$, $\sigma = 0$ or π and

$$\cosh\rho\cos t = \pm 1\,. \tag{8.2.7}$$

For example, for $\tilde{C} = 1$, this corresponds to a circular D-string at the boundary at $t = -\pi/2$ collapsing to the instantonic solution in $\rho = 0$ at $t = 0$, and then expanding again to a D-string reaching the boundary at $t = \pi/2$.

All of these solutions are restricted to the single covering of SL(2,$\mathbb{R}$). In the universal covering, t is decompactified and the picture is periodically repeated. The general solutions can be parametrized by a pair (σ, q), $q \in \mathbb{Z}$, or equivalently, the range of σ can be extended to $\sigma = ir + q\pi$ for dS_2 branes, $\sigma = \lambda + q\pi$ for H_2 branes or $\sigma = q\pi$ for point-like and light-cone branes.

Preparing for the discussions on one-point functions and *Cardy structure*, it is interesting to note that these parameters can be naturally identified with representations of the model. For instance, one can label the D-brane solutions as

$$\sigma = \frac{2\pi}{k-2}\left(j + \frac{1}{2} - w\frac{k-2}{2}\right)\,, \tag{8.2.8}$$

with $j = -\frac{1}{2} + is,\ s \in \mathbb{R}^+,\ w \in \mathbb{Z}$ for dS_2 branes, $j \in \left(-\frac{k-1}{2}, -\frac{1}{2}\right),\ w \in \mathbb{Z}$ for H_2 branes and finally $\sigma = n\pi,\ n \in \mathbb{Z}$ for the point-like and light-cone D-brane solutions.

The appearance of the level k in a classical regime could seem awkward. However, it is useful to recall that σ is just a parameter labeling the conjugacy classes, and the factor $k-2$ can be eliminated by simply redefining j through a change of variables. The important observation is that this suggests σ labels the exact solutions, *e.g* the one-point functions at finite k will be found to be parametrized exactly by (8.2.8) and in fact, in the semiclassical regime $k \to \infty$, the domain of σ does not change at all.

When ω is an outer automorphism, one can take $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ up to group translations. In this case, the twined conjugacy classes are given by

$$tr\ \omega g = 2\frac{X_2}{\ell} = 2C\,. \tag{8.2.9}$$

The world-volume geometry now describes an AdS_2 space for all C since

$$X_0^2 - X_1^2 + X_3^2 = \ell^2(1 + C^2)\,. \tag{8.2.10}$$

These are static open D-strings with endpoints fixed at the boundary. This is obvious in cylindrical coordinates, *i.e.*

$$\sinh\rho \sin\theta = \sinh r\,, \tag{8.2.11}$$

where we have renamed $C = \sinh r$. So, after decompactifying the time-like direction t, there is no need to extend the domain of r.

Let us end this brief review with a word of caution. In this section we have reviewed the twined conjugacy classes and, although branes wrap conjugacy classes, extra restrictions appear when studying the semiclassical or exact solutions. In particular, it was found in [90] that r becomes a positive quantized parameter at the semiclassical level.

8.3 One-point functions

In this section we summarize the results for one-point functions in maximally symmetric D-branes, obtained by applying the method of [74]. The solution for one-point functions in H_2 D-branes found in *loc. cit.* holds for integer level k. Here we work with an alternative expression, equivalent to the one obtained in [74], but with a different extension for generic $k \in \mathbb{R}$.

8.3.1 One-point functions for point-like instanton branes

To obtain the one-point functions for the point-like branes, we simply take the $\mathbb{Z}_{kN}$ orbifold action on the product of the one-point functions associated to D0 branes in the cigar [101] and to Neumann boundary conditions in the U(1) theories, respectively

$$\left\langle \Phi^{sl(2)/u(1)}_{j,n,\omega}(z,\bar{z})\right\rangle^{D0}_{\mathbf{s}} = \frac{\delta_{n,0}\,(-)^{r\omega}}{|z-\bar{z}|^{h^j_{nr}+\bar{h}^j_{nr}}}\frac{\Gamma\left(-j+\frac{k}{2}\omega\right)\Gamma\left(-j-\frac{k}{2}\omega\right)}{\Gamma\left(-2j-1\right)}$$

$$\times\left(\frac{k}{k-2}\right)^{\frac{1}{4}}\left(\frac{\sin[\pi b^2]}{4\pi}\right)^{\frac{1}{2}}\frac{\sin[\mathbf{s}(2j+1)]}{\sin[\pi b^2(2j+1)]}\frac{\Gamma\left(1+b^2\right)\ \nu^{1+j}}{\Gamma\left(1-b^2(2j+1)\right)}\,, \qquad (8.3.1)$$

and
$$\left\langle \Phi^{u(1)}_{\tilde{n},\tilde{\omega}}(z,\bar{z})\right\rangle^{\mathcal{N}}_{x_0} = \frac{\delta_{\tilde{n},0}e^{i\tilde{\omega}x_0}\left(\sqrt{k/2}R\right)^{\frac{1}{2}}}{|z-\bar{z}|^{\frac{k}{2}\tilde{\omega}^2}}\,.$$

Here $\mathbf{s}=\pi r b^2,\ r\in\mathbb{N},\, b^2=\frac{1}{k-2}$, ν was defined in (3.2.11), $\mathcal{N}$ refers to Neumann boundary conditions[6] and x_0 is the position of the D0 brane in the timelike direction. In the single covering of SL(2,$\mathbb{R}$), the only possibilities are $x_0=0$ and π, which represent the center of the group $\mathbb{Z}_2$ (see [91]). But in the universal covering, one can take $x_0=q\pi$ with $q\in\mathbb{Z}$ (see section 8.2).

To compare these one-point functions with those obtained in section 8.4, it is convenient to consider the conventions introduced in section 3.2.2 but with a different normalization in order to explicitly realize the relation between the spectral flow image of highest and lowest weight representations. The fields $\tilde{\Phi}^{j,w}_{m,\bar{m}}$ represent the spectral flow images of the primary fields $\tilde{\Phi}^{j,0}_{m,\bar{m}}$, *i.e* they are in correspondence with highest or lowest weight states depending if $w<0$ or $w>0$, and have $J^3_0,\ \bar{J}^3_0$ eigenvalues $M=m+\frac{k}{2}w,\ \bar{M}=\bar{m}+\frac{k}{2}w$. They are related to the vertex operators (3.2.20) as

$$\tilde{\Phi}^{j,w}_{m,\bar{m}}(z,\bar{z}) = (-)^w\sqrt{\nu^{-\frac{1}{2}-j}B(j)}\ \ V^{-1-j}_{n\omega\gamma p\tilde{\omega}}(z,\bar{z})\,. \qquad (8.3.2)$$

When looking for $w=0$ solutions, *i.e.* $\omega=-\tilde{\omega}$, one expects to reproduce the one-point

[6] In section 8.4 when explicitly constructing the coherent states we will consider Dirichlet gluing conditions. Here we take Neumann boundary conditions because this is the T dual version in the time direction.

functions of point-like D-branes in the H_3^+ model, which forces $x_0 = r\pi$. So,

$$\left\langle \tilde{\Phi}^{j,w}_{m,\bar{m}}(z,\bar{z}) \right\rangle_{\mathbf{s}} = \frac{\delta_{m,\bar{m}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}} \frac{\Gamma(1+j-m)\,\Gamma(1+j+m)}{\Gamma(2j+1)} \times \frac{i\sqrt{k}\;(-)^{w+1}}{2^{\frac{5}{4}}} \frac{\sin[\mathbf{s}\,((2j+1)-w(k-2))]}{\sqrt{\sin[\frac{\pi}{k-2}\,(2j+1)]}}\,, \tag{8.3.3}$$

with the parameter $\mathbf{s}$ labeling the positions of the instanton solutions.

8.3.2 One point-functions for H_2, dS_2 and light-cone branes

All of the H_2, dS_2 and light-cone branes can be constructed from a D2-brane in the cigar and taking Neumann boundary conditions in the U(1). They are simply related to each other by analytic continuation of a parameter labeling the scale of the branes. Here, we discuss in detail the case of the one-point functions of fields in discrete representations on H_2 branes in order to prepare the discussion for section 8.4. The approach developed in this section to construct the one-point functions gives the correlators associated to H_2 branes placed at the surface $X^3 = cons$ rather than that at $X^0 = cons$ presented in section 8.2, so we have to translate these solutions before comparing with the results of the previous section.

The one point-functions for the D2-branes in the cigar are given by [101]

$$\left\langle \Phi^{sl(2)/u(1)}_{j,n,\omega}(z,\bar{z}) \right\rangle^{D2}_{\tilde{\sigma}} = \frac{\frac{1}{2}\delta_{n,0}(-)^{\omega} e^{-i\tilde{\sigma}\omega(k-2)} \left(\frac{k-2}{k}\right)^{\frac{1}{4}}}{|z-\bar{z}|^{h^j_{nr}+\bar{h}^j_{nr}}} \Gamma(1+2j)\,\Gamma\left(1+\frac{1+2j}{k-2}\right)\nu^{\frac{1}{2}+j} \times \left(\frac{\Gamma\left(-j+\frac{k}{2}\omega\right)}{\Gamma\left(1+j+\frac{k}{2}\omega\right)} e^{i\tilde{\sigma}(1+2j)} + \frac{\Gamma\left(-j-\frac{k}{2}\omega\right)}{\Gamma\left(1+j-\frac{k}{2}\omega\right)} e^{-i\tilde{\sigma}(1+2j)} \right).$$

Notice that this differs from the result in [101] by the ω dependent phase $(-)^{\omega}\; e^{-i\tilde{\sigma}\omega(k-2)}$.[7] The

[7] This phase that we added by hand is required by the spectral flow symmetry, when used to construct the one-point functions for H_2 branes, which demands $\left\langle \tilde{\Phi}^{j,w}_{j,j} \right\rangle^{H_2} = \left\langle \tilde{\Phi}^{-\frac{k}{2}-j,w-1}_{\frac{k}{2}+j,\frac{k}{2}+j} \right\rangle^{H_2}$, in our conventions. The one-point function for D2 branes was constructed in [101] beginning from the parent H_3^+ model and was found to have some sign problems. We claim this phase cannot be deduced from the H_3^+ model because of the absence of spectral flowed states. It would be interesting to investigate the implications of this modification in the sign. Unfortunately, this information cannot be obtained from the w independent semiclassical limit of the one-point functions.

position of the D-brane over the $U(1)$ is again fixed by the one-point function of the H_3^+ model. We find

$$\left\langle \tilde{\Phi}^{j,w}_{m,\bar{m}}(z,\bar{z}) \right\rangle^{H_2,X^3}_{\tilde{\sigma}} = \frac{\delta_{m,\bar{m}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}} \frac{-1}{2^{\frac{5}{4}}\sqrt{i}(k-2)^{\frac{1}{4}}} \frac{\pi\ e^{-i\tilde{\sigma}w(k-2)}}{\sqrt{\sin\left[\frac{\pi}{k-2}(2j+1)\right]}}$$

$$\times \left(\frac{\Gamma(1+j-m)}{\Gamma(-j-m)} e^{-i\tilde{\sigma}(1+2j)} + \frac{\Gamma(1+j+m)}{\Gamma(-j+m)} e^{i\tilde{\sigma}(1+2j)} \right) . \qquad (8.3.4)$$

For fields in discrete representations with $m = -j + \mathbb{Z}_{\geq 0}$ and $j \notin \mathbb{Z}$, only one factor survives in the last line. Here $\tilde{\sigma}$ is a real parameter, determining the embedding of the brane in AdS_3 as $X^3 = \cosh\rho \sin t = \sin\tilde{\sigma}$. So, in order to compare with the solutions discussed in section 8.2, the identification $\tilde{\sigma} = \sigma + \frac{\pi}{2}$ and the global shift in the time-like coordinate on the cylinder, namely $t \to t + \frac{\pi}{2}$, must be perfomed. The latter simply adds a phase $e^{i\frac{\pi}{2}(M+\bar{M})}$ [8].

From the analysis of conjugacy classes, it is natural to relabel $\sigma = \frac{\pi}{k-2}(2j'+1) - w'\pi$, with $j' \in (-\frac{k-1}{2}, -\frac{1}{2})$, $w' \in \mathbb{Z}$ [9], and

$$\left\langle \tilde{\Phi}^{j,w}_{m,\bar{m}}(z,\bar{z}) \right\rangle^{H_2,X^0}_{\sigma(j',w')} = \frac{\delta_{m,\bar{m}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}} \frac{\Gamma(1+j+m)\Gamma(1+j-m)}{\Gamma(1+2j)}$$

$$\times \frac{-\pi\sqrt{-i}}{2^{\frac{5}{4}}(k-2)^{\frac{1}{4}}} \frac{(-)^w e^{\frac{4\pi i}{k-2}(j'+\frac{1}{2}-w'\frac{k-2}{2})(j+\frac{1}{2}-w\frac{k-2}{2})}}{\sqrt{\sin\left[\frac{\pi}{k-2}(2j+1)\right]}} . \qquad (8.3.5)$$

8.3.3 One-point functions for AdS_2 branes

For completeness, we display here the one-point functions for AdS_2 branes obtained in [74], in our conventions. These are constructed by gluing two one-point functions: one for a D1-brane in the coset model and another one with Dirichlet boundary conditions in the U(1) model. The result is

$$\left\langle \tilde{\Phi}^{j,w}_{m,\bar{m}}(z,\bar{z}) \right\rangle^{AdS_2}_{r} = \frac{\delta_{w,0}\delta_{m,-\bar{m}} e^{-i\frac{\pi}{4}} e^{in(\theta_0+x_0)} \left(\frac{k-2}{2}\right)^{\frac{1}{4}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}} \frac{\Gamma(-1-2j)}{\Gamma(-j-m)\Gamma(-j+m)}$$

[8] In fact, $J_0^3 + \bar{J}_0^3$ gives the energy in AdS_3 and so this combination is the generator of t translations.

