Gravedad en tres dimensiones y la conjetura AdS/CFT

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Resumen

En esta tesis son estudiados desarrollos recientes en gravedad en tres dimensiones. Luego de una descripción concisa de los hechos fundamentales de gravedad AdS_3, principalmente dentro del marco de AdS/CFT, se presentan los últimos intentos para calcular la función de partición de dicha teoría. En particular, presentamos en gran detalle una novedosa construcción de geometrías con singularidades cónicas, tanto estáticas como rotantes. Estas geometrías darían contribuciones adicionales a la función de partición de gravedad en tres dimensiones.

Por otro lado, el rol de la factorización holomorfa es enfatizado con el propósito de motivar la búsqueda de un dual gravitacional a una CFT holomorfa. Explicamos en detalle un posible ejemplo de Li, Maloney, Song y Strominger llamado gravedad quiral, el cual está definido en un punto especial de la gravedad topológicamente masiva con condiciones de contorno de Brown-Henneaux. Comentamos los pros y contras de tal teoría como un dual a una CFT holomorfa y también consideramos la teoría que aparece cuando se imponen condiciones de contorno más relajadas, llamada gravedad logarítmica, que sera dual a una CFT logarítmica. En este contexto, una solución exacta que obedece estas nuevas condiciones de borde, la cual fue encontrada por el autor y colaboradores, es descrita.

Finalmente, formulamos en detalle una propuesta original para una nueva gravedad quiral, la cual está definida en un punto especial de la teoría de Mielke-Baekler con condiciones de contorno de Brown-Henneaux. Aunque dicha dualidad pueda contenter soluciones con torsión, evita las complicaciones de tener un zoológico de soluciones como es el caso de gravedad quiral: cualquier solución en nuestra propuesta tiene curvatura y torsión constantes. La teoría de Mielke-Baekler genéricamente exhibe un álegbra asintótica dada por dos copias del álgebra de Virasoro. El análisis canónico en el punto quiral donde nuestra propuesta reside es presentado y se muestra como la mitad de los generadores de las simetrías asintóticas desaparecen y la carga central izquierda se anula. Por esto, la quiralidad es manifiesta.

Esta tesis está parcialmente basada en resultados que el autor ha publicado en las referencias [13, 14, 15, 24, 28, 35, 36].

Palabras clave: Gravedad en tres dimensiones; correspondencia AdS/CFT; teoría de Chern-Simons; gravedad cuántica; gravedad quiral.

Gravity in three dimensions and the AdS/CFT correspondence

Abstract

In this thesis, recent developments on three-dimensional gravity are studied. After a concise description of the fundamental facts on AdS_3 gravity, mainly within the context of AdS/CFT, recent attempts to compute the partition function of such theory are presented. In particular, we present in great detail a novel construction of geometries with conical singularities, both static and rotating. These are believed to give further contributions to the partition function of three-dimensional gravity.

On the other hand, the rôle played by holomorphic factorization in the dual conformal theory is emphasized in order to motivate the search for a gravitational dual to a holomorphic CFT. We explain in detail one possible example given by Li, Maloney, Song and Strominger, called chiral gravity, which is defined at a special point of topologically massive gravity with Brown-Henneaux boundary conditions. We comment on the pros and cons of such theory as a dual to a holomorphic CFT. We also consider the theory that appears when more relaxed boundary conditions are imposed, dubbed log gravity, presumably being dual to a logarithmic CFT. In this context, an exact solution obeying these new boundary conditions, which was found by the author and collaborators, is described.

Finally, we formulate in detail an original proposal for another chiral gravitational theory which is defined at a special point of Mielke-Baekler theory with Brown-Henneaux boundary conditions. Although it may contain solutions with torsion, it avoids the complications of having a wild zoo of solutions as chiral gravity does: Any solution in our proposal is of constant curvature and constant torsion. Mielke-Baekler theory generically exhibits an asymptotic symmetry algebra that can be cast in the form of two copies of the Virasoro algebra. The canonical analysis at the critical point where our proposal takes place is presented and it is shown how half of the generators of asymptotic symmetries disappear and the left central charge vanishes. Thus, chirality is manifest.

This thesis is partially based on results that the author has published in references [13, 14, 15, 24, 28, 35, 36].

Keywords: Three-dimensional gravity; AdS/CFT correspondence; Chern-Simons theory; quantum gravity; chiral gravity.

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Contents

Introduction

Probably the most important pending task in the high energy theoretical physics community is the unification of quantum physics with gravity. On one hand, it is hoped that once this is achieved, ambitious questions such as what happens in the center of a black hole or how the universe evolved from its origin will be satisfactorily answered. On the other hand, the self-building road to this unification promotes the emergence of new powerful physical ideas and mathematical tools, with diverse ramifications and possible uses. This is, without doubt, due to the fact that the task of unification of all forces is incredibly difficult, and because of this, one needs to resort to all kind of different approaches, most of them tested on physical models of partial resemblance to our universe, usually referred to as toy models. Then, if successful, these approaches have to be adapted to the physics of our world.

One such toy model is general relativity (GR) in three dimensions, i.e., the three-dimensional analogue to the theory of gravity we believe best describes our known universe, but in one less spatial dimension,

$$S = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}R.$$

This is the most promising "non-stringy" working ground to achieve a quantum version of general relativity [1]. The reason for such a claim stems in the non-existence of gravitons in three dimensions. In other words, every solution to Einstein equations looks locally the same. If one adds a cosmological constant Λ,

$$S = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}\left(R-2\Lambda\right),$$

there appears a dimensionless quantity $k \sim 1/(\sqrt{|\Lambda|}G)$ which can be used for perturbation theory, suggesting renormalizability of the theory. This is closely related to the fact, first discovered by Achúcarro and Townsend, that the theory can be written as a Chern-Simons action for some gauge group, depending on the sign of the cosmological constant [1, 2]. In this thesis we will only consider the case of a negative cosmological constant $\Lambda = -\ell^{-2}$ and so the solutions of general relativity will be locally Anti-de Sitter (AdS_3). Actually, we will loosely refer to this theory as AdS_3 gravity.

In Chapter 1 we will review in more detail the absence of local degrees of freedom and the relation to Chern-Simons theory. Also in that chapter, we will discuss a remarkable feature of three-dimensional gravity found by Brown and Henneaux: for some asymptotic boundary conditions, the canonical realization of asymptotic symmetries is isomorphic to two copies of the Virasoro algebra with central charge,

$$c = \frac{3\ell}{2G},$$

[1]We will always use natural units where G plays the part of Planck length scale in three dimensions.

which is the local conformal symmetry algebra of a quantum conformal field theory (CFT) in two dimensions [3]. Even more, we will explain how at the level of the action the imposition of asymptotic conditions yields a two-dimensional CFT known as Liouville theory. Although this could (unlikely) be a mere curiosity, we will see that, in the context of the AdS/CFT correspondence, it has a natural interpretation.

The space of solutions of AdS_3 gravity could seem at first sight everything but rich, given the fact that any solution is locally AdS_3. The parameterization of the space of solutions, for simple topologies, will be briefly commented in the first chapter. But in particular, an interesting solution does exist: Bañados, Teitelboim and Zanelli showed in 1992 that there exist black hole solutions (called BTZ black holes) in three-dimensional gravity with a negative cosmological constant [4]. This can be so because these geometries are made from identifications on AdS space [5]. Actually, there exist solutions that represent many BTZ black holes as well.

In the first lines of this introduction we mentioned that trying to answer difficult questions usually brings to life new powerful tools to reach the goal. Such is the case of the AdS/CFT correspondence, presented by Maldacena in the year of 1997 [6]. This is an idea that has spread in a large variety of fields in physics, from cosmology to solid-state physics, proving its enormous potential beyond doubt. This correspondence is a holographic statement, in the sense that it relates physics on a given space to some other physics on a space of at least one dimension less [7, 8]. Leaving technical details aside for a moment, let us just mention a few important facts about this conjecture. The first thing to be mentioned is that it relates a gravitational theory in the ten dimensions of $AdS_5 \times S^5$ to a CFT in 4 dimensions (it is often said that this CFT theory lives on the boundary of AdS_5, the latter often called "the bulk"). Besides, in the limit where the quantum gravity theory (string theory) is purely classical, and where one deals with just a gravitational theory, the CFT is a quantum field theory (QFT) with a large coupling constant. Conversely, when the string theory exhibits non-negligible quantum effects, the CFT becomes weakly coupled. In fact, this "strong-weak duality" is one of the most important features of Maldacena's conjecture, because it allows us to perform extremely difficult computations in a strongly coupled regime of a QFT by just resorting to its dual description in terms of a weakly coupled theory, and thus implementing perturbative methods. Another important aspect of Maldacena's conjecture is that local symmetries of the string theory turn out to match the global symmetries of the CFT, and vice versa. This is worth to be mentioned because this can be thought of as if the local gauge transformations in the boundary theory (the CFT) generate global gauge transformations for the gravitational theory. Therefore, an "innocent" transformation in the QFT such as a gauge transformation maps one solution to another in the gravitational theory on the bulk. This can be inferred from the result of Brown and Henneaux previously mentioned in the particular case of AdS_3/CFT_2, where the gravitational classical theory is AdS_3 gravity with $\ell/G \gg 1$.

In more technical detail, Maldacena's proposal claims that type IIB superstring theory in a curved spacetime (namely five-dimensional Anti-de Sitter times a 5−sphere) is dual to $\mathcal{N} = 4$ Super Yang-Mills Theory with gauge group $SU(N)$ in the limit of large N living on a four-dimensional Minkowski spacetime. Here, dual means that they describe the same physical system, as will be explained at the beginning of Chapter 2. We will also see there the precise prescription [9, 10] to extract physical information from the correspondence such as correlators in the CFT by means of classical computations in the bulk. At the end of the chapter a technique called *holographic renormalization*, which explains how to get the mean value of the CFT stress tensor by "renormalizing" the gravitational action, will be described. In particular, this is useful for computing conserved charges and the central charge.

Research on three-dimensional gravity with a negative cosmological constant experienced a boost in 2007 when, in the context of the AdS/CFT correspondence, Witten published a proposal for a family of holomorphic extremal CFTs, parameterized by the discrete central charges (c_R, c_L), that would be dual to AdS_3 gravity for different discrete values of ℓ/G [11]. The idea comes from looking at the Chern-Simons realization of gravity, with parity broken. The details will be made explicit in the first section of Chapter 3. Let us mention here that the holomorphic nature of the dual CFTs plus the requirement of being extremal (meaning that the lowest primary apart from the identity has dimension $k+1$, with $c = 24k$ and $k \in \mathbb{Z}$) allows to uniquely identify the partition function of the CFTs. The same year, Maloney and Witten computed, from the gravitational side only, the partition function of Euclidean AdS_3 gravity by summing all the contributions from smooth manifolds with a unique conformal boundary of genus one [12]. Their result fails to be equal to a partition function defined over a Hilbert space, and hence does not have a sensible physical interpretation. Nevertheless, when used their result to construct the holomorphic piece of a partition function of a holomorphic CFT, it is possible to reproduce the Bekenstein-Hawking entropy and the phenomena of Hawking-Page phase transitions between thermal AdS_3 and the BTZ black hole. This will be presented in Chapter 3. We end this chapter by discussing a paper from 2011 where some of the Brown-Henneaux states that enter in the partition function of Maloney and Witten are showed to be of null norm, therefore having to be removed from the spectrum [17]. The analysis, unfortunately, takes place only around AdS, so the consequences of extending it to the perturbations around every other manifold that Maloney and Witten consider is still an open issue.

Later on, in Chapter 4 we describe in detail work by the author and collaborators studying a set of singular geometries representing spinning massive particles in AdS_3 [13, 14, 15], which may need to be included in the partition function to restore its physical interpretation. Not only their construction by identifications - similar to those that make the BTZ black holes - is explained, but also in what kind of gravitational theories these geometries could appear. Furthermore, their stability is scrutinized in the context of super AdS_3 gravity [1, 2, 16]. The existence of BPS massive particles (with angular momentum and/or charge) supports the need for taking them into account when computing the partition function.

Chapter 5 is entirely devoted to present a conjecture dubbed chiral gravity, and topics related to it. Motivated by the work of Witten [11] and the nice holomorphic properties of the *ad hoc* holomorphic partition function of Maloney and Witten [12], Li, Song and Strominger proposed in 2008 a gravitational theory in three dimensions that would be dual to a holomorphic CFT with trivial left sector [18]. The gravitational theory they considered is topologically massive gravity (TMG), introduced by Deser, Jackiw and Templeton in 1982 [19, 20], at the point $\mu\ell = 1$. Here μ is the inverse of the coupling constant of the new term (that couples to pure gravity) in the TMG action. TMG is a third-order theory with a massive graviton of negative energy that makes it unstable. At the particular point $\mu\ell = 1$ in parameter space, the left central charge vanishes and the massive graviton tends to the left, pure-gauge graviton, so it seems that stability is restored. As shown soon after by Grumiller and Johansson [21], a new linear solution appears at the critical point considered in [18], with negative energy and a more relaxed fall-off behavior at infinity than that of Brown-Henneaux, opening the window to consider such *logarithmic* behavior at infinity. This graviton is now known as the log graviton. If one would only consider solutions to the linear equation at the critical point with Brown-Henneaux boundary conditions, then it turns out that there is indeed such a solution that spoils chirality, having negative left-charge [22]. The stability of chiral gravity, and thus its consistency, was a matter of intense debate within the high-energy theoretical physics community (we will give the appropriate references through Chapter 5).

Maloney, Song and Strominger later showed that, at second order in perturbation theory, any linear solution with Brown-Henneaux asymptotics has left vanishing charge, and that the solution in [22] does not actually satisfy the Brown-Henneaux boundary conditions at second order (so it would not be the linearization of any exact solution) [23]. All of this was discussed at the level of the linearized equations of motion. Considering the exact equations of motion of TMG at the point $\mu\ell = 1$, it was shown by the author and collaborators [24] that there are indeed solutions to the theory at the critical point that manifest the more-relaxed boundary conditions, the *log asymptotics* of [21]. Even more, the charges of such solutions satisfy the extremal relation $\ell M = J$. The upshot of this discussion is that chiral gravity is defined at present for Brown-Henneaux boundary conditions, while TMG at $\mu\ell = 1$ with the more general fall-off asymptotics is distinctively known as log gravity [21, 23, 25, 26]. In [23] they also computed the partition function of chiral gravity in the Euclidean sector, and showed that, in fact, it gives a holomorphic partition function with a sensible physical interpretation. Nevertheless, the non-existence of non-Einstein geometries with Brown-Henneaux asymptotics was assumed, but now we know that this hypothesis is not true [2], thanks to the solution reported in [27]. It was also assumed that there are no chiral (i.e., right-moving) gravitons with negative energy, what still remains to be proven.

In the final chapter, we propose, based on [28], a new gravitational theory in three dimensions that exhibits all the properties of [18] but by construction has no gravitons, since it comes from a Chern-Simons theory. Even more, the spectrum of exact solutions is under control. Such a theory is also defined at a particular point in parameter space of a more general theory: Mielke-Baekler theory of gravity [29, 30], which is the most general, Lorentz-invariant, second-order theory with torsion. Similarly as AdS_3 gravity, it can also be written as a Chern-Simons theory, thus manifesting its topological origin and the absence of gravitons. Also, for Brown-Henneaux boundary conditions, the asymptotic symmetry algebra is represented by two copies of the Virasoro algebra, with different left and right central extensions.

The exact solutions of Mielke-Baekler theory are geometries with constant curvature and torsion, provided certain parameters of the theory do not match. But when they do match, then the equations of motion degenerate into one equation and the constant quantity is a linear combination of the curvature and torsion. The chiral gravity proposed by us is defined by starting from the space of solutions of the general case, and taking the limit to the critical point. Then, every solution of our chiral gravity has by construction constant curvature and torsion, satisfy the extremal relation between mass and angular momentum, and, when Brown-Henneaux boundary conditions are imposed, the algebra of asymptotic symmetries is only spanned by half the generators of the general theory (which is a manifestation of the chirality of the theory). Furthermore, the left central charge vanishes. In addition, there is an arbitrariness on how the limit to the critical point is taken which does not affect all the properties just listed. We will see that there is a way of taking the limit where the torsion vanishes.

It is worth mentioning, in addition, another recent theory of gravity in three dimensions called new massive gravity (NMG), although we will not comment about it further in the thesis except briefly in Appendix G. This was defined by Bergshoeff, Hohm and Townsend in 2009 [31], and

[2]It is fair to say that the geometry presented in [27] suffers from having closed timelike curves, so wether it should be considered in the physical spectrum or not is not clear. In any case, it is clear now that there could be non-Einstein solutions that present the Brown-Henneaux asymptotic behavior and have no pathologies.

has the very nice property that although being a fourth-order theory, its linearization matches the unitary Fierz-Pauli action for a massive spin-2 field. In contrast to TMG, NMG is parity-invariant, so the left and right central charges coincide although they differ from the central charge of AdS_3 gravity. On the other hand, in resemblance to TMG, NMG supports very interesting solutions, such as the BTZ black holes, warped AdS geometries [32], and AdS-waves [33, 34]. Some of these appear only at a special point in parameter space, and some also present weakened asymptotic conditions as is the case with log gravity. There is no doubt that for higher-curvature theories, there are in general points in parameter space where the equations of motion exhibit some sort of degeneracy and new interesting solutions appear. One surprising example was shown to exist by the author and collaborators in [35], where a black hole solution which is not even asymptotically AdS (in any asymptotic sense) was presented. This solution exhibits a Lifshitz asymptotic behavior, meaning that there is a scaling symmetry that discriminates between the time coordinate and spatial coordinates. We have also shown that asymptotically Lifshitz black holes appear in quadratic corrections in the curvature of general relativity in higher dimensions [36]. This type of solutions are believed to describe holographically the physics of condensed matter Lifshitz fixed points in one less dimension [37]. This is an interesting proposal but is beyond the scope of this thesis.

The conventions we will use here are those of [38], and unless otherwise stated, Greek letters $\mu, \nu, \ldots = 0, 1, 2$ will be used for spacetime indices, while Latin letters like $i, j, k = 1, 2$ will represent the spatial components and $a, b, c = 0, 1, 2$ will be used for the tangent space indices. The antisymmetric symbol will satisfy $\epsilon_{012} = -1 = -\epsilon^{012}$ and the determinant of the vielbein, $\det e$, will be positive.

Chapter 1

General relativity in 2+1 dimensions

In this chapter we will make an account of several "old" results concerning general relativity in 2+1 dimensions with a negative cosmological constant. A short but precise treatment of them will make the way easier through the next chapters of this thesis and will also serve to motivate the recent approaches for constructing a quantum version of AdS_3 gravity.

1.1 Geometry of the solutions

General relativity in $d+1$ dimensions with a cosmological constant has the following action,

$$S = \frac{1}{16\pi G}\int d^{d+1}x \, (R - 2\Lambda)\,, \tag{1.1}$$

whose Euler-Lagrange equations of motion are,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.2}$$

or simply,

$$R_{\mu\nu} = \frac{2}{d-1}\Lambda g_{\mu\nu}. \tag{1.3}$$

So far this is quite general and does not give much information about the Riemann tensor. But in the particular case of three dimensions, the trace-free and conformally invariant part of the Riemann tensor called Weyl tensor vanishes identically. This means that in three dimensions the Riemman tensor can be algebraically expressed in terms of the Ricci tensor and scalar curvature. Thus, for any metric satisfying the equations (1.3),

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}), \tag{1.4}$$

and so any solution of gravity in three dimensions is either locally Minkowski, de Sitter, or Anti-de Sitter depending on Λ, which implies that *there are no local degrees of freedom in three-dimensional gravity.*

We will from now on focus on the case of a negative cosmological constant $\Lambda = -1/\ell^2$. So, the conclusion of the last paragraph is that any solution is locally diffeomorphic to AdS_3 space. This space can be defined in many ways, but a usual and actually useful way to do this is the following. Consider the space $\mathbb{R}^{2,2}$ which is the manifold $\mathbb{R}^4$ with a non-Riemannian metric of the form

$$X \cdot X = -(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2, \tag{1.5}$$

and then AdS_3 is the hypersurface in $\mathbb{R}^{2,2}$ such that,

$$\mathrm{AdS}_3 = \{X \in \mathbb{R}^{2,2} | X \cdot X = -\ell^2\}, \tag{1.6}$$

with the induced metric coming from $\mathbb{R}^{2,2}$. More than one global chart of AdS_3 can be obtained with clever parameterizations of this hypersurface and its metric can be described globally. For example, with the following coordinates,

$$\begin{aligned} x^0 &= \ell \cosh\rho \sin(t/\ell), \\ x^3 &= \ell \cosh\rho \cos(t/\ell), \\ x^1 &= \ell \sinh\rho \cos(\phi), \\ x^2 &= \ell \sinh\rho \sin(\phi), \end{aligned} \tag{1.7}$$

the metric becomes,

$$ds^2 = \ell^2 \left(-\cosh^2\rho d(t/\ell)^2 + \sinh^2\rho d\phi^2 + d\rho^2\right), \tag{1.8}$$

where $\rho \in [0, \infty)$, $t \in [0, 2\pi\ell)$ and $\phi \in [0, 2\pi)$. This space actually has closed timelike curves, so what it is common to do is to consider its universal covering by "unwrapping" the time coordinate: $t \in (-\infty, \infty)$. From now on we will always consider the universal covering although we will keep on calling it just AdS_3.

As we mentioned before, any solution has locally the form of (1.8), and so other solutions have to differ in global aspects, such as the topology of the manifold. We will come back to the description of different locally AdS_3 spacetimes in forthcoming sections. For now, let us finish this section by making some remarks on the six isometries of global AdS_3: these are generated by the action of the $SO(2, 2)$ group which is isomorphic to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. What we will see later is that solutions that are not globally diffeomorphic to AdS_3 have less isometries.

For further reading on the geometry of AdS_3 see for example [39, 40].

1.2 Connection to Chern-Simons theory

In the year 1986 two major results considering the description of three-dimensional gravity with a cosmological constant appeared. This and the following sections are devoted to explain them briefly.

The first of these results was presented by Achúcarro and Townsend [1], who showed that the supersymmetric extension of gravity in 2+1 dimensions can be written as a Chern-Simons theory (CS)[1] for the supersymmetric group $OSp(p|2; \mathbb{R}) \times OSp(q|2; \mathbb{R})$, with $\mathcal{N} = p + q$ supersymmetries. From then on, this was called the AdS supergroup. Their result can be briefly explained following Witten's posterior work [2] (see also the lectures by Zanelli for an excellent review on Chern-Simons supergravities and conventions [41]). We will just consider here the case where

[1]For a detailed introduction on differential geometry and the geometry of Chern-Simons theory see Appendices B and C, respectively.

all the non-gravitational fields are turned off and one is led then with just pure gravity in three dimensions. Then, the gauge group is $SO(2,2)$ which is locally isomorphic to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ or to $SO(2,1)\times SO(2,1)$.

First note that the action (1.1) can be written in first order formalism as [2],

$$S=-\frac{1}{16\pi G}\int \epsilon_{abc}\left(e^a\wedge R^{bc}+\frac{1}{3\ell^2}e^a\wedge e^b\wedge e^c\right), \tag{1.9}$$

where e^a $(a=0,1,2)$ is the *vielbein* and R^{ab} is (the ab component of) the curvature associated to the spin connection ω^{ab}. The equations of motion derived from the first-order action (1.9) are,

$$\begin{aligned} T^a &= de^a+\epsilon^a{}_{bc}\omega^b\wedge e^c=0 \\ R^{ab} &= d\omega^{ab}+\omega^a{}_c\wedge\omega^{cb}=-\frac{1}{\ell^2}e^a\wedge e^b, \end{aligned} \tag{1.10}$$

where the first one is obtained by varying with respect to the spin connection and the second one to the vielbien. They imply that the connection is the Levi-Civita connection and that the manifold is of constant negative curvature, such as the usual Einstein equations. Then, by using the dual spin connection $\omega^a=1/2\,\epsilon^a{}_{bc}\omega^{bc}$ one can construct two $sl(2,\mathbb{R})$ gauge fields A_R and A_L as,

$$A_R=w+\frac{1}{\ell}e=A_R^aJ_a^+,\quad A_L=w-\frac{1}{\ell}e=A_L^aJ_a^-, \tag{1.11}$$

where $\{J_a^\pm\}_{a=0,1,2}$ are the generators of two copies of the algebra $sl(2,\mathbb{R})$ (see Appendix A). The 2+1 gravity action (1.9) becomes,

$$S=\frac{\ell}{16\pi G}I_{\rm CS}[A_L]-\frac{\ell}{16\pi G}I_{\rm CS}[A_R], \tag{1.12}$$

with the Chern-Simons action being [3],

$$I_{\rm CS}[A]=\int {\rm tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right). \tag{1.13}$$

Then, at least at the level of the actions, Einstein gravity in three dimensions with a cosmological constant and Chern-Simons theory for the $SO(2,2)$ group are equal. This also applies for vanishing or positive cosmological constant with the groups $ISO(2,1)$ and $SO(3,1)$ respectively, but we will always stick to the case of a negative cosmological constant. The equations of motion coming from the Chern-Simons action (1.13) are $F^+=dA^++A^+\wedge A^+=0$ and the same for the minus copy, which can be read,

$$F^\pm=(R^a\pm\frac{1}{\ell}T^a+\frac{1}{2\ell^2}\epsilon^a{}_{bc}e^b\wedge e^c)J_a^\pm=0, \tag{1.14}$$

where $R^a=1/2\,\epsilon^a{}_{bc}R^{bc}$. So, these equations reproduce precisely equations (1.10).

[2] Here it is used that $\epsilon_{abc}e^a\wedge e^b\wedge e^c=-6\sqrt{-g}d^3x$.

[3] We are considering the algebra of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ group, which allows actually for two different traces. The one used here is given by $\langle J_a^+,J_b^+\rangle=\langle J_a^-,J_b^-\rangle=\frac{1}{2}\eta_{ab}$ and $\langle J_a^+,J_b^-\rangle=0$, which is the same trace of the two-dimensional representation for each copy. The other trace gives the "exotic Chern-Simons action" [11].

Now, a few points to be noted are called for. First of all, the equivalence of the actions (1.9) and (1.13) is valid modulo a boundary term of the form $\int_{\text{boundary}}(\omega_a \wedge e^a)$. Of course, the Euler-Lagrange equations are invariant under a total derivative in the Lagrangian, so this boundary term is not much of an issue, at least classically. A more important point comes from the fact that, in general, solutions of the Chern-Simons theory in three dimensions are flat connections ($F = 0$), one of which is locally the trivial one $A = 0$. This solution translates into a vanishing vielbein and therefore a non-invertible one. This cannot be allowed in a gravitational theory since it forbids to construct a volume form for the manifold as $\propto \text{tr}(e \wedge e \wedge e)$. Related to this, there are more issues in the claim that AdS$_3$ gravity is equivalent to Chern-Simons theory [42], although we will not comment on these. It is just enough for our purposes to state that these two theories can be regarded as equivalent in a somewhat weak sense: *they describe the same dynamics only perturbatively around a physically meaningful gravitational solution.* With physically meaningful we mean a metric that can be constructed from an invertible vielbein. Although weak, this equivalence can be a good place to grasp some of the features of quantum gravity in AdS$_3$. For example, when trying to quantize the Chern-Simons action by means of the path integral formalism, it turns out, thanks to its topological nature, that the levels in the action (in (1.13) this is $k = \ell/16\pi G$) must be quantized, for groups that are contractible to a compact subgroup, such as $SL(2, \mathbb{R})$ [11] (we will come back to this point in Section 3.1). Therefore, Chern-Simons theory *suggests* that the quantum version of AdS$_3$ gravity must have the ratio ℓ/G quantized. We will see that there are more reasons to believe this is the case from the AdS/CFT correspondence.

1.3 Asymptotic symmetries

The other important result within three-dimensional gravity that appeared 1986, this time by the hand of Brown and Henneaux, concerned the asymptotic boundary conditions and global charges of the theory. To present their work, we will follow their original paper [3].

In the Hamiltonian formulation of general relativity [38] the Hamiltonian is written as a combination of constraints $\mathcal{H}_\mu$, modulo a surface term $J[\xi]$,

$$H[\xi] = \int d^2x \xi^\mu \mathcal{H}_\mu + J[\xi], \tag{1.15}$$

where ξ are the allowed asymptotic symmetries and $J[\xi]$ (from now on *the charges*) are such that the variation of $H[\xi]$ with respect to the canonical variables is well defined for any ξ [43][4].

An asymptotic symmetry is a vector field such that the infinitesimal diffeomorphism that generates leaves invariant certain boundary conditions. In the case of AdS$_3$ gravity, Brown and Henneaux proposed to consider all the metrics that in a patch close to $\rho \to \infty$, in the coordinates of (1.8), behave as,

$$\begin{aligned} &h_{\rho\rho} = \mathcal{O}(e^{-2\rho}), \quad h_{\rho t} = \mathcal{O}(e^{-2\rho}), \quad h_{\rho\phi} = \mathcal{O}(e^{-2\rho}), \\ &h_{tt} = \mathcal{O}(1), \qquad\quad h_{t\phi} = \mathcal{O}(1), \qquad\quad h_{\phi\phi} = \mathcal{O}(1), \end{aligned} \tag{1.16}$$

where $h_{\mu\nu}$ is a perturbation with respect to (1.8) near the boundary and the dependence on the t and ϕ coordinates is arbitrary. The asymptotic symmetries are the solutions to the equations,

$$\mathcal{L}_\xi(g + h) = h, \tag{1.17}$$

[4]Note that the surface terms $J[\xi]$ are only defined up to a constant

which means that the vectors ξ transform locally AdS_3 metrics with the fall of conditions of (1.16) into themselves (generically changing the dependence on the coordinates t and ϕ as well as sub-sub-leading terms). In fact, the solutions to (1.17) are,

$$\begin{aligned}
\xi^{\rho} &= -\tfrac{1}{2}\left(U'(u) + V'(v)\right), \\
\xi^{u} &= U(u) + 2e^{-2\rho}V''(u), \\
\xi^{v} &= V(v) + 2e^{-2\rho}U''(v),
\end{aligned} \tag{1.18}$$

where $u = t/\ell + \phi$ and $v = t/\ell - \phi$ are light-cone coordinates at the boundary and $U(u)$ and $V(v)$ are arbitrary periodic functions. Because of this, the asymptotic symmetries can be expanded in (two) Fourier modes ξ^u_n and ξ^v_n and then the Hamiltonian charges $H[\xi]$ can be evaluated for each of these modes:

$$H[\xi^u_n] := L_n, \quad H[\xi^v_n] := \bar{L}_n. \tag{1.19}$$

Hamiltonian charges generate, via the Poisson bracket, infinitesimal variations on the canonical variables. The following step in [3] is to find the algebra with the Dirac bracket of the global charges $H[\xi] \approx J[\xi]$ on the surface of constrains. Actually, it will be interesting to see the algebra generated by the "Fourier modes" L_n and $\bar{L}_n$ defined in (1.19).