[9] The one-point functions for dS_2 branes are given by (8.3.4) with $j' \in \left\{-\frac{1}{2} + i\mathbb{R}^+\right\}$ and for light-cone branes, they are given by $\sigma = n\pi$, $n \in \mathbb{Z}$.

$$\times \cos\left(ir(j+\frac{1}{2})+m\pi\right)\Gamma\left(1-\frac{1+2j}{k-2}\right)\nu^{-\frac{1}{2}-j}\,, \qquad (8.3.6)$$

where θ_0 is related to the angles (in cylindrical coordinates) to which the branes asymptote when they get close to the boundary of AdS_3, x_0 is the location of the brane and r determines their scale. From the geometrical point of view, r seems to be an arbitrary real number, but as shown in [90], it becomes quantized at the semiclassical level.

8.4 Coherent states and relation to modular data

Boundary states play a fundamental role in understanding boundary conformal field theories. They store all the information about possible D-brane solutions and their couplings to bulk states. Following the ideas developed in section 8.1, one can study different gluing conditions for the left and right current modes, consistent with the affine algebra [104] as well as with the conformal symmetry via the Sugawara construction [105].

In the case of AdS_3, much of the progress reached in this direction is based on the analytic continuation from H_3^+ [93]. Gluing conditions were imposed as differential equations applied directly to find, with the help of certain *sewing contraints*, the one-point functions of maximally symmetric D-branes. It would be interesting to get the one-point functions of the AdS_3 model without reference to other models, but the approach used so far cannot be easily extended. In the first place, it was developed in the x-basis of the H_3^+ model, which is not a good basis for the representations of the universal covering of SL(2,$\mathbb{R}$). Suitable bases instead are the m- or t-basis [37, 59]. Moreover, there are still some open questions about the fusion rules of the AdS_3 model [43] which deserve further attention before analyzing the *sewing constraints*. Therefore, we will not compute the one-point functions in this way, but will give the first step in this direction by finding the explicit expressions for the Ishibashi states in the m-basis for all the representations of the Hilbert space of the bulk theory.

8.4.1 Coherent states for regular gluing conditions

Boundary states associated to dS_2, H_2, light-cone and point-like D-branes in AdS_3 must satisfy the following regular gluing conditions [92]

$$\begin{aligned}\left(J^3_n - \bar{J}^3_{-n}\right)|\mathbf{s}\rangle &= 0,\\ \left(J^\pm_n + \bar{J}^\mp_{-n}\right)|\mathbf{s}\rangle &= 0,\end{aligned} \tag{8.4.1}$$

where $\mathbf{s}$ labels the members of the family of branes allowed by the gluing conditions.

These constraints are linear and leave each representation invariant, so that the boundary states must be expanded as a sum of solutions in each representation. The solutions are coherent states, usually called Ishibashi states [105].

Let us begin introducing the following notation which will be useful in the subsequent discussions. Let

$$|j,w,\alpha;\mathrm{n},\mathrm{m}\rangle = |j,w,\alpha\rangle\,\{|\mathrm{n}\rangle\otimes\overline{|\mathrm{m}\rangle}\},\quad |j,w,+;\mathrm{n},\mathrm{m}\rangle = |j,w,+\rangle\,\{|\mathrm{n}\rangle\otimes\overline{|\mathrm{m}\rangle}\}, \tag{8.4.2}$$

denote orthonormal bases for $\hat{\mathcal{C}}^{\alpha,w}_j\otimes\hat{\mathcal{C}}^{\alpha,w}_j$ and $\hat{\mathcal{D}}^{+,w}_j\otimes\hat{\mathcal{D}}^{+,w}_j$, respectively. They satisfy[10]

$$\begin{aligned}\langle j,w,\alpha;\mathrm{n},\mathrm{m}|j',w',\alpha';\mathrm{n}',\mathrm{m}'\rangle &= \langle j,w,\alpha|j,w,\alpha\rangle \times \langle\mathrm{n}|\mathrm{n}'\rangle\times\overline{\langle\mathrm{m}|\mathrm{m}'\rangle}\\ &= \delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha')\ \epsilon_{\mathrm{n}}\delta_{\mathrm{n},\mathrm{n}'}\ \epsilon_{\mathrm{m}}\delta_{\mathrm{m},\mathrm{m}'},\end{aligned}$$

$$\begin{aligned}\left\langle j,w,+;\mathrm{n},\mathrm{m}|j',w',+;\mathrm{n}',\mathrm{m}'\right\rangle &= \langle j,w,+|j,w,+\rangle\left\langle\mathrm{n}|\mathrm{n}'\right\rangle\overline{\langle\mathrm{m}}\,\overline{|\mathrm{m}'\rangle}\\ &= \delta(j-j')\delta_{w,w'}\ \epsilon_n\delta_{\mathrm{n},\mathrm{n}'}\ \epsilon_{\mathrm{m}}\delta_{\mathrm{m},\mathrm{m}'},\end{aligned} \tag{8.4.3}$$

$\{|\mathrm{n}>\}$ is an orthonormal basis in $\hat{\mathcal{C}}^{\alpha,w}_j$ (or $\hat{\mathcal{D}}^{+,w}_j$) for which the expectation values of $J^3_n, J^\pm_n$ are real numbers and $\epsilon_{\mathrm{n}} = \pm 1$ is its norm squared. It is constructed by the action of the affine currents over the ket $|j,w,m=\alpha>= U_w|j,m=\alpha>$ $(|j,w,m=-j>)$.

[10] The separation between $|j,w,\alpha\rangle$ or $|j,w,+\rangle$ and $|\mathrm{n}\rangle,\overline{|\mathrm{m}\rangle}$ in different kets is simply a matter of useful notation for calculus and does not denote tensor product.

The Ishibashi states for continuous and discrete representations are found to be

$$|j,w,\alpha \gg = \sum_{\mathrm{n}} \epsilon_{\mathrm{n}} \bar{V} \, |j,w,\alpha;\mathrm{n},\mathrm{n}\rangle \quad \text{and} \quad |j,w,+ \gg = \sum_{\mathrm{n}} \epsilon_{\mathrm{n}} \bar{V} \, |j,w,+;\mathrm{n},\mathrm{n}\rangle \,, \tag{8.4.4}$$

respectively, where V is defined as the linear operator satisfying

$$\begin{aligned} V \prod_I J^{a_I}_{n_I} \, |j,w,m=-j\rangle &= \prod_I \eta_{a_I b_I} J^{b_I}_{n_I} \, |j,w,m=-j\rangle \,, \\ V \prod_I J^{a_I}_{n_I} \, |j,w,m=\alpha\rangle &= \prod_I \eta_{a_I b_I} J^{b_I}_{n_I} \, |j,w,m=\alpha\rangle \,, \end{aligned} \tag{8.4.5}$$

with $a = 1,2,3$, $\eta_{ab} = diag(-1,-1,1)$ and the bar denotes action restricted to the antiholomorphic sector. It is easy to see that this defines a unitary operator. The proof that they are solutions to (8.4.1) follows similar lines as those of [105]. As an example, let us consider an arbitrary base state $|j',w',\alpha';\mathrm{n}',\mathrm{m}'>$:

$$\begin{aligned} &< j',w',\alpha';\mathrm{n}',\mathrm{m}'|J^3_r - \bar{J}^3_{-r}|j,w,\alpha \gg = \\ &\delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha') \sum_{\mathrm{n}} \epsilon_{\mathrm{n}} \left\langle \mathrm{n}' \right| J^3_n \left|\mathrm{n}\right\rangle \overline{\langle \mathrm{m}' | \bar{V} | \mathrm{n} \rangle} - \epsilon_{\mathrm{n}} \left\langle \mathrm{n}' \right| \mathrm{n}\rangle \overline{\langle \mathrm{m}' | \bar{J}^3_{-n} \bar{V} | \mathrm{n}\rangle} = \\ &\delta(s-s')\delta_{w,w'}\delta(\alpha-\alpha') \sum_{\mathrm{n}} \epsilon_{\mathrm{n}} \left\langle \mathrm{n}' \right| J^3_n \left|\mathrm{n}\right\rangle \left\langle \mathrm{n}\right| V \left|\mathrm{m}'\right\rangle - \epsilon_{\mathrm{n}} \left\langle \mathrm{n}' \right| \mathrm{n}\rangle \left\langle \mathrm{n}\right| V J^3_n \left|\mathrm{m}'\right\rangle = 0 \,. \end{aligned}$$

The normalization fixed above for the Ishibashi states implies

$$\begin{aligned} \ll j,w,\alpha|e^{\pi i \tau \left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} e^{\pi i \theta (J^3_0 + \bar{J}^3_0)} |j',w',\alpha' \gg &= \delta(s-s') \, \delta_{w,w'} \, \delta(\alpha-\alpha') \, \chi^{\alpha,w}_j(\tau,\theta), \\ \ll j,w,+|e^{\pi i \tau \left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} e^{\pi i \theta (J^3_0 + \bar{J}^3_0)} |j',w',+ \gg &= \delta(j-j') \, \delta_{w,w'} \, \chi^{+,w}_j(\tau,\theta). \end{aligned} \tag{8.4.6}$$

8.4.2 *Cardy structure* and one-point functions for point-like branes

Assuming that after Wick rotation the open string partition function in AdS_3 reproduces that of the H_3^+ model and a generalized Verlinde formula, we show in this section that the one-point functions on localized branes in AdS_3 previously found in [74] (and reviewed in section 8.3) can be recovered. We also verify that the one-point functions on point-like and H_2 D-branes exhibit a *Cardy structure*. Usually, this structure is accompanied by a Verlinde formula for the representations appearing in the boundary spectrum. In fact, the *Cardy structure* is a natural solution to the *Cardy condition* when the Verlinde theorem holds. However, as we shall discuss, the latter does not hold in the AdS_3 WZNW model. The generalized Verlinde formula proposed in appendix E reproduces the fusion rules of the degenerate representations, but it gives contributions to the fusion rules of the discrete representations with an arbitrary amount of spectral flow, thus contradicting the selection rules determined in [40]. Nevertheless, we find a *Cardy structure*.

Boundary states

Worldsheet duality allows to write the one loop partition function for open strings ending on point-like branes labeled by $\mathbf{s}_1$ and $\mathbf{s}_2$ as

$$
\begin{aligned}
e^{-2\pi i\frac{k}{4}\frac{\theta^2}{\tau}}\; Z^{AdS_3}_{\mathbf{s}_1\mathbf{s}_2}(\theta,\tau,0) &= \left\langle \Theta\mathbf{s}_1\right|\tilde{q}^{H^{(P)}}\,\tilde{z}^{J_0^3}\left|\mathbf{s}_2\right\rangle \\
&= \sum_{w=-\infty}^{\infty}\int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj\; \mathcal{A}^{\mathbf{s}_1}_{(j,w)}\mathcal{A}^{\mathbf{s}_2}_{(j^+,w^+)}\chi^{+,w}_j(\tilde{\theta},\tilde{\tau},0)\;+\;ccr\,,
\end{aligned}
$$

where Θ denotes the worldsheet CPT operator in the bulk theory, $\tilde{q}=e^{2\pi i\tilde{\tau}}$, $\tilde{z}=e^{2\pi i\tilde{\theta}}$, $\tilde{\tau}=-1/\tau$, $\tilde{\theta}=\theta/\tau$, (j^+,w^+) refer to the labels of the (j,w)-conjugate representations, ccr denotes the contributions of continuous representations and $\mathcal{A}^{\mathbf{s}}_{(j,w)}$ are the Ishibashi coefficients of the boundary states.

The open string partition function for the "spherical branes" of the H_3^+ model was found in [89] for $\theta=0$ and extended to the case $\theta\neq 0$ in [101]. It reads

$$
Z^{H_3^+}_{\mathbf{s}_1\mathbf{s}_2}(\theta,\tau,0)=\sum_{J_3=|J_1-J_2|}^{J_1+J_2}\chi_{J_3}(\theta,\tau,0)\,, \tag{8.4.7}
$$

where $\mathbf{s}_i=\frac{\pi}{k-2}(1+2J_i)$ and $1+2J_i\in\mathbb{N}$. This reveals an open string spectrum of discrete degenerate representations.

The Lorentzian partition function is expected to reproduce that of the H_3^+ model after analytic continuation in θ and τ. Then, if we concentrate on the one-point functions of fields in discrete representations, we only need to consider the case $\theta + n\tau \notin \mathbb{Z}$. Thus, using the generalized Verlinde formula (see appendix E for details), namely

$$\sum_{J_3=|J_1-J_2|}^{J_1+J_2} \chi_{J_3}(\theta,\tau,0) = \sum_{w=-\infty}^{\infty} \int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj \; \frac{S_{J_1}{}^{j,w} S_{J_2}{}^{j,w}}{S_0{}^{j,w}} \; e^{2\pi i \frac{k}{4}\frac{\theta^2}{\tau}} \chi_j^{+,w}(\frac{\theta}{\tau},-\frac{1}{\tau},0)\,, \qquad (8.4.8)$$

we obtain the following expression for the coefficients of the boundary states:

$$\mathcal{A}^{\mathbf{s}}_{(j,w)} = f(j,w)\;(-)^w \sqrt{\frac{2}{i}} \left(\frac{2}{k-2}\right)^{\frac{1}{4}} \frac{\sin\left[\mathbf{s}\,(1+2j-w(k-2))\right]}{\sqrt{\sin\left[\frac{\pi}{k-2}\,(1+2j)\right]}}\,, \qquad (8.4.9)$$

defined up to a function $f(j,w)$ satisfying

$$f(j,w)\; f(-\frac{k}{2}-j,-w-1) = 1. \qquad (8.4.10)$$

One-point functions

To find the one-point functions associated to these point-like branes, let us make use of the definition (8.1.9) of the boundary states (see also [107])[11]:

$$\left\langle \Phi^{(H)}\left(|j,w,m,\bar{m}\rangle\,;z,\bar{z}\right)\right\rangle_{\mathbf{s}} = \left(\frac{d\xi}{dz}\right)^{\Delta_j} \left(\frac{d\bar{\xi}}{d\bar{z}}\right)^{\bar{\Delta}_j} \langle 0|\,\Phi^{(P)}\left(|j,w,m,\bar{m}\rangle\,;\xi,\bar{\xi}\right)|\mathbf{s}\rangle\;, \qquad (8.4.11)$$

where $\Phi^{(H)}\left(|j,w,m,\bar{m}\rangle\,;z,\bar{z}\right)$ ($\Phi^{(P)}\left(|j,w,m,\bar{m}\rangle\,;\xi,\bar{\xi}\right)$) is the bulk field of the boundary (bulk) CFT corresponding to the state inside the brackets[12], $z,\bar{z}$ denote the coordinates of the upper half plane and $\xi,\bar{\xi}$ those of the exterior of the unit disc.