To compute the Poisson bracket between two generators $H[\xi]$ and $H[\eta]$, Brown and Henneaux used a theorem earlier proved by themselves that states that this Poisson bracket is itself a well-defined generator [44], so it can be written as,

$$\{H[\xi], H[\eta]\} = H[\zeta] + K[\xi, \eta], \tag{1.20}$$

where K is a central extension independent of the canonical variables. Then, they showed that $\zeta = [\xi, \eta]$ asymptotically and so the algebra of the generators $H[\xi]$ form a central extension of the algebra of asymptotic symmetries. What is left to show is that this central extension is non-trivial, meaning that the extra term K in (1.20) cannot be absorbed by a redefinition of the generators. To achieve this, in [3], the authors interpret[5] the Dirac bracket $\{J[\xi], J[\eta]\}_{DB}$ as an infinitesimal change of the charge $J[\xi]$ under the deformation generated by $J[\eta]$, so,

$$\delta_\eta J[\xi] = \{J[\xi], J[\eta]\}_{DB} = \{H[\xi], H[\eta]\}|_{on-shell}. \tag{1.21}$$

Then, by virtue of equation (1.20), one gets,

$$\delta_\eta J[\xi] = J[[\xi, \eta]] + K[\xi, \eta]. \tag{1.22}$$

Now, by evaluating this expression on global AdS_3 at $t = 0$, and demanding that for this spacetime the charges $J[\xi]$ vanish[6], $\delta_\eta J[\xi] = K[\xi, \eta]$ where the l.h.s is the charge evaluated at the surface

[5]Notice that the phase space of this theory has dimension zero, since there are no degrees of freedom, so there is no way of proving equation (1.21) since once the constraints are imposed, the canonical variables, if exist, are defined on the boundary so one should somehow define the "remaining dynamics" at the boundary. The expression (1.21) can be regarded as a way of doing so.

[6]Remember that the charges $J[\xi]$ where defined up to a constant, so one needs to fix them for some geometry in order to get rid of this ambiguity.

deformed by η. Brown and Henneaux evaluated this explicitly for every mode L_n and $\bar{L}_n$ and got that these satisfy two copies of the Virasoro algebra,

$$\begin{aligned}
[L_n, L_m]_{DB} &= (n-m)L_{n+m} + \tfrac{c}{12}n(n^2-1)\delta_{m+n}, \\
\left[\bar{L}_n, \bar{L}_m\right]_{DB} &= (n-m)\bar{L}_{n+m} + \tfrac{c}{12}n(n^2-1)\delta_{m+n}, \\
\left[L_n, \bar{L}_m\right]_{DB} &= 0,
\end{aligned} \tag{1.23}$$

where

$$c = \frac{3\ell}{2G}. \tag{1.24}$$

This proves that the algebra of charges in 2+1 gravity with a negative cosmological constant is given by two copies of the Virasoro algebra with equal central charge given by (1.24).

1.4 Boundary dynamics

In this section we will explain a result by Coussaert, Henneaux and van Driel, which states that the dynamics of AdS_3 gravity can be classically described by Liouville field theory [45] (see also [46] for further details including the supersymmetric case).

Assuming the classical equivalence between three-dimensional gravity and Chern-Simons theory, these authors start from the action (1.12), which in Hamiltonian form, assuming a manifold of the form $\mathbb{R}\times\Sigma$, each of the two terms read [7],

$$I_{CS}[A] = \int_{\mathbb{R}\times\Sigma} dtdrd\phi \,\mathrm{tr}\left(\dot{A}_r A_\phi - \dot{A}_\phi A_r + 2A_0 F_{r\phi}\right), \tag{1.25}$$

where the coordinates are the ones in (1.6) with $r = \ell\sinh\rho$. If there is a boundary in Σ, then there also appears a boundary term of the form $\int_{\mathbb{R}\times\partial\Sigma} A_0 A_\phi$ which can be compensated by adding the same term with opposite sign in the initial action (since it is a boundary term, it does not change the equations of motion). From (1.25) one learns that A_0 is a Lagrange multiplier and the constraint is

$$F_{r\phi}|_\Sigma \approx 0. \tag{1.26}$$

In [48] it was first shown that when imposing this constraint in the action (1.25), what is left is a Wess-Zumino-Witten (WZW) theory [49] induced on the boundary depending on the topology of Σ. We review this fact following the logical steps in [45], where the simplest case $\Sigma = D^2$, the two-dimensional disk, is considered. The difference between [48] and [45] is that while the former uses the $(A_R)_0 = (A_L)_0 = 0$ boundary condition, the latter uses the Brown-Henneaux asymptotic conditions (1.16) adapted for the gauge connections,

$$A_R \simeq \begin{pmatrix} \frac{dr}{2r} & \mathcal{O}(1/r) \\ r dx^+ & -\frac{dr}{2r} \end{pmatrix}, \quad A_L \simeq \begin{pmatrix} -\frac{dr}{2r} & r dx^- \\ \mathcal{O}(1/r) & \frac{dr}{2r} \end{pmatrix}, \tag{1.27}$$

to leading order. The light-cone coordinates are again $x^+ = u = t/\ell + \phi$ and $x^- = v = t/\ell - \phi$. For specific details on the boundary conditions adapted for the Chern-Simons fields it is advisable to

[7] In [45] there are some typos and it is actually better to follow the careful computations in [47].

see [47], although beware that the label of the $sl(2,\mathbb{R})$ generators $+$ and $-$ seems to be interchanged with respect to the ones used in [45].

Conditions (1.27) have two important features: i) the light-cone components $(A_R)_-$ and $(A_L)_+$ are set to zero asymptotically and ii) $(A_R)_+^{(-)}$ and $(A_L)_-^{(+)}$ are not functions of the light-cone coordinates, where the superscripts $(\pm)$ indicate $sl(2,\mathbb{R})$ algebra indices [8]. Also, usual gauge conditions used in the WZW-related literature will be imposed:

$$(A_R)_+^{(3)} = (A_L)_-^{(3)} = 0, \tag{1.28}$$

which are, of course, compatible with the boundary conditions (1.27).

From Chern-Simons to Wess-Zumino-Witten

In order to make the action (1.25) an extremum with respect to an arbitrary variation of the gauge field, one has to add a boundary term of the form,

$$\int_{\mathbb{R}\times\partial\Sigma} dtd\phi\, \mathrm{tr} A_\phi^2. \tag{1.29}$$

From now on, the theory we are considering is defined by the sum of two actions, one left and one right, each of these given by the sum of (1.25) and (1.29).

The constraint (1.26) is easily solved on the disk by[9],

$$A_r = U\partial_r U^{-1}, \quad A_\phi = U\partial_\phi U^{-1}, \tag{1.30}$$

where $U(t,r,\phi)$ is a well-defined map $\mathbb{R}\times\Sigma \to SL(2,\mathbb{R})$. The case is the same for the other Chern-Simons field. Substituting solution (1.30) into the action with the boundary term included, we have,

$$\begin{aligned} I[U] &= \int_{\mathbb{R}\times\partial\Sigma} \mathrm{tr}\left[U\partial_t U^{-1}\, U\partial_\phi U^{-1} - \left(U\partial_\phi U^{-1}\right)^2\right] dtd\phi \\ &+ \frac{1}{3}\int_{\mathbb{R}\times\Sigma} \epsilon^{\mu\nu\rho}\ \mathrm{tr}\left(U\partial_\mu U^{-1}\, U\partial_\nu U^{-1}\, U\partial_\rho U^{-1}\right) d^3x. \end{aligned} \tag{1.31}$$

This action is known as the chiral WZW model and actually only depends on the boundary values of U. This is because the variation of the second term, $\Gamma[U]$, is a total derivative. Calling U_R and U_L the group elements for each Chern-Simons action, then it is shown in [45] that the group element $V = U_R^{-1}U_L$ is the field whose dynamics are governed by the sum of the two Chern-Simons terms, which actually gives rise to a non-chiral WZW theory. Thus, Brown-Henneaux boundary conditions, when applied to the Chern-Simons formulation of three-dimensional gravity, leave us with the usual WZW action,

$$I_{WZW}[g] = \int dtd\phi\, \mathrm{tr}(V\partial_+ V^{-1}\, V\partial_- V^{-1}) + \Gamma[V]. \tag{1.32}$$

[8]In (1.27) we are using the $sl(2,\mathbb{R})$ algebra generators shown in Appendix A.

[9]Here $U\partial_\mu U^{-1} = \partial_\mu \vec{f}\cdot X$, where $U = \exp(\vec{f}\cdot X)$, $\vec{f}$ is a function from the manifold to the real vector space of same dimension as the algebra and X is an arbitrary linear combination of the generators of the algebra.

The next step in [45] is to show that when imposing the remaining asymptotic conditions ii), plus the gauge fixing at the boundary (1.28), the action becomes the Liouville action. We will delay the discussion about the WZW theory for Chapter 6. Let us just mention here a neat way to define the currents associated to the WZW model for the case at hand following [47]. From the group elements,

$$g_R := U_R \exp(-\rho t^3), \quad g_L := U_L \exp(\rho t^3), \tag{1.33}$$

the currents are,

$$J_R^a = k \lim_{\rho\to\infty} \mathrm{tr}\left(t^a g_R \partial_\phi g_R^{-1}\right), \quad J_L^a = k \lim_{\rho\to\infty} \mathrm{tr}\left(t^a g_L \partial_\phi g_L^{-1}\right), \tag{1.34}$$

where t^a are the generators of the $sl(2,\mathbb{R})$ algebra and k is the level of one Chern-Simons term and then the action reads,

$$S = \frac{k}{4\pi} I[A_R] - \frac{k}{4\pi} I[A_L]. \tag{1.35}$$

This expression is equivalent to (1.12) with $k = \ell/(16\pi G)$ when one takes a different representation of the algebra from the one used in (1.12). This will be explained in detail in Section 3.1. Currents (1.34) are well defined and the boundary conditions (1.27) translate into,

$$J_R^- = k, \quad J_R^3 = 0; \quad J_L^+ = -k, \quad J_L^3 = 0. \tag{1.36}$$

Meanwhile, J_R^+ and J_L^- are arbitrary functions of the light-cone coordinates on the boundary. In [47], the right-side constraints of (1.36) are imposed after quantization using a BRST approach and it is seen that $J_R^- = k$ is sufficient to get rid of the ghost degrees of freedom, while $J_R^3 = 0$ is a gauge fixing term, as we already mentioned. After quantization, the authors of [47] put together both Chern-Simons actions.

On the other hand, constraints (1.36) are imposed at the level of the action in [45], once the field $g \in SL(2,\mathbb{R})$ is parameterized as,

$$g = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp(\frac{1}{2}\Phi) & 0 \\ 0 & \exp(-\frac{1}{2}\Phi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}. \tag{1.37}$$

Then, plugging this into the WZW action (1.32) and imposing the constraints (1.36) on Φ and the momenta[10] $\partial_-\beta$ and $\partial_+\alpha$, one arrives to the Liouville action,

$$S_{\text{Liouville}}[\Phi] \sim \int_{\mathbb{R}\times\partial\Sigma} dt d\phi \left(\frac{1}{2}\partial_+\Phi\partial_-\Phi + 2\exp(\Phi)\right), \tag{1.38}$$

with a pre factor depending on the level k. The main point is that, at the level of the action, one can reduce the Chern-Simons theory to Liouville theory by means of the Brown-Henneaux boundary conditions. This is not trivial, since many features of a theory depend on the equations of motion but are not present in the action[11]. Being Liouville theory a two-dimensional CFT, it has two copies of the Virasoro algebra as generators of local conformal symmetries, which are the residual symmetries of the Kac-Moody symmetries of WZW theory after imposing further

[10] To implement the constraints on the momenta one needs to add a particular boundary term (for fixed time) to the action. See [50] for details.

[11] Actually this fact is realized in Liouville theory: the action is not Weyl invariant, although the theory is conformally invariant! [51].

constraints. Thus, the result described in this section reproduces, from a different and highly non-trivial procedure, the outcome of Brown and Henneaux paper discussed in the previous section and even more it manifestly shows the underlying CFT at the boundary. However, Liouville theory seems to give only an effective collective description of the dual CFT, since the equivalence Chern-Simons $\leftrightarrow$ Liouville showed in [45] only works at the classical level. Furthermore, the effective central charge of Liouville theory is $c_{\text{eff}} = 1$ instead of $c = 3\ell/2G$, and so this says that we are dealing with only one degree of freedom (the Liouville field) collecting the whole dynamics of the microscopic dual theory [52].

1.5 ADM formulation and the space of solutions

Three-dimensional gravity has no local degrees of freedom, and for negative cosmological constant every solution must be locally AdS$_3$. These facts do not imply that the space of solutions, the phase space, consists of only one element, namely AdS$_3$. As will be explicitly shown in the next section, there are black hole solutions, suggesting that the phase space of AdS$_3$ gravity could be much more interesting than what we may naively expect. But first, let us see in this section what we can learn about the phase space of the theory from the ADM formalism [53]. The key point in the analysis is that the space of solutions is in one-to-one correspondence with the space of initial conditions. Thus, by parameterizing any possible initial condition for the *initial metric* g_{ij} and its conjugate momenta π_{ij}, the hole space of solutions is described.

To apply the ADM formalism let us assume that we are dealing with manifolds with the topology $M \approx \mathbb{R} \times \Sigma$, where the real line is identified with time and Σ is a two-dimensional surface where the initial conditions live. If Σ is non-compact then the analysis is quite more involved, but a better understanding has seemingly been reached recently by Scarinci and Krasnov [54]. For simplicity, we will focus only in the case of compact initial surfaces and follow the book of Carlip on three-dimensional gravity [55], just giving the main idea, leaving details aside.

The ADM decomposition of the three-dimensional metric reads,

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{1.39}$$

where N is the lapse function, N^i are the shift functions (in three dimensions there are only two) and g_{ij} are the components of the metric on the spatial surfaces Σ_t, for some t. The Hamiltonian of three-dimensional gravity, using this decomposition, reads,

$$H = \int_\Sigma d^2x \left(N\mathcal{H} + N_i \mathcal{H}^i \right), \tag{1.40}$$

with the Hamiltonian and momentum constraints being,

$$\begin{aligned} \mathcal{H} &= \frac{1}{\sqrt{\det g}} \left(\pi_{ij}\pi^{ij} - (\pi_{ij} g^{ij})^2 \right) - \sqrt{\det g}\,(R - 2\Lambda) \approx 0, && (1.41) \\ \mathcal{H}^i &= -2\nabla_j \pi^{ij} \approx 0, && (1.42) \end{aligned}$$

where every object is two-dimensional and constructed from g_{ij}. From (1.40), we see that the Hamiltonian vanishes when the constraints are imposed, as we already mentioned in Section 1.3. We also mentioned in that opportunity that surface terms arise in the expression of the Hamiltonian, as well. Those are not important in this discussion, since we are interested in solving the constraints (1.41) and (1.42), and in this way being able to discern which initial conditions are allowed.

Two crucial facts allow to (implicitly) solve the constraints in three-dimensional gravity (see [56] for more details in what follows). First, the fact that any metric in a compact surface Σ is conformal to a metric with constant intrinsic curvature σ, where $\sigma = 1$ for the two-sphere, $\sigma = 0$ for the torus, and $\sigma = -1$ for any surface of genus $g \geq 2$. So,

$$g_{ij} = e^{2\lambda} \tilde{g}_{ij}, \tag{1.43}$$

up to diffeomorphisms on Σ, where $\tilde{g}_{ij}$ are the components of a metric of constant curvature σ which fixes the function λ. The second fact, due to Fischer and Tromba [57], is that the space of constant curvature metrics on Σ modulo diffeomorphisms (connected to the identity) is a finite dimensional space diffeomorphic to the Teichmüller space of Σ, $\mathcal{T}(\Sigma)$, which is also diffeomorphic to $\mathbb{R}^{6g-6}$, for genus $g > 1$, and to the upper-half complex plane for genus 1. Therefore, a point in $\mathcal{T}(\Sigma)$ uniquely identifies a metric $\tilde{g}$ of constant curvature σ and all its diffeomorphic images [12]. On the other hand, it turns out that, using the constraints (1.41) and (1.42), the traceless and transverse part p_{ij} of the momenta π_{ij} is the only one that survives and plays the rôle of conjugate momenta to the coordinates $\{m_\alpha\}$ in $\mathcal{T}(\Sigma)$. The function λ defined in (1.43) is obtained [13] from the Hamiltonian constraint in terms of the canonical variables,

$$\lambda = \lambda(m, p, T), \tag{1.44}$$

where T is the York time-slicing [55]. Then, the reduced Hamiltonian reads,

$$H_{\text{red}}(m, p, T) = \int_{\Sigma_T} d^2x \sqrt{\det \tilde{g}}\; e^{2\lambda(m,p,T)}. \tag{1.45}$$

The lapse and shift functions can be obtained from Einstein equations once g_{ij} and π_{ij} are found. In general, the reduced Hamiltonian (1.45) has a complicated form, involving for example a square root of p^2 in the case of genus 1 [58], so its quantization does not seem straightforward.

1.6 The BTZ black hole

Until 1992, it was generally believed that AdS$_3$ gravity had solutions which were only global AdS$_3$ or manifolds with singularities [14], but any attempt to quantize it would give no information about the quantum physics of black holes. This view changed with the work of Bañados, Teitelboim and Zanelli [4], were they presented a spinning black hole solution of 2+1 general relativity with $\Lambda < 0$. Less than a year later, together with Henneaux, they described in detail the geometry of these black hole manifolds in [5], which we shall closely follow to summarize their analysis.

First of all, let us present the metric of the solution in "Schwarzschild coordinates",

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right) dt^2 + \left(\frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2 - J\, dr d\phi + r^2 d\phi^2, \tag{1.46}$$

[12] Actually, one should further impose constraints that make equivalent geometries related by "large diffeomorphisms", namely diffeomporphisms that are not connected to the identity. This takes the Teichmüler space and gives the moduli space.

[13] Existence and uniqueness of λ is guaranteed as far as the mean curvature of Σ, $\tau = g_{ij}\pi^{ij}/\sqrt{\det g}$, satisfies $\tau^2 > 0$ [56].

[14] Or even worst, manifolds with closed timelike curves [2].

where M and J are the mass and angular momentum respectively, and must satisfy, for $M > 0$, $|J| \leq M\ell$. The radius of the inner and outer horizons are,

$$r_{\pm}^2 = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}, \tag{1.47}$$

which leads to,

$$\ell^2 M = r_+^2 + r_-^2, \quad \ell J = 2r_+r_-. \tag{1.48}$$

We will see later that this coordinate system does not cover the whole manifold, but only one representative of the covering space of AdS_3 from which it is constructed.

The geometry of these black holes comes from a clever identification of (a subset of) AdS_3. To see this in detail let us first show how to embed the BTZ manifold in $\mathbb{R}^{2,2}$. Consider $X = (\bar{y}, \bar{x}, x, y) \in \mathbb{R}^{2,2}$ and so AdS_3 is the hypersurface,

$$-\bar{x}^2 - \bar{y}^2 + x^2 + y^2 = -\ell^2. \tag{1.49}$$

This metric submanifold has an $SO(2,2)$ isometry, i.e., the isometries of AdS_3, and the elements of the algebra of $SO(2,2)$ can be represented by vector fields,

$$J_{AB} = x_B\partial_A - x_A\partial_B, \tag{1.50}$$

where $X = (x_0, x_1, x_2, x_3) = (\bar{y}, \bar{x}, x, y)$. Any of these Killing vector fields generates a one-paremeter subgroup of isometries of AdS_3, by exponentiation,

$$P(\xi)_n : (\bar{y}, \bar{x}, x, y)|_{\mathrm{AdS}_3} \to e^{2\pi n\xi}(\bar{y}, \bar{x}, x, y)|_{\mathrm{AdS}_3}, \quad n \in \mathbb{Z}, \tag{1.51}$$

where $\xi = \xi^{AB}J_{AB}$. The identification is then achieved by identifying all points in the same orbit,

$$X \sim P(\xi)_n X, \quad X \in \mathrm{AdS}_3, \forall n \in \mathbb{Z}. \tag{1.52}$$

Given the fact that the norm of a Killing vector field ξ along its orbits is constant, thus if it is timelike in some region, i.e. $\xi \cdot \xi < 0$, then there will be closed timelike curves in that region after identification. Because of this, one needs to remove the region where $\xi \cdot \xi < 0$ from the manifold. What is left, before identifications, is then geodesically incomplete at the surface $\xi \cdot \xi = 0$. Let us call this space AdS_3'. Once the identifications are imposed, we are left with $\mathrm{AdS}_3'/\sim$, which is still geodesically incomplete, but the surface $\xi \cdot \xi = 0$ will play the rôle of a typical singularity inside a black hole, as it happens in the 3+1 case. Even more, the spacetime is *everywhere* smooth.

The specific Killing vector field that gives rise to the BTZ black hole with radii r_+ and r_- is,

$$\xi = \frac{r_+}{\ell}J_{12} - \frac{r_-}{\ell}J_{03}, \tag{1.53}$$

for $r_+ \neq r_-$. The extremal case is similar but its contruction invloves some subtleties; in this thesis it will be simply thought of as the limit $r_- \to r_+$. As mentioned earlier, the region where $\xi \cdot \xi < 0$ is first removed and then the identification is performed. According to [5], there are three types of regions with $\xi \cdot \xi > 0$:

$$\begin{aligned} &\text{Region I} : r_+^2 < \xi \cdot \xi, \\ &\text{Region II} : r_-^2 < \xi \cdot \xi < r_+^2, \\ &\text{Region III} : \xi \cdot \xi < r_-^2. \end{aligned} \tag{1.54}$$

Due to the fact that one actually starts from the universal covering of AdS_3, each of these regions has infinite connected pieces. Selecting adjacent regions of each type (one of each), I, II and III, there is a nice parameterization in terms of coordinates (t, r, ϕ). We give only one example for Region I (see [5] for more details):

$$\begin{aligned}
\bar{x} &= \sqrt{A(r)} \cosh \tilde{\phi}(t, \phi), \\
x &= \sqrt{A(r)} \sinh \tilde{\phi}(t, \phi), \\
y &= \sqrt{B(r)} \cosh \tilde{t}(t, \phi), \\
\bar{y} &= \sqrt{B(r)} \sinh \tilde{t}(t, \phi),
\end{aligned} \tag{1.55}$$

where,

$$\begin{aligned}
A(r) = \ell^2 \sqrt{\tfrac{r^2 - r_-^2}{r_+^2 - r_-^2}}, \quad B(r) = \ell^2 \sqrt{\tfrac{r^2 - r_+^2}{r_+^2 - r_-^2}}, \\
\tilde{t}(t, \phi) = \tfrac{1}{\ell}\left(r_+ \tfrac{t}{\ell} - r_- \phi\right), \quad \tilde{\phi}(t, \phi) = \tfrac{1}{\ell}\left(-r_- \tfrac{t}{\ell} - r_+ \phi\right).
\end{aligned} \tag{1.56}$$

In these coordinates, the metric becomes exactly as (1.46), but with $\phi \in (-\infty, \infty)$. The Killing vector field reads $\xi = \partial_\phi$, so the identification is just giving coordinate ϕ a period as $(t, r, \phi) \sim (t, r, \phi + 2\pi)$. This is actually the case for any of the Regions I, II, or III. The resulting manifold is everywhere smooth and with constant curvature because the identification is properly discontinuous (see Appendix B of [5]). In addition, the black hole has only two Killing vectors, ∂_t and ∂_ϕ, since every other isometry of AdS_3 turns out to be ill-defined after the identification. The Penrose diagram of the spinning BTZ black hole is depicted in Figure 1.1,

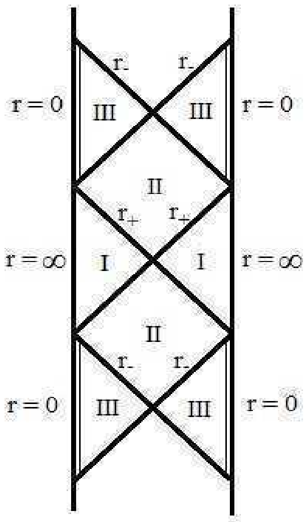

Figure 1.1: The Penrose diagram of the BTZ black hole with outer horizon r_+ and inner horizon r_- is displayed.

Since the BTZ black hole is included in the family of AdS_3 metrics of Brown-Henneaux, the mass and angular momentum can be calculated with the Hamiltonian charges of Section 1.3 with respect to the vectors ∂_t and ∂_ϕ respectively, and they give,

$$J[\partial_t] = M, \quad J[\partial_\phi] = J. \tag{1.57}$$

Taking into account what we discussed in the previous section, it is natural to ask what is the initial condition for the BTZ black hole. Actually, the answer to this question gives rise to even more surprising geometries, where many BTZ black holes coexist [59, 60, 61]. We will briefly explain this in Appendix D.

Thermodynamics

The thermodynamical properties of the BTZ black hole are quite similar to those of the Kerr black hole in 3+1 dimensions. First of all, its temperature can be computed from the surface gravity, and gives,

$$T = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+}. \tag{1.58}$$

Secondly, the entropy is easily obtained from the area law,

$$S = \frac{A}{4G} = \frac{2\pi r_+}{4G}. \tag{1.59}$$

The first law of thermodynamics is satisfied, with the energy being given by $E = M/8G$, as expected from the seminal work of Bardeen, Carter and Hawking [62]. All of these results can be obtained by different methods ranging from the Euclidean path integral approach to the computation of quantum Hawking radiation or by the uses of holographic methods (like Cardy's formula for the entropy). See [63] for further comments and references.

Chapter 2

A review of some aspects of the AdS/CFT correspondence

In this chapter we will make an account of some specific technology within the AdS/CFT correspondence that will be used later on in this thesis. First of all the correspondence as originally stated by Maldacena will be introduced. Then, we will briefly explain how to compute correlators using the duality. Finally, the technique of holographic renormalization will be discussed.

2.1 Maldacena's conjecture

The AdS/CFT correspondence, or Maldacena's conjecture [6], relates two very different theories in a precise way. In its best known version it claims that type IIB superstring theory (or supergravity) in $\mathrm{AdS}_5 \times S^5$ spacetime is dual (i.e., in some way equivalent) to $\mathcal{N} = 4$ Super Yang-Mills (SYM) $SU(N)$ gauge theory in four spacetime dimensions.

To be a little bit more precise, let us mention that in a non-Abelian gauge theory, as $\mathcal{N} = 4$ SYM, there is a well defined regime called the 't Hooft limit [64], that amounts to take the numbers of colors to infinity, $N \to \infty$, and the coupling constant to vanish, $g_{_{YM}} \to 0$, while keeping the so called 't Hooft coupling constant $\lambda = g^2_{_{YM}} N$ fixed. Taking this limit makes all the non-planar Feynman diagrams give $\mathcal{O}(1/N)$ contributions when compared to the planar diagrams, which are classified by the effective coupling λ. There is a well-known way to map Feynman diagrams to Riemann surfaces. In the case of the planar diagrams, the surface is a sphere, so the 't Hooft limit is interpreted as only taking into account interactions on the sphere, or in *stringy* language, tree-level string interactions: $g_s \sim \lambda/N \to 0$. This is the first "duality" relation which involves the YM coupling $g_{_{YM}}$ and the string coupling g_s, $g_s = g^2_{_{YM}}$.

There is another limit which is crucial for the AdS/CFT correspondence. Consider the Type IIB string theory which has closed and open strings propagating in flat spacetime. The open strings can be thought of as ending on D3-branes (called D-branes because Dirichlet boundary conditions are imposed) and thus represent deformations (excitations) of that object. The crucial limit is the low energy limit, $E \ll l_s^{-1}$, and then only massless modes are excited (here E is a typical energy of the system and l_s is the string length[1]). In the low energy regime, the closed string massless excitations are described by type IIB supergravity in ten dimensions, while the open

[1]In string theory, there is only one dimensionful parameter, $\alpha' = l_s^2$.

string massless excitations are described by $\mathcal{N}=4$ U(N) SYM in $3+1$ dimensions. Integrating out all the massive modes, one is led to Wilsonian action for the massless modes:

$$S = S_{bulk} + S_{brane} + S_{int}. \tag{2.1}$$

The action S_{bulk} is the effective action for the massless modes of the closed string, namely type IIB supergravity plus some higher derivative corrections, S_{brane} corresponds to the open string modes so it is $\mathcal{N}=4$ SYM in four dimensions plus higher derivative corrections, and S_{int} captures the interactions between the massless modes of closed and open strings. The coupling constant is given by G_N and has dimensions L^8 in ten dimensions, so the effective dimensionless coupling constant is $G_N E^8$ which goes to zero in the low energy limit (this limit is written as $\kappa^2 = 8\pi G_N \to 0$). The only thing that survives in the bulk part are the terms with no interaction: for the graviton, the action is,

$$S \sim \frac{1}{2\kappa^2}\int \sqrt{g}R \sim \int (\partial h)^2 + \kappa(\partial h)^2 h + \ldots, \tag{2.2}$$

where we have expanded as $g = \eta + \kappa h$, and so in the $\kappa \to 0$ limit the free part is the only one that survives. This means that in the bulk we have free massless modes. The interacting action, S_{int}, is proportional to κ, and thus vanishes. For the brane part, only the $\mathcal{N}=4$ SYM part survives. In conclusion, one is led with two decoupled systems: a CFT on the 3$-$brane, $\mathcal{N}=4$ SYM with gauge group U(N), and free closed strings in the ten-dimensional bulk. It is important to note that this picture is only valid when perturbation theory makes sense: $g_s N \sim \lambda \ll 1$.

A crucial point is that a similar decoupling limit for the D3-branes can be achieved from another perspective. Consider the same situation, type IIB string theory, but now we want to understand the low energy limit by regarding the D3-branes as gravitating objects. In other words, previously we considered open and closed strings in flat spacetime, now we will consider closed strings in a curved background generated by the D3-branes (which is where the open strings end). This description is valid in the semiclassical limit, where the strings are like point particles in a curved background, and so one should demand that $l_s \ll R$, where R is a typical length of the spacetime.

The D3-branes curve the spacetime because they have mass and charge: they solve the equations of motion of the low energy limit of closed strings, type IIB supergravity with a constant dilaton, $e^{\phi} = g_s$. The metric and 5-form field strength are given by,

$$\begin{aligned} ds^2 &= f^{-1/2}dx_{1,3}^2 + f^{1/2}(dr^2 + d\Omega_5^2), & (2.3)\\ F_5 &= (1+*)dt\wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1}, & (2.4)\\ f &= 1 + \frac{R^4}{r^4}, \quad R = 4\pi g_s N\sqrt{\alpha'} & (2.5)\end{aligned}$$

Here, Ω_5^2 is the metric of the S^5 and R is the radius of both AdS$_5$ and S^5. The metric has a Minkowski asymptotic region for $r \gg R$ and a throat for $r \simeq R$. The value of R is explained as follows. First, for dimensional reasons, R has to be proportional to $\sqrt{\alpha'}$. Then, g_s appears due to the appearance of a function of the dilaton in the equations of motion. Finally, $4\pi N$ is the proper normalization for *demanding* that the flux of F_5 in S^5 is exactly N. Then,

$$R^4 = 4\pi\lambda l_s^4. \tag{2.6}$$

Another way to see this is by saying $R^4 \sim G_N M \sim G_N N T_{D3}$, where the mass M per unit volume of the stack of D3-branes is the number of branes, N, times their tension, T_{D3}, which is

$1/g_s l_s^4$, according to their Born-Infeld description. Then, since $G_N \sim l_s^8 g_s^2$ (which comes from a a renormalization of the action), we have $R^4 \sim g_s N l_s^4$. We want to show that, in the low energy regime and in the near-horizon limit $r \ll l_s$, the modes localized there cannot interact with those far from the D3−branes, this leading to a decoupled description of closed strings, just as before.

The point is that in the low energy regime, modes far from the brane have long wavelengths which are much greater than the AdS radius R and therefore they do not see the throat produced by the D3−branes (which is of order R). On the other hand, if $r \ll l_s$, then $l_s \ll l_s^2/r = \alpha'/r := U^{-1}$. Understanding the coordinate U as a fixed typical energy scale, the limit which is often referred to in the literature, $\alpha' \to 0$, is actually $\alpha' \ll U^{-2}$. Then, it is straightforward to see that the harmonic function f goes to R^4/r^4 in this limit. The spacetime one is left with (in the near-horizon limit) is $\mathrm{AdS}_5 \times S^5$, and the strings living there cannot surpass the gravitational potential to reach the asymptotic flat region. This can be also thought of as follows: fixing the distance r to the branes and then decreasing α' implies larger and larger energies U to get to r. Then, every massless mode is excited near the throat, but they cannot propagate to the asymptotic flat region. In conclusion, one has, in this semiclassical description, free gravity in the asymptotic region and type IIB supergravity in $\mathrm{AdS}_5 \times S^5$. This is valid, as we said, when $l_s \ll R$, and so we can read that $1 \ll \lambda$.

Maldacena conjectured [6] that since the physical system is the same, both descriptions should be related, and since in both of them there is a decoupled free gravity sector, the correspondence should be between $\mathcal{N} = 4$ U(N) SYM in $3+1$ dimensions and type IIB supergravity in $\mathrm{AdS}_5 \times S^5$. While the former is valid in the $\lambda \ll 1$ regime, the latter is valid in the strong coupling regime $\lambda \gg 1$. This is why this correspondence is so useful: when the string theory is weakly coupled (no quantum string excitations) the gauge theory is strongly coupled, and viceversa.

2.2 Computing correlators within AdS/CFT

In this section we would like to give the concise recipes to compute some (Euclidean) correlators within the AdS/CFT correspondence.

Shortly after the appearance of [6], Witten [9] and independently Gubser, Klebanov and Polyakov [10] , gave a concise prescription for computing correlators in the boundary theory by means of a procedure based on fields with bulk dynamics. This procedure is defined in Euclidean space and can be shortly expressed as:

$$\frac{\delta}{\delta\Phi_0(x_1)\dots\delta\Phi_0(x_n)} Z_{\mathrm{bulk}}[\Phi_0]|_{\Phi_0=0} = \langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle_{\mathrm{CFT}}, \tag{2.7}$$

where Φ_0 denotes the boundary values of the all the bulk fields denoted by Φ, $Z_{\mathrm{bulk}}[\Phi_0]$ is the bulk on-shell (superstring or supergravity) action which depends on these boundary values, and $\mathcal{O}$ are a set of operators in the conformal conformal theory with the appropriate quantum numbers to couple to the boundary values Φ_0.

In order to be clear, let us consider a simple example [9]. Let ϕ be a free massless scalar field on AdS_5 (it could be the dilaton of type IIB string theory) and take the low-energy limit. The saddle-point configuration is the one that gives the major contribution to the partition function on the l.h.s. of (2.7). Consider the Klein-Gordon equation in the Euclidean Poincaré patch where the AdS_5 metric takes the form,

$$ds^2 = \frac{1}{x_0^2}\left((dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\right). \tag{2.8}$$

The solution to the Klein-Gordon equation of motion with boundary value ϕ_0 is,

$$\phi(x_0, \vec{x}) = C \int d^4x' \frac{x_0^4}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^4} \phi_0(\vec{x}'), \tag{2.9}$$

where $x_0 = 0$ is the locus of the boundary, $\vec{x}$ are the coordinates at the boundary where the CFT lives, and C is a normalization constant. Then, we evaluate this solution in the action for the scalar field:

$$I = 2C \int d^4x d^4x' \frac{\phi_0(\vec{x})\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^8}. \tag{2.10}$$

Clearly, taking two derivatives with respect to ϕ_0 following (2.7), gives a $2-$point function with the behavior $|\vec{x} - \vec{x}'|^{-2.\Delta}$, $\Delta = 4$, which is the expected one for a scalar operator of conformal weight[2] 4. Interesting enough, this behavior is known for $\lambda \ll 1$, and the aforementioned prescription in the supergravity regime predicts the same behavior for large coupling constant. This coincidence is rare and characteristic of supersymmetry-preserving correlators.

2.3 Holographic renormalization

Two years after the appearance of Maldacena's work [6], Balasubramanian and Kraus proposed a way to define the (boundary) stress tensor of the dual CFT for asymptotically AdS spacetimes [65]. Their approach is based on the classical quasi-local Brown-York stress tensor [66], with the inclusion of a boundary term in the action to make it well-defined and finite for metrics with AdS asymptotics. This procedure is now known as *holographic renormalization*. Although their construction is valid in any dimension and for a large class of gravitational theories, in this section we will restrict only to the case of pure three-dimensional gravity, since it captures all the basic features needed for later purposes.

The idea in short is to follow the principles outlined in the previous section for computing correlation functions, but in this case applied to determine the stress-energy tensor $1-$point function, by varying the on-shell action with respect to the boundary metric γ [3]. In other words, we think of γ as the current that couples to the stress tensor of the dual CFT on the boundary. The boundary stress tensor is defined as,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-\det\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}}, \tag{2.11}$$

where $\gamma_{\mu\nu}$ is the $\mu\nu$-component of the boundary metric, and S is the *improved* gravitational action evaluated on some specific solution. We say improved because one has to add a *counterterm* to get rid of the ultraviolet divergence that arises when evaluating the action on, say, AdS_3, because of the boundary at infinity. This will be explicitly shown shortly. As we said, the AdS/CFT interpretation of (2.11) is that it gives the mean value of the CFT stress tensor, $\langle T_{\mu\nu}\rangle_{CFT}$. On the other hand, one can understand (2.11) as giving the stress tensor of a two-dimensional field theory defined over a surface with metric γ.

[2] For AdS_{d+1}, the result is the same but with conformal weight d.

[3] To be precise, the boundary metric is only defined up to Weyl transformations, so one needs to think of the boundary metric as a conformal class of metrics. We will speak of the boundary metric anyway to keep the language simple.

The action S is given by,

$$S = \frac{1}{16\pi G}\int_M d^3x\sqrt{-\det g}\,(R-2\Lambda) + \frac{1}{8\pi G}\int_{\partial M} d^2x\left(\sqrt{-\det\gamma}K + L_{ct}\right), \tag{2.12}$$

where K is the trace of the extrinsic curvature with components $K_{\mu\nu}$, and L_{ct} is the lagrangian of the counterterm that needs to be added in order to have a finite stress tensor at the boundary. The term with the extrinsic curvature is the Gibbons-Hawking term needed to have a well-defined variational principle for the action of general relativity. Analogously, for other gravitational theories, one may need to include other terms that make the action have a well-defined functional variation with respect to the metric.

From (2.12) the stress energy tensor (2.11) that is obtained reads,

$$T^{\mu\nu} = \frac{1}{8\pi G}\left(K^{\mu\nu} - K\gamma^{\mu\nu}\right) - \frac{2}{\sqrt{-\det\gamma}}\frac{\delta}{\delta\gamma_{\mu\nu}}\int_{\partial M} d^2x L_{ct}. \tag{2.13}$$

By demanding that the counterterm only depends on local, diffeomorphism-invariant quantities, and that the stress tensor is finite when taking the boundary to infinity (in particular evaluated for AdS_3), the only possibility is,

$$L_{ct} = \frac{1}{\ell}\sqrt{-\det\gamma}. \tag{2.14}$$

Then, the boundary stress tensor for AdS_3 gravity reads,

$$T^{\mu\nu} = \frac{1}{8\pi G}\left(K^{\mu\nu} - K\gamma^{\mu\nu} - \frac{1}{\ell}\gamma^{\mu\nu}\right). \tag{2.15}$$

Charges

With the knowledge of (2.15) one can compute the charges of any asymptotically AdS_3 Einstein geometry, by using the Brown-York charge,

$$Q[\xi] = \int_{\partial\Sigma} dx\sqrt{\sigma}\left(u^\mu T_{\mu\nu}\xi^\nu\right), \tag{2.16}$$

where the integral is performed over the spacelike surface $\partial\Sigma$ at the boundary ∂M, u is a unit timelike vector on ∂M normal to $\partial\Sigma$ and ξ is an asymptotic symmetry. From expression (2.16), the charges of the BTZ are precisely,

$$Q[\partial_t] = \frac{M}{8G}, \quad Q[\partial_\phi] = \frac{J}{8G}, \tag{2.17}$$

in perfect agreement with the Hamiltonian charges (1.57) modulo the $1/8G$ factor which is a matter of convention. Note that AdS_3 in global coordinates has parameters $M=-1$ and $J=0$, so it has non-zero energy; but when using a non-global patch, like in (2.8), this corresponds to parameters $M=J=0$, which gives a vanishing energy. Of course, one needs to use a patch that covers the entire boundary to get a truthful answer from (2.16), mainly because ξ needs to be well-defined all over the boundary.

Central charge

Now that we have a precise way to obtain the mean value of the stress tensor of the dual CFT, let us try to extract information about this theory. For this, we can analyze how the boundary stress tensor transforms when a Brown-Henneaux asymptotic diffeomorphism like (1.18) is performed. The simplest way to accomplish this goal is the one pursued in [65], where they use the Poincaré patch (which can be thought of as the limit $r_+ \to 0$ of the non-spinning BTZ black hole while unwrapping the angular coordinate ϕ) since one is only interested in the local behavior of the stress tensor. In Poincaré coordinates the boundary stress tensor (2.11) vanishes,

$$T_{\mu\nu}|_{\text{Poincaré}} = 0. \tag{2.18}$$

In particular, in light-cone coordinates u and v,

$$T_{uu} = T_{vv} = 0, \quad \text{Poincaré patch.} \tag{2.19}$$

Now, consider the Poincaré metric,

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - r^2 dudv, \tag{2.20}$$

with $2r \sim \ell \exp(\rho)$ near the boundary. If we perform a Brown-Henneaux transformation of the form,

$$\begin{aligned} u &\to u + \xi^u, \\ v &\to v + \xi^v, \\ \rho &\to \rho + \xi^\rho, \end{aligned} \tag{2.21}$$

with the components of ξ given by (1.18), then the metric (2.20) gets modified to,

$$ds^2 \to \frac{\ell^2}{r^2}dr^2 - r^2 dudv - \frac{\ell^2}{2}(\partial_u^3 U)du^2 - \frac{\ell^2}{2}(\partial_v^3 V)dv^2. \tag{2.22}$$

Now, the stress tensor associated to this new metric is,

$$T_{uu} = -\frac{\ell}{16\pi G}\partial_u^3 U, \quad T_{vv} = -\frac{\ell}{16\pi G}\partial_v^3 V, \tag{2.23}$$

which means that the boundary stress tensor is not a tensor in the bulk, since it transforms with a quantum anomalous term, which in CFT language is $-(c/24\pi)\partial_u^3 U$ and the same for the left-moving sector. Thus, one can read of from these expressions the central charge being,

$$c = \frac{3\ell}{2G}, \tag{2.24}$$

coinciding with the one of Brown and Henneaux (1.24). It is not surprising that the boundary stress tensor is not a tensor in the bulk and only at the boundary, since its definition is not generally covariant: it is defined in terms of tensors at the boundary.

It is also possible to reproduce the central charge from another quantum effect: the conformal anomaly,

$$T^\mu_\mu = -\frac{c}{24\pi}R. \tag{2.25}$$

What is needed to do this is to reexpress the trace of the extrinsic curvature in the definition of the boundary stress tensor (2.11) in terms of the 2-dimensional intrinsic curvature of the boundary metric. To do this, the authors in [65] first evaluate the trace T^μ_μ in a particular background,

$$ds^2 = \frac{\ell^2}{r^2} dr^2 + \gamma_{\mu\nu} dx^\mu dx^\nu, \tag{2.26}$$

with arbitrary boundary metric γ and then use the Fefferman-Graham expansion [67] which gives an asymptotic expansion on the radial coordinate of this metric using Einstein equations,

$$\gamma = r^2 \gamma^{(0)} + \gamma^{(2)} + \dots, \tag{2.27}$$

with the property,

$$\mathrm{tr}\left[\left(\gamma^{(0)}\right)^{-1} \gamma^{(2)}\right] = \frac{\ell^2 r^2}{2} R. \tag{2.28}$$

Then, the trace of the boundary stress tensor can be put in terms of the intrinsic curvature and the central charge of the boundary theory, and (2.24) is reobtained by a different procedure.

Chapter 3

General relativity in 2+1 dimensions reconsidered

3.1 Witten's holographic proposal of 2007

In 2007, Edward Witten proposed a map between 2+1 general relativity with a negative cosmological constant for different values of ℓ/G and a family of particular CFTs [11], in the spirit of the AdS/CFT correspondence.

First of all, he claimed that the definition of a quantum version of AdS_3 gravity with fixed ℓ/G should be given in terms of the dual two-dimensional CFT. So his work [11] is devoted to find such CFTs, by taking hints both from the spectrum of the family of BTZ black holes as well as from the perturbative equivalence between gravity and Chern-Simons theory.

The plan then is to vary the value of the central charge (actually there will be two of these, as will be explained shortly), $c = 3\ell/2G$, and for each value define a two-dimensional CFT dual to the gravitational theory. But an immediate issue arises. Indeed, Zamolodchikov's c-theorem, states that the central charge must be constant in any family of CFTs parameterized by a continuous parameter. Thus, the central charge, which parameterizes the members of the CFTs Witten is looking for, must take discrete values. Otherwise, the central charge would be continuously parameterizing the family of CFTs but at the same time it would not be constant. The ratio ℓ/G must thereby take discrete values in order for Witten's proposal to make sense. In Zamolodchikov's theorem, it is assumed that there is an $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$-invariant vacuum, which semiclassicaly would correspond to global AdS_3.

Hints from Chern-Simons theory

The quantization of the central charge will be motivated by the topological quantization of the level in front of the Chern-Simons action. As was reviewed in Section 1.2, the gravitational action can be written in terms of one Chern-Simons action for the $SO(2,2)$ group or by a linear combination of two Chern-Simons actions for the $SO(2,1) \times SO(2,1)$ group [1],

$$I = \frac{k}{4\pi} I_{CS}(A_L) - \frac{k}{4\pi} I_{CS}(A_R). \tag{3.1}$$

Here, $k = \ell/16G$ is the level. If one considers different non-vanishing [2] levels for the left and right sectors, k_L and k_R, then at the classical level the space of solutions remains the same, and so the theory is the same. This means that the action now takes the form,

$$I = \frac{k_L}{4\pi} I_{CS}(A_L) - \frac{k_R}{4\pi} I_{CS}(A_R), \tag{3.2}$$

with $k_L + k_R = \ell/8G$ to recover the Einstein-Hilbert action. The remaining part, proportional to $k_R - k_L$, has a term with the Chern-Simons 3-form for the spin connection ω and a term of the form $\int e_a \wedge T^a$. Despite this, it is important to stress out that the equations of motion remain the same, so the torsion vanishes.

Let us review the quantization of the level of a Chern-Simons theory just for the case of the $SO(2,1)$ group,

$$I = \frac{k}{4\pi} \int_W \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{3.3}$$

where A is a connection for the group. This is actually the case only if the bundle is trivial. If not, then A is only a local section in the bundle. To give A a global meaning (as a global section of the line bundle), one can pick an extension of W in one more dimension, to get a four-manifold M such that $\partial M = W$, and extend the line bundle where A is defined to four dimensions, although there is no unique way of doing this (but this can always be done). Then the action is,

$$I_M = \frac{k}{4\pi} \int_M \mathrm{tr}\left(F \wedge F \right), \tag{3.4}$$

where $F = dA + A \wedge A$ is the curvature. If we pick another extension to, say, a four-manifold M', then we can repeat the above procedure and compare both presumable equivalent actions by gluing the four-manifolds M and M' along the boundary to form a manifold Y without boundary, with action

$$I_M - I_{M'} = \frac{k}{4\pi} \int_Y \mathrm{tr}\left(F \wedge F \right). \tag{3.5}$$

If A would be a connection for the $U(1)$ gauge group, then this integral would give $\pi k c_1$, where $c_1 \in \mathbb{Z}$ is the first Chern-class of the bundle. Then, by demanding that the action be defined modulo 2π times an integer (so the path integral makes sense), we would get that $k/2$ is an integer. But in our case we have the group $SO(2,1)$ instead of $U(1)$. The fact that helps finishing the train of thoughts is that $SO(2,1)$ is contractible onto its maximal compact subgroup $SO(2)$ which is isomorphic to $U(1)$. So the quantization in the case of $U(1)$ translates into the case we are interested in, which is $SO(2,1)$. The only detail left aside is that we must pick a trace for the $SO(2,1)$ theory, and this gives an additional factor 2, which translates in the condition

[1] In [11] the trace is taken over the three-dimensional representation of the group $SO(2,1)$, which means that for each copy $\mathrm{tr}(J_a J_b) = -2\eta_{ab}$, this gives an extra factor of -4 when compared to the two-dimensional trace used in [2] and in Section 1.2.

[2] The case where one of the levels vanishes will be the key point in Chaper 6.

that k should be an integer [3]. If we consider two actions like in (3.2), with different levels, the quantization condition reads,

$$k_L, k_R \in \mathbb{Z}. \tag{3.6}$$

Then, given the fact that for $k_R = k_L = \ell/16\pi G$ we have $c_L = c_R = 3\ell/2G$, it reads that in this case $(c_L, c_R) = (24k_L, 24k_R)$. Then, for the case with different left- and right-moving levels the left central charge should depend only on the left level and the same for the right sector, so one expects the same relation,

$$(c_L, c_R) = (24k_L, 24k_R), \quad k_L, k_R \in \mathbb{Z}. \tag{3.7}$$

Holomorphic factorization

In the previous subsection we have seen that Chern-Simons theory suggests that the left and right central charges satisfy (3.7). Now we will see what should be demanded from the two-dimensional CFTs to have such a quantization of the central charges.

The ground state of a CFT in two dimensions with left- and right-moving sectors has energy $(-c_L/24, -c_R/24)$. Then, modular invariance imposes the condition that the difference of these energies is an integer, which would mean, according to the dictionary (3.7), that $k_L - k_R$ is an integer. Since both k_L and k_R are integers, their difference is integral.

Now, if holomorphic factorization is imposed for the partition function on the torus,

$$Z(\tau) = \mathrm{tr}\left(e^{2\pi i\tau L_0}\right) \mathrm{tr}\left(e^{-2\pi i\bar{\tau}\bar{L}_0}\right), \tag{3.8}$$

with τ being the modulus, each factor in the partition function is holomorphic if the ground state energies $-c_L/24$ and $-c_R/24$ are integers. That is, both c_L and c_R are integer multiples of 24, i.e., relation (3.7) must hold. Thus, we conclude that holomorphic factorization is a good requirement to achieve the condition (3.7) hinted from Chern-Simons theory for the group $SO(2,1) \times SO(2,1)$. This also points into the direction that this was the correct group to consider and not any of its coverings.

Spectrum

Another important request for the CFTs is that they do not have a Kac-Moody algebra enhancing the conformal algebra. This is because any of these additional fields would have to be interpreted as interacting with the gravitational fields, and then one would not be treating the case of pure gravity. So, having said this, the question to answer is what kind of requirement should these CFTs fulfill in order to guarantee there are no Kac-Moody fields? Witten's proposal is that a reasonable thing to ask is that there is no primary field of lower dimension than $k+1$, and that the lowest dimension of some primary should be exactly $k+1$. This determines the partition function uniquely. This CFTs are called extremal, and it is not known if they exist for generic integer k.

Taking into account the result of Brown and Henneaux, interpreted as the existence of non-trivial Virasoro symmetries around asymptotically AdS$_3$ solutions, the way Witten infers the spectrum of the CFT is the following. The vacuum would represent AdS$_3$ spacetime, primaries

[3] If we would use the group $SL(2,\mathbb{R})$ which covers twice the group $SO(2,1)$ we would get $4k \in \mathbb{Z}$. See [11] for further details.

of dimension $k+1$ or greater would create BTZ black holes (since they are the ones with defined energy $L_0 \geq 1$), while the Virasoro operators would give other quantum excitations. The semiclassical limit would correspond to $k \to \infty$, since this means $\ell \gg G$.

Partition functions

Now, we focus on the task of motivating the partition functions on the torus for each k, only looking at the holomorphic factor. As we commented before, the vacuum state has an energy $L_0 = -c/24 = -k$, so its contribution to the partition function is q^{-k}, where $q = e^{2\pi i\tau}$ and τ is the modulus of the torus. Then, "close" to the vacuum we have the Brown-Henneaux excitations that come from applying the operators L_{-n} to the vacuum, with $n > 1$ (i.e., the descendants of the vacuum). The partition function taking into account the vacuum plus these excitations, is

$$Z_0(q) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n}. \tag{3.9}$$

This is not modular invariant. But modular invariance should be restored when one sums all the contributions from the other primaries and its descendants. Then, in [11] Witten made the assumption that any primary that would create a BTZ black hole has $L_0 \geq 1$, and thus the corresponding contributions to the partition function are of order q:

$$Z(q) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n} + \mathcal{O}(q). \tag{3.10}$$

This result has the great advantage (and that is why the assumption $L_0 \geq 1$ is important) that modular-invariant partition functions of this form are unique. In fact, Witten finally gets the following partition functions expressed in powers of q,

$$\begin{aligned} Z_1(q) &= q^{-1} + 196884q + \ldots, \\ Z_2(q) &= q^{-2} + 1 + 49287520q + \ldots, \\ Z_k(q) &= q^{-k} + \ldots + (\sim \text{number of states at } L_0 = 1)\, q + \ldots, \end{aligned} \tag{3.11}$$

where by "$\sim$ number of states at $L_0 = 1$" we mean the number of primaries with $L_0 = 1$ put together in some representation of the symmetry group of the CFT (which are the ones we are interested in), minus some descendants that may have just that energy (but those contributions are generically negligible). So for example, it is conjectured that there is a unique extremal CFT, with a monster group symmetry, constructed by Frenkel, Lepowsky and Meurman [68], for the case $k = 1$. This is the one that Witten takes as the dual of gravity at $c = 24$. Its degeneracy for $L_0 = 1$ is 196884 (although one state is not a primary) which gives an entropy of $\ln 196883 \cong 12.19$ which compared to the Beckenstein-Hawking entropy 4π is not that close. The agreement is improved for larger k, as one should expect for the proposal to make sense.

Leaving aside the many assumptions in Witten's proposal and taking it seriously, then one should be really concerned by the fact that there are no known holomorphic extremal CFTs for $k \geq 1$. Even more, soon after [11] appeared, Gaiotto showed that there are no extremal CFTs with $c = 48$ and with Monster symmetry [69]. Furthermore, Gaberdiel and Keller gave strong evidence [70], but no proof, that there shall not exist extremal CFTs for $k \geq 42$, which would be fatal for the proposal since the semiclassical limit would not exist.

3.2 The partition function of Maloney and Witten

Maloney and Witten undertook the task of summing explicitly all the contributions corresponding to classical configurations to the partition function, in the Euclidean sector [12]. Their idea was to fix the boundary topology to a torus with given modulus parameter τ, and fill it with smooth Euclidean geometries. They classified these geometries, considered also the Brown-Henneaux contributions and summed all up. The resulting partition function, however, cannot be given a physical meaning, as will be explained shortly. Even so, by demanding this partition function to be holomorphically factorized, interesting results can be reached.

Classical geometries

Maloney and Witten started by classifying all the Euclidean geometries that are locally AdS_3 with some restrictions. The first condition these geometries should obey is that their boundary Σ must be conformal to a torus of parameter $\tau = (\theta + i\beta)/2\pi$. They also ask for Σ to be the only boundary of the three-manifold M. The locally AdS_3 manifold is constructed by discrete identifications on AdS_3: $M = \mathrm{AdS}_3/\Gamma$, where Γ is a discrete subgroup of the isometry group $SO(3,1)$ [4]. Actually, one has to substract from AdS_3 the points where Γ does not act discretely (if any). Anyway, in [12] the authors studied the different ways in which Γ can act on Σ, demanding that Σ has genus 1. There are basically two possibilities: either having cusp geometries or not (see Section 2.1 of [12] for further details). Giving the fact that cusp geometries have actually two boundaries, the conformal one and the one where the cusp is located, these are ruled out. Then, one is led with a set of manifolds, which could have conical singularities or not. Maloney and Witten also discard manifolds with conical singularities with the justification that these are interpreted as massive particles and they want to consider the most minimalistic theory (if exists) of pure gravity.

The set of manifolds they end up with can be finally labeled by two co-prime numbers c, d. In brief, the construction goes as follows. The discrete group Γ is isomorphic to $\mathbb{Z}$ and is generated by an $SL(2,\mathbb{C})$ matrix of the form,

$$W = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \tag{3.12}$$

with $|q| < 1$. Then, Σ is $\{z \in \mathbb{C} - \{0\}\}$ modulo the action of Γ, where W acts on the complex plane as,

$$w \to w + \tau, \quad \tau = \frac{\log q}{2\pi i}, \tag{3.13}$$

with $z = \exp(2\pi i w)$ and then $w \sim w + 1$. This generates a torus in the complex plane. But the modulus of the torus is defined up to an $SL(2,\mathbb{Z})$ transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{3.14}$$

Then, for any choice of a, b, c, d one obtains an admissible manifold, although there may be two or more choices that give the same manifold. Only two of the integers a, b, c, d are relevant, since for example by fixing c and d, then a and b can be obtained from the relation $ad - bc = 1$ and the shift $(a, b) \to (a, b) + t(c, d), t \in \mathbb{Z}$, which leaves q invariant. So, as already mentioned, the

[4] Note that since we are treating the Euclidean case here, the isometries of global AdS_3 are no longer those of $SO(2,2)$ but $SO(3,1)$.

manifolds are labeled by a pair of co-prime integers c and d and are referred to as $M_{c,d}$. Given a fixed τ there is a modular transformation that takes any manifold $M_{c,d}$ to any other. These are diffeomorphisms that are not connected to the identity, i.e., large diffeomorphisms.

The simplest of these manifolds is $M_{0,1}$, *thermal* AdS$_3$, which is obtained from Euclidean AdS$_3$,

$$ds^2 = \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \tag{3.15}$$

with $-\infty < t < \infty$, $0 \leq \phi < 2\pi$, $0 \leq \rho < \infty$, and then identifying,

$$\phi + it \sim \phi + it + 2\pi \sim \phi + it + 2\pi\tau. \tag{3.16}$$

In this case the coordinate ϕ is the one contractible in the bulk. If one performs a $(c, d) = (1, 0)$ transformation this means $\tau \to -1/\tau$ and then the identification is,

$$\phi + it \sim \phi + it + 2\pi \sim \phi + it - 2\pi/\tau. \tag{3.17}$$

To reobtain an expression like (3.16) one should replace $\phi+it \to (\phi+it)/\tau$ and then the contractible coordinate is a combination of ϕ and t, as it happens in the Euclidean BTZ black hole. Thus, $M_{1,0}$ is the Euclidean version of the BTZ black hole. In general, $M_{c,d}$ are referred to as $SL(2, \mathbb{Z})$ black holes. It is not likely that these black holes have a physically meaningful Lorentzian continuation. For this reason, it cannot be assured that the Euclidean partition function is the partition function of standard (Lorentzian) pure AdS$_3$ gravity. But given the fact there are many examples where the Euclidean approach leads to the correct thermodynamical quantities of a gravitational solution, it is worth exploring this approach.

Partition function

The task now is to compute the genus-one partition function of pure (Euclidean) AdS$_3$ gravity. Let $Z_{c,d}(\tau)$ be the contribution of the $M_{c,d}$ geometry at fixed τ. Since all the geometries are related by a modular transformation,

$$Z_{c,d}(\tau) = Z_{0,1}\left(\frac{a\tau + b}{c\tau + d}\right), \tag{3.18}$$

one needs to compute only the contribution of one geometry, say $M_{0,1}$, and then proceed to sum over the modular group:

$$Z(\tau) = \sum_{c,d} Z_{0,1}\left(\frac{a\tau + b}{c\tau + d}\right). \tag{3.19}$$

The sum is over the co-prime integers c and d. Formally, this partition function is modular invariant by construction. But in practice, one has to show that it converges, and if it does not, then modular invariance is generically lost.

We see from (3.19) that an important step is to compute the contribution $Z_{0,1}$ to the partition function. This is,

$$Z_{0,1}(\tau) = \mathrm{tr}\left(-\beta H + i\theta J\right), \tag{3.20}$$

with the trace taken over a Hilbert space of fluctuations over AdS$_3$, and H and J are two commuting operators in this Hilbert space. If one considers only the classical contribution $Z^{(0)}_{0,1}(\tau)$

of thermal AdS$_3$ by evaluating the Einstein-Hilbert action with the Gibbons-Hawking term, one gets,

$$Z^{(0)}_{0,1}(\tau) = |q|^{-2k}, \tag{3.21}$$

with $k = \ell/16G$. From an AdS/CFT correspondence point of view this is understood by the same arguments of the previous section. If there is a CFT dual to pure gravity, then it is natural to identify $H = L_0 + \bar{L}_0$ and $J = L_0 - \bar{L}_0$. Accepting that $M_{0,1}$ should be thought of as the ground state, it has $L_0 = \bar{L}_0 = -c/24$ which reproduces (3.21).

According to Maloney and Witten [12], the analysis of Brown-Henneaux [3] shows that there are actually more states that need to be taken into account in the partition function of the $M_{0,1}$ geometry. From the AdS/CFT point of view, these correspond to descendants of the vacuum, as was discussed in the previous section. But from a purely gravitational side, one has to analyze the phase space of solutions taking into account the Brown-Henneaux diffeomorphisms. The way they do it is by considering the phase space $\mathcal{M}$ as the space of classical solutions that obey the Brown-Henneaux conditions (1.16), modulo diffeomorphisms that vanish fast enough at infinity ("fast enough" ends up meaning that they act as isometries at infinity). This can be thought of as the homogeneous space G/H, where G is the conformal group with central charge $c = 3\ell/2G$;

$$G = \widehat{\mathrm{diff}}\, S^1 \times \widehat{\mathrm{diff}}\, S^1 \tag{3.22}$$

To see this, recall the two periodic functions that generate the asymptotic symmetries in (1.18). The subgroup H of G is the group that leaves a solution fixed under its action, so if the solution is, as in this case, $M_{0,1}$, then H is the isometry group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Quantization of a homogeneous space G/H should give a Hilbert space and since there is an $H-$invariant point, the Hilbert space should have an eigenstate of the action of H (do not confuse here the subgroup H with the Hamiltonian mentioned at the beginning of this subsection). It turns out that the representation of the Virasoro group $\widehat{\mathrm{diff}}\, S^1$ that gives this Hilbert space in which L_0 is bounded from below and with an invariant $SL(2,\mathbb{R})$ element is uniquely determined by the central charge $c = 24k$. It is precisely the one discussed in the previous section and familiar from two-dimensional CFT:

$$\prod_{n=2}^{\infty} L_{-n}^{a_n} |\Omega\rangle, \quad a_n \in \mathbb{N}, \tag{3.23}$$

where $|\Omega\rangle$ is the vacuum satisfying,

$$\left(L_n + k\delta_{n,0}\right)|\Omega\rangle = 0, \quad n \geq -1. \tag{3.24}$$

The energy of the states (3.23) is

$$E = -k + \sum_{n=2}^{\infty} n a_n. \tag{3.25}$$

Then, the same lines of the previous section follow and the partition function for the $M_{0,1}$ geometry with the addition of its Brown-Henneaux states is,

$$Z_{0,1}(\tau) = |q|^{-2k} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}. \tag{3.26}$$

Note that this is one-loop exact, and can be seen as follows: First recall that in perturbation theory, an effective action is defined as summarizing all the loop contributions,

$$Z = \exp(-gI_{\text{eff}}), \tag{3.27}$$

where g is the coupling parameter that is used for perturbation theory and in our case $g = k$. This effective action contains a classical piece plus quantum corrections which go as inverse powers of k,

$$I_{\text{eff}} = I_{\text{classical}} + \sum_{r=1}^{\infty} k^{-r} I_r. \tag{3.28}$$

Then, one needs to compare this expression with (3.26), and one sees that there is only one factor that goes like $\exp(\propto k)$, giving the classical contribution, the rest being a one-loop correction. Therefore, the perturbative analysis around $Z_{0,1}$ is one-loop exact.