Conformal invariance forces the *l.h.s.* of (8.4.11) to be

$$\left\langle \Phi^{(H)}\left(|j,w,m,\bar{m}\rangle\,;z,\bar{z}\right)\right\rangle_{\mathbf{s}} = \frac{\mathcal{B}(\mathbf{s})^{j,w}_{m,\bar{m}}}{|z-\bar{z}|^{\Delta_j+\bar{\Delta}_j}}\,, \qquad (8.4.12)$$

[11]Strictly speaking, this identity is valid on a Euclidean worldsheet. However, it is appropriate to use it here since we want to explore the relation of our results with those of the Euclidean model defined in [74] where the coefficients of the one-point functions are assumed to coincide with those of the Lorentzian AdS_3.

[12]Here $|j,w,m,\bar{m}>$ is a shorthand notation for $|j,w,m> \otimes |j,w,\bar{m}>$ and it must be distinguished from the orthonormal basis introduced in section 8.4.1.

where the z-independent factor $\mathcal{B}(\mathbf{s})^{j,w}_{m,\bar{m}}$ is not fixed by the conformal symmetry. The solution (8.4.4), (8.4.5) implies

$$\mathcal{B}(\mathbf{s})^{j,w}_{m,\bar{m}} = (-)^{j+m}\delta_{m,\bar{m}}\mathcal{A}^{\mathbf{s}}_{j,w} \quad , \tag{8.4.13}$$

from which the spectral flow symmetry determines $f = 1$. [13]

It is important to note that the normalization used here differs from the one usually considered in the literature. Our normalization is such that the spectral flow image of the primary operator corresponding to the state $|j, w, m, \bar{m} >$ is normalized to 1. In particular, it implies the following operator product expansions

$$\begin{aligned} J^3(\zeta)\Phi^{(P)}\left(|j,w,m,\bar{m}\rangle ; \xi, \bar{\xi}\right) &= \frac{m+\frac{k}{2}w}{\zeta-\xi}\Phi^{(P)}\left(|j,w,m,\bar{m}\rangle ; \xi, \bar{\xi}\right) + \ldots \\ J^{\pm}(\zeta)\Phi^{(P)}\left(|j,w,m,\bar{m}\rangle ; \xi, \bar{\xi}\right) &= \frac{\sqrt{-j(1+j)+m(m\pm 1)}}{(\zeta-\xi)^{1\pm w}}\Phi^{(P)}\left(|j,w,m\pm 1,\bar{m}\rangle ; \xi, \bar{\xi}\right) \\ &\quad + \ldots \end{aligned} \tag{8.4.15}$$

Comparing with the OPE of $\tilde{\Phi}^{j,w}_{m,\bar{m}}\left(\xi, \bar{\xi}\right)$ which coincides with (3.2.16) we obtain the following relation, valid for $m = \bar{m} \in -j + \mathbb{Z}_{\geq 0}$,

$$\tilde{\Phi}^{j,w}_{m,\bar{m}}(\xi,\bar{\xi}) = \Omega\ (-)^{j+m}\frac{\Gamma\left(1+j-m\right)\Gamma\left(1+j+m\right)}{\Gamma\left(1+2j\right)}\ \Phi^{(P)}\left(|j,w,m,\bar{m}\rangle ; \xi, \bar{\xi}\right) \ , \tag{8.4.16}$$

where Ω is the normalization of $\Phi^{j,w}_{-j,-j}$. We find perfect agreement between the expressions (8.4.13) and (8.3.3) for one-point functions, as long as $\Omega = -\sqrt{\frac{-ik(k-2)}{16}}$.

[13] In fact, the identification $< \Phi^{(H)}\left(|j,w,m=\bar{m}=-j\rangle ; z,\bar{z}\right) > = < \Phi^{(H)}\left(|j',w',m'=\bar{m}'=j'\rangle ; z,\bar{z}\right) >$ where $j' = -\frac{k}{2} - j,\ w' = w+1$ together with (8.4.13) requires $f(j,w) = f(-\frac{k}{2}-j, w+1)$ which implies

$$f(j,w) = f(j,w+2) \tag{8.4.14}$$

So it is sufficient to find $f(j,0)$ and $f(j,1)$. It is immediate from this constraint and (8.4.10) that $f(j,0)^2 = f(j,1)^2 = 1$. But as f must be continuous in j, it must be j independent, $f(j,w) = f(w) = \pm 1$ (*i.e.* the same sign for every w). To choose the upper or the lower sign is simply a matter of convention.

8.4.3 *Cardy structure* in H_2 branes

In section 8.3 we reviewed the results for the one-point functions in maximally symmetric D-branes obtained by applying the method of [74]. From the one for fields in discrete representations on H_2 branes we find the following Ishibashi coefficients (see (8.3.5) and (8.4.16))

$$\mathcal{A}_{(j,w)}^{\sigma'\equiv(j',w')} = \frac{\pi}{\sqrt{k}}\left(\frac{2}{k-2}\right)^{\frac{3}{4}} \frac{(-)^{w} e^{\frac{4\pi i}{k-2}(j'+\frac{1}{2}-w'\frac{k-2}{2})(j+\frac{1}{2}-w\frac{k-2}{2})}}{\sqrt{\sin\left[\frac{\pi}{k-2}(2j+1)\right]}} \sim (-)^{w'} \frac{S_{j,w}{}^{j',w'}}{\sqrt{S_0{}^{j,w}}}\,, \quad (8.4.17)$$

which exhibit a clear compromise between microscopic and modular data. It gives the following degeneracy in the open string spectrum

$$\mathcal{N}_{j_1,w_1\ j_2,w_2}{}^{j_3,w_3} \sim \delta_{\mathcal{D}_j^{+,w}} \sum_{m=-\infty}^{\infty} \delta\left(j_2+j_3-j_1-(w_2+w_3-w_1)\frac{k-2}{2}+m\right)\,,$$

where the divergent integral $\int_0^1 d\lambda \frac{e^{-2\pi i\left(m+\frac{1}{2}\right)\lambda}}{2i\sin(\pi\lambda)}$ has been replaced by its principal value, $\frac{1}{2}$. $\delta_{\mathcal{D}_j^{+,w}}$ is one when j is within the unitary bound and vanishes if it is not. $\sim$ means up to the factor $\frac{-i\,2\,\pi^2(-)^{w_3}}{k(k-2)}$ which would imply that either the one point function computed in [74] has a subtle modification to ensure integer coefficients in the boundary spectrum or such a factor should be supplemented by the contribution of other representations appearing in the boundary spectrum (associated to open string) not contained in the AdS_3 WZNW spectrum (associated to closed strings) which transform under the modular group in terms of the spectral flow images of the principal lowest weight representations.

Contrary to what happens in RCFT with maximally symmetric D-branes, the Boundary CFT spectrum does not coincide with the fusion rules of the Bulk CFT. The consequence of this observation is the failure of the Verlinde Theorem.

8.4.4 Coherent states for twined gluing conditions

The gluing conditions defining the coherent states $|j, w \gg$ for AdS_2 branes [92], frequently called twisted boundary conditions, are

$$\left(J_n^3 + \bar{J}_{-n}^3\right)|j, w \gg = 0, \qquad \left(J_n^{\pm} + \bar{J}_{-n}^{\pm}\right)|j, w \gg = 0\,. \tag{8.4.18}$$

These constraints are highly restrictive. As we show below, coherent states satisfying these conditions can only be found for representations where the holomorphic and antiholomorphic sectors are conjugate of each other, *i.e.* only for $w = 0$, $\alpha = 0, \frac{1}{2}$ continuous representations in the AdS_3 model.

Let us assume $|j, w \gg$ is an Ishibashi state associated to the spectral flow image of a discrete or continuous representation. The spectral flow transformation (3.2.2) allows to translate the problem of solving (8.4.18) to that of solving

$$\left(J_n^3 + \bar{J}_{-n}^3 + kw\delta_{n,0}\right)|j\rangle^w = 0\,, \qquad \left(J_n^{\pm} + \bar{J}_{-n\mp 2w}^{\pm}\right)|j\rangle^w = 0\,, \tag{8.4.19}$$

where $|j\rangle^w = U_{-w}\bar{U}_{-w}|j, w \gg$ is in an unflowed representation[14].

The special case $n = 0$ in (8.4.19) implies $2\alpha + kw \in \mathbb{Z}$ and $-2j + kw \in \mathbb{Z}$ for continuous and discrete representations, respectively. In particular, for $w = 0$ continuous representations there are two solutions with $\alpha = 0, \frac{1}{2}$, given by

$$|j, 0, \alpha \gg = \sum_{\mathrm{n}} \epsilon_n \bar{U}\,|j, w, \alpha; \mathrm{n}, \mathrm{n}\rangle\,, \tag{8.4.20}$$

where the antilinear operator U is defined by

$$U \prod_I J_{n_I}^{a_I}\,|j, w = 0, m = \alpha\rangle = \prod_I -J_{n_I}^{a_I}\,|j, w = 0, m = -\alpha\rangle\;. \tag{8.4.21}$$

[14]Notice that in the case $w = -\bar{w}$ discussed in [92] for the single covering of SL(2,$\mathbb{R}$), one gets (8.4.18) with the unflowed $|j >^{w,\bar{w}}$ state replacing $|j, w \gg$ instead of (8.4.19). Then, once an Ishibashi state is found for $w = -\bar{w} = 0$, the solutions for generic representations with $w = -\bar{w}$ are trivially obtained applying the spectral flow operation, and coherent states in arbitrary spectral flow sectors are found. This fails in AdS_3 and thus the discussion in *loc. cit.* does not apply here, except for $w = 0$ discrete or $w = 0$, $\alpha = 0, \frac{1}{2}$ continuous representations.

It can be easily verified that this defines an antiunitary operator and it is exactly the same Ishibashi state found in SU(2) [105].

To understand why there are no solutions in other modules, let us expand the hypothetical Ishibashi state in the orthonormal base $|j, w, \zeta > \{|\mathrm{n} > \otimes \overline{|\mathrm{m} >}\}$, with $\zeta = \alpha$ or $+$ and $|\mathrm{n} >$, $\overline{|\mathrm{m} >}$ eigenvectors of J_0^3, L_0 and $\bar{J}_0^3, \bar{L}_0$ respectively. The constraint that Ishibashi states are annihilated by $L_0 - \bar{L}_0$ forces $|\mathrm{n} >, \overline{|\mathrm{m} >}$ to be at the same level. But taking into account that all modules at a given level are highest or lowest weight representations of the zero modes of the currents (with the only exception of $w = 0$ continuous representations) and the fact that the eigenvalues of the highest (lowest) weight operators decrease (increase) after descending a finite number of levels, the first equation in (8.4.18) with $n = 0$ has no solution below certain level. This implies that below that level there are no contributions to the Ishibashi states and so, using for instance the constraint $(J_1^a + \bar{J}_{-1}^a)|j, w \gg = 0$, it is easy to show by induction that no level contributes to the coherent states.

The coherent states defined above are normalized as

$$\ll j, 0, \alpha| e^{\pi i \tau \left(L_0 + \bar{L}_0 - \frac{c}{12}\right)} e^{\pi i \theta (J_0^3 - \bar{J}_0^3)} |j', 0, \alpha' \gg \;=\; \delta(s - s')\delta(\alpha - \alpha')\; \chi_j^{\alpha,0}(\tau, \theta), \qquad (8.4.22)$$

for $\alpha = 0, \frac{1}{2}$. The fact that it is only possible to construct Ishibashi states associated to $w = 0$ continuous representations is again in agreement with the one-point functions found in [74] and the conjecture in [99] that only states in these representations couple to AdS_2 branes.

Chapter 9

Conclusions and perspectives

Now it is time to summarize and to discuss the results obtained during the doctoral activity and to indicate possible directions to continue these investigations.

String theory in anti de Sitter spaces has occupied a central place in the last twenty years. The initial motivation was to learn more about string theory with Lorentzian target space with a non trivial timelike direction. Indeed this is one of the simplest theories, if not the simplest, with this property and it took not too much time to realize the complexity and richness of the model still under study. A few years later from the initial investigations by O'Raifeartaigh *et al* [36], the *Maldacena conjecture* taught us that string theory with target AdS spaces are not only interesting models to learn more about string theory with the aim of applying it to more realistic scenarios, but it is also an incredible tool which allows us to analyze non perturbative regimes of gauge theories, with applications from quark gluon plasma and quantum gravity of black holes to condensed matter systems.

In this thesis we have concentrated on some aspects of the worldsheet theory of strings in AdS_3. This is described by a WZNW model with non compact group which implies that the Conformal Field Theory is non Rational. RCFT was extensively studied in the literature and was completely solved but unfortunately the situation is much more involved in the case of non RCFT where many of the properties which helped in the resolution of the former do not hold, *e.g.* the conformal blocks cannot be solved from purely algebraic methods, because the fact that three point functions do not vanish does not necessarily imply that the conjugate of the third field will be in the fusion rules of the first two. The proofs of the factorization and crossing symmetry of four point functions are of considerable complexity. The spectrum contains sectors where L_0 is not bounded from below, and some properties like the Cardy formula, determining the one point functions of the Boundary CFT, and the Verlinde Theorem, which determines the fusion rules of the Bulk CFT and the Open string spectrum of the Boundary CFT, do not work any more.

Some non RCFTs, like Liouville theory and the H_3^+ model were successfully solved, both the Bulk and the Boundary CFTs. Instead, even though much is known today about the AdS_3 WZNW model, its spectrum was completely determined, there is great confidence on the one point functions of maximally symmetric and some symmetry breaking D-branes and on the structure constants, it has not been completely solved as the crossing symmetry has not yet been established.

We made the following contributions to the the resolution of the AdS_3 WZNW model.