In [12], the sum (3.19) is performed using (3.26) and it turns out to be divergent, so a regularization procedure is proposed and the result is a partition function which cannot be written as a sum of exponential functions with positive integral coefficients. The most likely reason for this failure is that there are more geometries to consider in order for the partition function to make physical sense. For example one could consider massive particles propagating in AdS_3, which are solutions to the theory of pure gravity (if one removes the points of the worldline from the manifold) or the one with a gauge coupling to a point-like source [13]. We will comment on this in the next section.

Despite their non-physical result, Maloney and Witten went on a little further and proposed a holomorphically factorized partition function of the form,

$$Z = \left(\sum_{\gamma \in \mathcal{W}} q^{-k} \prod_{n=2}^{\infty} \frac{1}{1-q^n}\Big|_{\gamma} \right) \left(\sum_{\gamma' \in \mathcal{W}} \bar{q}^{-k} \prod_{n=2}^{\infty} \frac{1}{1-\bar{q}^n}\Big|_{\gamma'} \right), \tag{3.29}$$

where $\mathcal{W}$ is the space of geometries $M_{c,d}$ and $\bar{q}$ is the complex conjugate of q. Clearly this proposal is based on the partition function (3.19) together with (3.26), but with a different sum for the holomorphic and anti-holomorphic factors to make the whole expression (3.29) holomorphically factorized. With this new partition function, Maloney and Witten compute the microcanonical entropy of the BTZ black hole and get the usual Bekenstein-Hawking entropy plus logarithmic corrections in the horizon area [12]. They also show a sort of Hawking-Page phase transition between thermal AdS and the BTZ black hole for $k \to \infty$. These results, which are expected to be reproduced in a sensible quantum theory of gravity in three dimensions, depend strongly on the holomorphic form of (3.29). Then, this *holomorphic feature*, although seemingly not present in pure AdS_3 gravity, could be an important ingredient to ask for in other toy models of gravity in three dimensions. This will be the main motivation for the remaining chapters, where we will describe some new theories of gravity in three dimensions where the left and right sectors are disentangled.

3.3 Null states contributions in the partition function

In 2011, Castro, Hartman and Maloney published an interesting work where they reconsider the states that should be summed up to construct the partition function of gravitational theories, and in particular in pure AdS_3 gravity [17]. They claim that when $c < 1$, there is at least one non-zero state that is taken into account which has actually vanishing norm, and thus cannot be part of the physical spectrum. In this section let us explain the analysis in [17] for the case of pure three-dimensional gravity.

Castro et. al. consider small fluctuations around AdS_3, δg, and in particular those given by infinitesimal diffeomorphisms $\delta_\xi g$, generated by a Brown-Henneaux vector field ξ. They introduce a product in the space of fluctuations in terms of the symplectic current ω (do not confuse with the spin connection),

$$(\delta_1 g, \delta_2 g) := \int_\Sigma \omega(g, \delta_1 g^*, \delta_2 g), \tag{3.30}$$

where Σ is an *initial hypersurface*. In the case at hand, g is AdS_3, and in [17] they are implicitly assuming that definition (3.30) is i) convergent and ii) independent of Σ, both depending strongly on the boundary conditions. It is believed that the Brown-Henneaux boundary conditions make definition (3.30) well defined.

On the other hand, from the geometrical formulation of Hamiltonian mechanics it is suggestive to make the following identification [80],

$$\delta J[\xi] := \int_\Sigma \omega(g, \delta g, \delta_\xi g), \tag{3.31}$$

which states that the variation of the charge $J[\xi]$ of the metric g can be computed from the symplectic current ω evaluated at the solution g and at both variations δg and $\delta_\xi g$. The expression (3.31) comes from the definition of a Hamiltonian function in the geometrical formulation of mechanics: given a symplectic manifold (M, Ω) and a vector field X on it, this field is said to be Hamiltonian if there exists a function H on M such that $i_X \Omega = dH$. Conversely, a function H is said to be Hamiltonian if there exists a vector field X such that $i_X \Omega = dH$. Vector X gives the evolution of the Hamiltonian system defined by the symplectic manifold and H. In (3.31) the manifold is the set of metric solutions, the symplectic structure is $\Omega = \int \omega$, the vector fields are the perturbations δg which are tangent to the space of solutions, and the exterior derivative is δ (see [81] for a nice and simple introduction of this).

Now, calling *boundary graviton* ξ to the infinitesimal diffeomorphism generated by ξ and using both (3.30) and (3.31), we have,

$$||\xi||^2 = \delta_{\xi^*} J[\xi]. \tag{3.32}$$

But a charge $J[\xi]$ generates diffeomorphisms on-shell by the Dirac bracket. Then, using this fact,

$$||\xi||^2 = \{J[\xi^*], J[\xi]\}_{DB}. \tag{3.33}$$

This bracket has already been presented: it is given by the Virasoro algebra, thanks to the analysis of Brown and Henneaux introduced in Section 1.3. Again, by defining $L_n := J[\xi_n]$, where ξ_n is the $n-$th Fourier mode of the vector ξ in the light-cone coordinate u (see the lines after (1.18)), and the same for the coordinate v, one gets two copies of Virasoro algebra (1.23) with central charge $c = 3\ell/2G$. In short, Castro et. al. found a simple way of computing the norm of the boundary gravitons by resorting to the analysis of Brown and Henneaux.

The state of zero energy and angular momentum is AdS_3 and is represented by $|0\rangle$. It is therefore annihilated by L_0 and $\bar{L}_0$. Because it is also assumed to be the lowest energy state, it must be annihilated by all the Virasoro modes L_n and $\bar{L}_n$ for $n > 0$. In other words, AdS_3 is a weight-zero primary. According to [17], single-particle states are then obtained as [5],

$$L_{-n}|0\rangle, \quad n > 1, \tag{3.34}$$

while multi-particle states are constructed as,

$$L_{-n_1} \dots L_{-n_k}|0\rangle, \quad n_i > 0. \tag{3.35}$$

Then, single particle states have norm given by (3.33) and multi-particle states are given the same kind of norm:

$$||\chi||^2 := \langle 0|[\chi^*, \chi]|0\rangle, \tag{3.36}$$

where χ represents a multi-particle state. The authors of [17] define conjugation of the Virasoto mode L_n with respect to (3.33) and gives $L^*_{-n} = L_n$, although strictly speaking one should use the product (3.30) to do so.

The question is if there are boundary gravitons of null norm and the answer depends on the central charge: only if $c < 1$ a null state appears. For example, for $c = 1/2$, the multi-particle state $\chi_{-6}|0\rangle$, with,

$$\chi_{-6} = L_{-6} + \frac{22}{9}L_{-4}L_{-2} - \frac{31}{36}L^2_{-3} - \frac{16}{27}L^3_{-2}, \tag{3.37}$$

has zero norm. Generically, the states with zero norm are multi-particle states. Whether these null states are of physical relevance or just a mathematical artifact will not be totally clear until a better understanding of the quantum version of pure AdS_3 gravity is reached. If the analysis of Castro et. al. is correct, then for fixed central charge (less than one) there may be many null states that must be removed from the spectrum. Sadly, this fact would only remove microstates instead of adding them to reach a huge number that would make up the entropy of a three-dimensional black hole.

To finish this section, let us briefly comment on the partition functions found in [17] for three-dimensional gravity. We have seen that when taking into account the full Verma module (with the exception of $L_{-1}|0\rangle$) the result of the contribution to the partition function of AdS_3 is (3.26). According to [17], this seems to be correct for $c > 1$, but for $c < 1$ there are more states to subsract from the spectrum. In general, the partition function associated to states close to AdS_3 will be the vacuum character of a minimal model CFT [82].

[5]The state that comes from applying L_{-1} to AdS_3 has zero norm since this mode is generated by an isometry of AdS_3.

Chapter 4

Massive particles

We have seen in the previous section that taking into account some smooth AdS_3 geometries, the partition function of AdS_3 gravity is not what one would call the partition function of a quantum theory. It is clear that if the quantum version of three-dimensional (Euclidean) gravity exists, then it is likely that one should consider other geometries that could contribute to the partition function. One class of these geometries has conical singularities, which describe massive point-particles sailing through AdS_3. In this section we will describe their Lorentzian geometry, how they can be constructed by identifications (similar to the ones of the BTZ black holes), and also how they can be embedded in a supersymmetric Chern-Simons gravity (see [41] for a review on these gravitational theories) with a coupling to a (Dirac delta) source that makes explicit the interpretation of the solutions as point particles. What follows is based on some original work by the author together with collaborators [13, 14, 15] and is developed in the context of Chern-Simons AdS_3 gravity, as reviewed in Section 1.2.

4.1 Static 0-brane in brief

Let us start by constructing what we will call a static 0-brane, having in mind generalizations to higher-dimensional extended objects. Using the parameterization of $\mathbb{R}^{2,2}$,

$$\begin{aligned} x^0 &= A\cos\phi_{03}\,, \qquad & x^1 &= B\cos\phi_{12}\,, \\ x^3 &= A\sin\phi_{03}\,, \qquad & x^2 &= B\sin\phi_{12}\,, \end{aligned} \tag{4.1}$$

the AdS_3 space corresponds to the surface,

$$x\cdot x = -(x^0)^2 - (x^3)^2 + (x^1)^2 + (x^2)^2 = B^2 - A^2 = -\ell^2\,. \tag{4.2}$$

The AdS_3 metric then reads,

$$ds^2 = \frac{\ell^2\,dB^2}{B^2+\ell^2} - \left(B^2+\ell^2\right)\,d\phi_{03}^2 + B^2 d\phi_{12}^2\,. \tag{4.3}$$

Note that by unwrapping ϕ_{03}, the global covering of AdS_3 in polar coordinates is obtained and closed timelike curves are eliminated. The static 0-brane can be constructed as a topological defect in the (1-2)-plane by introducing a deficit in the ϕ_{12} angle,

$$\phi_{12} = a_0\,\phi_0\,, \quad \phi_0 \simeq \phi_0 + 2\pi\,, \quad 0 < a_0 \le 1\,. \tag{4.4}$$

The periodicity of ϕ_{12} is thus $2\pi a_0$. The metric, with this identification, reads

$$ds_0^2 = \frac{dr^2}{\frac{r^2}{\ell^2} + a_0^2} - \left(a_0^2 + \frac{r^2}{\ell^2}\right) dt_0^2 + r^2 d\phi_0^2 \,, \tag{4.5}$$

where $r = a_0 B$ and $t_0 = \ell\, \phi_{03}/a_0$. Compared with the BTZ static black hole (1.46), the parameter $-a_0^2 = M$ is seen to play the rôle of the (negative) mass. The spacetime geometry given by (4.5) has constant Riemann curvature and vanishing torsion everywhere, except at $r = 0$. On the other hand, the conical singularity at $r = 0$ is the locus of the static 0-brane, where infinite curvature is concentrated. This Dirac delta distribution has support in the center of the $r\phi$-plane, that is the (1-2)-plane in the ambient space $\mathbb{R}^{2,2}$, and we shall call it the 2-manifold Σ_{12}. Including the singular point, the AdS curvature, $F = dA + A \wedge A$, can then be written as $F = j$, where the source is given by the current 2-form

$$j = -2\pi a_0\, \delta(\Sigma_{12})\, J_{12} \,. \tag{4.6}$$

This is the source that would correspond to a massive particle sitting at the center of the Σ_{12} plane with mass $-a_0^2$. We have defined the Dirac delta distribution 2-form that is coordinate-independent,

$$\delta(\Sigma_{12}) \quad = \quad \delta(x^1)\delta(x^2)\, dx^1 \wedge dx^2 = \frac{1}{2\pi}\, \delta(r)\, dr \wedge d\phi \,. \tag{4.7}$$

Of course, we have not yet justified that $F = j$ is the equation of motion that has to be satisfied. We will postpone this for later when we discuss the coupling to the Chern-Simons AdS$_3$ gravity and the stability of these solutions. For now, let us take at face value that the equation we want to solve is precisely $F = j$.

4.2 Spinning 0-brane

In the *non-extremal* case, the massive spinning 0-brane can be obtained from the global AdS metric (4.3) by an identification produced by a linear combination of the Killing vectors $\partial_{\phi_{03}}$ and $\partial_{\phi_{12}}$. This procedure closely follows the one described in [5] to construct the BTZ black hole. Consider two simultaneous angular deficits in the x^0x^3- and x^1x^2-planes in $\mathbb{R}^{2,2}$,

$$\phi_{03} \sim \phi_{03} + 2\pi b \,, \qquad \phi_{12} \sim \phi_{12} + 2\pi a \,. \tag{4.8}$$

The choice of real constants a and b is not unique since these parameters can be shifted by integers. We choose $a, b \in (0, 1]$ and note that $a = b = 1$ means that there is no angular deficit. This identification in $\mathbb{R}^{2,2}$ corresponds to an identification in AdS$_3$ space by some angle ϕ, what can be made explicit by redefining the coordinates as

$$\phi_{03} = b\phi + \frac{u_1 t}{\ell} \,, \qquad \phi_{12} = a\phi + \frac{u_2 t}{\ell} \,, \tag{4.9}$$

where ϕ is periodic with period 2π. The transformation is invertible if $au_1 - bu_2 \neq 0$. Then, $\phi \simeq \phi + 2\pi$ induces the following identification in $\mathbb{R}^{2,2}$,

$$x^A \sim \begin{pmatrix} \cos 2\pi b & 0 & 0 & -\sin 2\pi b \\ 0 & \cos 2\pi a & -\sin 2\pi a & 0 \\ 0 & \sin 2\pi a & \cos 2\pi a & 0 \\ \sin 2\pi b & 0 & 0 & \cos 2\pi b \end{pmatrix} x^A \,, \tag{4.10}$$

or, in an infinitesimal form,

$$x^A \sim x^A + \xi^A\,, \qquad \xi^A = \left(-2\pi b\, x^3, -2\pi a\, x^2, 2\pi a\, x^1, 2\pi b\, x^0\right)\,. \tag{4.11}$$

The identification is produced by the Killing vector,

$$\begin{aligned}\xi &= \xi^A \partial_A = 2\pi a\, J_{12} - 2\pi b\, J_{03}\,,\\ &= 2\pi a\, \partial_{\phi_{12}} + 2\pi b\, \partial_{\phi_{03}}\,,\end{aligned} \tag{4.12}$$

where $J_{AB} = x_A \partial_B - x_B \partial_A$ are generators of rotations in $\mathbb{R}^{2,2}$ (as in (1.50)), and,

$$\partial_{\phi_{12}} = \frac{\partial x^A}{\partial \phi_{12}}\,\partial_A = J_{12}\,, \qquad \partial_{\phi_{03}} = \frac{\partial x^A}{\partial \phi_{03}}\,\partial_A = -J_{03}\,. \tag{4.13}$$

According to the classification given in [5], this Killing vector is of the type $\mathbf{I}_c$ when $a \neq b$ $(a, b \neq 0)$, and of type $\mathbf{II}_b$ when $a = b \neq 0$. Thus, this geometry belongs to a sector topologically different from the BTZ black hole, produced by identifications of type $\mathbf{I}_b$ $(r_+ \neq r_-,\ r_\pm \neq 0)$ and type $\mathbf{II}_a$ $(r_+ = r_- \neq 0)$.

Using the ortho-normality of the tangent vectors, $\partial_A \cdot \partial_B = \eta_{AB}$, with η the metric of $\mathbb{R}^{2,2}$, the norm of the Killing vector ξ reads,

$$\|\xi\|^2 = (2\pi)^2 \left[\left(a^2 - b^2\right) B^2 - b^2 \ell^2\right]\,, \tag{4.14}$$

which is valid for any a and b. However, notice that if $a^2 \leq b^2$, the norm $\|\xi\|^2$ is a timelike vector, and since it identifies different points in our geometry, it would lead to closed timelike curves. Therefore, we assume $a^2 > b^2$,

$$\|\xi\|^2 = (2\pi)^2 \left(a^2 - b^2\right)\left(B^2 - B_*^2\right)\,, \qquad B_* = \frac{b\ell}{\sqrt{a^2 - b^2}}\,. \tag{4.15}$$

After implementing the identification $\phi \sim \phi + 2\pi$, the metric (4.3) becomes

$$\begin{aligned} ds^2 &= \frac{\ell^2 dB^2}{B^2 + \ell^2} + (a^2 - b^2)(B^2 - B_*^2) d\phi^2 \\ &\quad - \left(\frac{u_1^2 - u_2^2}{\ell^2}\, B^2 + u_1^2\right) dt^2 - 2\left(\frac{a u_2 - b u_1}{\ell}\, B - u_1 b \ell\right) d\phi\, dt\,. \end{aligned} \tag{4.16}$$

In sum, the metric (4.16) is obtained from global AdS by the identification (4.10) which means simultaneous identifications of the embedding angles ϕ_{03} and ϕ_{12}. Alternatively, this metric with $u_1 = a$ and $u_2 = b$ can be re-interpreted as the static 0-brane with angular defect a_0, whose spin is introduced by boosting the azimuthal angle ϕ with the velocity $0 \leq v < 1$, similarly to the construction in [71] for the spinning charged black hole.

Indeed, if $b^2 < a^2$, we can always write $a^2 - b^2 \equiv a_0^2 > 0$, where $a_0 \in (0, 1]$, and parameterize the constants a and b in terms of a hyperbolic angle Ξ, as $a = a_0 \cosh \Xi$ and $b = a_0 \sinh \Xi$. Then, Ξ can be re-interpreted as the rapidity of some Lorentz transformation with velocity $v = \tanh \Xi$, $0 \leq v < 1$, so that the original angular deficits are related to the boost velocity as,

$$a = \frac{a_0}{\sqrt{1 - v^2}}\,, \qquad b = \frac{v a_0}{\sqrt{1 - v^2}}\,. \tag{4.17}$$

Now, the relations (4.9) with $u_1 = a$ and $u_2 = b$ are equivalent to a single Lorentz boost $(t_0, \phi_0) \to (t, \phi)$ acting on the static 0-brane [1],

$$t = \frac{t_0 - \ell v \phi_0}{\sqrt{1-v^2}}\,, \qquad \phi = \frac{\phi_0 - \frac{v}{\ell}\, t_0}{\sqrt{1-v^2}}\,. \tag{4.18}$$

Note that this interpretation is possible only for $b^2 < a^2$ $(v \neq 1)$, which means that the brane is not extremal, and allows for the identification in the (t, ϕ)-plane as $(t, \phi) \sim (t, \phi + 2\pi)$. The limit $v = 0$ recovers the static brane $(a = a_0,\ b = 0)$. Thus, b is related to the angular momentum of the brane.

Although the original static metric describes a manifold with a conical singularity, the addition of angular momentum makes the manifold regular, its curvature being constant everywhere, similar to the case in [72]. This spacetime, however, has a *causal horizon*, the surface $B = B_*$, where the norm of the Killing vector field (4.15) vanishes. The component $g_{\phi\phi} = \|\xi\|^2$ of the metric also vanishes there, though its positivity should guarantee that ϕ is an angle.

In the exterior region, $B > B_*$, the vector field ξ is space-like and the causal structure is well-defined. In the interior region, $B < B_*$, ξ becomes timelike allowing closed timelike curves. This region of spacetime can be removed by introducing a new radial coordinate r, such that $g_{\phi\phi} = r^2$ is always non-negative, or

$$B^2 = \frac{r^2}{a_0^2} + B_*^2\,, \tag{4.19}$$

where the above transformation is valid for $B \geq B_*$. The removed region corresponds to $0 < B < B_*$ $(g_{\phi\phi} < 0)$, and then the boundary $g_{\phi\phi} = r^2 = 0$ is shrunk to a point in the r-ϕ plane. Note that the identification $B = B_*$ as a single point in the r-ϕ plane is not produced by a Killing vector. The region of interest, $B > B_*$, excludes the original singularity where the static 0-brane lays $(B = 0)$. In spite of this, in what follows we shall show that there is a singular behavior at $r = 0$ due to the identification of this circle (at fixed time) with a point. If this identification had not been performed, the resulting spacetime would have been regular, but with a geometry similar to the one of the spinning BTZ black hole at $r = 0$ (see Appendix B of [5]).

Once the region with closed timelike curves has been removed, the metric can be recast into the ADM form,

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 \left(d\phi + N^\phi\, dt \right)^2\,, \tag{4.20}$$

where the lapse and shift functions are,

$$N^2 = a^2 + b^2 + \frac{r^2}{\ell^2} + \frac{\ell^2 a^2 b^2}{r^2}\,, \qquad N^\phi = -\frac{ab\ell}{r^2}\,. \tag{4.21}$$

Note that, with this parameterization, the metric is well-defined in the limit $a^2 = b^2$, although the extremal case is obtained by a different identification in AdS space [73].

Even though the Lagrangian governing the dynamics of this 0-brane has not been introduced, the existence of a non-trivial angular momentum can be established by calculating the angular

[1] Actually, one should say that it is global AdS_3 the one which is boosted in order to later impose the periodic identification $\phi \sim \phi + 2\pi$, since the 0−brane already comes from a precise identification on ϕ_0. We will not make this distinction again, since it seems more clear the picture of a boosted 0−brane, although not mathematically accurate.

velocity, $\Omega = -g_{\phi t}/g_{\phi\phi}$, at the circle $r = Const$,

$$\Omega = -N^{\phi} = \frac{\ell ab}{r^2} \neq 0\,. \tag{4.22}$$

The geometry of this asymptotically locally AdS spacetime is an analytic continuation of the $2+1$ black hole, where the parameters a, b are continuations of the horizons $r_{\pm}$. By comparison with the $2+1$-black hole, the real parameters a and b can be related to the mass, M, and angular momentum, J, of the BTZ solution (1.46) as,

$$a \pm b = \sqrt{-M \pm \frac{J}{\ell}}\,. \tag{4.23}$$

Thus, the metric (4.20) describes a BTZ-like 0-brane with a negative mass parameter, $M = -(a^2+b^2) < 0$, boosted with respect to the static brane as $M = \frac{1+v^2}{1-v^2}\, M_0$. The parameter $b \neq 0$ is related to the angular momentum, $J = 2ab\ell$.

4.3 Sources for a spinning 0-brane

Let us summarize what we have done so far. An identification by a Killing vector of the pseudo-sphere $x \cdot x = -\ell^2$ embedded in $\mathbb{R}^{2,2}$ amounts to a single identification by $\xi = 2\pi(a\partial_{12} + b\partial_{03})$ in AdS$_3$. The resulting manifold $\mathbb{M}'$ =AdS$_3/\xi$ is described by the metric (4.16). Since the identification is made by an isometry and is properly discontinuous, except at $B = 0$, $\mathbb{M}'$ has constant curvature for $B > 0$ and thus there is no curvature singularity. This manifold, however, contains closed timelike curves in the region $B < B_*$ $(r < 0)$. Therefore, in order to have a causally well-defined spacetime, the region $B < B_*$ must be removed by cutting along the surface $B = B_*$, defined by $\|\xi\| = 0$. Then all points that satisfy $B = B_*$ at fixed time are identified, producing a new manifold, $\mathbb{M}$, which has a naked singularity at $B = B_*$.

Another way of constructing the same geometry is the following. Take the AdS$_3$ spacetime (4.3) and remove a portion of space $B < B_*$ (for some arbitrary B_*). Then identify the points of the resulting space with (4.10), where now a and b satisfy the relation with B_* given by (4.15). In this way, no closed timelike curves are produced since the region where they would appear was already cut out from space, and also there is no singularity. Now, the origin of the manifold is actually the circle $B = B_*$ $(r = 0)$ for fixed time, so by identifying all those points –which is *not* done by a Killing vector– a curvature singularity appears at the origin. The question we want to analyze is what happens with the Chern-Simons curvature F at the point $r = 0$ of $\mathbb{M}$ after this last identification.

The position of the source responsible for the singularity in spacetime is determined by the surface where the norm of the Killing vector vanishes. In the case at hand, we thus expect a source with a Dirac delta-like distribution of the form $\delta(||\xi||) \sim \delta(r)$. As shown below, this is indeed the case and there are no stronger singularities on the manifold, such as $\delta(r)/r$, or $\partial_r\delta(r)$, etc.

In general, the singularity appears because the 1-form $d\phi$ is not exact on the whole manifold $\mathbb{M}$. Namely, at $r = 0$, where ϕ is not defined, it is not true that $dd\phi = 0$. Thus, we shall assume that $dd\phi = \Delta(r)\, dr \wedge d\phi$, where $\Delta(r)$ is some distribution that can be thought of as zero when $r \neq 0$ and infinite when $r = 0$. The static 0-brane, for example, has $\Delta(r) = \delta(r)$. The same should be expected for the spinning case, as will be confirmed below.

In order to identify the source and nature of the singularity, one can construct the AdS connection using the vielbein and the spin-connection for the metric (4.16) in the region of interest $B > B_*$ ($r > 0$),

$$\begin{aligned} A &= \frac{\partial_r B\, dr}{\sqrt{B^2+\ell^2}}\, J_{13} + \frac{1}{\ell}\left[B\,(bJ_{01} + aJ_{23}) + \sqrt{B^2+\ell^2}\,(bJ_{03} - aJ_{12})\right] d\phi \\ &\quad + \frac{1}{\ell^2}\left[B\,(aJ_{01} + bJ_{23}) + \sqrt{B^2+\ell^2}\,(aJ_{03} - bJ_{12})\right] dt\,. \end{aligned} \tag{4.24}$$

Then, the curvature is,

$$F = \left[\frac{1}{\ell}\sqrt{\frac{r^2+\ell^2a^2}{a^2-b^2}}\,(b\,J_{03} - a\,J_{12}) + \frac{1}{\ell}\sqrt{\frac{r^2+\ell^2b^2}{a^2-b^2}}\,(a\,J_{23} + b\,J_{01})\right]\Delta(r)\, dr\wedge d\phi\,. \tag{4.25}$$

This form of F vanishes everywhere, with the possible exception at $r = 0$. It has been claimed that spinning branes require derivatives of the Dirac delta function (or $\delta(r)/r$) [74, 75]. Those arguments rely on the condition of vanishing torsion everywhere but, as discussed below, this requirement is not met by the present solution. In fact, as can be observed from (4.25), the torsional parts (along J_{03} and J_{23}) have the same singular behavior as the curvature terms (along J_{01} and J_{12}). Thus, the configurations analyzed in [74, 75], not obtained by identifications, correspond to different solutions if compared to the ones considered here.

First let us note that the functional dependence on r in $[\ldots]$ in equation (4.25) is regular at $r = 0$. Then, requiring that the static 0-brane is recovered in the limit $b \to 0$ implies that $\Delta(r) = \delta(r)$ and leads to the source $j = F$ where,

$$j = \frac{1}{\sqrt{a^2-b^2}}\left[ab\,(J_{03} + J_{23}) + b^2\,J_{01} - a^2\,J_{12}\right]\,\delta(r)\, dr\wedge d\phi\,. \tag{4.26}$$

Clearly, this form of the source reproduces the static limit, but not the extremal one. The distribution $\Delta(r)$ can be calculated directly using the definition of the Riemann curvature, *i.e.*, by parallel transport of a Lorentz vector V^a along an infinitesimal contour around the point $r = 0$. In this way, the Riemannian curvature piece of the source can be evaluated (see Appendix E), with the result,

$$j_{\text{curvature}} = \frac{2\pi}{\sqrt{a^2-b^2}}\left(b^2 J_{01} - a^2 J_{12}\right)\delta(r)\, dr\wedge\frac{d\phi}{2\pi}\,. \tag{4.27}$$

The torsional piece of the source (along the generators J_{a3}) cannot be obtained by the same parallel transport. The remaining components, along the generators J_{a3} in (4.26), correspond to the torsional part of the source,

$$j_{\text{torsion}} = \frac{2\pi ab}{\sqrt{a^2-b^2}}\,(J_{03} + J_{23})\,\delta(r)\, dr\wedge\frac{d\phi}{2\pi}\,. \tag{4.28}$$

It is then plain to see that the source carries no singularities stronger than $\delta(r)$.

The above method to calculate j is not easily generalized to higher dimensions because it computes directly only the Riemann curvature and the number of differential equations that need to be solved grows with the dimension. The most precise way to signal the presence of the source is to use the fact that F is locally flat, so the AdS connection has the form,

$$A = g^{-1}dg\,, \tag{4.29}$$

where g is a group element of AdS$_3$, i.e., of $SO(2,2)$. By solving this equation in g with the AdS connection given by (4.24), we obtain

$$g(t,B,\phi) = g_0\, e^{-\phi_{12}J_{12}} e^{\phi_{03}J_{03}} e^{p(B)\,J_{13}}\,, \tag{4.30}$$

where $\phi_{12} = a\phi + bt/\ell$, $\phi_{03} = b\phi + at/\ell$, we denote,

$$p(B) = \sinh^{-1}(B/\ell), \tag{4.31}$$

and g_0 is a constant element of the AdS group.

A non-trivial holonomy appears because g is not single-valued. Indeed, the holonomy $g|_{\phi=2\pi}\, g^{-1}|_{\phi=0}$ is generated by the Killing vector (4.12) as $e^{-\xi}$, up to a conjugation by a constant group element g_0. The result does not depend on the value of the coordinates B and t.

In order to actually calculate the curvature, one can look at the quantity $\oint_{\mathcal{C}_*} A$, where $\mathcal{C}_*$ is a small circle of radius $B \gtrsim B_*$ around the causal horizon at constant t. We obtain,

$$\begin{aligned}
\oint_{\mathcal{C}_*} A &= -\frac{1}{\ell}\,(A_*\,J_{12} - B_*\,J_{23}) \oint_{\mathcal{C}_*} d\phi_{12} + \frac{1}{\ell}\,(B_*\,J_{01} + A_*\,J_{03}) \oint_{\mathcal{C}_*} d\phi_{03} \\
&= -\frac{2\pi a}{\ell}\,(A_*\,J_{12} - B_*\,J_{23}) + \frac{2\pi b}{\ell}\,(B_*\,J_{01} + A_*\,J_{03})\,,
\end{aligned} \tag{4.32}$$

where $A_* = \sqrt{B_*^2 + \ell^2}$. On the other hand, if Σ_* is a surface whose boundary is $\mathcal{C}_*$, in a similar fashion it can be shown that $\int_{\Sigma_*} A \wedge A = 0$, so that,

$$\int_{\Sigma_*} F = \int_{\Sigma_*} dA = \oint_{\mathcal{C}_*} A\,. \tag{4.33}$$

Note that a non-trivial result in the AdS curvature comes from $F = g^{-1}ddg \approx dd\phi \neq 0$. Thus, due to the holonomy centered at $\|\xi\| = 0$, or $B = B_*$ $(r = 0)$ in the $B\phi$-plane, we have,

$$\int_{\Sigma_*} F = \frac{2\pi}{\sqrt{a^2 - b^2}}\,\left(abJ_{03} + ab\,J_{23} + b^2\,J_{01} - a^2\,J_{12}\right)\,. \tag{4.34}$$

Then, from (4.25), $\Delta(r) = \delta(r)$, and from $j = F$ the source reads,

$$j = \frac{2\pi}{\sqrt{a^2 - b^2}}\,\left[-a^2 J_{12} + b^2 J_{01} + ab\,\left(J_{03} + J_{23}\right)\right]\,\delta(r)\,dr \wedge \frac{d\phi}{2\pi}\,, \tag{4.35}$$

in perfect agreement with the result obtained before by other method. Note that had we not identified every point of constant t and $B = B_*$, then we would have only computed the holonomy associated with $\mathcal{C}_*$, but this would not be the curvature at $B = B_*$ since this would not be a point but a circle at fixed t. Also note that when $b = 0$, this current reproduces the static brane result, $j_{\text{static}} = -2\pi a\,J_{12}\,\delta(r)\,dr \wedge \frac{d\phi}{2\pi}$. Furthermore, when $a = 0$, the source simply vanishes. (Vanishing a and b are physically equivalent to $a = b = 1$ in our choice of the range of these parameters.)