We determined the operator algebra as presented in chapter 4. Performing the analytic continuation of the expressions in the Euclidean H_3^+ model proposed in [41, 42] and adding spectral flow structure constants, we obtained the OPE of spectral flow images of primary fields in the Lorentzian theory. We have argued that the spectral flow symmetry forces a truncation in order to avoid contradictions and we have shown that the consistent cut amounts to the closure of the operator algebra on the Hilbert space of the theory as only operators outside the physical spectrum must be discarded and moreover, every physical state contributing to a given OPE is also found to appear in all possible equivalent operator products. The fusion rules obtained in this way are consistent with results in [40], deduced from the factorization of four-point functions of $w = 0$ short strings in the boundary conformal field theory, and contain in addition operator products involving states in continuous representations. A discussion of the relation between our results and some conclusions in [40] can also be found in chapter 4. Several consistency checks have been performed in this chapter and the OPE displayed in items 1. to 3. of section 4.2.5 can then be taken to stand on solid foundations.

Given that scattering amplitudes of states of string theory on AdS_3 should be obtained from correlation functions in the AdS_3 WZNW model, our results constitute a step forward towards the construction of the scattering S-matrix in string theory on Lorentzian AdS_3 and to learn more about the dual conformal field theory on the boundary through AdS/CFT, in the spirit of [40]. Indeed an important application of our results would be to construct the S-matrix of long strings in AdS_3 which describes scatterings in the CFT defined on the Lorentzian two-dimensional boundary. In particular, the OPE of fields in spectral flow non trivial continuous representations obtained here sustains the expectations in [40] that short and long strings should appear as poles in the scattering of asymptotic states of long strings.

The full consistency of the fusion rules should follow from a proof of factorization and crossing symmetry of the four-point functions. An analysis of the factorization of amplitudes involving states in different sectors of the theory was presented in chapter 5. As an interesting check we have shown that using the factorization ansatz and OPE obtained in the previous chapter we can reproduce the spectral flow selection rules of four and higher point functions determined in [40]. We illustrated in one example that the amplitudes must factorize as expected in order to avoid inconsistencies, *i.e.* only states according to the fusion rules determined in chapter 4 must propagate in the intermediate channels.

In chapter 6 we have computed the characters of the relevant representations of the AdS_3 model on the Lorentzian torus and we showed that contrary to other proposals these seem to preserve full information on the spectrum. The price to pay is that characters have now a subtler definition as distributional objects. In the following chapter we studied their modular transformations. We fully determined the generalized S matrix, which depends on the sign of τ, and showed that real modular parameters are crucial to find the modular maps, which implies that if one employs the standard method for Lorentzian models and Wick rotate the characters, these not only loose information on the spectrum but also do not transform properly under modular transformations.

We have seen that the characters of continuous representations transform among themselves under S while both kinds of characters appear in the S transformation of the characters of discrete representations. An important consequence of this fact is that the Lorentzian partition function is not modular invariant (and the departure from modular invariance is not just the sign appearing in (7.2.2)). The analytic continuation to obtain the Euclidean partition function (which must be invariant) is not fully satisfactory. Following the road of [37] and simply discarding the contact terms, one recovers the partition function of the H_3^+ model obtained in [55]. But even though modular invariant, this expression has poor information about the spectrum. Not only the characters of the continuous representations vanish in all spectral flow sectors but also those of the discrete representations are only well defined in different regions of the moduli space, depending on the spectral flow sector, so that it makes no mathematical sense to sum them in order to find the modular S transformation. An alternative approach was followed in [73], where an expression for the partition function was found starting from that of the SL(2,$\mathbb{R}$)/U(1) coset computed in [116] and using path integral techniques. Although formally divergent, it is modular invariant and allows to read the spectrum of the model[1]. It was shown that the partition function obtained in [55, 37] is recovered after some formal manipulations.

In the following chapter we computed the full set of Ishibashi states for all the gluing maps leading to maximally symmetric D-branes and to all the representations of the spectrum. The treatment of the boundary states presented here differs from previous works in related models. While we have expressed them as a sum over Ishibashi states, in other related models such as H_3^+ [93], Liouville [117] or the Euclidean black hole [101], the boundary states have been expanded, instead, in terms of primary states and their descendants. The coefficients in the latter expansions directly give the one-point functions of the primary fields. For instance, in the H_3^+ model, the gluing conditions were imposed in [93] not over the Ishibashi states but over the one-point functions. One of the reasons why this approach seems more suitable for H_3^+ is the observation that the expectation values used to fix the normalization of the Ishibashi states diverge in the hyperbolic model. As we have seen, this is not the case in AdS_3. Another difficulty in applying the standard techniques to H_3^+ models comes form the fact that the spectrum does not factorize as holomorphic times antiholomorphic sectors.

We found that there is a one to one map between the representation of the models and the

[1] The spectrum was also obtained from a computation of the Free Energy in [39].

family of maximally symmetric D-branes. Thanks to this observation we then showed that the one-point functions of fields in discrete representations coupled to point-like and to H_2 branes are determined by the generalized modular S matrix, as usual in RCFT. We also found a generalized Verlinde formula which gives the fusion rules of the degenerate representations of $sl(2)$ appearing in the spectrum of open strings attached to the point-like D-branes of the model.

There are some natural directions to follow in the future. The most important and most difficult one is to prove the crossing symmetry of four point functions. We hope that this will resolve the problem of the breaking of analyticity and that it can be used to put the fusion rules on a firmer mathematical ground.

Another interesting line of investigation is to better understand how the information is lost in the procedure implemented in [73] when reproducing the H_3^+ partition function and to explore if it is possible to find an analytic continuation of the Lorentzian partition function computed here leading to the integral expression obtained in *loc.cit.* (or an equivalent one), in a controlled way in which the knowledge on the spectrum is not removed.

The Verlinde like formula was shown to hold for generic θ, τ far from $\theta + n\tau \in \mathbb{Z}$. It would be interesting to study the extension to generic θ, τ which requires to consider the S matrix block (7.2.24). Furthermore, one could also study the modular transformations of the characters of other degenerate representations and their spectral flow images and explore the validity of generalized Verlinde formulas in these cases. It will be very interesting to also compute the boundary spectrum of H_2, dS_2, light cone and point like D-branes independently of the H_3^+ and the coset model. This can shed light on the puzzle which arises when considering the open/closed duality which gives negative degeneracies in the open string spectrum of the H_2 branes.

The original contributions discussed along this thesis are based on the author's publications [43, 49]. Other investigations performed during the doctoral period were published in [108, 109]. In [108] we discussed the stability of certain systems of branes with non vanishing background gauge fields in a flat target space. In [109] we performed a dimensional reduction of Double Field Theory (DFT) [110, 111, 112] and we found that it reproduces the four dimensional bosonic electric sector of gauged $N = 4$ supergravity [113]. We showed that the standard NS-NS (non)-

geometric fluxes (H, ω, Q, R) can be identified with the gaugings of the effective action, and that the string d-dimensional background can be decoded from the double twisted 2d-torus. The fluxes obey the standard T-duality chain and satisfy Jacobi identities reproducing the results of [114]. In this way, the higher dimensional origin of the string fluxes can be traced to the new degrees of freedom of DFT. Thus, the formalism of DFT allows to describe effective field theories in an intermediate stage between supergravity and the full quantum string theory description. Even though this mechanism was successfully implemented in WZNW models [115], it seems unnatural to introduce dual coordinates to winding because there are no non trivial cycles in AdS_3.

Appendix A

Basic facts on CFTs

A.1 Correlation functions

Correlation functions play a central role in any Quantum Field Theory, as they are the physical objects that connect the theory with measured quantities. Computing correlators in non trivial curved spaces usually involve arduous calculations, but fortunately in many situations the symmetries of the specific model one is analyzing can be exploited to help in the computations. For instance, the global conformal symmetry completely fixes the coordinate dependence of two and three point functions and it determines that of four point functions up to functions of the anharmonic ratios, which are coordinate combinations invariant under the global conformal group. There are two linearly independent ratios in generic spacetime dimension, *e.g.* we may consider

$$x = \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{34}}{\mathbf{x}_{13} \cdot \mathbf{x}_{24}} \,, \qquad y = \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{34}}{\mathbf{x}_{23} \cdot \mathbf{x}_{14}}, \tag{A.1.1}$$

where the notation $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ was used.

As is well known, in the particular case of two dimensional CFT there is an enhancement of the symmetry group. The conformal algebra is now a local symmetry and it is generated by all Virasoro generators L_n ($n \in \mathbb{Z}$) in contrast with the global conformal symmetry of higher dimensional theories generated by $\{L_{-1}, L_0, L_1\}$. If one replaces the two dimensional vectors $\mathbf{x}_i$ by complex coordinates $x_i, \bar{x}_i$ one rapidly realizes that the conformal group is nothing else than the group of holomorphic and antiholomorphic maps. In this basis the global invariant combinations are the analogous of (A.1.1) replacing the vectors by the complex variables and the dot product by complex multiplication. An important difference holding in the two dimensional case is that only one of the anharmonic ratios is linearly independent because four points require an extra constraint to lie in the same plane. Indeed it is easily found that $y = \frac{x}{1-x}$ and $\bar{y} = \frac{\bar{x}}{1-\bar{x}}$.

Even in the simplest non trivial correlators, the three point functions, their coefficients C_{mnp} (see (A.1.4)) are not fixed by conformal symmetry. Dynamical inputs are required, like *crossing symmetry* as well as the full local conformal symmetry, not just the global one, to completely determine them, as will be clear in section A.3. Genus zero correlation functions of arbitrary higher order can be written in a factorized form when the operator algebra and the three point function coefficients are known and so the theory is said to be completely solved [75].

Let us consider an arbitrary two point function of primary fields ϕ_m. As is well known, global conformal symmetry fixes it to be

$$< \phi_m(z_1, \bar{z}_1)\phi_n(z_2, \bar{z}_2) >= \begin{cases} 0 \quad , & \Delta_m \neq \Delta_n \;\; or \;\; \bar{\Delta}_m \neq \bar{\Delta}_n, \\ \frac{C_{mn}}{z_{12}^{2\Delta}\bar{z}_{12}^{2\bar{\Delta}}} \; , & \Delta_m = \Delta_n = \Delta, \bar{\Delta}_m = \bar{\Delta}_n = \bar{\Delta}. \end{cases} \tag{A.1.2}$$

C_{mn} is simply the normalization of the primary states defined as the asymptotic states generated by the primary fields, *i.e*

$$\begin{aligned} |\Delta_n > &= \phi_n(0,0)|0> \;, \\ < \Delta_n| &= \lim_{z,\bar{z}\to\infty} z^{2\Delta_n}\bar{z}^{2\bar{\Delta}_n} < 0|\phi_n(z,\bar{z}) \;. \end{aligned} \tag{A.1.3}$$

Since correlators are invariant under field permutations, C_{mn} must be symmetric and so it is always possible to find a basis where C_{mn} is diagonalizable and, if desired, simply a Kronecker delta [1] when the fields are properly normalized. In order to avoid confusion, the reader has to bear in mind that (for historical reasons) this will not be the convention we will be using when we refer to the AdS_3 WZNW model. In order to change to a self conjugate basis we need the field redefinition, $\phi_m^{\pm} = (\delta_m{}^n \pm \mathcal{C}_m{}^n)\phi_n$, $\mathcal{C}$ being the charge conjugation matrix, followed by the

[1] Or a combination of Kronecker and Dirac delta functions in the case of continuous conformal families.

appropriate normalization.

Concerning higher point correlation functions, global conformal symmetry forces three point functions of primary fields to be

$$< \phi_m(z_1,\bar{z}_1)\phi_n(z_2,\bar{z}_2)\phi_p(z_3,\bar{z}_3) >= \frac{C_{mnp}}{\prod_{i<j} z_{ij}^{\Delta_{ij}}\ \bar{z}_{ij}^{\bar{\Delta}_{ij}}}, \tag{A.1.4}$$

where $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$, $k \neq i, j$. C_{mnp}, not determined by global conformal symmetry, is the so called three point function coefficient determining the structure constants. As commented above the coordinate dependence of the four point function is not completely fixed. These are given by

$$< \phi_{m_1}(z_1,\bar{z}_1)\dots\phi_{m_4}(z_4,\bar{z}_4) >= \frac{G_{34}^{21}(z,\bar{z})}{\prod_{i<j} z_{ij}^{\Delta_i+\Delta_j-\frac{\Delta}{3}}\ \bar{z}_{ij}^{\bar{\Delta}_i+\bar{\Delta}_j-\frac{\bar{\Delta}}{3}}}, \tag{A.1.5}$$

where z is the anharmonic ratio defined as x in (A.1.1) and $\Delta = \sum_{i=1}^{4} \Delta_i$.

It is interesting to note that global conformal transformations can be used to fixed $z_1 \to \infty, z_2 = 1, z_4 = 0$ and so $z_3 = z$. So that the knowledge of the correlation function in these points completely fixes the four point function with primary fields inserted in any other points. Indeed $f(z,\bar{z})$, which completely determines the correlator, can be alternatively defined as

$$G_{34}^{21}(z,\bar{z}) = \lim_{z_1',\bar{z}_1'\to\infty} {z_1'}^{2\Delta_1} {\bar{z}_1'}^{2\bar{\Delta}_1} < \phi_{m_1}(z_1',\bar{z}_1')\phi_{m_2}(1,1)\phi_{m_3}(z,\bar{z})\phi_{m_4}(0,0) > . \tag{A.1.6}$$

A.2 Operator Product Expansion

Scale invariance requires that the Operator Product Expansion (OPE) must have the following structure

$$\phi_{m_1}(z,\bar{z})\phi_{m_2}(0,0) = \sum_p \sum_{\{k,\bar{k}\}} C_{12}^{p\{k,\bar{k}\}} z^{\Delta_p-\Delta_1-\Delta_2+K} \bar{z}^{\bar{\Delta}_p-\bar{\Delta}_1-\bar{\Delta}_2+\bar{K}} \phi_p^{\{k,\bar{k}\}}(0,0) \tag{A.2.1}$$

where $K = \sum k_i$ and $\{k_i\}$ denotes an arbitrary collection of non negative integers $\{k_1, k_2, \dots k_N\}$, such that $\phi_p^{\{k,\bar{k}\}}(z,\bar{z})$ denotes the descendant field

$$\phi_p^{\{k,\bar{k}\}}(z,\bar{z}) = L_{-k_1} \dots L_{-k_N} \bar{L}_{-\bar{k}_1} \dots \bar{L}_{-\bar{k}_{\bar{N}}} \phi_p(z,\bar{z}) \tag{A.2.2}$$

So the operator algebra is determined when all the conformal weights, Δ_p, and OPE coefficients, $C_{12}^{p\{k,\bar{k}\}}$, are known.