In the next section, we show that three-dimensional spinning 0-branes can be stable.

4.4 BPS spinning 0-branes

AdS 0-branes can be constructed in any gravity theory with negative cosmological constant where global AdS is an exact solution. In order to study their stability, one needs to know about the dynamics of the theory and analyze fluctuations around the solution. We thus need to provide a bulk Lagrangian, which we shall take to be CS AdS supergravity. This is a gauge theory whose supergroup is $OSp(p_1|2)\times OSp(p_2|2)$, and it contains $\mathcal{N}=p_1+p_2$ supersymmetries [1]. Apart from the vielbein and spin-connection (1.11), the super AdS connection contains additional bosonic components, that we call the bosonic CS matter.

Besides the Chern-Simons Lagrangian, we couple a term like [76, 77],

$$I_{\text{source}}=\int_M \mathrm{tr}(A\wedge j), \tag{4.36}$$

which results in the equation of motion,

$$F=j. \tag{4.37}$$

This is why in the previous sections we were building a geometry with a curvature proportional to a Dirac delta distribution. This assured that the solution is sourced by a point-like object.

A static 0-brane without bosonic CS matter is possibly unstable since it breaks all supersymmetries. Inclusion of $U(1)$ matter can stabilize the brane and turn it into a BPS state by preserving some supersymmetries in an extremal, charged, static case. Here we show that an extremal spinning 0-brane can be a BPS state even without bosonic matter, as first found in [78] in the special case of $p_1=p_2=1$. In general, the number of preserved supersymmetries is determined by the number of Killing spinors $\epsilon_I^{\pm}$, each component '+' or '−' transforming as a vector in $OSp(p|2)$, labeled by the indices $I=1,\dots,p$.

To find these spinors, we will make use of the static case analyzed in detail in [13], because both cases (spinning and static) have locally the same form (the details of the analysis on the stability of the static 0−brane are shown in Appendix F). Thus, in the background of an uncharged 0-brane and with a suitable representation of the generators, the Killing spinor equation for each copy of $OSp(p|2)$ has the form,

$$D_{\pm}(A)\epsilon_I^{\pm}=\left[d-\left(\frac{1}{4}\epsilon^{a}{}_{bc}\,\omega^{bc}\pm\frac{1}{2\ell}\,e^{a}\right)\Gamma_a\right]\epsilon_I^{\pm}=0\,, \tag{4.38}$$

where Γ_a are three-dimensional matrices satisfying the Clifford algebra (we consider only one of the two inequivalent representations of Γ-matrices, $c=1$ [73, 79]). The vielbein and spin-connection are given in (4.24). Then, a general solution for the Killing spinor is [13],

$$\epsilon_I^{\pm}=e^{\pm\frac{1}{2}\,p(B)\Gamma_1}e^{\pm\frac{1}{2}(\phi_{12}-\phi_{03})\Gamma_0}\chi_I^{\pm}\,, \tag{4.39}$$

where $p(B)$ is defined in (4.31) and $\chi_I^{\pm}$ is a constant spinor fulfilling the chirality projection,

$$\Gamma_0\,\chi_I^{\pm}=i\chi_I^{\pm}\,. \tag{4.40}$$

This Killing spinor is also *globally* well-defined if it satisfies periodic or anti-periodic boundary conditions for $\phi\simeq\phi+2\pi$, *i.e.*, $\phi_{12}\simeq\phi_{12}+2\pi a$ and $\phi_{03}\simeq\phi_{03}+2\pi b$. In consequence, $\epsilon_I^{\pm}(\phi+2\pi)=\pm\epsilon_I^{\pm}(\phi)$ implies the extremality condition,

$$a-b=n\in\mathbb{Z}\,. \tag{4.41}$$

When $b = 0$ (static case), the only possibility to have this condition satisfied is for global AdS ($a = 1$). For the spinning 0-brane, $a, b \in (0, 1)$, the BPS configuration can exist even without additional bosonic matter, because the angular momentum plays the role of a $U(1)$ field. This is an accident of three dimensions only, where the metric admits the limit $a \to b$, even though the brane constructions for $a = b$ and $a \neq b$ differ [73].

These BPS states preserve $\mathcal{N}/2$ supersymmetries; a half is projected out by the condition (4.40). The result can be generalized to include charged 0-branes, as well.

Everything in this chapter can be generalized to higher dimensions. We showed in [13, 14] that static codimension-two branes can be constructed in a similar fashion, resorting to identifications of AdS space. Some of these, when charged under the corresponding AdS supergroup, are shown to be stable configurations. This was proven by solving the Killing spinor equations. In the particular case of BPS 2-branes in five dimensions we showed that the Bogomol'nyi bound is saturated for some specific geometries. The introduction of angular momentum in higher dimensions makes the construction of codimension-two branes quite more involved. This was studied in detail in [15], where it was shown that different intersecting BPS branes with one or more angular momenta can be constructed.

Chapter 5

The chiral gravity conjecture

5.1 Chiral gravity

In this section we will describe a conjecture proposed by Li, Song and Strominger in 2008 [18], in which there is a gravitational theory that admits AdS_3 solutions, and its dual would be a chiral CFT, with only right-moving Virasoro modes. This is in the spirit of Witten's idea of a holomorphically factorized theory. Remember that in section 3.2 we saw that holomorphic factorization allowed for computing the Bekenstein-Hawking temperature and Hawking-Page phase transition from the holomorphic partition function. In the present case, in addition, the left sector would be trivial and the partition function would only have one holomorphic factor. This conjecture was dubbed chiral gravity.

To explain it in detail, we will need to introduce the particular gravitational theory in which the conjecture relies upon. It has AdS_3 as a possible vacuum and is called topologically massive gravity. This was introduced in the '82 by Deser, Jackiw and Templeton [19], and we describe it next.

Topologically massive gravity

Topologically massive gravity (TMG) [19, 20] with a negative cosmological constant, which we review here within the context of [18], is defined by the following action,

$$S = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}\left(R+\frac{2}{\ell^2}\right) + \frac{1}{32\pi G\mu}\int_M d^3x\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\rho\nu}+\frac{2}{3}\Gamma^{\gamma}_{\mu\rho}\Gamma^{\sigma}_{\nu\gamma}\right), \tag{5.1}$$

where μ is a coupling constant with mass units. This action is not parity invariant and is diffeomorphism-invariant modulo a boundary term, so the equations of motion are actually expressed by tensors,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{1}{\ell^2}g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{5.2}$$

where $C_{\mu\nu}$ is the Cotton tensor, given by

$$C_{\mu\nu} = \varepsilon_{(\mu}^{\ \ \alpha\beta}\nabla_\alpha R_{\beta\nu)}. \tag{5.3}$$

This is a theory of gravity with third-order derivatives in the metric, which makes it much more complicated that general relativity. Nevertheless, it shares some important features with its

$G\mu \to \infty$ cousin. For example, it admits as solutions every solution of AdS_3 gravity, since for any locally AdS_3 geometry, the Cotton tensor vanishes identically [1]. In particular, the spectrum of BTZ black holes are present as well as the Brown-Henneaux excitations, although a careful revision of the charges is called for, since the constraints of the theory are modified due to the Chern-Simons term in (5.1). For instance, this term induces a gravitational anomaly [83] and the central charges must obey $c_L - c_R = -3/\mu G$, and then assuming $c_L + c_R = 3\ell/G$,

$$c_L = \frac{3\ell}{2G}\left(1 - \frac{1}{\mu\ell}\right), \quad c_R = \frac{3\ell}{2G}\left(1 + \frac{1}{\mu\ell}\right). \tag{5.4}$$

This can also be seen from a precise analysis *à la* Brown-Henneaux of the canonical realization of the asymptotic symmetries in TMG [84] with Brown-Henneaux boundary conditions (although a more-relaxed set of boundary conditions can be imposed and will be discussed later).

It is important to mention, for future reference, that the theory can also be defined by a first order action, in a similar way as (1.9),

$$\begin{aligned} S_{\mathrm{TMG}} \;&=\; -\frac{1}{16\pi G}\int_{\Sigma_3} \epsilon_{abc}\; R^{ab}\wedge e^c - \frac{\Lambda}{16\pi G}\int_{\Sigma_3} \frac{1}{3}\epsilon_{abc}\; e^a \wedge e^b \wedge e^c \\ &+\; \frac{1}{16\pi G\mu}\int_{\Sigma_3} (\omega_a \wedge d\omega^a + \frac{1}{3}\epsilon_{abc}\omega^a\wedge\omega^b\wedge\omega^c) + \frac{1}{16\pi G\mu}\int_{\Sigma_3} \lambda_a \wedge T^a \;, \end{aligned} \tag{5.5}$$

where the torsion 2-form $T^a = \frac{1}{2}T^a_{\mu\nu}\; dx^\mu \wedge dx^\nu$ is defined by,

$$T^a = de^a + \omega^{ab}\wedge e_b \;.$$

The first two terms in the gravitational action (5.5) correspond to the Einstein-Hilbert and the cosmological terms, with Newton constant G and cosmological constant $\Lambda = -\ell^{-2}$. The third contribution in (5.5) is the so-called *exotic* gravitational Chern-Simons term, which is purely made of the spin connection ω. Also, there is a fourth term in the action, which includes the torsion and a Lagrange multiplier with Lorentz-components λ_a. The Lagrange multiplier is actually a vector-valued 1-form $\lambda^a = \lambda^a_\mu\; dx^\mu$, whose inclusion in the action implements the constraint of vanishing torsion $T^a = 0$. The theory has a mass scale μ, which turns out to be the mass of the gravitons of the theory [19, 20].

The equations of motion coming from the action above are,

$$\epsilon_{abc}\left(R^{bc} + \frac{1}{l^2}e^b\wedge e^c\right) - \frac{1}{\mu}D\lambda_a \;=\; 0\;, \tag{5.6}$$

$$R^a + \frac{1}{2}\epsilon_{abc}\lambda^b\wedge e^c \;=\; T^a\;, \tag{5.7}$$

$$T^a \;=\; 0\;, \tag{5.8}$$

where the 2-form $D\lambda_a = d\lambda_a + \omega_{ab}\wedge\lambda^b$ is the covariant derivative of the Lagrange multiplier. These equations correspond to varying action (5.5) with respect to the dreibein, the spin connection,

[1] This can be seen from the definition of the Cotton tensor (5.3) or from the fact that any locally AdS_3 metric is conformally flat (as is understood from a stereographic projection [60]) and this also makes the Cotton tensor to vanish identically. In other words, the Cotton tensor in three dimensions (5.3) plays the part of the Weyl tensor.

and the Lagrange multiplier, respectively. Notice that this is different from what happens in three-dimensional general relativity, where the equation of motion $T^a = 0$ comes from varying the Einstein-Hilbert action with respect to the spin connection instead. For a concise review of TMG in the first order formalism we refer to the recent papers [85, 86].

Using equation (5.8) above, one may write the set of field equations as follows,

$$\epsilon_{abc}\left(R^{bc} + \frac{1}{l^2}e^b \wedge e^c\right) - \frac{1}{\mu}D\lambda_a = 0\ , \tag{5.9}$$

$$R^a + \frac{1}{2}\epsilon_{abc}\lambda^b \wedge e^c = 0\ , \tag{5.10}$$

and from (5.10), which is an algebraic equation, one solves for λ and replace it back in (5.10) to obtain the Cotton tensor made of $D\lambda$. This defines TMG in the form we know it [19, 20].

A major difference of TMG with respect to pure gravity, as already mentioned, is that it is a third-order theory with solutions that may not be locally AdS_3. In particular, this hints towards the appearance of gravitons, in contrast to general relativity in three dimensions. Indeed, apart from the two usual pure-gauge gravitons, there is one degree of freedom: a propagating massive graviton of mass μ. The energy of this graviton is actually negative for $\mu\ell > 1$ [18]. In contrast, the energy of the BTZ black holes is positive when $\mu\ell > 1$, so they are unstable under the massive graviton perturbation. At the special value $\mu\ell = 1$ peculiar things occur and this is the case where chiral gravity takes place.

Chirality at $\mu\ell = 1$

In 2008, Li, Song and Strominger considered the particular scenario where $\mu\ell = 1$, for the action (5.1) of TMG. They proposed that there is a dual description given by a holomorphic CFT, based on several hints coming from the gravitational side which will be described in this subsection. They called this conjecture chiral gravity.

As was discussed in the previous subsection, the central charges of TMG, both from a holographic point of view [83] or from the algebra of asymptotic symmetries [84], receive a correction and are given by (5.4). In the case at hand, with $\mu\ell = 1$, the left central charge vanishes,

$$c_L = 0, \quad c_R = \frac{3\ell}{G}, \quad \mu\ell = 1. \tag{5.11}$$

This is the first hint in the direction to a holomorphic CFT on the boundary [18]. Nevertheless, it is important to check that the left Virasoro modes have actually a trivial action on physical quantities, which is actually the case, as we will see shortly.

On the other hand, in a generic point of parameter space, BTZ black holes have mass and angular momentum given by,

$$H[\partial_t] := M_{\,\mathrm{TMG}} = M + \frac{1}{\mu\ell^2}J, \quad H[\partial_\phi] := J_{\mathrm{TMG}} = J + \frac{1}{\mu}M, \tag{5.12}$$

so in chiral gravity [18],

$$J_{\mathrm{TMG}} = \ell M_{\mathrm{TMG}}, \quad \mu\ell = 1. \tag{5.13}$$

This means that every BTZ black hole satisfies the extremality condition on its conserved charges when $\mu\ell = 1$. From a holographic point of view, if we identify a BTZ with a primary state in the

CFT, then this means that this state has conformal weights $(h, 0)$, i.e., it is purely right-moving. Again, this hints towards a holomorphic CFT with no left sector.

Now we go a little bit deeper and analyze the spectrum of gravitons close to AdS_3. This was done in detail in [18] and we simply review their results. As usual, by considering a metric perturbation around AdS_3 as $g = \bar{g} + h$ and then fixing the gauge, in [18] arrive to the third-order linear equation,

$$\mathcal{D}^L \mathcal{D}^R \mathcal{D}^M h = 0, \tag{5.14}$$

where,

$$\left(\mathcal{D}^{L/R}\right)_\mu^{\ \nu} := \delta_\mu^\nu \pm \ell \epsilon_\mu^{\ \alpha\nu} \bar{\nabla}_\alpha, \quad \left(\mathcal{D}^M\right)_\mu^{\ \nu} := \delta_\mu^\nu + \frac{1}{\mu} \epsilon_\mu^{\ \alpha\nu} \bar{\nabla}_\alpha. \tag{5.15}$$

These operators are mutually commuting, so equation (5.14) has three linearly independent solutions: the left/right pure-gauge gravitons of AdS_3 gravity, $h^{L/R}$, which are annihilated by the operators $\mathcal{D}^L$ or $\mathcal{D}^R$, and a massive graviton h^M which satisfies $\mathcal{D}^M h^M = 0$. In [18] the different linear solutions are constructed and is shown that they obey the Brown-Henneaux boundary conditions (1.16). They also show that the solution of the massive graviton tends to the solution for the left graviton when $\mu\ell \to 1$, as can be anticipated from the form of the operators (5.15). This gives the authors of [18] a reason to claim that since there are no longer massive gravitons (since they are equivalent to the pure-gauge left gravitons), the theory at $\mu\ell = 1$ is stable, in contrast for what happens for $\mu\ell > 1$.

Actually, at the chiral point, there appears a new massive graviton, which satisfies $\left(\mathcal{D}^M\right)^2 h^M = 0$ and $\mathcal{D}^M h^{\log} \neq 0$, and is linearly independent of the left and right gravitons. But a main difference is present in this new graviton, it has a fall-off behavior at infinity weaker than that of the Brown-Henneaux conditions [21]. Actually, its behavior near the boundary is logarithmic in the coordinate $2r = \ell e^\rho$, for $\rho \to \infty$. This fact led the authors of [21] to call this graviton the *log graviton*, $h^{\log}$. They further showed that this mode has negative energy, thus making chiral gravity as defined in [18] unstable, and that the existence of the log graviton fits in what would be the gravitational dual of a logarithmic CFT. Next section will be devoted to this logarithmic mode and the dual log CFT.

If one wants to impose the stricter boundary conditions of Brown-Henneaux, then the discussion about the stability of chiral gravity becomes even more interesting, since there actually is a linearized mode that satisfies these boundary conditions [22]. Of course, it is not an eigenmode of the L_0 or $\bar{L}_0$ operators, but it can be constructed from the log graviton as,

$$X_{\mu\nu} = (\bar{L}_{-1} \psi^{\log})_{\mu\nu} + \mathcal{L}_\zeta \bar{g}_{\mu\nu}, \quad \mathrm{Re}\, \psi^{\log} = h^{\log}, \tag{5.16}$$

where ζ is the generator of a diffeomorphism such that,

$$\zeta^t, \zeta^\phi = \text{constant} \times e^{-i(v+2u)-4\rho} [y(t,\rho) + \mathcal{O}(1)]. \tag{5.17}$$

This mode, dubbed the GKP mode, has negative energy, as the log graviton does, so it would also spoil the stability of chiral gravity. The stability of chiral gravity was a matter of intense debate from the appearance of [18]. See [22, 25, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97] for some of the relevant discussions. We will come back to this in Section 5.4.

The discussion about chiral gravity was mainly about its spectrum, as it is crucial to establish the consistency of the whole construction. Consequently, the field content of the theory was analyzed in extent, both at the linearized level and at the level of exact solutions. On the one hand, in what regards to linearized solutions, the discussion is summarized in [23], where it was understood that at $\mu l = 1$ two different set of boundary conditions are admissible: the one proposed by Brown and Henneaux in [3], and the weakened version proposed by Grumiller and Johansson in [21, 25, 94]; and depending on which of these asymptotics is chosen, the resulting theory happens to exhibit different properties. In particular, the boundary conditions proposed in [21, 25, 94] permit asymptotic behaviors like,

$$
\begin{aligned}
g_{tt} &\simeq -\frac{r^2}{\ell^2} + \mathcal{O}(\log(r)) \ , \qquad g_{\phi t} \simeq \mathcal{O}(\log(r)) \ , \\
g_{rr} &\simeq \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}) \ , \qquad g_{\phi\phi} \simeq r^2 + \mathcal{O}(\log(r)) \ ,
\end{aligned}
\tag{5.18}
$$

which are certainly weaker than (1.16), taking $r = \ell e^{\rho}$ close to the boundary.

An important point is wether the logarithmic asymptotics of the linearized solution of [21] can be actually realized in an exact solution of the theory. The answer is positive, since in [24] we showed that there is actually a geometry that exhibits the logarithmic fall-off at infinity and that solves the equations (5.2) when $\mu\ell = 1$. This solution will be reviewed in the Section 5.3.

But first, let us mention another way to notice that TMG exhibits special features at $\mu\ell = 1$ with an exact solution describing *pp*-waves in AdS$_3$ [33, 34]. Consider the exact solution,

$$
ds^2 = -r^2 F(u,r) du^2 - 2r^2 du\, dv + \frac{l^2}{r^2} dr^2 \ , \tag{5.19}
$$

which corresponds to a non-linear solution of the equations of motion (5.2) whose physical interpretation is that of a *pp*-wave sailing the AdS$_3$ spacetime. AdS$_3$ spacetime written in Poincaré coordinates corresponds to $F(u,r) = 0$, with $u = t/\ell + \phi$ and $v = t/\ell - \phi$, so that the front of the wave corresponds to the surfaces $u = v = \text{const}$. The function $F(u,r)$ gives the profile of the wave, which takes the form $F(u,r) = (r/\ell)^{\mu\ell - 1} f(u)$. This function satisfies the scalar wave equation on AdS$_3$, namely $(\Box - m_{\text{eff}}^2)F(u,r) = 0$, where $\Box$ stands for the D'Alembert operator in AdS$_3$, and the effective mass m_{eff} is given by $m_{\text{eff}}^2 = \mu^2(1 - \mu^{-2}\ell^{-2})$. That is, the profile function $F(u,r)$ behaves as a scalar mode of the space on which the non-linear wave solution is propagating. Then, one immediately notices that in the limit $\mu\ell \to 1$ such scalar mode becomes massless. One can also verify that non-linear solutions (5.19) develop a logarithmic falling-off behavior at the boundary; namely, solutions like $\sim \log(r/\ell)$ arise at $\mu\ell = 1$.

5.2 Log Gravity

As already mentioned, the spectrum of TMG at $\mu\ell = 1$ is not that simple, and solutions with a weaker fall-off behavior at infinity than that of Brown-Henneaux may arise (see the discussion before (5.18)). In this section we will focus in the first example of this type of (linear) solution, which was found by Grumiller and Johansson [21] short after the initial proposal of [18]: the aforementioned log graviton and the dual log CFT.

The log graviton $h^{\log}$, as was explained before, is a solution of the linearized equations of TMG (5.14) at $\mu\ell = 1$ that satisfies $\mathcal{D}^M \mathcal{D}^M h^{\log} = 0$ and $\mathcal{D}^M h^{\log} \neq 0$. Not only this, but it has the logarithmic asymptotic behavior discussed previously and carries negative energy [21]. Let us briefly comment on the *logarithmic properties* of this graviton. First, let ψ represent primary fields of weight $(h, \bar{h})$ (with respect to the $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ algebra generators) and then take

$\mathrm{Re}\,\psi = h$, where h represents the solutions of (5.14) (do not confuse with the weight h of the primaries). ψ satisfies the same linear equations as the gravitons of [18]. The log graviton is obtained by taking the following limit,

$$\psi^{\log} := \lim_{\mu\ell\to 1} \frac{\psi^M - \psi^L}{\mu\ell - 1} = f(t,\phi)\psi^L, \tag{5.20}$$

with $f(t,\phi) = -it - \ln\cosh\rho$. In terms of the action of $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ algebra generators, we have,

$$L_0\psi^{\log} = 2\psi^{\log} + \frac{1}{2}\psi^L, \quad \bar{L}_0\psi^{\log} = \frac{1}{2}\psi^L, \quad L_1\psi^{\log} = \bar{L}_1\psi^{\log} = 0. \tag{5.21}$$

So we see that $\psi^{\log}$ does not have well defined weights. The representation of the generators L_0 and $\bar{L}_0$ in terms of their action on $\psi^{\log}$ and ψ^L gives,

$$L_0 = \begin{pmatrix} 2 & 1/2 \\ 0 & 2 \end{pmatrix}, \quad \bar{L}_0 = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}, \tag{5.22}$$

which is the typical Jordan normal form for two logarithmic partners in a log CFT [21]. So the terminology *log graviton* is further justified, not only because its asymptotics, but also because $\psi^{\log}$ plays the role of logarithmic partner of ψ^L.

Even more, in [26] is shown that the 2-point and 3-point correlators between the dual operators to the left and log gravitons satisfy the relations that make up a log CFT (with vanishing left central charge). To be more precise, let us call $\mathcal{O}^L$ and $\mathcal{O}^{\log}$ the left flux of the stress-energy tensor and its log companion. Thinking of them as the dual operators to the left and log gravitons respectively and using the typical rules of the AdS/CFT correspondence to obtain $n-$point functions, in [26, 98] obtain,

$$\begin{aligned} \langle \mathcal{O}^L(z)\mathcal{O}^L(0)\rangle &= 0, \\ \langle \mathcal{O}^L(z)\mathcal{O}^{log}(0)\rangle &= \tfrac{b_L}{2z^4}, \\ \langle \mathcal{O}^{log}(z)\mathcal{O}^{log}(0)\rangle &= -\tfrac{b_L \ln\left(m_L^2|z|^2\right)}{z^4}, \end{aligned} \tag{5.23}$$

where $b_L = -3\ell/G$ plays the role of an anomalous central charge and m_L is an arbitrary constant that can be set to one by a redefinition of the operator $\mathcal{O}^{\log}$. Similar but more cumbersome and qualitative expressions are obtained for the 3-point functions.

5.3 A logarithmic rotating solution

We had already mentioned the possibility of considering logarithmic boundary conditions, but these where mainly discussed at the linearized level. Wether the spectrum of TMG at $\mu\ell = 1$ contains exact (as opposed to linear) solutions exhibiting such asymptotic behavior is a non-trivial and important question. In [24] the author together with collaborators showed that there are in fact solutions of this type and we describe them in this section.

At the chiral point $\ell\mu = 1$ one finds a vacuum solution of TMG, whose metric reads,

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2(N_\phi(r)dt - d\phi)^2 + N_k^2(r)(dt - \ell d\phi)^2, \tag{5.24}$$

where,

$$N^2(r) = \frac{r^2}{\ell^2} - \kappa^2 M + \frac{\kappa^4 M^2 \ell^2}{4r^2}, \qquad N_\phi(r) = \frac{\kappa^2 M \ell}{2r^2}, \tag{5.25}$$

and,

$$N_k^2(r) = k \log((r^2 - \kappa^2 M \ell^2/2)/r_0^2), \tag{5.26}$$

where k [2] and r_0 are two real arbitrary constants, and $\kappa^2 = 8\pi G$. Metric (5.24) represents an exact solution of topologically massive gravity that emerges at the chiral point. The Cotton tensor associated to this solution is proportional to k, so it is a genuine solution of TMG in the sense that it does not solve Einstein equations, except for the particular case $k = 0$ where the metric becomes the extremal BTZ black hole. For all values of k the metric is clearly circularly symmetric and static, and thus compatible with $SO(2) \times \mathbb{R}$ symmetry.

In its ADM form, the metric reads,

$$ds^2 = -\mathcal{N}_\perp^2(r) dt^2 + \frac{dr^2}{N^2(r)} + \mathcal{R}^2(r)(d\phi - \mathcal{N}_\phi(r) dt)^2, \tag{5.27}$$

where we have defined,

$$\mathcal{N}_\perp^2(r) = N^2(r) - r^2 N_\phi^2(r) - N_k^2(r) + \mathcal{R}^2(r) \mathcal{N}_\phi^2(r), \tag{5.28}$$

and,

$$\mathcal{R}^2(r) = r^2 + \ell^2 N_k^2(r), \qquad \mathcal{N}_\phi(r) = \mathcal{R}^{-2}(r)(r^2 N_\phi(r) + \ell N_k^2(r)). \tag{5.29}$$

Metric (5.24) is actually nicely behaved. Despite the abstruse form of the off-diagonal component $g_{\phi t}$, the determinant of the metric is $\det g = -r^2$, and the metric is Lorentzian for all values of the radial coordinate r. The metric seems to present a horizon at $r^2 = \kappa^2 M \ell^2/2$. Nevertheless, for $k \neq 0$ the metric in its form (5.24) is not defined for $r^2 \leq \kappa^2 M \ell^2/2$ (for $k = 0$ region $r^2 < \kappa^2 M \ell^2/2$ would correspond to the interior of the BTZ black hole). Let us analyze this aspect together with the geodesic structure in more detail: At $r^2 = \kappa^2 M \ell^2/2$, function N_k^2 diverges while N^2 vanishes. Then, by analyzing the geodesic equation for massive particles, one observes that the divergence of N_k^2 contributes to the radial effective potential with a term like $\sim -(k/r^2)\log(r^2 - \kappa^2 M \ell^2/2)$. This means that, for $k > 0$, massive particles are scattered back when they approach $r^2 = \kappa^2 M \ell^2/2$, and this means that, at least for positive k, the "horizon is not actually there". In fact, for $k > 0$ the circle $r^2 = \kappa^2 M \ell^2/2$ turns out to be located at infinite geodesic distance from any point. For $k < 0$ the geodesic distance to a point at $r^2 = \kappa^2 M \ell^2/2$ turns out to be finite. However, by taking a look at the angular component of the geodesic equation one realizes that the trajectories of massive particles wind indefinitely around the circle defined by $r^2 = \kappa^2 M \ell^2/2$ and thus these geodesics cannot be extended across this circle [99].

From (5.24) we also notice that g_{tt} vanishes at $r^2 = \kappa^2 M \ell^2 + k l^2 \log((r^2 - \kappa^2 M \ell^2/2)/r_0^2)$, and this always happens if $k \leq 0$. In particular, we know that for the extremal BTZ (i.e. $k = 0$) the radius $r = \kappa^2 M \ell^2$ defines its ergosphere [5]. For $k > 0$, however, metric function g_{tt} only vanishes if the parameters satisfy

$$\kappa^2 M \geq 2k(1 - \log(\ell^2 k/r_0^2)). \tag{5.30}$$

For instance, let us consider the case $M = 0$, for which the metric (5.24) takes the simple form

$$ds^2 = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2}(d\phi^2 - dt^2) + k \log(\frac{r^2}{\ell^2})(dt - d\phi)^2 = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} dx^+ dx^- + k \log(\frac{r^2}{\ell^2})(dx^-)^2, \tag{5.31}$$

[2] Do not confuse this k with the level of the Chern-Simons actions that appeared before several times.

where we defined $x^{\pm} = \phi \pm t$, we absorbed a factor ℓ in ϕ, and fixed r_0. From this expression we observe that if $k < 0$ the component g_{tt} vanishes at $r^2 = -2|k|\ell^2 \log(r/r_0)$, and that $g_{\phi\phi}$ may also vanish depending on r_0. On the other hand, if $k > 0$ then the component $g_{\phi\phi}$ vanishes at $r^2 = -2k\ell^2 \log(r/r_0)$, and g_{tt} may also vanish.

Now, let us move on and discuss the asymptotic behavior of (5.24). In the large r limit, metric (5.24) takes the asymptotic form,

$$g_{tt} = -\frac{r^2}{\ell^2} + \mathcal{O}(\log(r)) + \mathcal{O}(1), \qquad g_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}), \tag{5.32}$$

$$g_{\phi\phi} = r^2 + \mathcal{O}(\log(r)) + \mathcal{O}(1), \qquad g_{\phi t} = \mathcal{O}(\log(r)) + \mathcal{O}(1). \tag{5.33}$$

We observe from this large r expansion that this solution is not asymptotically AdS_3 according to the definition given by Brown and Henneaux in [3]. Nevertheless, (5.24) does still obey the weakened AdS_3 asymptotic (5.18) proposed by Grumiller and Johansson in [21, 25, 94]. These weakened boundary conditions were discussed within the context of chiral gravity, and these were shown to be consistent with conformal asymptotic symmetry. In turn, this would permit to define a consistent stress-tensor in the boundary. Our solution can be thought of as a realization of the boundary conditions of [21, 25, 94].

Conserved charges and boundary terms

Because the off-diagonal term in (5.24) grows logarithmically $\sim 2k\log(r)$ at large distance [3], it turns out that metric (5.24) is not asymptotically AdS_3 in the sense of [3]. However, we can still proceed to compute conserved charges of this solution by holographic methods. After all, the solution is still asymptotically AdS_3 in the sense of the boundary conditions recently proposed in [21, 25, 94]. Then, we can resort to the method of defining an effective stress-tensor induced on the boundary ∂M, as in the case of asymptotically locally AdS_3 solutions [65] (see also the seminal paper [66]). We will use what we saw in Section 2.3 for the case of TMG and the logarithmic solution of [24].