In order to get more information on the later, let us consider the asymptotic behavior of the following particular three point function of primary fields

$$\begin{aligned} < \Delta_r|\phi_{m_1}(z,\bar{z})|\Delta_{m_2} > &= lim_{w,\bar{w}\to\infty} w^{2\Delta_r}\bar{w}^{2\bar{\Delta}_r} < \phi_r(w,\bar{w})\phi_{m_1}(z,\bar{z})\phi_{m_2}(0,0) > \\ &= \frac{C_{rm_1m_2}}{z^{\Delta_1+\Delta_2-\Delta_r}\bar{z}^{\bar{\Delta}_1+\bar{\Delta}_2-\bar{\Delta}_r}} \end{aligned} \tag{A.2.3}$$

Then, after inserting the OPE (A.2.1) and using the orthogonality in the Verma module we conclude

$$C_{12}^p \equiv C_{12}^{p,\{0,0\}} = C_{pm_1m_2} \tag{A.2.4}$$

so that the leading contribution in the operator algebra is simply given by the three point function of primary fields.[2]

Following similar steps with insertion of descendant fields, one can determine the other coefficients. The correlations with descendant fields are obtained form those of the primaries by repeatedly applying Virasoro generators, and they are non vanishing only if the correlator with highest weight is not vanishing. So, it is natural to propose the factorization ansatz

$$C_{12}^{p,\{k,\bar{k}\}} = C_{12}^p\ \beta_{12}^{p\{k\}}\ \bar{\beta}_{12}^{p\{\bar{k}\}}. \tag{A.2.6}$$

[2] In the convention we will use for primary fields of the AdS_3 WZNW model, the fields will not be self conjugate and so (A.2.4) must be replaced by

$$C_{12}^p \equiv C_{12}^{p,\{0,0\}} = C_{p^*m_1m_2}, \tag{A.2.5}$$

where p^* denotes the representation conjugate to the one labeled by p.

The coefficients $\beta_{12}^{p\{k\}}$ can be recursively determined as functions of conformal weights and the central charge. Let us show how it works.

Of course, by definition, $\beta_{12}^{\{0\}} = 1$. To find the other coefficients, notice that

$$\begin{aligned} \phi_1(z,\bar{z})|\Delta_2,\bar{\Delta}_2> &= \phi_1(z,\bar{z})\phi_2(0,0)|0> \\ &= \sum_p z^{\Delta_p-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_1-\bar{\Delta}_2}|z,\Delta_p>|\bar{z},\bar{\Delta}_p>, \end{aligned} \tag{A.2.7}$$

where

$$\begin{aligned} |z,\Delta_p> &= \sum_{\{k\}} z^k \beta_{12}^{p\{k\}} L_{-k_1}\dots L_{-k_N}|\Delta_p> \\ &\equiv \sum_{N=0}^{\infty} z^N |N,\Delta_p> \end{aligned} \tag{A.2.8}$$

Applying the Virasoro mode L_n $(n>0)$ on (A.2.7) and using the commutator of Virasoro modes with primary fields (3.1.18)

$$\begin{aligned} L_n\phi_1(z,\bar{z})|\Delta_2,\bar{\Delta}_2> &= [L_n,\phi_1(z,\bar{z})]|\Delta_2,\bar{\Delta}_2> \\ &= \left[z^{n+1}\partial_z + z^n(n+1)\Delta_1\right]\phi_1(z,\bar{z})|\Delta_2,\bar{\Delta}_2> \end{aligned} \tag{A.2.9}$$

one obtains

$$\begin{aligned} L_n|N,\Delta_p> &= 0 \quad , & N &= 1,2,\dots,n-1 \\ L_n|N,\Delta_p> &= (n\Delta_1+\Delta_p-\Delta_2+N-n)|N-n,\Delta_p>, & N &= n,n+1,\dots . \end{aligned} \tag{A.2.10}$$

The first, non trivial, case is $|N=1,\Delta_p>= \beta_{12}^{p\{1\}}L_{-1}|\Delta_p>$, satisfying

$$\begin{aligned} L_1|1,\Delta_p> &= (\Delta_p+\Delta_1-\Delta_2)|\Delta_p> \\ &= \beta_{12}^{p\{1\}}[L_1,L_{-1}]|\Delta_p> \\ &= 2\Delta_p\beta_{12}^{p\{1\}}|\Delta_p> \end{aligned} \tag{A.2.11}$$

from which

$$\beta_{12}^{p\{1\}} = \frac{\Delta_p + \Delta_1 - \Delta_2}{2\Delta_p}. \tag{A.2.12}$$

At the next level, $|2, \Delta_p >= \beta_{12}^{p\{1,1\}} L_{-1}^2 |\Delta_p > + \beta_{12}^{p\{2\}} L_{-2} |\Delta_p >$. Now, acting with L_1, and L_2 on $|2, \Delta_p >$ and using the relations

$$\begin{aligned}
[L_1, L_{-1}^2] &= 4L_{-1}L_0 + 2L_{-1} \ , \\
[L_1, L_{-2}] &= 3L_{-1} \ , \\
[L_2, L_{-1}^2] &= 6L_{-1}L_1 + 6L_0 \ , \\
[L_2, L_{-2}] &= 4L_0 + \frac{c}{12} \ ,
\end{aligned} \tag{A.2.13}$$

one finds

$$\begin{aligned}
\beta_{12}^{p\{1,1\}} &= \frac{18\Delta_p(\Delta_p + 2\Delta_1 - \Delta_2) - 3(\Delta_p + \Delta_1 - \Delta_2 + 1)(\Delta_p + \Delta_1 - \Delta_2)(4\Delta_p + \frac{c}{2})}{6\Delta_p \left[18\Delta_p + (4\Delta_p + \frac{c}{2})(2 + 4\Delta_p)\right]}, \\
\beta_{12}^{p\{2\}} &= \frac{(2 + 4\Delta_p)(\Delta_p + 2\Delta_1 - \Delta_2) + 3(\Delta_p + \Delta_1 - \Delta_2 + 1)(\Delta_p + \Delta_1 - \Delta_2)}{18\Delta_p + (4\Delta_p + \frac{c}{2})(2 + 4\Delta_p)}.
\end{aligned} \tag{A.2.14}$$

At level N, one has $P(N)$ different coefficients $\beta_{12}^{p\{k\}}$, but this is exactly the number of ways $|N, \Delta_p >$ can be brought to zero level by acting with Virasoro modes, $L_n, \ n = 1, 2, \ldots, N$. The number of unknowns equals the number of equations and so the problem admits solution.

Notice that knowledge of C_{mnp} and the $\beta_{12}^{p\{k\}}$ coefficients determines the three point functions containing arbitrary descendant fields.

The knowledge of the coefficients of the three point functions of primary fields, C_{mnp}, as well as the knowledge of the conformal weights, Δ_n, and the central charge, c, completely determines the operator algebra, and with it any higher order correlation function can be written, at least in a factorized form, in terms of three point functions. In this sense the theory is completely determined once the central charge, conformal weights and three point function coefficients of primary fields are known. We will come back to this issue in section A.3.

In the case of a WZNW model, the coefficients of the correlation functions also depend on parameters labeling states of representations of the algebra. Nevertheless, new constraints must be satisfied. For instance, from the invariance of the correlation functions under the action of the group G of the WZNW model the following Ward identity must be satisfied

$$\sum_{i=1}^{n} t_i^a < \phi_{m_1}(z_1) \dots \phi_{m_n}(z_n) >= 0, \tag{A.2.15}$$

where the index i in t_i^a denotes action on ϕ_{m_i}. *E.g.* invariance under the action of t^3 in WZNW models with $sl(2)$ symmetry requires the sum of the eigenvalues of J_0^3 to vanish in any correlation function.

The combination of the Ward identities with the Sugawara construction gives rise to a partial differential equation to be satisfied by the correlators. It plays a fundamental role in WZNW models and is usually referred to as the Knizhnik-Zamolodchikov (KZ) equation:

$$\left(\partial_{z_i} + \frac{1}{k + \mathfrak{g}_c} \sum_{j \neq i} \frac{\sum_a t_i^a \otimes t_j^a}{z_{ij}} \right) < \phi_{m_1}(z_1) \dots \phi_{m_n}(z_n) >= 0. \tag{A.2.16}$$

We will use it when considering the factorization of four point functions (see section 5.3). The reason why this is particularly important in the case of four point functions is because in this case the partial differential equation becomes an ordinary differential equation, since only one degree of freedom (the anharmonic ratio) is not fixed by global conformal symmetry.

Extra constraints follow with the introduction of null states in the correlators. Let us suppose that the descendant field generated by the action of certain chain of generators on a primary field is a null vector. The correlation function with such a null field vanishes but by using the Wick theorem such a correlator can be expressed in terms of correlators where the currents act on the other primary fields giving rise to differential equations for the correlators. In fact, the usage of the *fusion relations* in four point functions with degenerate fields (*i.e*, primary fields with null descendants) was the key to solve the structure constants of the Liouville theory [118, 119] as well as the ones of the H_3^+ model [41].

A.3 The bootstrap approach

As we commented in section A.1, coordinates in 4-point functions can be set to $(\infty, 1, x, 0)$, x being the anharmonic ratio defined in (A.1.1). Notice that $G^{21}_{34}(x,\bar{x})$ defined in last section can also be written as

$$G^{21}_{34}(x,\bar{x}) =< \Delta_1, \bar{\Delta}_1|\phi_2(1,1)\phi_3(x,\bar{x})|\Delta_4, \bar{\Delta}_4 > \tag{A.3.1}$$

After inserting the operator algebra (A.2.1) one ends with

$$G^{21}_{34}(x,\bar{x}) = \sum_p C^p_{34} C^p_{12} A^{21}_{34}(p|x,\bar{x}), \tag{A.3.2}$$

where we introduced the *partial waves*, $A^{21}_{34}(p|x,\bar{x})$

$$\begin{aligned} A^{21}_{34}(p|x,\bar{x}) &= \sum_p (C^p_{12})^{-1} x^{\Delta_p - \Delta_3 - \Delta_4} \bar{x}^{\bar{\Delta}_p - \bar{\Delta}_3 - \bar{\Delta}_4} < \Delta_1, \bar{\Delta}_1|\phi_2(1,1)\psi_p(x,\bar{x}|0,0)|0 > \\ &= \mathcal{F}^{21}_{34}(p|x)\ \bar{\mathcal{F}}^{21}_{34}(p|\bar{x}), \end{aligned} \tag{A.3.3}$$

wherein

$$\psi_p(x,\bar{x}|0,0) = \sum_{\{k,\bar{k}\}} \beta^{p\{k\}}_{34}\ \bar{\beta}^{p\{\bar{k}\}}_{34}\ x^K \bar{x}^{\bar{K}} \phi^{\{k,\bar{k}\}}_p(0,0), \tag{A.3.4}$$

the *conformal blocks* are given by

$$\mathcal{F}^{21}_{34}(p|x) = x^{\Delta_p - \Delta_3 - \Delta_4} \sum_{\{k\}} \beta^{p\{k\}}_{34} x^K \frac{< \Delta_1|\phi_2(1) L_{-k_1} \dots L_{-k_N}|\Delta_p >}{< \Delta_1|\phi_2(1)|\Delta_p >} \tag{A.3.5}$$

and similarly for the antiholomorphic ones.

As will be clear below the purpose of factorizing four point functions as in (A.3.2) is because the *conformal blocks* are completely determined when conformal weights and the central charge are known.

The general explicit expression for *conformal blocks* is not known. Just in a few cases, (*e.g.* minimal models) one has a closed expression. The strategy is to expand it in power series

$$\mathcal{F}^{21}_{34}(p|x) = x^{\Delta_p - \Delta_3 - \Delta_4} \sum_{\ell=0}^{\infty} \mathcal{F}_\ell x^\ell, \tag{A.3.6}$$

where obviously $\mathcal{F}_0 = 1$. The next coefficient, $\mathcal{F}_1$ is

$$\begin{aligned} \mathcal{F}_1 &= \beta^{p\{1\}}_{34} \frac{< \Delta_1|\phi_2(1)L_{-1}|\Delta_p >}{< \Delta_1|\phi_2(1)|\Delta_p >} \\ &= \frac{(\Delta_p + \Delta_2 - \Delta_1)(\Delta_p + \Delta_3 - \Delta_4)}{2\Delta_p}, \end{aligned} \tag{A.3.7}$$

where $\beta^{p\{1\}}_{12}$ was computed in (A.2.12) and the commutator of a primary with a Virasoro mode (see (3.1.18)) was used.

The situation becomes rapidly tedious for higher coefficients. For instance $\mathcal{F}_2$ is an involved combination of conformal weights and the central charge, which is three lines long.

The invariance of correlation functions under field permutations imposes a series of relations over *conformal blocks*. For instance, after the conformal transformation $z \to z^{-1}$ one ends with

$$\sum_p C^p_{21}\ C^p_{34}\ \mathcal{F}^{21}_{34}(p|x)\ \bar{\mathcal{F}}^{21}_{34}(p|\bar{x}) = \sum_q C^q_{24}\ C^q_{31}\ \mathcal{F}^{24}_{31}\left(q|x^{-1}\right)\ \bar{\mathcal{F}}^{24}_{31}\left(q|\bar{x}^{-1}\right). \tag{A.3.8}$$

On the other hand, after $z \to 1 - z$, one finds

$$\sum_p C^p_{21}\ C^p_{34}\ \mathcal{F}^{21}_{34}(p|x)\ \bar{\mathcal{F}}^{21}_{34}(p|\bar{x}) = \sum_q C^q_{41}\ C^q_{32}\ \mathcal{F}^{41}_{32}(p|1-x)\ \bar{\mathcal{F}}^{41}_{32}(p|1-\bar{x}). \tag{A.3.9}$$

These conditions are a realization of the *crossing symmetry* and are usually referred to as the *bootstrap equations*, being the sole dynamical input required to solve the theory, as they can, in principle, be exploited to find the three point function coefficients, C_{mnp} and the conformal dimensions, Δ_n.