Consider the action with the boundary term,

$$\begin{aligned} I_G &= \frac{1}{2\kappa^2}\int_M d^3x\sqrt{-g}\left(R + \frac{2}{\ell^2}\right) + \frac{1}{\kappa^2}\int_{\partial M} d^2y\sqrt{-\gamma}K \\ &+ \frac{1}{4\kappa^2\mu}\int_M d^3x\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\nu\rho} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right), \end{aligned} \tag{5.34}$$

where $K = \mathrm{tr}K = K^i_i$ is the trace of the extrinsic curvature K_{ij}. Here, we see the Gibbons-Hawking term B appears. This action can be expressed in terms of Gaussian coordinates $ds^2 = d\eta^2 + \gamma_{ij}dx^idx^j$, with $K_{ij} = \frac{1}{2}\partial_\eta\gamma_{ij}$. This reads [83, 21],

$$\begin{aligned} I_G &= \frac{1}{2\kappa^2}\int_M d^2y\ d\eta\ \sqrt{-\gamma}(R^{(2)} + K^2 - \mathrm{tr}(K^2) + \frac{2}{\ell^2}) + \\ &+ \frac{1}{4\kappa^2\mu}\int_M d^2y\ d\eta\ \epsilon^{ij}(-2K^{\ell}_i\partial_\eta K_{jl} + \Gamma^l_{in}\partial_\eta\Gamma^n_{jl} \\ &+ 2K^n_k\Gamma^l_{in}\Gamma^k_{jl} + K^l_n\partial_j\Gamma^n_{il} + \Gamma^l_{jn}\partial_iK^n_l), \end{aligned} \tag{5.35}$$

[3] It is worth pointing out that this logarithmic next-to-leading order is not the one considered in [100, 101]; c.f. the leading behaviour of $g_{\phi t}$.

where $\text{tr}K^2 = K_i^j K_j^i$. In this expression, the Gibbons-Hawking term does not appear because it cancels against a total derivative coming from the bulk contribution. Expression (5.35) turns out to be an action for the metric γ_{ij}, which corresponds to the induced metric in the boundary. The stress-tensor T^{ij} associated to the boundary manifold [66] is then obtained by varying (5.35) with respect to γ_{ij} and evaluating it on-shell; namely, $\delta I_G = \frac{1}{2}\int_{\partial M} d^2x\sqrt{-\gamma}T^{ij}\delta\gamma_{ij}$. The conserved charges computed with this stress-tensor (see (5.38) below) diverge and then it is necessary to regularize the action by adding an appropriate counter-term [83, 65]. Such counter-term turns out to be a cosmological constant term in the boundary; namely

$$\Delta I_G = -\frac{1}{\ell 8\pi G}\int d^2y\sqrt{-\gamma}, \tag{5.36}$$

which only depends on geometric quantities of the boundary, not affecting the equations of motion in the bulk.

Including the counter-term (5.36), and in the case of asymptotically AdS_3 spaces, the boundary stress-tensor takes the form,

$$2\kappa^2 T^{ij} = 2(K^{ij} - \gamma^{ij}\text{tr}K - \frac{1}{\ell}\gamma^{ij}) + \frac{1}{\mu}\epsilon^{k(i}(\gamma^{j)l}\partial_\eta K_{kl} + 2\partial_\eta K_k^{j)}). \tag{5.37}$$

This expression can be used to compute conserved charges associated to isometries on the boundary ∂M. One is mainly concerned with the conserved charges that are associated to Killing vectors ∂_t and ∂_ϕ, which correspond to the mass and the angular momentum respectively. To define the charges it is convenient to make use of the ADM formalism adapted to the boundary ∂M. As we saw in Section 2.3 the charges can be computed as [66],

$$Q[\xi] = \int ds\xi^i u^j T_{ij}, \tag{5.38}$$

where ds is the volume element of the constant-t surfaces at the boundary, u is a unit vector orthogonal to the constant-t surfaces, and ξ is the Killing vector that generates the isometry in $\partial\mathcal{M}$. The Brown-York charges (5.38) are supposed to be related to the ones of Regge and Teitelboim (1.15) $J[\xi]$.

To see how it works, let us consider the BTZ solution. It is straightforward to compute the mass and the angular momentum of (1.46) following the recipe described above. The mass and the angular momentum of BTZ black hole in TMG are then given by

$$M_{\text{BTZ}} = M + \frac{J}{\ell^2\mu}, \qquad J_{\text{BTZ}} = J + \frac{M}{\mu}, \tag{5.39}$$

respectively. It is well known [102, 83] that this result differs from the charges of the same solution for GR, which are recovered if $1/\mu = 0$. In particular, these values for the mass and angular momentum in TMG imply that at the chiral point $\mu = 1/\ell$ all the BTZ black holes in TMG fulfill the relation $J_{\text{BTZ}} = \ell M_{\text{BTZ}} = \ell M + J$. More specifically, if $J = -\ell M$ at the chiral point both the mass and the angular momentum vanish.

Then, we can use the same idea to compute the mass and angular momentum of (5.24). It yields

$$M_{(k)} = \frac{3k}{4G}, \qquad J_{(k)} = -\frac{3\ell k}{4G}. \tag{5.40}$$

This is consistent with the fact that (5.24) is a perturbation of the extremal BTZ black hole with $J = -\ell M$ at the chiral point $\mu = 1/\ell$. Recall that BTZ black holes with bare parameters obeying $J = -\ell M$ in chiral gravity have zero mass and zero angular momentum, and then we interpret it as the *ground state* for (5.24). Notice that, as long as Newton constant is positive, the BTZ black hole in TMG have positive mass, and our solution (5.24) has also positive mass for $k > 0$. Conversely, if we adopt the *wrong sign* for Newton constant (what amounts to change $G \to -G$ in (5.1) but keeping GM unchanged) then the BTZ black hole turns out to have negative mass, while (5.24) has positive mass for $k < 0$.

Before concluding this section, let us mention that at the point $\ell\mu = -1$ one also finds a vacuum solution of TMG with the form

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2(N_\phi(r)dt + d\phi)^2 + N_k^2(r)(r^2 - \kappa^2 M\ell^2/2)(dt + \ell d\phi)^2. \qquad (5.41)$$

Unlike solution (5.24), this metric tends to that of the extremal BTZ black hole when r approaches the horizon $r^2 = \kappa^2 M\ell^2/2$. The off-diagonal term in (5.41), however, grows in more drastic way, behaving like $\sim 2kr^2 \log r$ at large distances.

Also, a charged solution at the chiral point exists, and it has a form like (5.24) and (5.41) with its charge associated to k. The electromagnetic field comes both from an Abelian Yang-Mills term and a Chern-Simons term $F \wedge A$.

5.4 Refined chiral gravity conjecture

The heated discussion on the consistency at the linearized level of the holographic proposal called chiral gravity cooled down when Maloney, Song and Strominger reconsidered the effect that comes from taking into account different boundary conditions [23]. Their work starts with a study of the linearized solutions to TMG at the chiral point $\mu\ell = 1$ and then they compute the Euclidean partition function. Let us in this section summarize this.

First of all, in [23], they redefine chiral gravity as TMG at the chiral point with Brown-Henneaux boundary conditions. By doing this, any logarithmic solution is left out from the spectrum of chiral gravity by definition. Then, it remains to see if this boundary conditions are consistent with chirality, in the sense that once imposed, every left generator vanishes on-shell (and thus generates vanishing Dirac brackets on the physical space). This is what Maloney et. al. showed in [23]. They considered the covariant formalism of Barnich and Brandt for conserved charges in gauge theories [103, 104], where the expression for the charges is [4],

$$Q[\xi] = -\frac{1}{16\pi G}\oint_{\partial\Sigma} *\mathcal{F}, \qquad (5.42)$$

where $\mathcal{F}$ is a two-form called *superpotential* and for TMG was computed in [106]. What is important to say is that it comes from a perturbation of the equations of motion, $E^{(1)}$, and depends on an asymptotic Killing vector field, ξ, as,

$$\xi^\mu E^{(1)}_{\mu\nu} dx^\nu = *d*\mathcal{F}. \qquad (5.43)$$

[4]For further details on this formalism and references see the PhD thesis of G. Compère [105].

Note that this formalism only uses the linearized equations of motion as opposed to the Hamiltonian formalism. Nevertheless, the charges computed using the covariant formalism match those obtained from the Hamiltonian formalism [107].

The explicit expression for the charges of TMG at the chiral point is,

$$\begin{aligned} Q[\xi] &= \frac{1}{32\pi G\ell}\oint_{\partial\Sigma} d\phi\left[\, V(x^-)\left(-2\partial_\rho^2 h_{--} + 4\partial_\rho h_{--} + 2\partial_\rho h_{-+} - 4h_{-+} + \frac{e^{2\rho}}{4}h_{\rho\rho}\right)\right.\\ &+ \left. U(x^+)\left(8h_{++} - 8\partial_\rho h_{++} + 2\partial_\rho^2 h_{++} + 2\partial_\rho h_{-+} - 4h_{-+} + \frac{e^{2\rho}}{4}h_{\rho\rho}\right)\right], \end{aligned} \tag{5.44}$$

where we have used the same coordinates and notation of Section 1.3, with $u = x^+$ and $v = x^-$, but so far the Brown-Henneaux boundary conditions (1.18) have not been imposed. If they are actually considered [5], then

$$Q[\xi] = \frac{1}{4\pi\ell G}\oint_{\partial\Sigma} d\phi\, U(x^+)h_{++}, \quad \text{for chiral gravity,} \tag{5.45}$$

so it is evident that any left-moving asymptotic diffeomorphism ($U(x^+) = 0$) has vanishing charge, and chirality is manifest.

It turns out that the boundary charge (5.42) depends on the second-order perturbation to the metric, $h^{(2)}$, while the bulk charge, obtained from integrating by parts the former, depends on the first-order perturbation (quadratically). The GKP graviton, mentioned when discussing the first attempt to define chiral gravity, respects the Brown-Henneaux boundary conditions to first order in the metric perturbation and has a left bulk charge given by,

$$E_L^{\text{GKP}} = -\frac{\ell}{12G} < 0, \quad \text{bulk computation,} \tag{5.46}$$

where its first-order perturbation was used. Since the result for the charge must be the same when computed with the boundary expression and this expression depends on the second-order perturbation, it must be that the second-order perturbation does not obey the Brown-Henneaux boundary conditions. Indeed, in [23] was explicitly shown that,

$$h_{--}^{(2)\text{GKP}} = \frac{\ell^2\rho}{3} + \ldots, \tag{5.47}$$

which violates the Brown-Henneaux boundary conditions. So, the GKP mode is ruled out from the linearized spectrum of the theory and stability is almost assured. What would need to be proven still, is that there is no linearized chiral solution with right negative energy. So far, there is no proof of this claim. What is clear from the analysis in [23], is that any linearized solution needs to be studied at least to second order.

What was also shown in [23] is that any linearized solution to chiral gravity, understood as any solution to the linearized equations of motion of TMG at $\mu\ell = 1$ and with Brown-Henneaux boundary conditions to all-order in the metric perturbation, is also a solution of the linearized Einstein equations. This hints towards the thought that *any exact solution* to chiral gravity is a solution of pure gravity, meaning that the Brown-Henneaux boundary conditions would be strong enough to match the space of solutions of both theories. Also, in [23] they showed that the only stationary and axially symmetric exact solution of chiral gravity is the BTZ black hole, thus reinforcing the idea that the space of exact solutions of chiral gravity and pure gravity is the same. Maloney et. al. assumed this claim to compute the partition function of chiral gravity.

[5] An asymptotic constraint, $2\partial_\rho h_{-+} - 4h_{-+} + \frac{e^{2\rho}}{4}h_{\rho\rho} = 0$, needs also to be considered to get (5.45).

Partition function

The partition function of chiral gravity obtained in [23] stems on the same procedure as that of [12], which was explained in Section 3.2. Here we will only point out some specific assumptions needed to do such calculation.

The Euclidean form of the TMG equations of motion is,

$$G_{\mu\nu} + i\ell C_{\mu\nu} = 0, \tag{5.48}$$

where the i appears in the second term due to the non-invariance under parity transformations of the Chern-Simons term in (5.1) and the Wick rotation $t \to t_E = it$. Since the tensors appearing in (5.48) are precisely those of the Lorentzian theory, any *real* Euclidean metric will have real Einstein tensor and real Cotton tensor. This implies that both terms in (5.48) must vanish separately. If the assumption in [23] that any solution of chiral gravity has vanishing Cotton tensor and real analytic continuation, then one only needs to compute the partition function with the contributions from Einstein geometries. Even more, since the linearized Hilbert space is only right-moving, there is only one holomorphic sector to take into account. Also, by assuming that every linearized solution is obtained by descendants of the highest-weight representation of the right-Virasoro algebra, the whole program of Maloney and Witten discussed in Section (3.2) applies. The absence of the left sector turns out to give a sensible partition function of the form [6],

$$Z(\tau) = \sum_{\Delta=-k}^{\infty} N(\Delta) q^{\Delta}. \tag{5.49}$$

Although we now know that there are in fact solutions of chiral gravity which are not Einstein geometries [27] [7], the result of Maloney, Song, Strominger is very interesting. It means that if there was a theory with only one copy of Virasoro generators and the same space of solutions as pure gravity, then its partition function would be that computed by Maloney and collaborators in [23]. Such a theory will be defined and study in the following chapter.

[6] See [12, 23] for the detailed derivation of expression (5.49).

[7] The geometry found by de Buyl et. al. suffers from having closed timelike curves, which may be a good reason to leave it out from the spectrum of chiral gravity. In any case, the assumption in [23] is not generally true and geometries without pathologies and that would contradict such an assumption could exist.

Chapter 6

A chiral gravity with torsion

This chapter is devoted to present a new proposal by the author of this thesis and collaborators, for a chiral theory of gravity in three dimensions [28]. The proposal takes place in a similar theory as TMG, not being parity invariant thanks to a Chern-Simons term for the spin connection, but with an important difference: it is a Chern-Simons theory in three dimensions, called Mielke-Baekler theory [30, 108]. As such, any linear analysis is trivial and thus the question of instabilities due to negative-energy massive gravitons does not apply. Also, the space of solutions, by definition of the proposal, is under control. This is the main advantage of the theory we will present now.

6.1 Mielke-Baekler theory of gravity

A different construction from that of the original proposal in [18] for a chiral theory of gravity in three dimensions is possible. In order to give a detailed account of this proposal, we need first to describe the theory from where it comes, which is that of considering the three-dimensional case of the Lovelock-Cartan theories constructed in [29], also known as Mielke-Baekler theory [30]. The action of the theory can be written as follows,

$$S_{\mathrm{MB}} = \frac{1}{16\pi G} S_1 + \frac{\Lambda}{16\pi G} S_2 + \frac{1}{16\pi G\mu} S_3 + \frac{m}{16\pi G} S_4 \ , \tag{6.1}$$

where the four terms are,

$$S_1 = 2\int_{\Sigma_3} e_a \wedge R^a \ , \qquad S_2 = -\frac{1}{3}\int_{\Sigma_3} \epsilon_{abc}\, e^a \wedge e^b \wedge e^c \ , \tag{6.2}$$

$$S_3 = \int_{\Sigma_3} (\omega_a \wedge d\omega^a + \frac{1}{3}\epsilon_{abc}\, \omega^a \wedge \omega^b \wedge \omega^c) \ , \quad S_4 = \int_{\Sigma_3} e_a \wedge T^a \ . \tag{6.3}$$

Here, again, we see that in addition to the Einstein-Hilbert action, S_1, and the cosmological constant term, S_2, we have the exotic Chern-Simons gravitational term, S_3, together with the term S_4 that involves the torsion explicitly. In fact, there are actually two stages at which one introduces torsion here: first, this is done by treating the ($SO(2,1)$-components of the) dreibein, e^a, and the spin connection, ω^a, as independent fields, following in this way the standard formulation à la Einstein-Cartan. In the case of general relativity, the Palatini formulation teaches us that considering e^a and ω^a as independent variables does not introduce any substantial difference for the classical theory, as the Einstein equations are recovered by varying the Einstein-Hilbert action with respect to e^a, while the vanishing torsion constraint follows from varying the action with respect to ω^a. However, when the exotic gravitational Chern-Simons term is present in the action, the fact of treating e^a and ω^a as independent geometrical entities does make an important difference.

A second stage at which one introduces torsion in the theory is by adding the term S_4 when writing the action. Such term includes the torsion explicitly, and, in contrast to (5.5), it does not involve a Lagrange multiplier that fixes the torsion to zero but it couples the torsion to the dreibein directly. The term S_4 is dubbed 'translational Chern-Simons term' and, as it happens with the exotic Chern-Simons term S_3, it can be also associated to a topological invariant in four dimensions: While S_3 is thought of as the term whose (dimensionally extended) exterior derivative gives the Pontryagin 4-form density $R^{ab} \wedge R_{ab}$ in four dimensions, the exterior derivative of the translational term S_4 gives the Nieh-Yan 4-form density $T^a \wedge T_a - e^a \wedge e^b \wedge R_{ab}$, [109, 110]. In this sense, all the terms involved in the action (6.1) are of the same sort [41]. Mielke-Baekler theory is the most general, second-order, Lorentz-invariant theory of gravity in three-dimensions with torsion.

The equations of motion coming from (6.1) are

$$R^a - \frac{\Lambda}{2}\, \epsilon^{a}_{\ bc}\, e^b \wedge e^c + m\, T^a \;=\; 0\ , \tag{6.4}$$

$$T^a + \frac{1}{\mu}\, R^a + \frac{m}{2}\, \epsilon^{a}_{\ bc}\, e^b \wedge e^c \;=\; 0\ . \tag{6.5}$$

The first one comes from varying S_{MB} with respect to the dreibein, while the second one comes from varying it with respect to the spin connection. One actually sees that in the case $m = 1/\mu = 0$ the theory agrees with Einstein gravity, for which $R^{ab} \sim e^a \wedge e^b$ and $T^a = 0$. In the special case $m = \mu$ with $\Lambda = -m^2$, the two equations of motion (6.4) and (6.5) coincide and the theory exhibits a degeneracy. We will analyze this special case in Section 5. In the generic case, the theory has four coupling constants, which provide three dimensionless ratios, and the four characteristic length scales G, $\sqrt{\Lambda}$, μ^{-1}, and m^{-1}.

Generic case

The two equations of motion (6.4)-(6.5) are independent equations provided $m \neq \mu$; so let us consider such case first. Rearranging these equations, one finds,

$$R^a = \frac{\mu}{2}\frac{\Lambda + m^2}{\mu - m}\, \epsilon^{a}_{\ bc}\, e^b \wedge e^c\ , \tag{6.6}$$

$$T^a = \frac{1}{2}\frac{m\mu + \Lambda}{m - \mu}\, \epsilon^{a}_{\ bc}\, e^b \wedge e^c\ . \tag{6.7}$$

These equations, provided $m \neq \mu$, express the fact that the solutions of the theory have constant curvature and constant torsion. From (6.6)-(6.7) one immediately identifies two special cases. When $m\mu = -\Lambda$ ($m \neq \mu$), equation (6.7) implies that torsion vanishes, and thus (6.6) becomes Einstein equations. A second special case is $m^2 = -\Lambda$ ($m \neq \mu$), where it is the spacetime curvature what vanishes; this is usually called the *teleparallel theory*.

Even though equation (6.6) implies that the solutions of the theory have to be of constant curvature, the space has torsion, so that the affine connection is not necessarily a Levi-Civita connection. Then, seeing whether the solutions of the theory actually correspond to Einstein manifolds or not requires a little bit more of analysis: To actually see this, it is convenient to write the spin connection ω^a as the sum of a torsionless contribution $\tilde{\omega}^a$ and the contorsion $\Delta\omega^a$; namely,

$$\omega^a = \tilde{\omega}^a + \Delta\omega^a \ , \tag{6.8}$$

where $\tilde{\omega}^a$ is indeed the Levi-Civita connection. Then, from (6.7) one obtains,

$$\Delta\omega^a = \frac{1}{2}\frac{m\mu+\Lambda}{m-\mu}\ e^a \ , \tag{6.9}$$

and from (6.6) one finally gets,

$$\tilde{R}^{ab} = d\tilde{\omega}^{ab} + \tilde{\omega}^a{}_c \wedge \tilde{\omega}^{cb} = -\frac{1}{2l^2}\ e^a \wedge e^b \ , \tag{6.10}$$

which expresses that solutions are indeed Einstein manifolds, where the effective cosmological constant is given by,

$$l^{-2} = \frac{1}{4}\left(\frac{m\mu+\Lambda}{m-\mu}\right)^2 + \frac{\Lambda\mu + m^2\mu}{m-\mu} \ . \tag{6.11}$$

In the case $m = 1/\mu = 0$ one finds $l^{-2} = -\Lambda = \ell^{-2}$.

6.2 Black holes and torsion

Mielke-Baekler theory admits asymptotically AdS_3 black holes as exact solutions. In fact, it can be seen that equations of motion (6.4)-(6.5) are satisfied by the BTZ metric (1.46), provided the space also presents torsion [111]. The presence of non-vanishing torsion, however, does not represent an actual "hair" since the strength of T^a is fixed by (6.7) and so there is no additional parameter to characterize the geometry. Then, the only two parameters of the black hole solutions are still M and J, and for the Mielke-Baekler theory, the mass and angular momentum of the black hole being related to the coupling constants in the following way [111],

$$\mathcal{M} = M\left(1 + \frac{1}{2}\frac{m\mu+\Lambda}{m\mu-\mu^2} + \frac{J}{Ml^2\mu}\right) \ , \qquad \mathcal{J} = J\left(1 + \frac{1}{2}\frac{m\mu+\Lambda}{m\mu-\mu^2} + \frac{M}{J\mu}\right) \ . \tag{6.12}$$

The ADM values of general relativity are recovered in the case $m = 1/\mu = 0$.

Black hole thermodynamics is also affected by the presence of torsion. The entropy of the BTZ black holes in Mielke-Baekler theory can be computed from the Euclidean action [112] or the Cardy formula [113], and is given by,

$$S_{\text{BH}} = \frac{\pi r_+}{2G}\left(1 + \frac{1}{2}\frac{m\mu+\Lambda}{m\mu-\mu^2} - \frac{1}{\mu l}\frac{r_-}{r_+}\right) \ , \tag{6.13}$$

While the first term in (6.13) reproduces the Bekenstein-Hawking area law, contributions proportional to $1/\mu$ give deviations from the result of general relativity. It will be discussed below how the black hole entropy (6.13) is recovered from CFT methods through holography.

6.3 Central charges

In this section, we focus on the computation of the central charges corresponding to the asymptotic algebra. Seen from the holographic point of view, these central charges turn out to be those of the dual conformal field theory. To calculate these central charges it is convenient to discuss first the Chern-Simons formulation of the theory (6.1).

Chern-Simons formulation and Hamiltonian reduction

Mielke-Baekler theory admits to be expressed as a sum of two Chern-Simons actions [108, 114, 115],

$$S_{\rm CS}=\frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)-\frac{\hat{k}}{4\pi}\int \mathrm{tr}\left(\hat{A}\wedge d\hat{A}+\frac{2}{3}\hat{A}\wedge \hat{A}\wedge \hat{A}\right)\ , \tag{6.14}$$

where the corresponding $sl(2,\mathbb{R})-$connections are given by,

$$A^a=\omega^a+\lambda\ e^a\ ,\qquad \hat{A}^a=\omega^a+\hat{\lambda}\ e^a\ , \tag{6.15}$$

with coefficients,

$$\lambda=-\frac{1}{2}\frac{m\mu+\Lambda}{m-\mu}+\frac{1}{l}\ ,\qquad \hat{\lambda}=-\frac{1}{2}\frac{m\mu+\Lambda}{m-\mu}-\frac{1}{l}\ ; \tag{6.16}$$

whereas the coupling constants read,

$$k=\frac{l}{4G}\left(1+\frac{1}{\mu l}+\frac{1}{2}\frac{m\mu+\Lambda}{m\mu-\mu^2}\right)\ ,\qquad \hat{k}=\frac{l}{4G}\left(1-\frac{1}{\mu l}+\frac{1}{2}\frac{m\mu+\Lambda}{m\mu-\mu^2}\right)\ . \tag{6.17}$$

The supra index in (6.15) is playing the rôle of an algebra index, to be contracted with the $3+3$ generators of the $sl(2,\mathbb{R})\oplus sl(2,\mathbb{R})$ algebra [1]. This is analogous to the standard Chern-Simons realization of three-dimensional gravity of Section 1.2 and, in fact, in the case $m=1/\mu=0$ the realization of [1, 2] is recovered. Now, the equations of motion of the theory read,

$$F=0,\qquad \widehat{F}=0\ , \tag{6.18}$$

where F and $\widehat{F}$ are the curvatures corresponding to the gauge fields A and $\hat{A}$ respectively.

Having the theory written in its form (6.14), one can compute the central charges by following the procedure originally introduced in [45, 46]. This amounts to implementing the constraints and the Brown-Henneaux asymptotic boundary conditions at the level of the Chern-Simons actions, by reducing them first to two chiral Wess-Zumino-Witten (WZW) actions, and then using the asymptotic conditions again to reduce some degrees of freedom of the latter. This procedure was reviewd in Section 1.4 and eventually gives the central charges of the boundary two-dimensional conformal field theory through the Hamiltonian reduction of the WZW theory, as in [45, 46]. Nevertheless, despite the analysis here is very similar to that of three-dimensional Einstein gravity, it is worth noticing that, in contrast to the case where no exotic Chern-Simons term is included, the full action is not exactly the difference of two chiral WZW actions with the same level $k=\hat{k}$.

[1] We are using the trace corresponding to the two-dimensional representation as in Section 1.2. In the references sometimes the three-dimensional representation is used and this changes the value of the levels by a factor 4, as was the case in Section 3.1.

The exotic term actually unbalances the two chiral contributions. In turn, Hamiltonian reduction must be performed in each piece separately.

A consistent set of AdS_3 boundary conditions for the theory with torsion, compatible with those of Brown and Henneaux, are the ones proposed in [108, 116, 117]

$$\begin{aligned} e_t^0 &\simeq \frac{r}{l} + \mathcal{O}(1/r) \ , & e_r^0 &\simeq \mathcal{O}(1/r^4) \ , & e_\phi^0 &\simeq \mathcal{O}(1/r) \ , \\ e_t^1 &\simeq \mathcal{O}(1/r^2) \ , & e_r^1 &\simeq \frac{l}{r} + \mathcal{O}(1/r^3) \ , & e_\phi^1 &\simeq \mathcal{O}(1/r^2) \ , \\ e_t^2 &\simeq \mathcal{O}(1/r) \ , & e_r^2 &\simeq \mathcal{O}(1/r^4) \ , & e_\phi^2 &\simeq r + \mathcal{O}(1/r) \ . \end{aligned} \tag{6.19}$$

From equation (6.7), one obtains the asymptotic behavior for the components of the spin connection; namely

$$\begin{aligned} \omega_t^0 &\simeq \frac{\alpha r}{2l} + \mathcal{O}(1) \ , & \omega_r^0 &\simeq \mathcal{O}(1/r^4) \ , & \omega_\phi^0 &\simeq -\frac{r}{l} + \mathcal{O}(1) \ , \\ \omega_t^1 &\simeq \mathcal{O}(1/r^2) \ , & \omega_r^1 &\simeq \frac{\alpha l}{2r} + \mathcal{O}(1/r^3) \ , & \omega_\phi^1 &\simeq \mathcal{O}(1/r^2) \ , \\ \omega_t^2 &\simeq -\frac{r}{l^2} + \mathcal{O}(1/r) \ , & \omega_r^2 &\simeq \mathcal{O}(1/r^4) \ , & \omega_\phi^2 &\simeq \frac{\alpha r}{2} + \mathcal{O}(1/r) \ , \end{aligned} \tag{6.20}$$

where $\alpha = (m\mu + \Lambda)/(m - \mu)$.

Then, following the procedure developed in [45, 46] and explained in Section 1.4, one verifies that implementing some of the asymptotic conditions (6.19)-(6.20) amounts to define a boundary action, consisting of two copies of the chiral WZW model (see Section 1.4 together with [45, 46] for details, and see also [47] for a very nice discussion). The WZW theory has $SL(2,\mathbb{R})_k \times SL(2,\mathbb{R})_{\hat{k}}$ affine Kac-Moody symmetry, which is generated by the currents (1.34)[2],

$$J^i(z) = \sum_{n\in\mathbb{Z}} J_n^i \ z^{-n-1} \ , \quad \bar{J}^i(z) = \sum_{n\in\mathbb{Z}} \bar{J}_n^i \ \bar{z}^{-n-1} \ , \qquad i = 1,2,3 \ ,$$

with the boundary variables $z = t/\ell + i\phi$, $\bar{z} = t/\ell - i\phi$. The modes obey the Kac-Moody current algebra,

$$[J_m^+, J_n^-] = -2J_{n+m}^3 - \frac{k}{2} n \, \delta_{m+n,0} \ , \qquad [J_m^3, J_n^\pm] = \pm J_{n+m}^\pm \ , \qquad [J_m^3, J_n^3] = \frac{k}{2} n \delta_{m+n,0} \ ,$$

with $J_n^\pm = J_n^1 \pm i J_n^2$, where k is a central element; analogously for the anti-holomorphic counterpart $\bar{J}_n^i$ with $\hat{k}$. Then, Sugawara construction [82] gives the Virasoro generators in terms of the Kac-Moody generators; namely,

$$L_m = \frac{h_{ij}}{k-2} \sum_{n\in\mathbb{Z}} J_{m-n}^i \, J_n^j \ , \qquad \bar{L}_m = \frac{h_{ij}}{k-2} \sum_{n\in\mathbb{Z}} \bar{J}_{m-n}^i \, \bar{J}_n^j \ , \tag{6.21}$$

where h_{ij} is the Cartan-Killing bilinear form of $sl(2,\mathbb{R})$ and the -2 in the denominator stands for the Coxeter number of $SL(2,\mathbb{R})$. Then, we have the stress-tensor,

$$T(z) = \sum_{n\in\mathbb{Z}} L_n \ z^{-n-2} \ , \qquad \overline{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \ \bar{z}^{-n-2} \ , \tag{6.22}$$

[2] In the standard WZW theory the currents are defined with a derivative with respect to the complex variables z and $\bar{z}$, as opposed to (1.34), and this is what we follow from now on.

whose modes realize the Virasoro algebra,

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{k}{4(k-2)} m(m^2-1)\, \delta_{m+n,0} \ , \tag{6.23}$$

that gives the central charge $c = 3k/(k-2)$, and analogously for the anti-holomorphic counterpart replacing L_n by $\bar{L}_n$ and k by $\hat{k}$, yielding $\widehat{c} = 3\hat{k}/(\hat{k}-2)$. These are not yet the central charges of the boundary CFT as it still remains to impose some of the boundary conditions (6.19)-(6.20). It is possible to verify that implementing the whole set of asymptotic boundary conditions (6.19)-(6.20) amounts to fixing the constraints $J^+(z) \equiv k$ and $\bar{J}^+(\bar{z}) \equiv \hat{k}$ (see Section 1.4 together with [47]). This condition requires an improvement of the stress-tensor of the sort $T(z) \to T(z) + \partial J^3(z)$, as it demands the current $J^+(z)$ to be a dimension-zero field. This is equivalent to shifting $L_n \to L_n - (n+1)J^3_n$, and the same for $\bar{L}_n$, which results in a shifting of the value of the central charges c and $\widehat{c}$. The central charges now become $c_R = 3k/(k-2) + 6k$ and $c_L = 3\hat{k}/(\hat{k}-2) + 6\hat{k}$, and for large k, $\hat{k}$ one gets the standard result $c_R \simeq 6k$ and $c_L \simeq 6\hat{k}$. Then, one finds,

$$c_L = \frac{3l}{2G}\left(1 - \frac{1}{\mu l} + \frac{1}{2}\frac{m\mu + \Lambda}{m\mu - \mu^2}\right) \ , \qquad c_R = \frac{3l}{2G}\left(1 + \frac{1}{\mu l} + \frac{1}{2}\frac{m\mu + \Lambda}{m\mu - \mu^2}\right) \ , \tag{6.24}$$

together with (6.11). One rapidly verifies that this result agrees with the ones obtained in the literature [108, 114, 116, 117, 118].