Indeed, let us suppose that the *conformal blocks* are known for generic conformal weights, and suppose that the theory we are interested in has N conformal families, *e.g.* a RCFT, then we have N^3 (C_{mnp}) + N (Δ_n) unknowns. These have to be contrasted with the N^4 conditions which follow from a naive counting. There is no proof that the problem admits solution in a generic case, but in many situations, *e.g* the minimal models, the bootstrap equations were completely solved. This road to solve the three point function coefficients and the conformal weights is usually denoted as the *bootstrap aproach*. This method was developed in the seminal work [75] and the interested reader will find there the application to some interesting examples.

Appendix B

Analytic structure of W_1

The purpose of this appendix is to study the analytic structure of W_1. In particular, we are specially interested in possible zeros appearing in W_1 which are not evident in the expression (4.2.25), but are very important in our definition of the OPE.

Let us recall some useful identities relating different expressions for $G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix}$ [120],

$$G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix} = \frac{\Gamma(b)\Gamma(c)}{\Gamma(e-a)\Gamma(f-a)} G\begin{bmatrix} e-a, f-a, u \\ u+b, u+c \end{bmatrix}, \tag{B.0.1}$$

$$G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix} = \frac{\Gamma(b)\Gamma(c)\Gamma(u)}{\Gamma(f-a)\Gamma(e-b)\Gamma(e-c)} G\begin{bmatrix} a, e-b, e-c \\ e, a+u \end{bmatrix}, \tag{B.0.2}$$

where u is defined as $u = e+f-a-b-c$. Using the permutation symmetry among a, b, c and e, f, which is evident from the series representation of the hypergeometric function ${}_3F_2$, seven new identities may be generated. In what follows we use these identities in order to obtain the greatest possible amount of information on W_1.

Consider for instance C^{12} defined in (4.2.22). Using (B.0.1), it can be rewritten for $j_1 =$

$-m_1+n_1$, with n_1 a non negative integer, as

$$
\begin{aligned}
C^{12} &= \frac{\Gamma(-N)\Gamma(-j_{13})\Gamma(-j_{12})\Gamma(1+j_2+m_2)}{\Gamma(-j_3-m_3)} \\
&\times \sum_{n=0}^{n_1} \begin{pmatrix} n_1 \\ n \end{pmatrix} \frac{(-)^n}{\Gamma(n-2j_1)} \frac{\Gamma(n-j_{12})}{\Gamma(-j_{12})} \frac{\Gamma(n+1+j_{23})}{\Gamma(1+j_{23})} \frac{\Gamma(1+j_3-m_3)}{\Gamma(1+j_3-m_3-n_1+n)} . \qquad \text{(B.0.3)}
\end{aligned}
$$

Using (B.0.2) instead of (B.0.1), one finds an expression for C^{12} equal to (B.0.3) with $j_3 \to -1-j_3$.

There is a third expression in which C^{12} can be written as a finite sum for generic j_2, j_3. This follows from (4.2.22), using the identity obtained from (B.0.2) with $(e \leftrightarrow f)$. This expression is explicitly invariant under $j_3 \to -1-j_3$.

Consider for instance (B.0.3). All quotients inside the sum are such that the arguments in the $\Gamma-$functions of the denominator equal those in the numerator up to a positive integer, except for the one with $\Gamma(n-2j_1)$ which is regular and non vanishing for $\mathrm{Re}\, j_1 < -\frac{1}{2}$. Then, each quotient is separately regular. Eventually, some of them may vanish, but not for all values of n. In particular, for $n=0$ the first two quotients equal one. The last factor may vanish for $n=0$, but for $n=n_1$ it equals one. However, particular configurations of j_i, m_i may occur such that one of the first two quotients vanishes for certain values of n, namely $n=n_{min}, n_{min}+1, \ldots, n_1$, and the last one vanishes for other special values, namely $n=0,1,\ldots,n_{max}$. Thus, if $n_{max} \geq n_{min}$, all terms in the sum cancel and C^{12} vanishes as a simple zero. In fact, let us consider for instance both $1+j_{23}=-p_3$ and $1+j_3-m_3=1+n_3$, with p_3, n_3 non negative integers. This requires $\Phi^{j_2,w_2}_{m_2,\overline{m}_2} \in \mathcal{D}^{-,w_2}_{j_2}$ and $j_3 = j_1-j_2-1-p_3 = m_3+n_1-n_2-1-p_3$, which impose $p_3 < n_1$ and allow to rewrite the sum in (B.0.3) as

$$
\sum_{n=0}^{p_3} \frac{1}{n!} \frac{n_1!}{(n_1-n)!} \frac{p_3!}{(p_3-n)!} \frac{\Gamma(n-j_{12})}{\Gamma(-j_{12})} \frac{1}{\Gamma(n-2j_1)} \frac{n_3!}{\Gamma(1+n_3-n_1+n)} . \qquad \text{(B.0.4)}
$$

Finally, taking into account that $1+n_3-n_1+n = -n_2-(p_3-n) \leq 0$, for $n=0,1,\ldots,p_3$, the sum vanishes as a simple zero. A similar analysis for $j_{12}=p_3 \geq 0$ and $1+j_3-m_3 = 1+n_3 \geq 1$ shows that no zeros appear in this case when $\Phi^{j_2,w_2}_{m_2,\overline{m}_2}$ is the spectral flow image of a primary field.

From the expression obtained for C^{12} by changing $j_3 \to -1-j_3$, one finds zeros again for $\Phi^{j_2,w_2}_{m_2,\overline{m}_2} \in \mathcal{D}^{-,w_2}_{j_2}$. These appear when both $j_3 = j_2 - j_1 + p_3$ and $j_3 = -m_3 - 1 - n_3$ hold simultaneously.

Finally, repeating the analysis for the sum in the third expression for C^{12}, *i.e.* that explicitly symmetric under $j_3 \to -1-j_3$, one finds the same zeros as in the previous cases.

Let us now consider the analytic structure of $W_1 := D_1 C^{12}\overline{C}^{12}$. Expression (4.2.25) together with the discussions above allow to rewrite W_1 as

$$W_1(j_i;m_i,\overline{m}_i) = \frac{(-)^{m_3-\overline{m}_3+\overline{n}_1}\pi^2\gamma(-N)}{\gamma(-2j_1)\gamma(1+j_{12})\gamma(1+j_{13})}\frac{\Gamma(1+j_2+m_2)}{\Gamma(-j_2-\overline{m}_2)}\frac{\Gamma(1+j_3+m_3)}{\Gamma(-j_3-\overline{m}_3)}E_{12}\overline{E}_{12}\,, \quad \text{(B.0.5)}$$

where E_{12} is given by $\Gamma(-2j_1)$ times (B.0.4). E_{12} has no poles but it may vanish for certain special configurations if $\Phi^{j_2,w_2}_{m_2,\overline{m}_2} \in \mathcal{D}^{-,w_2}_{j_2}$, namely $n_2 < n_1 - p_3$ and $j_3 = m_3 + n_3$ or $j_3 = -m_3 - 1 - n_3$, with $n_3 = 0,1,2,\ldots$, where $p_3 = -1 - j_{23}$ in the former and $p_3 = j_{13}$ in the latter. The same result applies to $\overline{E}_{12}$, changing n_i by $\overline{n}_i$. Obviously one might find, using other identities, new zeros for special configurations. This could be a difficult task, because the series does not reduce to a finite sum in general. Fortunately, it is not necessary for our purposes.

Appendix C

The Lorentzian torus

In this appendix we present a description of the moduli space of the torus with Lorentzian metric[1]. Although it can be easily obtained from the Euclidean case, we include it here for completeness.

Consider the two dimensional torus with worldsheet coordinates σ^1, σ^2 obeying the identifications

$$(\sigma^1, \sigma^2) \cong (\sigma^1 + 2\pi n, \sigma^2 + 2\pi m), \qquad n, m \in \mathbb{Z}. \tag{C.0.1}$$

By diffeomorphisms and Weyl transformations that leave invariant the periodicity, a general two dimensional Lorentzian metric can be taken to the form

$$ds^2 = (d\sigma^1 + \tau_+ d\sigma^2)(d\sigma^1 + \tau_- d\sigma^2), \tag{C.0.2}$$

where τ_+, τ_- are two real independent parameters. Recall that the metric of the Euclidean torus, namely $ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$, is degenerate for $\tau \in \mathbb{R}$ since $det\ g = (\tau - \tau^*)^2$. In contrast, here it is degenerate for $\tau_- = \tau_+$.

The linear transformation

$$\tilde{\sigma}^1 = \sigma^1 + \tau^+ \sigma^2, \quad \tilde{\sigma}^2 = \tau^- \sigma^2\,, \qquad \tau^\pm = \frac{\tau_- \pm \tau_+}{2}, \tag{C.0.3}$$

[1]Tori in $1+1$ dimensions have been considered previously in [121] - [124] in the context of string propagation in time dependent backgrounds.

takes (C.0.2) to the Minkowski metric. The new coordinates obey the periodicity conditions

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \cong (\tilde{\sigma}^1 + 2\pi n + 2\pi m\tau^+, \tilde{\sigma}^2 + 2\pi\tau^- m)\,, \qquad n, m \in \mathbb{Z}\,, \tag{C.0.4}$$

while the light-cone coordinates $\tilde{\sigma}_\pm = \tilde{\sigma}^1 \pm \tilde{\sigma}^2$, obey

$$\tilde{\sigma}_\pm \cong \tilde{\sigma}_\pm + 2\pi n + 2\pi m\tau_\mp\,. \tag{C.0.5}$$

In the Euclidean case, there are in addition global transformations that cannot be smoothly connected to the identity, generated by Dehn twists. A twist along the a cycle of a Lorentzian torus preserves the metric (C.0.2) but changes the periodicity to

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \cong (\tilde{\sigma}^1 + 2\pi n + 2\pi m(1+\tau^+), \tilde{\sigma}^2 + 2\pi m\ \tau^-)\,, \qquad n, m \in \mathbb{Z}, \tag{C.0.6}$$

or

$$\tilde{\sigma}_\pm \cong \tilde{\sigma}_\pm + 2\pi n + 2\pi m\ (\tau_\mp + 1)\,. \tag{C.0.7}$$

Thus it gives a torus with modular parameters $(\tau'_+, \tau'_-) = (\tau_+ + 1, \tau_- + 1)$. A twist along the b cycle leads to the following periodicity conditions

$$(\tilde{\sigma}^1, \tilde{\sigma}^2) \cong (\tilde{\sigma}^1 + 2\pi n(1+\tau^+) + 2\pi m\ \tau^+, \tilde{\sigma}^2 + 2\pi n\ \tau^- + 2\pi m\ \tau^-), \qquad n, m \in \mathbb{Z}, \tag{C.0.8}$$

or

$$\tilde{\sigma}_\pm \cong \tilde{\sigma}_\pm + 2\pi n(1+\tau_\mp) + 2\pi m\ \tau_\mp\,. \tag{C.0.9}$$

As in the Euclidean case, this is equivalent to a torus with $(\tau'_+, \tau'_-) = (\frac{\tau_+}{\tau_+ + 1}, \frac{\tau_-}{\tau_- + 1})$ and conformally flat metric. But there is a crucial difference. In the Euclidean case, the overall conformal factor multiplying the flat metric is positive definite, namely $\frac{1}{(1+\tau)(1+\tau^*)}$. On the contrary, in the Lorentzian torus, the conformal factor $\frac{1}{(1+\tau_-)(1+\tau_+)}$ is not positive definite and so, it cannot be generically eliminated through a Weyl transformation.

Defining the modular S transformation as $S\tau_\pm = -\frac{1}{\tau_\pm}$, we can write $\tau'_\pm = \frac{\tau_\pm}{1+\tau_\pm} = TST\ \tau_\pm$, and then the problem can be reformulated in the following way. The T transformation works as in the Euclidean case. Instead, under a modular S transformation, the torus defined by (C.0.1) and (C.0.2) is equivalent to a torus with the same periodicities but with the following metric (after diffeomorphisms and Weyl rescaling)

$$ds^2 = sgn(\tau_- \tau_+) \left(d\sigma'^1 + \tau_+ d\sigma'^2\right) \left(d\sigma'^1 + \tau_- d\sigma'^2\right) . \tag{C.0.10}$$

Appendix D

The mixing block of the S matrix

In this appendix we sketch the computation of the off-diagonal block of the S matrix mixing the characters of continuous and discrete representations.

A useful identity

It is convenient to begin displaying a useful identity.