It is important to point out that, even in the case $m = 0$, where the action of the theory does not contain S_4, these values for the central charges do not coincide with those of TMG. This is because, as mentioned earlier, both theories differ not only because of the inclusion of S_4 in the action, but also because of the Lagrange multiplier in TMG that assures the vanishing of the torsion. In fact, if $m = 0$, and taking (6.11) into account, one finds $c_L = (3/2G\mu)(\sqrt{1+\mu^2 l^2} - 1)$, $c_R = (3/2G\mu)(\sqrt{1+\mu^2 l^2} + 1)$, which coincides with (5.4) only at first order in $1/\mu$. On the other hand, if $m \neq 0$ and $1/\mu = 0$, the central charges above simply become $c_L = c_R = 3l/(2G)$. This does not imply that the value of m dissapears from the expressions since (6.11) depends on m and, thus, when $1/\mu = 0$, the effective cosmological constant is given by $-l^{-2} = \Lambda + m^2$. This can be simply seen by taking a glance at the equations of motion and noticing that replacing $1/\mu = 0$ in (6.4)-(6.5) makes the curvature disappear from (6.5), while inducing at the same time a redefinition of the cosmological constant in (6.4).

Quantization conditions

So, we have central charges (6.24). These are the central elements of the canonical realization of the asymptotic AdS$_3$ isometry algebra [118], and from the AdS/CFT conjecture point of view these are the charges of the dual CFT. As was commented in Section 3.1, modular invariance of such CFT demands $(c_L - c_R)/24 = (8G\mu)^{-1} \in \mathbb{Z}$, giving a quantization condition for the parameters in the action. Besides, even before resorting to the dual CFT description, one may argue that the central charges have to be quantized. Indeed, quantization of the $SL(2,\mathbb{R})$ Chern-Simons coefficient imposes conditions on $c_R = 6k$ and $c_L = 6\hat{k}$ as well (see Section 3.1). For instance, already in the case $1/\mu = 0$, one finds $(16G\sqrt{-\Lambda})^{-1} \in \mathbb{Z}$.

As Witten pointed out in [11], and was reviewed in Section 3.1, the quantization of the central charge (and not only of the difference $c_L - c_R$) is also natural from the point of view of the dual conformal field theory. This is because of the Zamolodchikov c-theorem [119], which states the impossibility of having a family of CFTs with an $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ invariant vacuum parameterized by a continuous value of the central charge. In turn, consistency of the theory, provided one assumes the AdS/CFT conjecture, demands the dimensionless ratios constructed by the different coupling constants of the theory to take special values for the bulk theory to be well defined at the quantum level.

Furthermore, one could also ask whether there is a way to understand these quantization conditions from the point of view of the microscopic theory. To analyze this, one could think of embedding the three-dimensional gravity action, including the exotic Chern-Simons term, in a bigger consistent theory, like string theory. Even though a complete description of it has not yet been accomplished (see [120] for an attempt), one can consider a toy example to see how it would work. For instance, let us play around with the $\mathcal{O}(R^4)$ M-theory terms, which are those that supplement the eleven-dimensional supergravity action. Among such higher-curvature terms one finds couplings between the 3-form $A = A_{\mu\nu\rho}\ dx^\mu \wedge dx^\nu \wedge dx^\rho$ with the curvature tensor $R^{ab}{}_{\mu\nu} = e^a{}_\alpha e^b{}_\beta R^{\alpha\beta}{}_{\mu\nu}$.

One such term is of the form $\int_{\Sigma_{11}} A\wedge\mathrm{tr}(R\wedge R)\wedge\mathrm{tr}(R\wedge R)$, together with other terms (the trace is taken over the indices in the tangent bundle $a, b, ...$). Then, one can think of a compactification of the form[3],

$$\Sigma_{11} = \Sigma_3 \times M_4 \times X_4, \tag{6.25}$$

with $F = dA$ having flux on M_4, and asking X_4 to have non-trivial signature (non-vanishing Pontryagin invariant). Integrating by parts the higher-order term written above, one finds a contribution of the form,

$$-\int_{\Sigma_3} (\omega_a \wedge d\omega^a + \frac{\epsilon_{abc}}{3}\omega^a \wedge \omega^b \wedge \omega^c) \int_{M_4} F \int_{X_4} R_{ab} \wedge R^{ab}, \tag{6.26}$$

so that the exotic gravitational term appears here, being the effective three-dimensional coupling $(8\pi G\mu)^{-1} \sim \sigma_{(X_4)} N_{(M_4)}$, where $\sigma_{(X_4)}$ is the signature of X_4 and $N_{(M_4)}$ is the flux of F. This sketches how a (yet to be found) microscopic realization could yield the quantization condition for $c_R - c_L$.

Black hole entropy

Now, before concluding the discussion on the central charges, let us consider a quick application of the result (6.24). The values of the central charges derived above provide us with a tool to compute the black hole entropy microscopically. This was discussed in [112, 113, 121] for the case of the theory with torsion, and it follows the well-known procedure originally proposed by Strominger in [122]. This amounts to considering the Cardy formula [123] of the dual CFT. In a two-dimensional CFT, Cardy's formula gives an asymptotic expression for the growing of density of states for fixed L_0 and $\bar{L}_0$. It follows from modular invariance and some general hypothesis about the spectrum of the theory. The formula for the microcanonical entropy, representing the logarithm of the number of degrees of freedom for given values of $\mathcal{M}$ and $\mathcal{J}$, reads

$$S_{\mathrm{CFT}} = 2\pi\sqrt{\frac{c_L}{12}\left(\mathcal{M}l - \mathcal{J}\right)} + 2\pi\sqrt{\frac{c_R}{12}\left(\mathcal{M}l + \mathcal{J}\right)}\ , \tag{6.27}$$

[3]This suggestion was actually given to G. Giribet by B.S. Acharya.

where the conserved charges associated to Killing vectors ∂_t and ∂_ϕ, namely the mass and the angular momentum, are identified with the Virasoro generators $L_0 + \overline{L}_0$ and $L_0 - \overline{L}_0$ respectively. Resorting to equations (6.12), (1.47) and (6.24), one actually verifies that (6.27) exactly reproduces the black hole entropy (6.13); see [112, 113, 121, 114, 115].

6.4 Chiral limit

In this section we present, based on the previous discussion of Mielke-Baekler theory, a limit taken in the space of solutions which gives a sensible gravitational dual to a holomorphic CFT. It possesses all the desired features of a chiral gravitational theory and the price to pay is the non-vanishing torsion of the solutions.

Degeneracy in Mielke-Baekler theory

Now, let us consider the special case $\mu = m = \sqrt{-\Lambda}$. As said before, in this case the equations of motion (6.4) and (6.5) coincide and the Mielke-Baekler theory develops a kind of degeneracy as the equations of motion only give,

$$R^a + \mu T^a + \frac{\mu^2}{2}\epsilon^a_{\ bc} e^b \wedge e^c = 0 \ . \tag{6.28}$$

Certainly, this equation is not sufficiently restrictive unless one specifies additional information, e.g. about the torsion. On the other hand, if $\mu = m$ equations (6.4) and (6.5) cannot be generically written in the form (6.6) and (6.7). In fact, $\mu = m = \sqrt{-\Lambda}$ is a singular point of the theory. This is why in order to analyze this point it is necessary to take the limit carefully proposing a consistent prescription. A particular consistent way this limit can be taken is to actually consider the form (6.6) and (6.7) for the equations of motion, namely,

$$R^a = \frac{\mu}{2}\frac{\Lambda + m^2}{\mu - m}\,\epsilon^a_{\ bc}\, e^b \wedge e^c \ , \qquad T^a = \frac{1}{2}\frac{m\mu + \Lambda}{m - \mu}\,\epsilon^a_{\ bc}\, e^b \wedge e^c \ , \tag{6.29}$$

define $1 - m/\mu = \varepsilon$, and then take the limit ε going to zero in such a way that the equations (6.29) remain well defined. For this to be consistent one has to consider the limit $1 - m/\mu = \varepsilon \to 0$ together with the limit $\Lambda + m^2 = \mathcal{O}(\varepsilon) \to 0$. Then, if the torsion is set to zero, (6.28) would require $-1/l^2$ to coincide with the constant Λ appearing in the Lagrangian, and so one finds that $m + \Lambda/\mu$ identically vanishes. In turn, the limit $1 - m/\mu \to 0$ is consistent with (6.29) and one eventually obtains,

$$R^a = \frac{\Lambda}{2}\,\epsilon^a_{\ bc}\, e^b \wedge e^c, \qquad T^a = 0 \ ; \tag{6.30}$$

that is, Einstein equations. In this limit one also finds that the central charges (6.24) become,

$$c_L = 0 \ , \qquad\qquad c_R = \frac{3l}{G} \ , \tag{6.31}$$

with $l^{-2} = -\Lambda$. Finally, to take the analogy with the model of [18] one step further, one may notice that, at this special point, all the black hole solutions of the theory fulfill the extremal relation,

$$l\mathcal{M} = \mathcal{J} \ , \tag{6.32}$$

which can be seen from expressions (6.12). On the other hand, it seems clear that we could have also taken the $\mu \to m$ limit in such a way that the torsion would not vanish at the critical point, obtaining, instead of (6.30), the following,

$$T^a = \frac{1}{2}\beta\,\epsilon^a{}_{bc}\,e^b \wedge e^c, \tag{6.33}$$

with arbitrary value β. In this case, the effective cosmological constant is given by,

$$-l^{-2} = \Lambda\left(1 + \frac{\beta}{2\sqrt{-\Lambda}}\right)^2, \tag{6.34}$$

and then we end up having a non-vanishing torsion at the critical point. That is, the point $\mu = m = \sqrt{-\Lambda}$ is a degenerate point of Mielke-Baekler theory and such degeneracy gets realized by the ambiguity in the choice of β, which is fixed only after a particular prescription for the limit is adopted. The most general limit which makes equations (6.29) well-defined along this procedure is,

$$1 - \frac{\mu}{m} = \varepsilon, \quad m + \frac{\Lambda}{\mu} = -\beta\varepsilon, \quad \varepsilon \to 0. \tag{6.35}$$

This implies that,

$$\Lambda + m^2 = -\varepsilon m(m + \beta), \tag{6.36}$$

and equation (6.28) is automatically satisfied for any β.

The choice $\beta = 0$ gives a theory similar to that pursued in [18]. Besides, it is clear from (6.28) that at the degenerate point the theory neither gives information about the curvature nor about the torsion, but about the combination $R^a + \mu T^a$. Then, the only equation of motion written in the Chern-Simons form turns out to be $F = 0$, which is a field equation for $A^a = \omega^a + \mu\, e^a$. Here, it is worth emphasizing that, at $m = \mu = \sqrt{-\Lambda}$, the theory defined by (6.28) and that defined by (6.29) are not equivalent. In fact, while equations (6.29) in the limit $m \to \mu \to \sqrt{-\Lambda}$ still define a theory with constant curvature and constant torsion, equation (6.28) only gives information about the quantity $R^a + \mu T^a$. It is (6.28), and not (6.29), the model that corresponds to a single Chern-Simons field theory. The theory defined by (6.29) together with (6.35) is the one that defines the chiral gravity with torsion and is a subset of the theory defined by (6.28).

The singular point, as we will shortly analyze in the next subsection within the canonical formalism, gives a particular combination of the coupling constants for which some of the would be degrees of freedom simply decouple (the situation here is a bit more cumbersome since these theories have no local degrees of freedom on their own). If the microscopic Lagrangian of the theory is fine tuned to those values that would lead to the critical point, one should simply make a field redefinition from scratch and the theory becomes a Chern-Simons theory for a single $SL(2,\mathbb{R})$, whose geometrical meaning is unclear. However, whatever approach to this problem is chosen, it seems more natural to embed the Mielke-Baekler Lagrangian into a bigger picture, so that the singular point is eventually approached to from the generic non-degenerate situation. As such, it is natural to give a prescription to reach the singular point that smoothly interpolates with the generic case, where both the curvature and the torsion are constant. Still, there is some freedom within this prescription, which is reflected in the parameter β in (6.33). The choice $\beta = 0$ is special in that it makes the theory closely reminiscent to chiral gravity [18].

To understand the indefinition in the parameter β, it is worth studying the map between different geometries and how it behaves at the degenerate point: In Mielke-Baekler theory there is a natural way to establish a map between geometries which are solutions of the theory (6.1) for different values of the coupling constants [124]. That is, one can perform a linear transformation of the fields like,

$$\omega^a \to \omega^a + \beta\ e^a\ , \qquad e^a \to e^a\ . \tag{6.37}$$

and find that this transformation induces a transformation of the four coupling constants that appear in the action. To give an example of how it works, it is sufficient to consider the Lagrangian of the theory in the particular case in which its coupling constants satisfy the relation $\mu m = -\Lambda$. In this case, a transformation like (6.37) generates the following transformation of the coupling constants,

$$G \quad \to \quad \widetilde{G} = \frac{G\mu}{\mu+\beta}\ , \tag{6.38}$$

$$\mu \quad \to \quad \widetilde{\mu} = \mu + \beta\ , \tag{6.39}$$

$$m \quad \to \quad \widetilde{m} = \frac{m\mu + 2\mu\beta + \beta^2}{\mu+\beta}\ , \tag{6.40}$$

$$\Lambda \quad \to \quad \widetilde{\Lambda} = \frac{\Lambda\mu - 3m\mu\beta - 3\mu\beta^2 - \beta^3}{\mu+\beta}\ . \tag{6.41}$$

The case we started with already satisfied the special condition $\mu m = -\Lambda$, and provided it also satisfied $\mu = m$ one finds that the transformed coupling constants obey $\widetilde{\mu}\widetilde{m} = -\widetilde{\Lambda}$ and $\widetilde{\mu} = \widetilde{m}$ as well. That is, the special condition $m^2 = \mu^2 = -\Lambda$ appears to be a fixed point of the β-transformation (6.37); in fact, after the transformation one finds $\widetilde{\mu}^2 = \widetilde{m}^2 = -\widetilde{\Lambda} = (\beta+\mu)^2$. And we see that this transformation generates (constant) torsion $T^a \sim \beta\ \epsilon^a{}_{bc} e^b \wedge e^c$ from a configuration with vanishing torsion. The combination that remains invariant is, precisely, $R^a + \mu T^a + \frac{\mu^2}{2}\epsilon^a{}_{bc} e^b \wedge e^c \to R^a + \tilde{\mu} T^a + \frac{\tilde{\mu}^2}{2}\epsilon^a{}_{bc} e^b \wedge e^c$, as in (6.28). This explains the degenerate point appearing as a fixed point of (6.37).

Analogy with chiral gravity

Equations (6.30), (6.31) and (6.32) are actually evocative of what happens in chiral gravity. The point $\mu = m = \sqrt{-\Lambda}$ corresponds to the point of the space of parameters where the Chern-Simons coupling $\hat{q}$ vanishes. In turn, the theory consists of a single Chern-Simons action (see [125] for a brief comment about the relation between the singular point $\hat{q} = 0$ and the chiral point of [18]; cf. [115]). When $\hat{q} = 0$ the left-handed degrees of freedom are left unspecified; however, we have just argued that one could consistently demand the torsionless condition $T^a = 0$ when approaching the singularity, what would add an additional equation where the left-handed field is involved.

We have just identified a special (singular) point of Mielke-Baekler theory at which the theory behaves pretty much like chiral gravity of [18]. That is, it gives a model of three-dimensional gravity that fulfills the following properties:

a) Once suitable asymptotically AdS_3 boundary conditions are imposed, the asymptotic symmetry group turns out to be generated by one (right-handed) Virasoro algebra with central charge $c_R = 3l/G$, while the central charge of the left-handed part c_L vanishes.

b) The theory can be written as a single $SL(2,\mathbb{R})$ Chern-Simons term, as the other copy of the bulk action decouples in the limit, being proportional to c_L.

c) The BTZ black holes have mass $\mathcal{M}$ and angular momentum $\mathcal{J}$ that obey the relation $l\mathcal{M} = \mathcal{J}$, no matter the values that the parameters M and J of the solution take.

d) The theory has no local degrees of freedom, as it corresponds to a special case of the Mielke-Baekler theory.

e) If the limit $\mu \to m \to \sqrt{-\Lambda}$ is taken as in (6.35), where equations (6.29) are well defined, all the solutions of the theory at the special point have constant curvature and torsion. If the limit is defined with $\beta = 0$, we always have vanishing torsion and the space of solutions is constituted by Einstein manifolds, *i.e.*, locally AdS_3 spaces.

Nevertheless, besides the resemblance between the chiral model obtained from the degenerate case of Mielke-Baekler theory and the chiral gravity of [18], it is worth emphasizing that both constructions are radically different. For instance, in what regards to the property *e)* listed above: in the proposal of [18], there are non-Einstein solutions [27].

In the next subsection, we will analyze the canonical structure of the theory and how it changes at the degenerate point.

Canonical analysis

In the previous section we discussed a degenerate point of the Mielke-Baekler theory of gravity in AdS_3 space and we proposed a prescription to approach this point in the space of parameters. Now, let us briefly discuss the canonical structure of the theory. Our discussion will follow the approach and notation of references [118, 126], but paying special attention to the analysis of the constrained system in order not to miss the difference between the critical and the non-critical cases.

The Hamiltonian analysis of the theory starts by slicing the three-dimensional spacetime manifold, separating the temporal components from the spatial ones, and defining a configuration space. The coordinates of this configuration space (henceforth denoted by q) are the components e_a^μ and ω_a^μ. Explicitly we can write the canonical momenta associated to these as follows,

$$\pi_a^0 = 0 \ , \quad \Pi_a^0 = 0 \ , \quad \pi_a^i = \mu\varepsilon^{ij}\left(\omega_{aj} + m\ e_{aj}\right) \ , \quad \Pi_a^i = \varepsilon^{ij}\left(\omega_{aj} + \mu\ e_{aj}\right) \ , \tag{6.42}$$

which correspond to e_a^0, ω_a^0, e_a^i and ω_a^i, respectively, where the notation is such that $i,j = 1,2$ refer to the spatial part of the spacetime indices. The canonical momenta are indeed defined with respect to the action (6.1) times $16\pi G\mu$. These relations define the primary constraints of the theory; namely,

$$\phi_a^0 \equiv \pi_a^0 \ , \quad \Phi_a^0 \equiv \Pi_a^0 \ , \quad \phi_a^i \equiv \pi_a^i - \mu\varepsilon^{ij}\left(\omega_{aj} + m e_{aj}\right) \ , \quad \Phi_a^i \equiv \Pi_a^i - \varepsilon^{ij}\left(\omega_{aj} + \mu e_{aj}\right) \ . \tag{6.43}$$

Then, the primary Hamiltonian density is,

$$\mathcal{H}_T = e_0^a\,\mathcal{H}_a + \omega_0^a\,\mathcal{K}_a + \dot{e}_0^a\,\phi_a^0 + \dot{\omega}_0^a\,\Phi_a^0 + \dot{e}_i^a\,\phi_a^i + \dot{\omega}_i^a\,\Phi_a^i \ , \tag{6.44}$$

where the dot stands for time derivatives and, following the notation used in [118],

$$\mathcal{H}^a \;\; = -\mu \left(m\, T^a_{ij} + R^a_{ij} - \Lambda\, \varepsilon^a{}_{bc} e^b_i e^c_j \right) \varepsilon^{ij} \ , \tag{6.45}$$

$$\mathcal{K}^a \;\; = - \left(\mu T^a_{ij} + R^a_{ij} + m\, \mu\, \varepsilon^a{}_{bc} e^b_i e^c_j \right) \varepsilon^{ij} \ . \tag{6.46}$$

The dynamics of the theory is generated by $\mathcal{H}_T$, while the time derivatives of the coordinates that accompany the constraints play the rôle of Lagrange multipliers that fix them to zero. The structure of the Hamiltonian is, in general, given by $\mathcal{H}_T = \tilde{\mathcal{H}} + \dot{q}^I\, \phi_I$. That is, the actual Hamiltonian is given by the sum of the canonical Hamiltonian and the contributions coming from the constraints. The Poisson structure arises from imposing canonical constraints on coordinates and momenta through the Lie bracket $\{,\}$. The constraints $\phi_I = 0$ reduce the original phase space to the physical one, and consistency of the theory demands the constraints to be preserved through the dynamical evolution of the system in the reduced phase space. This requires $\dot{\phi}_J$ to weakly vanish,

$$\dot{\phi}_J = \{\mathcal{H}_T, \phi_J\} = \{\tilde{\mathcal{H}}, \phi_J\} + \dot{q}^I \{\phi_I, \phi_J\} \approx 0 \ . \tag{6.47}$$

In our case, we have,

$$\dot{\phi}^0_a = -\mathcal{H}_a \ , \qquad \dot{\Phi}^0_a = -\mathcal{K}_a \ ,$$

$$\dot{\phi}^i_a = 2\mu m \epsilon^{ji} \left(\partial_j e_{a0} - \epsilon_{ab}{}^c \left(\omega^b_j - \frac{\Lambda}{m} e^b_j \right) e_{c0} \right) + 2\mu \epsilon^{ji} \left(\partial_j \omega_{a0} - \epsilon_{ab}{}^c \left(\omega^b_j + m e^b_j \right) \omega_{c0} \right)$$

$$+ 2\mu \epsilon^{ji} \left(m \dot{e}_{aj} + \dot{\omega}_{aj} \right) ,$$

$$\dot{\Phi}^i_a = 2\mu \epsilon^{ji} \left(\partial_j e_{a0} - \epsilon_{ab}{}^c \left(\omega^b_j + m e^b_j \right) e_{c0} \right) + 2 \epsilon^{ji} \left(\partial_j \omega_{a0} - \epsilon_{ab}{}^c \left(\omega^b_j + \mu e^b_j \right) \omega_{c0} \right)$$

$$+ 2 \epsilon^{ji} \left(\mu \dot{e}_{aj} + \dot{\omega}_{aj} \right) \ ,$$

The first line above expresses the fact that that $\mathcal{H}_a$ and $\mathcal{K}_a$ are secondary constraints, while the second and third lines give equations that allows to find the values of $\dot{e}_{aj}$ and $\dot{\omega}_{aj}$ that set these expressions to zero. Solving these equations is always possible except when the determinant of the system is zero, what precisely occurs when $m = \mu$. Leaving the critical case aside for a moment, one can continue the analysis and verify that the secondary constraints are actually consistent: the non-trivial Poisson brackets for $m \neq \mu$ are,

$$\{\phi^i_a, \phi^j_b\} = -2m\mu\, \varepsilon^{ij}\, \delta_{ab} \ , \quad \{\phi^i_a, \Phi^j_b\} = -2\mu\, \varepsilon^{ij}\, \delta_{ab} \ , \quad \{\Phi^i_a, \Phi^j_b\} = -2\, \varepsilon^{ij}\, \delta_{ab} \ , \tag{6.48}$$

with,

$$\{\phi^i_a, \bar{\mathcal{H}}_b\} = \varepsilon_{ab}{}^c \left(\frac{\Lambda + m\mu}{m - \mu}\, \phi^i_c + \mu \frac{\Lambda + m^2}{\mu - m}\, \Phi^i_c \right) \ , \tag{6.49}$$

$$\{\phi^i_a, \bar{\mathcal{K}}_b\} = \{\Phi^i_a, \bar{\mathcal{H}}_b\} = -\varepsilon_{ab}{}^c \phi^i_c \ , \quad \{\Phi^i_a, \bar{\mathcal{K}}_b\} = -\varepsilon_{ab}{}^c \Phi^i_c \ , \tag{6.50}$$

and,

$$\{\bar{\mathcal{H}}_a, \bar{\mathcal{H}}_b\} = \varepsilon_{ab}{}^c \left(\frac{\Lambda + m\mu}{m-\mu}\, \bar{\mathcal{H}}_c + \mu \frac{\Lambda + m^2}{\mu - m}\, \bar{\mathcal{K}}_c \right) , \tag{6.51}$$

$$\{\bar{\mathcal{H}}_a, \bar{\mathcal{K}}_b\} = -\varepsilon_{ab}{}^c \bar{\mathcal{H}}_c \ , \quad \{\bar{\mathcal{K}}_a, \bar{\mathcal{K}}_b\} = -\varepsilon_{ab}{}^c \bar{\mathcal{K}}_c \ . \tag{6.52}$$

There is of course a $\delta^2(\vec{x} - \vec{y})$ implicit in all these formulas. After substituting the expression for the multipliers back into the total Hamiltonian, one can integrate by parts to rearrange the factors that accompany the canonical variables, that instead of $\mathcal{H}$ and $\mathcal{K}$ are now,

$$\bar{\mathcal{H}}_a = \mathcal{H}_a - \left(\partial_i \phi_a^i - \epsilon_{ab}{}^c\, \omega_i^b\, \phi_c^i\right) - \epsilon_{ab}{}^c\, e_i^b \left(\frac{\Lambda + m\mu}{m - \mu}\, \phi_c^i + \mu \frac{\Lambda + m^2}{\mu - m} \Phi_c^i \right) ,$$

$$\bar{\mathcal{K}}_a = \mathcal{K}_a - \left(\partial_i \Phi_a^i - \epsilon_{ab}{}^c\, \omega_i^b\, \Phi_c^i\right) + \epsilon_{ab}{}^c\, e_i^b\, \Phi_c^i \ .$$

In contrast, at the critical point the theory exhibits a dynamical pathology. The reason is that, when $m = \mu$, a new symmetry appears, and this must be properly taken into account when analyzing the constraints. What happens when going from the generic case to the critical case $m = \mu$ is that two of the momenta become proportional to each other, namely $\pi_a^i = \mu\, \Pi_a^i$, and consequently the respective constraints happen to carry the same information. This is basically because at such point of the space of parameters the coordinates e^a and ω^a play symmetric rôles in the action. As mentioned before, at the singular point one of the Chern-Simons actions drops out and one is left with a single action describing the dynamics of the field $A^a = \omega^a + \mu e^a$. In order to take this symmetry (between the rôle played by e^a and ω^a) into account, one can replace the constraint ϕ_a^i by the new one $\psi_a^i \equiv \phi_a^i - \mu \Phi_a^i$, in such a way that the constraints turn out to be given by,

$$\dot{\phi}_a^0 = -\mathcal{J}_a \ , \quad \dot{\Phi}_a^0 = -\mathcal{J}_a \ , \quad \dot{\psi}_a^i = 0 \ , \tag{6.53}$$

$$\dot{\Phi}_a^i = 2\epsilon^{ij} \left(\partial_j A_{a0} - \epsilon_{ab}{}^c A_j^b A_{c0} \right) + 2\epsilon^{ji} \left(\mu \dot{e}_{aj} + \dot{\omega}_{aj} \right) \ , \tag{6.54}$$

where the last equation can always be solved. At the critical point, $\mathcal{J}_a = \mathcal{H}_a/\mu = \mathcal{K}_a = -\epsilon^{ij} F_{ij}^a$, where $\mu = m = \sqrt{-\Lambda}$. The non-zero Poisson brackets in the critical case are,

$$\{\Phi_a^i, \Phi_b^j\} = -2\varepsilon^{ij} \delta_{ab} \ , \quad \{\Phi_a^i, \bar{\mathcal{J}}_b\} = -\varepsilon_{ab}{}^c \Phi_c^i \ , \quad \{\bar{\mathcal{J}}_a, \bar{\mathcal{J}}_b\} = -\varepsilon_{ab}{}^c \bar{\mathcal{J}}_c \ , \tag{6.55}$$

where,

$$\bar{\mathcal{J}}_a = \mathcal{J}_a - \left(\partial_i \Phi_a^i - \epsilon_{ab}{}^c A_i^b \Phi_c^i \right) . \tag{6.56}$$

The difference between the critical point $m = \mu$ and the generic case can be summarized easily by counting the amount of constraints of first class (FC) and of second class (SC) that appear in each case. Namely,

	Primary	Secondary
FC	ϕ_a^0, Φ_a^0	$\bar{\mathcal{H}}_a$, $\bar{\mathcal{K}}_a$
SC	ϕ_a^i, Φ_a^i	—

	Primary	Secondary
FC	ϕ_a^0, Φ_a^0, ψ_a^i	$\bar{\mathcal{J}}_a$
SC	Φ_a^i	—

We see that at the critical point one primary constraint of second class is promoted to the first class[4]. This indicates that a new symmetry appears at the critical point; namely,

$$\delta_\xi e^a_i = \xi^a_i \ , \qquad \delta_\xi \omega^a_i = -\mu\, \xi^a_i \ , \tag{6.57}$$

which certainly leaves A^a_i invariant. Also, two secondary constraints of the first class, $\mathcal{H}$ and $\mathcal{K}$, collapse to one, denoted by $\mathcal{J}$. Note that $\mathcal{H}$ and $\mathcal{K}$ are the only ones with spatial derivatives and so need boundary terms to make them differentiable, giving rise to central extensions following the analysis of Brown and Henneaux of [3] explained in Section 1.3. Thus, once they collapse to $\mathcal{J}$ at the critical point, half of the generators of asymptotic symmetries vanish, as needs to be the case to have a manifest chirality on the boundary.

It is worth noticing that the prescription for going from the general case to the critical point defined through the limiting procedure (6.35) can be applied to the set of commutators (6.48)-(6.51), along with the change $\phi^i_a \to \psi^i_a = \phi^i_a - \mu\Phi^i_a$ and $\mathcal{J}_a = \mathcal{H}_a/\mu = \mathcal{K}_a$, to obtain the set of commutation relations of the critical case.

[4]Notice that the number of degrees of freedom in each case remains to be zero: #coordinates +#momenta $-2 \times$ #FC$-$#SC.

Conclusions

In this thesis we have reviewed the main results up to now in the context of quantization of general relativity in three dimensions with a negative cosmological constant via the AdS/CFT correspondence. Besides, in Chapters 3, 4, 5 and 6, some of the original contributions by the author and collaborators in these topics have been explained in detail.

In the first two chapters we have presented the necessary facts about three-dimensional gravity and AdS/CFT. In particular, we noted the influence of Brown-Henneaux analysis to immediately see the rôle of the asymptotic symmetries in the path to quantization, giving an explicit way to reach Liouville theory as an effective classical dual CFT. With the knowledge of the existence of "Virasoro gravitons" living on the boundary of AdS_3, we have explained the proposal of Witten [11] for a family of holomorphic extremal CFTs dual to gravity, and also the computation of the partition function of AdS_3 gravity by Maloney and Witten [12]. As we mentioned at that point, this partition function fails to be identified with the trace over a Hilbert space of the exponential of a Hamiltonian operator. One of the most probable reasons for this is that there are more geometries to consider in the partition function, such as point-particle geometries, i.e. conical singularities. Motivated by this and by the need of a complete understanding of the rôle of such singularities in the quantum theory, we investigated BPS solutions to Chern-Simons AdS supergravities in three dimensions [13, 14, 15]. What we described in Chapter 4 is that from a similar construction as the one for the BTZ black holes, it is possible to obtain singular geometries that describe spinning massive particles sailing in AdS_3, where the singularity in the curvature is a Dirac delta distribution. It is worth mentioning that these singularities are not calling for a cosmic censor: They correspond to point-like objects, with the geometry around them being smooth. Then, by a coupling in the action of the matter source to the gauge field of the form $\int A \wedge j$, the equation of motion is $F = j$ and the fact of the curvature being a Dirac delta distribution means that the source that is coupled is indeed a massive particle. These geometries, although they describe particles, seem like sensible contributions to the partition function of pure gravity, since they can be thought of with the singularity curve removed, and, even more, they fill in the energy gap between global AdS_3 and the BTZ black holes, with masses between $-1/8G$ and 0.