Let $h(x;\epsilon_0) = \frac{1}{1-e^{2\pi i(x+i\epsilon_0)}}$, with $x \in \mathbb{R}$, be the distribution defined as the weak limit $\epsilon_0 \to 0$ and $G(x;\epsilon_1,\epsilon_2,\epsilon_3,\dots)$ a generalized function having simple poles outside of the real line[1], defined as the weak limit $\epsilon_i \to 0, i = 1,2,3,\dots$. The non vanishing infinitesimals ϵ_i are allowed to depend on the x coordinate and they all differ from each other in an open set around each simple pole. Then, the following identity holds (in a distributional sense):

$$\begin{aligned}\frac{1}{1-e^{2\pi i(x+i\epsilon_0)}}G(x;\epsilon_1,\epsilon_2,\epsilon_3,\dots) &= \frac{1}{1-e^{2\pi i(x+i\tilde{\epsilon}_0)}}G(x;\epsilon_1,\epsilon_2,\epsilon_3,\dots)\\ &+ \sum_{x_i^{\downarrow}}\delta(x-x_i^{\downarrow})G(x;\epsilon_1-\epsilon_0,\epsilon_2-\epsilon_0,\epsilon_3-\epsilon_0,\dots)\\ &- \sum_{x_i^{\uparrow}}\delta(x-x_i^{\uparrow})G(x;\epsilon_1-\epsilon_0,\epsilon_2-\epsilon_0,\epsilon_3-\epsilon_0,\dots), \qquad \text{(D.0.1)}\end{aligned}$$

where $\tilde{\epsilon}_0$ is a new infinitesimal parameter, $x_i^{\downarrow}$ ($x_i^{\uparrow}$) is the real part of the pulled down (up) poles,

[1] $G(x;0,0,0,\dots)$ not necessarily has only simple poles. In the most general case, it will have poles of arbitrary order.

i.e. those poles where $\epsilon_0(x_i^{\downarrow}) < 0 < \tilde{\epsilon}_0(x_i^{\downarrow})$ $(\tilde{\epsilon}_0(x_i^{\uparrow}) < 0 < \epsilon_0(x_i^{\uparrow}))$. Of course, here $x_i^{\downarrow}$, $x_i^{\uparrow} \in \mathbb{Z}$, but (D.0.1) can be trivially generalized to other functionals having simple poles, the only change being that the residue has to multiply each delta function.

The proof of this identity follows from multiplying (D.0.1) by an arbitrary test function $(f(x) \in C_0^{\infty})$ and integrating over the real line.

As an example, let us consider the simplest case $G = 1$, $\epsilon_0 = 0^+$, $\tilde{\epsilon}_0 = 0^-$, where one recovers the well known formula

$$\frac{1}{1 - e^{2\pi i(x+i0^+)}} = \frac{1}{1 - e^{2\pi i(x+i0^-)}} - \sum_{m=-\infty}^{\infty} \delta(x+m)\,. \tag{D.0.2}$$

The mixing block

Let us first consider the modular transformation of the elliptic theta function

$$\begin{aligned} \frac{1}{i\vartheta_{11}(\theta + i\epsilon_2^w, \tau + i\epsilon_1)} \quad &\rightarrow \quad \frac{1}{i\vartheta_{11}(\frac{\theta}{\tau} + i\epsilon_2^w, -\frac{1}{\tau} + i\epsilon_1)} \equiv \frac{1}{i\vartheta_{11}(\frac{\theta + i{\epsilon'_2}^w}{\tau + i\epsilon'_1}, -\frac{1}{\tau + i\epsilon'_1})} \\ &= \frac{-sgn(\tau) e^{-\pi i \frac{\theta^2}{\tau}} e^{-sgn(\tau) i \frac{\pi}{4}}}{\sqrt{|\tau|}} \frac{1}{\vartheta_{11}(\theta + i{\epsilon'_2}^w, \tau + i\epsilon'_1)}, \end{aligned} \tag{D.0.3}$$

$$\begin{cases} \epsilon'_1 = \tau^2 \epsilon_1 \,, \\ {\epsilon'_2}^w = \tau\left(\epsilon_2^w + \theta\epsilon_1\right) \,, \end{cases} \tag{D.0.4}$$

and ϵ_1, ϵ_2^w satisfy (6.1.2). The identity (7.2.21) was used in the last line of (D.0.3) and the limits $\epsilon'_1, {\epsilon'_2}^w \to 0$ were taken where it is allowed.

Let us now concentrate on the last term in (D.0.3). It is explicitly given by (6.1.6), where now the ϵ's are replaced by $\epsilon'_1, {\epsilon'_3}^{n,w}, {\epsilon'_4}^{n,w}$ satisfying $\epsilon'_1 > 0$,

$${\epsilon'_3}^{n,w} \begin{cases} < 0 \;, \theta - n\tau \leq -1 - w \\ > 0 \,,\; \theta - n\tau \geq -w \end{cases} , \qquad {\epsilon'_4}^{n,w} \begin{cases} < 0 \;, \theta + n\tau \geq -w \\ > 0 \,,\; \theta + n\tau \leq -1 - w \end{cases} , \qquad \tau < 0, \tag{D.0.5}$$

$${\epsilon'_3}^{n,w} \begin{cases} < 0 \;, \theta - n\tau \geq -w \\ > 0 \,,\; \theta - n\tau \leq -1 - w \end{cases} , \qquad {\epsilon'_4}^{n,w} \begin{cases} < 0 \;, \theta + n\tau \geq -w \\ > 0 \,,\; \theta + n\tau \leq -1 - w \end{cases} , \qquad \tau > 0\,. \tag{D.0.6}$$

By comparing with (6.1.7) and using (D.0.1), one finds, for instance in the case $w < 0, \tau < 0$, after a straightforward but tedious computation, the following identity:

$$\begin{aligned}
\frac{1}{i\vartheta_{11}(\theta + i{\epsilon'_2}^w, \tau + i\epsilon'_1)} &= \frac{1}{i\vartheta_{11}(\theta + i{\epsilon_2}^w, \tau + i\epsilon_1)} \\
&- \frac{1}{\eta^3(\tau + i\epsilon_1)} \left[e^{-i\pi\theta} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{-w-1} (-)^n e^{\pi i \tau n(1+n)} \delta(-\theta + n\tau + m) \right. \\
&\left. + \; e^{i\pi\theta} \left(\sum_{n=1}^{-w-1} \sum_{m=w+1}^{\infty} - \sum_{n=-w}^{\infty} \sum_{m=-\infty}^{w} \right) (-)^n e^{\pi i \tau n(1+n)} \delta(\theta + n\tau + m) \right] .
\end{aligned}$$

Repeating the same analysis for the other cases one finds, for arbitrary w,

$$\begin{aligned}
\frac{1}{i\vartheta_{11}(\theta + i{\epsilon'_2}^w, \tau + i\epsilon'_1)} &= \frac{1}{i\vartheta_{11}(\theta + i{\epsilon_2}^w, \tau + i\epsilon_1)} \\
&+ \left[\sum_{n=-\infty}^{w} \begin{cases} \sum_{m=-\infty}^{w} \delta(\theta - n\tau + m), \tau < 0 \\ \sum_{m=1+w}^{\infty} \delta(\theta - n\tau + m), \tau > 0 \end{cases} \right. \\
&\left. - \sum_{n=1+w}^{\infty} \begin{cases} \sum_{m=1+w}^{\infty} \delta(\theta - n\tau + m), \tau < 0 \\ \sum_{m=-\infty}^{w} \delta(\theta - n\tau + m), \tau > 0 \end{cases} \right] \frac{(-)^{n+m} e^{2i\pi\tau\frac{n^2}{2}}}{\eta^3(\tau + i\epsilon_1)}
\end{aligned}$$

Using (7.2.18) and summing or subtracting delta function terms like in (6.1.8) and (6.1.9), in order to construct the characters of discrete representations, one finds

$$\begin{aligned}
\chi_j^{+,w}(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0) &= e^{-2\pi i \frac{k}{4}\frac{\theta^2}{\tau}} sgn(\tau) \\
&\times \left\{ \sum_{w'=-\infty}^{\infty} \int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj' \sqrt{\frac{2}{k-2}} (-)^{w+w'+1} e^{\frac{4\pi i}{k-2}\left(j'+\frac{1}{2}-w'\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} \chi_{j'}^{+,w'}(\theta, \tau, 0) \right. \\
&+ \sum_{w',n,m \,\in\, \mathcal{I}(\tau)} \int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj' \sqrt{\frac{2}{k-2}} (-)^{w+1} e^{\frac{4\pi i}{k-2}\left(j'+\frac{1}{2}-w'\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} \\
&\left. \times \; \frac{e^{-\frac{2\pi i}{k-2}\tau(j'+\frac{1}{2}-w'\frac{k-2}{2})^2} e^{-2\pi i\theta(j'+\frac{1}{2}-w'\frac{k-2}{2})}}{\eta^3(\tau + i\epsilon_1)} (-)^{n+m} e^{2\pi i\tau\frac{n^2}{2}} \delta(\theta - n\tau + m) \right\} ,
\end{aligned}$$

where $\sum_{w',n,m \,\in\, \mathcal{I}(\tau)}$ is expected to reproduce the contribution from the continuous representations and is explicitly given by

$$\sum_{w',n,m\,\in\,\mathcal{I}(\tau)} \equiv -\sum_{w'=-\infty}^{w-1}\sum_{n=1+w'}^{w}\sum_{m=-\infty}^{\infty}+\sum_{w'=1+w}^{\infty}\sum_{n=1+w}^{w'}\sum_{m=-\infty}^{\infty}$$
$$+\sum_{w'=-\infty}^{\infty}\left(\sum_{n=-\infty}^{w}\left\{\begin{array}{l}\sum_{m=-\infty}^{w}\\ \sum_{m=1+w}^{\infty}\end{array}\right. -\sum_{n=1+w}^{\infty}\left\{\begin{array}{l}\sum_{m=1+w}^{\infty}\\ \sum_{m=-\infty}^{w}\end{array}\right.\right)$$
$$=\sum_{w'=-\infty}^{\infty}\left(\sum_{n=-\infty}^{w'}\left\{\begin{array}{l}\sum_{m=-\infty}^{w}\\ \sum_{m=1+w}^{\infty}\end{array}\right. -\sum_{n=1+w'}^{\infty}\left\{\begin{array}{l}\sum_{m=1+w}^{\infty}\\ \sum_{m=-\infty}^{w}\end{array}\right.\right)$$
$$=\sum_{n=-\infty}^{\infty}\left(\sum_{w'=n}^{\infty}\left\{\begin{array}{l}\sum_{m=-\infty}^{w}\\ \sum_{m=1+w}^{\infty}\end{array}\right. -\sum_{w'=-\infty}^{n-1}\left\{\begin{array}{l}\sum_{m=1+w}^{\infty}\\ \sum_{m=-\infty}^{w}\end{array}\right.\right),$$

where the upper lines inside the brackets hold for $\tau < 0$ and the lower ones for $\tau > 0$. In the last line we have exchanged the order of summations. The sum over w' together with the integral over j', the spin of the states in discrete representations, match together to give, after analytic continuation, the integral over s', the imaginary part of the spin of the states in the principal continuous representations:

$$\sum_{w'=n}^{\infty}\int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj' e^{\frac{4\pi i}{k-2}\left(j'+\frac{1}{2}-w'\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} e^{-\frac{2\pi i}{k-2}\tau(j'+\frac{1}{2}-w'\frac{k-2}{2})^2} e^{-2\pi i\theta(j'+\frac{1}{2}-w'\frac{k-2}{2})}$$
$$=\int_{-\infty}^{0} d\lambda e^{\frac{4\pi i}{k-2}\left(\lambda-n\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} e^{-\frac{2\pi i}{k-2}\tau(\lambda-n\frac{k-2}{2})^2} e^{-2\pi i\theta(\lambda-n\frac{k-2}{2})}$$
$$=\left\{\begin{array}{ll} i\int_0^{\infty} ds'\; e^{\frac{4\pi i}{k-2}\left(-is'-n\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} e^{-\frac{2\pi i}{k-2}\tau(-is'-n\frac{k-2}{2})^2} e^{-2\pi i\theta(-is'-n\frac{k-2}{2})}, & \tau < 0\ , \\ -i\int_0^{\infty} ds'\; e^{\frac{4\pi i}{k-2}\left(is'-n\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)}\; e^{-\frac{2\pi i}{k-2}\tau(is'-n\frac{k-2}{2})^2}\; e^{-2\pi i\theta(is'-n\frac{k-2}{2})}\ , & \tau > 0\ . \end{array}\right.$$

After a similar analysis for the terms in the sum $\sum_{w'=-\infty}^{n-1}$ and relabeling the dummy index $n \to w'$, one finds the following contribution from the continuous series

$$\sum_{w'=-\infty}^{\infty}\; i\sqrt{\frac{2}{k-2}}\int_0^{\infty} ds'(-)^{w+w'+1}\left[\sum_{m=-\infty}^{w} e^{\frac{4\pi i}{k-2}\left(-is'-w'\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} e^{-2\pi i m\left(\frac{1}{2}+is'+w'\frac{k-2}{2}\right)}\right.$$

$$- \sum_{m=1+w}^{\infty} e^{\frac{4\pi i}{k-2}\left(is'-w'\frac{k-2}{2}\right)\left(j+\frac{1}{2}-w\frac{k-2}{2}\right)} e^{-2\pi i m\left(\frac{1}{2}-is'+w'\frac{k-2}{2}\right)} \Bigg] \frac{e^{2\pi i\tau\left(\frac{s'^2}{k-2}+\frac{k}{4}w'^2\right)}}{\eta^3(\tau+i\epsilon_1)} \delta(\theta - w'\tau + m)$$

Finally, using (7.2.13), with the appropriate relabeling and performing the sum over m (which then simply reduces to a geometric series) one gets

$$\sum_{w'=-\infty}^{\infty} \int_0^{\infty} ds' \int_0^1 d\alpha' \mathcal{S}_{j,w}{}^{s',\alpha',w'} \chi_{s'}^{\alpha',w'}(\theta,\tau,0), \tag{D.0.7}$$

with

$$\mathcal{S}_{j,w}{}^{s',\alpha',w'} = -i\sqrt{\frac{2}{k-2}} e^{-2\pi i\left(w'j - w\alpha' - ww'\frac{k}{2}\right)} \left[\frac{e^{\frac{4\pi}{k-2}s'\left(j+\frac{1}{2}\right)}}{1+e^{-2\pi i(\alpha'-is')}} + \frac{e^{-\frac{4\pi}{k-2}s'\left(j+\frac{1}{2}\right)}}{1+e^{-2\pi i(\alpha'+is')}} \right]. \tag{D.0.8}$$

It is interesting to note that (repeated indices denote implicit sum)

$$\begin{aligned} \mathcal{S}_{j,w}{}^{s_1,\alpha_1,w_1} \;\; \mathcal{S}_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'} &= -\, \mathcal{S}_{j,w}{}^{j_1,w_1} \;\; \mathcal{S}_{j_1,w_1}{}^{s',\alpha',w'} \\ &= \frac{(-)^{w+w'+1}}{2\pi} \sum_{m=-\infty}^{\infty} \left[\frac{1}{\frac{1}{2}+\alpha'-is'-m} + \frac{1}{\frac{1}{2}+\alpha'+is'-m} \right] \\ &\quad \times \delta\left(j - \alpha' - (w+w')\frac{k-2}{2} + m \right). \end{aligned} \tag{D.0.9}$$

The first line implies $\mathcal{S}^2{}_{j,w}{}^{s',\alpha',w'} = 0$.

To show that $(\mathcal{S}T)^3{}_{j,w}{}^{s',\alpha',w'} = 0$ is a bit more involved. This block is explicitly given by

$$\mathcal{S}_{j,w}{}^{s_1,\alpha_1,w_1} \left[(T\mathcal{S}T\mathcal{S}T)_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'} \right] + \mathcal{S}_{j,w}{}^{j_1,w_1} \left[(T\mathcal{S}T\mathcal{S}T)_{j_1,w_1}{}^{s',\alpha',w'} \right]. \tag{D.0.10}$$

The first term above coincides with the first one in (D.0.9). This is a consequence of (7.4.2), which implies $(T\mathcal{S}T\mathcal{S}T)_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'} = \mathcal{S}_{s_1,\alpha_1,w_1}{}^{s',\alpha',w'}$. So, in order for this block to vanish it is sufficient to show that the term inside the second bracket is exactly the $\mathcal{S}$ matrix mixing block.

The factor inside the last bracket splits into the sum

$$
\begin{aligned}
& {T_{j_1,w_1}}^{j_2,w_2}{\mathcal{S}_{j_2,w_2}}^{s_3,\alpha_3,w_3}{T_{s_3,\alpha_3,w_3}}^{s_4,\alpha_4,w_4}{\mathcal{S}_{s_4,\alpha_4,w_4}}^{s_5,\alpha_5,w_5}{T_{s_5,\alpha_5,w_5}}^{s',\alpha',w'} \\
+ \quad & {T_{j_1,w_1}}^{j_2,w_2}{\mathcal{S}_{j_2,w_2}}^{j_3,w_3}{T_{j_3,w_3}}^{j_4,w_4}{\mathcal{S}_{j_4,w_4}}^{s_5,\alpha_5,w_5}{T_{s_5,\alpha_5,w_5}}^{s',\alpha',w'} . \qquad \text{(D.0.11)}
\end{aligned}
$$

These terms are very difficult to compute separately because each one gives the integral of a Gauss error function. So, we show here how the sums can be reorganized in order to cancel all the intricate integrals when summing both terms and one ends with the mixing block ${\mathcal{S}_{j_1,w_1}}^{s',\alpha',w'}$. In fact, after some few steps, the first line can be expressed as

$$
\begin{aligned}
& \sqrt{\frac{2}{k-2}}\int_0^\infty ds \ \Bigg\{ {\tilde{S}_{j_1,w_1}}^{s_2,\alpha_2,w_2}\Bigg[\sum_{w=-\infty}^{0} e^{-i\frac{\pi}{4}}e^{-\frac{2\pi i}{k-2}\left[-is-w\frac{k-2}{2}-(j_1+\frac{1}{2})+is_2\right]^2}e^{2\pi i w\left(\alpha_2+\frac{1}{2}-is_2\right)} \\
- \quad & \sum_{w=1}^{\infty} e^{-i\frac{\pi}{4}}e^{\frac{2\pi i}{k-2}\left[-is-w\frac{k-2}{2}-(j_1+\frac{1}{2})+is_2\right]^2}e^{2\pi i w\left(\alpha_2+\frac{1}{2}-is_2\right)}\Bigg] + (s_2\to -s_2)\Bigg\}, \qquad \text{(D.0.12)}
\end{aligned}
$$

where we have introduced $\quad {\tilde{S}_{j_1,w_1}}^{s_2,\alpha_2,w_2} = -i\sqrt{\frac{2}{k-2}}\ e^{-2\pi i\left(w_2 j_1 - w_1\alpha_2 - w_1 w_2\frac{k}{2}\right)}\ e^{\frac{4\pi s_2}{k-2}\left(j_1+\frac{1}{2}\right)}$.

On the other hand, the second line in (D.0.11) takes the form

$$
\begin{aligned}
& \sum_{w=-\infty}^{\infty}\sqrt{\frac{2}{k-2}}\int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj\ e^{i\frac{\pi}{2}}e^{-\frac{2\pi i}{k-2}\left[j+\frac{1}{2}-w\frac{k-2}{2}-(j_1+\frac{1}{2})+is_2\right]^2}e^{2\pi i w\left(\alpha_2+\frac{1}{2}-is_2\right)} \\
& \qquad \times \quad \frac{{\tilde{S}_{j_1,w_1}}^{s_2,\alpha_2,w_2}}{1+e^{-2\pi i(\alpha_2-is_2)}} \quad + \quad (s_2\to -s_2)\,. \qquad \text{(D.0.13)}
\end{aligned}
$$

Now notice that, for $w\leq -1$, the integral over j can be replaced by an integral over $-\frac{k-1}{2}+is$ minus an integral over $-\frac{1}{2}+is$ with $s\in[-\infty,0]$. For $w\geq 1$, the original integral splits into the same two integrals, but now with $s\in[0,\infty]$. Adding these terms to (D.0.12) one ends, after some extra contour deformations in the remaining integrals, with $S_{j_1,w_1}^{s',\alpha',w'}$ and we can conclude that ${(\mathcal{S}T)^3_{j,w}}^{s',\alpha',w'} = 0$.

Appendix E

A generalized Verlinde formula

As is well known, the Verlinde theorem allows to compute the fusion coefficients in RCFT as:

$$\mathcal{N}_{\mu\nu}{}^{\rho} = \sum_{\kappa} \frac{S_{\mu}{}^{\kappa} S_{\nu}{}^{\kappa} \left(S_{\rho}{}^{\kappa}\right)^{-1}}{S_{0}{}^{\kappa}} \,, \tag{E.0.1}$$

where the index "0" refers to the representation containing the identity field. In the case of the fractional level admissible representations of the $\widehat{sl(2)}$ affine Lie algebra, the negative integer fusion coefficients obtained from (E.0.1) in [125] were interpreted as a consequence of the identification $j \to -1-j$ in [82][1], where it was also shown that fusions are not allowed by the Verlinde formula if the fields involved are not highest- or lowest-weight. Applications to other non RCFT were discussed in [45], where generalizations of the theorem were proposed for certain representations in the Liouville theory, the H_3^+ model and the SL(2,$\mathbb{R}$)/U(1) coset.

In order to explore alternative expressions in the AdS_3 model, let us consider the more tractable finite dimensional degenerate representations. From the results for the characters obtained in section 7.2.4, it is natural to propose the following generalization of the Verlinde formula[2]

[1]Interestingly, it was shown in a recent detailed study of the $\hat{sl}(2)_{k=\frac{1}{2}}$ model [126], that the origin of the negative signs is the absence of spectral flow images of the admissible representations in the analysis of [82].

[2]A similar expression was obtained in [45] for the H_3^+ model applying the Cardy ansatz.

$$\sum_{J_3} \mathcal{N}_{J_1 J_2}{}^{J_3} \;\; \chi_{J_3}(\theta,\tau,0) = \sum_{w=-\infty}^{\infty} \int_{-\frac{k-1}{2}}^{-\frac{1}{2}} dj \;\; \frac{S_{J_1}{}^{j,w} S_{J_2}{}^{j,w}}{S_0{}^{j,w}} \;\; e^{2\pi i \frac{k}{4}\frac{\theta^2}{\tau}} \chi_j^{+,w}(\frac{\theta}{\tau}, -\frac{1}{\tau}, 0)\,, \tag{E.0.2}$$

which holds for generic (θ,τ) far from the points $\theta + n\tau \in \mathbb{Z}, \forall n \in \mathbb{Z}$. In order to prove it, notice that, in the region of the parameters where we claim it holds, one can neglect the $\epsilon' s$ and contact terms on both sides of the equation and show that the fusion coefficients $\mathcal{N}_{J_1 J_2}{}^{J_3}$ coincide with those obtained in the H_3^+ model, namely

$$\mathcal{N}_{J_1 J_2}{}^{J_3} = \begin{cases} 1 & |J_1 - J_2| \leq J_3 \leq J_1 + J_2, \\ 0 & \text{otherwise}\,. \end{cases} \tag{E.0.3}$$

Let us denote the *r.h.s.* of (E.0.2) as $I(J_1,J_2)$ and rewrite it as (see (7.2.16))

$$\begin{aligned} I(J_1,J_2) &= \sqrt{\frac{2}{k-2}} \frac{e^{\frac{2\pi i}{k-2}\left(\frac{k-2}{2}\right)^2 \frac{\theta^2}{\tau}}}{\sqrt{i\tau}\; i\vartheta_{11}(\theta,\tau)} \int_{-\infty}^{\infty} d\lambda \;\; \frac{e^{\frac{2\pi i}{k-2}\frac{\lambda^2}{\tau}} e^{2\pi i \frac{\theta}{\tau}\lambda}}{e^{\pi i \sqrt{\frac{2}{k-2}}\lambda} - e^{-\pi i \sqrt{\frac{2}{k-2}}\lambda}} \\ &\times \left[e^{\frac{2\pi i}{k-2} N_1 \lambda} + e^{-\frac{2\pi i}{k-2} N_1 \lambda} - e^{\frac{2\pi i}{k-2} N_2 \lambda} - e^{-\frac{2\pi i}{k-2} N_2 \lambda} \right], \end{aligned} \tag{E.0.4}$$

where $N_1 = 2(J_1 + J_2 + 1)$ and $N_2 = 2(J_1 - J_2)$. Changing $\lambda \to -\lambda$ in the second and fourth terms, we get

$$I(J_1,J_2) = I(N_1) - I(N_2)\ , \qquad I(N_i) = \tilde{I}(N_i,\theta,\tau) + \tilde{I}(N_i,-\theta,\tau)\ , \tag{E.0.5}$$

with

$$\tilde{I}(N_i,\theta,\tau) = \frac{\sqrt{\frac{2}{k-2}}}{\sqrt{i\tau}\; i\vartheta_{11}(\theta,\tau)} \int_{-\infty}^{\infty} d\lambda \; \frac{e^{\frac{2\pi i}{k-2}\frac{1}{\tau}\left(\lambda + \theta\frac{k-2}{2}\right)^2} e^{\pi i \sqrt{\frac{2}{k-2}} N_i \lambda}}{e^{\pi i \sqrt{\frac{2}{k-2}}\lambda} - e^{-\pi i \sqrt{\frac{2}{k-2}}\lambda}}\,. \tag{E.0.6}$$

The divergent terms in this expression cancel in the sum (E.0.5).

Without loss of generality, let us assume $J_1 \geq J_2$. To perform the λ-integral in (E.0.6), it is convenient to split the cases with odd and even N_i. Writing $N_i + 1 = 2m_i$, $m_i \in \mathbb{N}$, in the first case we get

$$\tilde{I}(N_i, \theta, \tau) = \sum_{L=0}^{m_i-1} \frac{e^{\frac{-2\pi i}{k-2}\tau L^2} e^{-2\pi i \theta L}}{i\vartheta_{11}(\theta,\tau)} - \frac{e^{\pi i \frac{k-2}{2}\frac{\theta^2}{\tau}}}{\sqrt{i\tau}\; i\vartheta_{11}(\theta,\tau)} \int_{-\infty}^{\infty} d\lambda \; \frac{e^{\pi i \frac{\lambda^2}{\tau}} e^{2\pi i \sqrt{\frac{k-2}{2}}\frac{\theta\lambda}{\tau}}}{1 - e^{2\pi i \sqrt{\frac{2}{k-2}}\lambda}}\,, \qquad \text{(E.0.7)}$$

where the second term diverges. For even N_i, take $N_i + 2 = 2n_i$ with $n_i \in \mathbb{N}$, and then

$$\begin{aligned} \tilde{I}(N_i, \theta, \tau) \;&=\; \sum_{L=0}^{n_i-1} \frac{e^{\frac{-2\pi i}{k-2}\tau\left(L-\frac{1}{2}\right)^2} e^{-2\pi i \theta\left(L-\frac{1}{2}\right)}}{i\vartheta_{11}(\theta,\tau)} \\ &\quad - \frac{e^{\pi i \frac{k-2}{2}\frac{\theta^2}{\tau}}}{\sqrt{i\tau}\; i\vartheta_{11}(\theta,\tau)} \int_{-\infty}^{\infty} d\lambda \; \frac{e^{\pi i \frac{\lambda^2}{\tau}} e^{2\pi i \sqrt{\frac{k-2}{2}}\frac{\theta\lambda}{\tau}} e^{-\pi i \sqrt{\frac{2}{k-2}}\lambda}}{1 - e^{2\pi i \sqrt{\frac{2}{k-2}}\lambda}}\,, \end{aligned} \qquad \text{(E.0.8)}$$

where again the second term diverges.

Notice that N_1 and N_2 are either both even or odd, and since the divergent term is the same in $I(N_1)$ and $I(N_2)$, it cancels in the sum $I(J_1, J_2)$. Thus, putting all together we get

$$I(J_1, J_2) = \sum_{J_3=J_1-J_2}^{J_1+J_2} \frac{-e^{\frac{-2\pi i}{4(k-2)}\tau(2J_3+1)^2} 2\, \sin\left(\pi i \theta(2J_3+1)\right)}{\vartheta_{11}(\theta,\tau)} = \sum_{J_3=J_1-J_2}^{J_1+J_2} \chi_{J_3}(\theta,\tau,0)\,. \qquad \text{(E.0.9)}$$

where we have defined $J_3 = L - \frac{1}{2}$ for odd N_1 and N_2 and $J_3 = L - 1$ for even N_1 and N_2.

From a similar analysis of the case $J_2 > J_1$, we obtain (E.0.2) and (E.0.3).

In conclusion, consistently with the assumption that correlation functions of fields in degenerate representations in the H_3^+ and AdS_3 models are related by analytic continuation, the generalized Verlinde formula (E.0.2) reproduces the fusion rules of degenerate representations previously obtained in the Euclidean model. However, even if it is not expected to reproduce the fusion rules of continuous representations [82], applying it for discrete representations also fails.

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