The stability of such geometries was established in the context of the supersymmetric extension of three-dimensional gravity: super AdS_3 gravity. It was shown that for any number of supersymmetries, there are BPS spinning and/or charged 0-branes that are supersymmetric when their parameters satisfy certain quantization conditions. This reinforces the idea of including such stable geometries in the computation of the partition function, at least of the supersymmetric theory. The actual inclusion is an interesting open problem.

In any case, when we reviewed [11] and [12], the holomorphic factorization hypothesis was emphasized as a feature that would help in the understanding of dualities in three dimensions, even if it is not *a priori* present in pure AdS_3 gravity. Motivated by this, the conjecture by Li, Maloney, Song and Strominger [18, 23], called chiral gravity conjecture, was explained in detail. It states that TMG at the critical point $\mu\ell = 1$ with Brown-Henneaux boundary conditions is dual to a holomorphic CFT, with vanishing left central charge, and whose partition function has a sensible quantum interpretation. Nevertheless, we saw that at least two assumptions on the spectrum of chiral gravity are still not clear to be valid: 1) the absence of right-moving ghosts is not demonstrated yet, and 2) the absence of non-Einstein solutions to chiral gravity with Brown-Henneaux asymptotics is now known to be false [27]. But to be precise, these non-Einstein solutions suffer from pathologies such as closed-timelike curves and also they have zero conserved charges, contributing minimally to the partition function. It is not still clear wether they should be included in the physical spectrum. In addition, we commented on the possibility of relaxing the asymptotic conditions of chiral gravity, giving rise to what is now known as log gravity. This is a much more complicated theory, since a zoo of exact solutions with boundary conditions with slower fall-off behavior as that of Brown-Henneaux is known to exist. We explained in detail one of such *log solutions*, which was first presented by the author and collaborators in [24]. It is an extremal geometry, in the sense that it possesses non-vanishing mass and angular momentum which satisfy the relation $\ell M = J$, and its vacuum is the extremal BTZ black hole where both charges vanish.

In the last chapter, we presented an alternative for a chiral gravity, in the context of Mielke-Baekler theory of gravity in asymptotically AdS_3 spacetime with torsion. We have reviewed the computation of central charges of the asymptotic algebra, which turn out to be the central charges of the dual CFT_2. The result we obtained agrees with the central charges obtained in the literature by employing different methods [108, 114, 115, 116, 117]. It was observed that a special point of the space of parameters exists, at which one of the central charges vanishes. This point was compared with the chiral point of topologically massive gravity, and the analogies between both models were pointed out. This is a singular point for the Mielke-Baekler theory, where the theory exhibits degeneracy. We analyzed this at the level of the space of solutions, in the Chern-Simons formulation, and in the canonical approach. In the Chern-Simons formulation this critical point appears as the point of the space of parameters at which one of the two $SL(2,\mathbb{R})$ actions drops out. This point was recently mentioned by Witten [125] within the context of the analytically extended theory, where the connection with the chiral gravity of [18] was already mentioned. It was one of our motivations to make this connection with chiral gravity more explicit.

One of the aspects one observes here is that several features of the dual conformal field theory do not seem to depend on the precise prescription adopted to reach the singular point of the Mielke-Baekler theory. This raises the question as to whether the relevant physical information is independent of the way one approaches $m = \mu = \sqrt{-\Lambda}$. Despite the fact that quantities in the geometric realization do actually depend on how the limit is taken, this possibly reflecting that the theory becomes in essence non-geometrical, it seems plausible that all these geometries are different realizations of the same theory, and, likely, the way of making sense out of Mielke-Baekler theory at the point it exhibits degeneracy is, in fact, resorting to the dual description in terms of a chiral CFT.

Appendix A

$sl(2,\mathbb{R})$ algebra and close relatives

The $SO(2,2)$ Lie group is the set of four-by-four matrices of determinant one that leave invariant the quadratic form (1.5), together with the product of matrices. It is a double cover of $SO(2,1)\times SO(2,1)$ [11], with each piece being double-covered by the group $SL(2,\mathbb{R})$. Thus, $SO(2,2)$ is locally isomorphic to $SO(2,1)\times SO(2,1)$ which is locally isomorphic to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. This implies that their corresponding algebras are the same. The $sl(2,\mathbb{R})$ algebra is generated by,

$$t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{A.1}$$

which satisfy the Lie algebra product,

$$[t_+, t_-] = 2t_3, \quad [t_3, t_\pm] = \pm t_\pm. \tag{A.2}$$

Note that by performing the isomorphism $t_\pm = t_2 \pm t_1$, the generators $\{t_1, t_2, t_3\}$ satisfy $[t_i, t_j] = \epsilon_{ij}{}^k t_k$, where $\epsilon_{123} = -1$ and there is a Minkowski metric $\eta = \mathrm{diag}(-1,1,1)$, so with this unfortunate notation $\eta_{11} = -1$. These generators give,

$$\mathrm{tr}(t_i t_j) = \frac{1}{2}\eta_{ij}. \tag{A.3}$$

In order to make contact with the indices $a = 0,1,2$ used for example in Section 1.2, it is enough to make the redefinition $123 \to 012$ of the notation in this appendix.

Appendix B

Notions of differential geometry

This part is entirely based on [127] and is intended to give an account of some mathematical objects of differential geometry as well as building foundations to construct gauge theories, such as Chern-Simons theory, which is present all along the thesis.

Vectors

Given a manifold M, a vector field v over M is defined as a function from $C^\infty(M)$ to itself, and must satisfy the following properties which define a Derivation,

$$\begin{aligned} v(f+g) &= v(f)+v(g) \\ v(\alpha f) &= \alpha v(f) \\ v(fg) &= v(f)g+fv(g), \end{aligned}$$

where $\alpha \in \mathbb{R}$ and $f, g \in C^\infty(M)$. Let's call $\mathrm{Vect}(M)$ to the set of all vector fields in M.

Now, a tangent vector of M at p, is a function $v_p : C^\infty(M) \to \mathbb{R}$ that satisfies,

$$\begin{aligned} v_p(f) &= v(f)(p) \\ v_p(f+g) &= v_p(f)+v_p(g) \\ v_p(\alpha f) &= \alpha v_p(f) \\ v_p(fg) &= v_p(f)g(p)+f(p)v_p(g). \end{aligned}$$

Let us denote T_pM the set of all tangent vector of M at p. We call the Tangent Bundle TM the union of all $T_p(M)$.

A curve $\gamma(t)$ in M is a function $\gamma : \mathbb{R} \to M$. One would like to have a vector in $T_{\gamma(t)}M$ that is tangent to the curve, so we call it $\gamma'(t) : C^\infty(M) \to \mathbb{R}$, and define it as a

$$\gamma'(t)(f) := \frac{d}{dt} f(\gamma(t)) \tag{B.1}$$

Pullback and pushforward

If M and N are manifolds, let $\phi : M \to N$ and $f : N \to \mathbb{R}$. We call $\phi * f : M \to \mathbb{R}$ the pullback of f by ϕ and define it by,

$$\phi^* f := f \circ \phi. \tag{B.2}$$

So a function $f : N \to \mathbb{R}$, by means of $\phi : M \to N$, can be used to give another function from M to $\mathbb{R}$. Pulling back is an operation,

$$\phi^* : C^\infty(N) \to C^\infty(M). \tag{B.3}$$

Now let us define the pushforward operation. It will take a tangent vector in p, $v_p \in T_pM$, and give another tangent vector but in $q = \phi(p)$, $v_q \in T_qN$. This is defined as,

$$(\phi_* v_p)(f) := v_p(\phi^* f). \tag{B.4}$$

The pushforward sends a vector in M to a vector in N. If γ is a curve in M, then the curve $\phi \circ \gamma(t)$ has tangent vector $\phi_*\gamma'(t)$. Let's prove it,

$$\begin{aligned} (\phi \circ \gamma)'(t)(f) &= \frac{d}{dt}\big(f[\phi \circ \gamma(t)]\big) = \frac{d}{dt}\big(f \circ \phi[\gamma(t)]\big) && \text{(B.5)} \\ &= \gamma'(f \circ \phi) = \gamma'(\phi^* f) = (\phi_*\gamma'(t))(f). && \text{(B.6)} \end{aligned}$$

Vector bundles

We call $\mathrm{End}(V)$ to the set of all linear maps from V to itself. It can be shown that it is isomorphic to $V \otimes V^*$,

$$\mathrm{End}(V) \cong V \otimes V^*. \tag{B.7}$$

If $v \otimes z \in V \otimes V^*$, then the endomorphism acts on $w \in V$ as,

$$w \mapsto z(w)v. \tag{B.8}$$

The space $\mathrm{End}(V)$ is in fact a vector space, where we define,

$$\begin{aligned} (\alpha T)(v) &= \alpha T(v) \\ (S + T)(v) &= S(v) + T(v), \end{aligned}$$

where $\alpha \in \mathbb{R}(\mathbb{C})$ and $T, S \in \mathrm{End}(V)$. It is convenient to introduce a basis for $\mathrm{End}(V)$. Let e_i be a basis for V and e^j for its dual V^*. Then a good basis for $\mathrm{End}(V)$ is e_i^j with the property

$$e_i^j e_k = \delta_k^j e_i, \tag{B.9}$$

since now, any vector $v = v^i e_i$ gets mapped as,

$$T(v) = T_j^i\, e_i^j\, v^k e_k = T_j^i\, v^j\, e_i. \tag{B.10}$$

A vector bundle is just a bundle whose fiber is a vector space. Then, for each point p in the base manifold M, one has a fiber E_p that is in fact a vector space. We also have a dual vector bundle E^*, which can be constructed in the following way: for each p, take the dual of the vector sapce E_p, E_p^*. Now, take the union of all E_p^* for $p \in M$. The projection $\pi : E^* \to M$ maps each E_p^* to the corresponding p. It can be show that E^* is in fact a vector bundle.

One can define the endomorphism bundle of a vector bundle E as $\mathrm{End}(E) = E \otimes E^*$. This is a good definition since sections of $\mathrm{End}(E)$ do determine vector bundles morphisms form E to itself: note that the fibre of $\mathrm{End}(E)$ over any $p \in M$ is just the same as $\mathrm{End}(E_p)$, therefore a section T of $\mathrm{End}(E)$ defines a map from E to itself, sending a vector $v \in E_p$ to $T(p)v \in E_p$. This is a vector bundle morphism. As a result, any section T of $\mathrm{End}(E)$ acts on any section of E pointwise, giving a new section Ts of E, as

$$(Ts)(p) = T(p)s(p). \tag{B.11}$$

Thus, T is a function $T : \Gamma(E) \to \Gamma(E)$. This function is $C^\infty(M)$-linear, meaning,

$$T(fs) = fT(s), \tag{B.12}$$

with $f \in C^\infty(M)$.

G-bundles and stuff

Let us concentrate now in the specific case when E is a G-bundle, with G some Lie group. This means that there is an open cover $\{U_\alpha\}$ of M such that E is built by gluing together trivial bundles $U_\alpha \times V$, where V is a vector space on which G has a representation ρ.

Given a section T of $\mathrm{End}(E)$, we will say that $T(p) \in \mathrm{End}(E_p)$ lives in G if it is of the form $\rho(g)$, for some $g \in G$. We say it leaves in $\mathfrak{g}$ if it is of the form $d\rho(x)$, for some $x \in \mathfrak{g}$. Going one step forward, we say that T lives in $\mathfrak{g}$ if $T(p)$ leaves in it for all $p \in M$. And if $T(p)$ lives in G for all p, we say that T is a *gauge transformation*. Let us call $\mathcal{G}$ the set of all gauge transformations which is in fact a group, with products and invserses given by,

$$\begin{aligned} (TS)(p) &= T(p)S(p) \\ T^{-1}(p) &= T(p)^{-1}. \end{aligned}$$

Now we will define the concept of "connection", which allows to differentiate sections along some direction in the base space. A connection D on M assigns to each vector field v on M a funcion D_v form $\Gamma(E)$ to $\Gamma(E)$, satisfying,

$$\begin{aligned} D_v(\alpha s) &= \alpha D_v s \\ D_v(t+s) &= D_v t + D_v s \\ D_v(fs) &= v(f)s + fD_v s \\ D_{v+w} s &= D_v s + D_w s \\ D_{fv} s &= fD_v s, \end{aligned}$$

for $\alpha \in \mathbb{R}(\mathbb{C})$, $f \in C^\infty(M)$, $t, s \in \Gamma(E)$, $v, w \in \mathrm{Vect}(M)$. We call $D_v s$ the covariant derivative of s in the direction v.

Let us see what a connection does to a section (given some vector field) in local coordinates. Let us call $\{\partial_\mu\}$ the basis of vector fields over some open set $U \subset M$ and let e_i be a basis for section of $E|_U$. We will use the abbreviation $D_{\partial_\mu} = D_\mu$. Note that $D_\mu e_j \in \Gamma(E|_U)$, so it must be equal to some linear combination of the basis elements e_i, with coefficients being functions over U. We will call this coefficients $A^i_{\mu j}$,

$$D_\mu e_j = A^i_{\mu j} e_i. \tag{B.13}$$

Now, the covariant derivative of s along v is,

$$\begin{aligned} D_v s &= D_{v^\mu \partial_\mu} s = v^\mu D_\mu (s^i e_i) && \text{(B.14)} \\ &= v^\mu (\partial_\mu s^i e_i + s^i D_\mu e_i) && \text{(B.15)} \\ &= v^\mu (\partial_\mu s^i e_i + s^i A^j_{\mu i} e_j) && \text{(B.16)} \\ &= v^\mu (\partial_\mu s^i + s^j A^i_{\mu j}) e_i && \text{(B.17)} \\ &= v^\mu (D_\mu s)^i e_i. && \text{(B.18)} \end{aligned}$$

The functions $A^i_{\mu j}$ are called the components of the vector potential. The last term we got when computing the covariant derivative,

$$A^j_{\mu i} v^\mu s^i e_j, \tag{B.19}$$

is a section of $E|_U$. Note that is $C^\infty(U)$-linear in v and s, so one would (correctly) think that the vector potential is an *End*(E)*-valued 1-form* on U. In other words, it is a section of the bundle $\mathrm{End}(E|_U) \otimes T^*U$. Let us see that if we define the vector potential this way, we get (B.19). So we define the vector potential (on U),

$$A := A^j_{\mu i} e_j \otimes e^i \otimes dx^\mu. \tag{B.20}$$

Now for some vector field $v \in \mathrm{Vect}(U)$,

$$A(v) = A^j_{\mu i} v^\mu e_j \otimes e^i,$$

and for some section $s \in \Gamma(E|_U)$,

$$\begin{aligned} A(v)s &= A^j_{\mu i} v^\mu e_j \otimes e^i s \\ A(v)s &= A^j_{\mu i} v^\mu s^i e_j. \end{aligned}$$

Here is an important property: given some connection D^0 on E, for each connection D on E there exists a vector potential A such that,

$$D_v s = D^0_v s + A(v)s, \tag{B.21}$$

for every $v \in \mathrm{Vect}(M)$ and every $s \in \Gamma(E)$. The demonstration is by construction. If D and D^0 are connections on E, then it is easy to show that the difference between them is $C^\infty(M)$-linear in any vector v and any section s, so it is in fact an $\mathrm{End}(E)$-valued 1-form. The vector potential that suits each connection D is just the one we get when taking the difference $D - D^0$. The connection D^0 is usually chosen to be the one with cero vector potential in some local trivialization:

$$D^0_v s = v(s^j) e_j. \tag{B.22}$$

This connection D^0 is called the “standard flat connection”.

We say that D is a G-connection if in local coordinates the components $A_\mu \in \mathrm{End}(E)$ live in $\mathfrak{g}$. This definition is coordinate-independent since if A_μ lives in $\mathfrak{g}$, then $A'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} A_\mu$ will do too.

We now define a gauge transformation over a G-connection. Let D be a G -connection on E and g a gauge transformation, then, the claim is that there exists a new G-connection D' that we will call the gauge-transformed from D by g, defined by,

$$D'_v(s) = g D_v (g^{-1} s). \tag{B.23}$$

It is not difficult to prove that D' is indeed a G-connection.

Let us see how the vector potential is transformed. Recall that $D' = D^0 + A'$, so we compare this to

$$
\begin{aligned}
D'_v(s) &= gD_v(g^{-1}s) \\
&= gv((g^{-1}s)^j)e_j + gA(v)g^{-1}s \\
&= gv((g^{-1})^j_i s^i)e_j + gA(v)g^{-1}s \\
&= gv((g^{-1})^j_i)s^i e_j + g(g^{-1})^j_i v(s^i)e_j + gA(v)g^{-1}s \\
&= gv((g^{-1})^i_j)s^j e_i + gg^{-1}v(s^i)e_i + gA(v)g^{-1}s \\
&= gv((g^{-1})^i_j)s^j e_i + D^0_v(s) + gA(v)g^{-1}s \\
&= D^0_v(s) + gv(g^{-1})s + gA(v)g^{-1}s,
\end{aligned}
$$

where we have used how the basis elements of $\mathrm{End}(E)$ act on elements of $\Gamma(E)$ and in the last step we defined

$$v(g^{-1}) = v((g^{-1})^j_i)e^i_j. \tag{B.24}$$

Thus, we can write,

$$A'_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1}. \tag{B.25}$$

If the G-connection D' is obtained from the G-connection D, then it is said that they are gauge equivalent.

As an example, let us consider the case of a $U(1)$-bundle, that is, electromagnetism. We will take the bundle to be trivial, so we assume $E = M \times \mathbb{C}$. Since $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$ (canonically isomorphic), then the vector potential is a complex-valued 1-form. We think of the fiber $\mathbb{C}$ as the fundamental representation ($\rho : U(1) \to GL(\mathbb{C})$) of $U(1)$, so E is a $U(1)$-bundle. If the connection D is a $U(1)$-connection, then the components A_μ of the vector potential must live in $\mathfrak{u}(1)$. Since this Lie algebra is the set of purely imaginary numbers, A_μ is a purely imaginary function. In other words, $A = i\mathcal{A}$, with $\mathcal{A}$ a real-valued 1-form.

Curvature, in brief

The curvature two-form is defined as,

$$F(u,v)s = D_u D_v s - D_v D_u s - D_{[u,v]}s. \tag{B.26}$$

The third term makes the curvature vanish (for all vectors v and u) when comes from a flat connection over a trivial bundle, with fiber V, where a section is just a function $f : M \to V$,

$$f(v,u) = vuf - uvf - [v,u]f = 0. \tag{B.27}$$

On the other hand, a connection with vanishing curvature for all vectors v and u and sections s, is said to be flat.

An important property of F is that is $C^\infty(M)$-linear in each argument (vectors and sections), so it is really an $\mathrm{End}(E)$-valued two-form.

Defining $F_{\mu\nu} := F(\partial_\mu, \partial_\nu)$ (which is antisymmetric by definition), one can show, using a basis for sections, that

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{B.28}$$

where internal indices i, j, k are suppressed. Then, F can be expressed as,

$$F = \frac{1}{2} F_{\mu\nu} \otimes dx^\mu \wedge dx^\nu. \tag{B.29}$$

Generalization of exterior derivatives

First, we define an E-valued p-form as a section of $E \otimes \Lambda^p T^* M$. Let us now define the wedge product of an E-valued form[1] $s \otimes w$ and an ordinary form μ on M,

$$(s \otimes w) \wedge \mu := s \otimes w \wedge \mu, \tag{B.30}$$

where s is a section of E and w is a differential form on M.

We know so far how to covariantly differentiate a section $T \in \Gamma(E)$. Now, we wish to define something similar for a E-valued differential form. First, we define an exterior covariant derivative d_D of a section s as the 1-form satisfying,

$$d_D s(v) = D_v(s), \quad \forall v \in \mathrm{Vect}(M). \tag{B.31}$$

In local coordinates on some open set, this is

$$d_D s = D_\mu s \otimes dx^\mu. \tag{B.32}$$

Now, we define an exterior covariant derivative on E-valued forms, as

$$d_D(s \otimes w) = d_D s \wedge w + s \otimes dw. \tag{B.33}$$

Locally, if we express the E-valued form s as $s_I \otimes dx^I$ (where I goes over several indices), we have,

$$d_D(s_I \otimes dx^I) = D_\mu s_I \otimes dx^\mu \wedge dx^I. \tag{B.34}$$

We will show that d_D^2 of any E-valued form η is proportional to F. First, we need to define a wedge product between an End(E)-valued form $T \otimes w$ and an E-valued form $s \otimes \mu$,

$$(T \otimes w) \wedge (s \otimes \mu) := T(s) \otimes (w \wedge \mu). \tag{B.35}$$

Now, let us calculate $d_D^2 \eta$, where $\eta = s_I \otimes dx^I$ is an E-valued form,

$$\begin{aligned}
d_D^2 \eta &= d_D(d_D s_I \otimes dx^I) \\
&= d_D(D_{nu} s_I \otimes dx^\nu \wedge dx^I) \\
&= D_\mu D_\nu s_I \otimes dx^\mu \wedge dx^\nu \wedge dx^I \\
&= \frac{1}{2}[D_\mu, D_\nu] s_I \otimes dx^\mu \wedge dx^\nu \wedge dx^I \\
&= \frac{1}{2} F_{\mu\nu} s_I \otimes dx^\mu \wedge dx^\nu \wedge dx^I \\
&= (\frac{1}{2} F_{\mu\nu} \otimes dx^\mu \wedge dx^\nu) \wedge (s_I \otimes dx^I) \\
&= F \wedge \eta.
\end{aligned}$$

[1]Any E-valued form can be expressed as a (not necessarily unique) sum of elements of the form $s \otimes w$. Locally, there is only a unique way.

We now want to define a connection on End(E) starting from one on E. Then, we will define an exterior covariant derivative on End(E)-valued forms, just as we did on E-valued forms. So, by inspiration on Leibnitz rule,

$$v(\lambda(s)) = (D_v^*\lambda)(s) + \lambda D_v(s), \tag{B.36}$$

where $s \in \Gamma(E)$, $\lambda \in \Gamma(E^*)$ and v is some vector field on M. Now that we have a connection D^* on E^*, we define a connection[2] on End$(E) \cong E \otimes E*$,

$$(D \otimes D^*)_v(s \otimes s^*) = (D_v s) \otimes s^* + s \otimes (D_v^* s^*). \tag{B.37}$$

One can go further and see what a connection D on End(E) actually looks like,

$$\begin{aligned}
(D_vT)(s) &= D_v(T^i_j e_i \otimes e^j)(s)\\
&= v(T^i_j)(e_i \otimes e^j)(s) + T^i_j D_v(e_i \otimes e^j)(s)\\
&= v(T^i_j)e_i s^j + T^i_j(D_v e_i)s^j + T^i_j e_i \otimes D^*_v(e^j)(s)\\
&= v(T^i_j)e_i s^j + T^i_j(D_v e_i)s^j + T^i_j(e_i v(s^j) - e_i \otimes e^j(D_v(s)))\\
&= v(T^i_j)e_i s^j + T^i_j D_v(s^j e_i) - T^i_j e_i \otimes e^j(D_v(s))\\
&= D_v(T^i_j s^j e_i) - T(D_v s)\\
&= D_v(Ts) - T(D_v s).
\end{aligned}$$

So, the connection D on End(E) is

$$(D_vT)(s) = D_v(Ts) - T(D_v s). \tag{B.38}$$

Now, we define an exterior covariant derivative on an End(E)-valued form $T \otimes w$ just as we did with E-valued forms,

$$d_D(T \otimes w) = d_D T \otimes w + T \otimes dw, \tag{B.39}$$

where $(d_D T)(v) = D_v T$. Locally,

$$d_D(T \otimes w) = D_\mu T \otimes dx^\mu \wedge w. \tag{B.40}$$

Let us see how d_D acts on a wedge product between an End (E)-valued p-form $w = T_I \otimes dx^I$ and an (E)-valued form $\mu = s_J \otimes dx^J$, with D a connection on E,

$$\begin{aligned}
d_D(w \wedge \mu) &= d_D((T_I \otimes dx^I) \wedge (s_J \otimes dx^J))\\
&= d_D(T_I s_J \otimes dx^I \wedge dx^J)\\
&= D_\mu(T_I s_J) \otimes dx^\mu \wedge dx^I \wedge dx^J\\
&= ((D_\mu T_I)s_J + T_I D_\mu s_J) \otimes dx^\mu \wedge dx^I \wedge dx^J\\
&= (D_\mu T_I \otimes dx^\mu \wedge dx^I) \wedge (s_J \otimes dx^J) + (-1)^p (T_I \otimes dx^I) \wedge (D_\mu s_I \otimes dx^\mu \wedge dx^J)\\
&= d_D w \wedge \mu + (-1)^p w \wedge d_D \mu.
\end{aligned} \tag{B.41}$$

We have made an abuse of language when calling also D to the connection acting on T_I, which is a section of End(E), from the forth equality above.

[2]Actually, this is valid for any product $E \otimes E'$, where E' is another vector bundle over M.

Let us prove Bianchi identity: $d_D F = 0$. On one hand, $d_D^3\eta = d_D^2(d_D\eta) = F \wedge d_D\eta$, for any E-valued form η. On the other hand,

$$\begin{aligned} d_D^3\eta &= d_D(d_D^2\eta) \\ &= d_D(F \wedge \eta) \\ &= d_D F \wedge \eta + F \wedge d_D\eta, \end{aligned}$$

so, since this is valid for any η,

$$d_D F = 0. \tag{B.42}$$

So far we have defined the wedge product of End(E)-valued forms with E-valued forms. Here we define the wedge product of an End(E)-valued form $S \otimes w$ and another one $T \otimes \mu$,

$$(S \otimes w) \wedge (T \otimes \mu) := ST \otimes w \wedge \mu. \tag{B.43}$$

Similar to (B.41), we have for an End(E)-valued p-form w and an End(E)-valued form μ,

$$d_D(w \wedge \mu) = d_D w \wedge \mu + (-1)^p w \wedge d_D\mu. \tag{B.44}$$

Let us define the *graded commutator* between an End(E)-valued p-form w and an End(E)-valued q-form μ,

$$[w, \mu] := w \wedge \mu - (-1)^{pq}\mu \wedge w = -(-1)^{pq}[\mu, w]. \tag{B.45}$$

Let su consider the case where E admits a flat connection D^0. This happens for all trivial bundles, so D^0 exists at least locally. We define

$$d := d_{D^0}, \tag{B.46}$$

for the exterior covariant derivative of E-valued or End(E)-valued forms with respect to the connection D^0. Note that, although an abuse of notation, it has some sense since $d^2 = 0$, just as the exterior derivative of ordinary differential forms. Given any connection D on E, it can be written as $D^0 + A$, so let us prove that if w is an E-valued form, then $d_D w = dw + A \wedge w$,

$$\begin{aligned} d_D w &= D_\mu w_I \otimes dx^\mu \wedge dx^I \\ &= (D^0_\mu + A_\mu) w_I \otimes dx^\mu \wedge dx^I \\ &= dw + A \wedge w. \end{aligned}$$

Likewise, for $\eta = \eta_I \otimes dx^I$ an End(E)-valued p-form, we can prove that $d_D\eta = d\eta + [A, \eta]$:

$$\begin{aligned} d_D\eta &= [D_\mu, \eta_I] \otimes dx^\mu \wedge dx^I \\ &= [D^0_\mu + A_\mu, \eta_I] \otimes dx^\mu \wedge dx^I \\ &= d\eta + A \wedge \eta - \eta_I A_\mu \otimes dx^\mu \wedge dx^I \\ &= d\eta + A \wedge \eta - (-1)^p \eta \wedge A \\ &= d\eta + [A, \eta]. \end{aligned}$$

Let us prove a useful identity, $[A, A \wedge A] = 0$:

$$[A, A \wedge A] = A \wedge (A \wedge A) - (A \wedge A) \wedge A = 0, \tag{B.47}$$

because of the property of associativity that enjoys the product of End(E)-valued forms.

We had already seen that $d_D^2 w = F \wedge w$ for some E-valued form w. If one computes this quantity again, but using the result (B.47), one gets,

$$d_D^2 w = (dA + A \wedge A) \wedge w, \tag{B.48}$$

so this looks like one can say,

$$F = dA + A \wedge A. \tag{B.49}$$

This is indeed the case: if one writes this in components, recovers (B.28). The Bianchi identity can be again recovered using (B.49) and (B.47).

Now let us see how the curvature transforms under gauge transformations,

$$\begin{aligned} F'(u,v)(s) &= D'_u D'_v s - D'_v D'_u s - D'_{[u,v]} s \\ &= g(D_u D_v - D_v D_u - D_{[u,v]})(g^{-1}s) \\ &= gF(u,v)g^{-1}s, \end{aligned}$$

so

$$F' = gFg^{-1}. \tag{B.50}$$

This product is well defined: one should think of the gauge transformations as End(E)-valued 0-forms. It not difficult to show the following results: if η is an E-valued form, then

$$d_{D'}\eta = g d_D(g^{-1}\eta). \tag{B.51}$$

If T is a section of End(E), then

$$D'_v T = \mathrm{Ad}(g) D_v(\mathrm{Ad}(g^{-1})T), \tag{B.52}$$

where $\mathrm{Ad}(g)T = gTg^{-1}$. Finally, if η is an End(E)-valued form, then

$$d_{D'}\eta = \mathrm{Ad}(g) d_D(\mathrm{Ad}(g^{-1})\eta). \tag{B.53}$$

To end this subsection, let us define a trace of End(E) as a linear map $\mathrm{tr} : \mathrm{End}(E) \to \mathbb{R}$ such that,

$$\mathrm{tr}(v \otimes \lambda) := \lambda(v), \tag{B.54}$$

so if T is a section of End(E), then

$$\mathrm{tr}(T) = T^i_j \mathrm{tr}(e_i \otimes e^j) = T^i_i, \tag{B.55}$$

so we get the usual trace. This naturally defines a function tr:$M \to \mathbb{R}$. Even more, we can define the trace of an End(E)-valued form, which is an ordinary form, like this,

$$\mathrm{tr}(T \otimes w) = \mathrm{tr}(T) w. \tag{B.56}$$

Appendix C

Notions of gauge theories

What a variation means in differential geometry

We will be interested in taking variations with respect to a vector potential, so let us define what this means. A plausible transformation of the vector potential A is adding to it s times an End(E)-valued 1-form δA,

$$A_s = A + s\delta A. \tag{C.1}$$

If Z is a function of the vector potential A, then its variation is defined by,

$$\delta Z := \frac{d}{ds} Z(A_s)\Big|_{s=0}. \tag{C.2}$$

When we say $\delta Z = 0$, we mean that this variation vanishes for all δA.

Yang-Mills acion

The Yang-Mills action is,

$$S_{YM}[A] = \frac{1}{2}\int_M \mathrm{tr}(F \wedge \star F), \tag{C.3}$$

where $\star$ is the Hodge dual[1]. We explicitly say that this action is a function of the vector potential because F depends on the connection D, and we can sweep all of them by choosing a fixed connection D^0 and adding to it every End(E)-valued 1-form A. So the vector potential can be thought of being the actual dynamical field.

In order to calculate the variation of the Yang-Mills action, we need to know first several things. For example, the variation of the curvature, so we point towards that direction. We saw that if D^0 is the flat connection, then $F = dA + A \wedge A$. But if the bundle does not admit a flat connection, we need a more general formula for F. Let us suppose that D^0 is some connection (not necessarily flat) and call d_0 its exterior covariant derivative. Then,

$$d_0^2 w = F_0 \wedge w, \tag{C.4}$$

[1] Here the star operator acts on an End(E)-valued 2-form, so this is defined as the Hodge dual acting on the differential 2-form sector, not seeing the End(E)-valued section.

for any (E)-valued form w and F_0 being the curvature of D^0. On the other hand, $d_D^2 w = F \wedge w$. So let us keep this in mind, and recalculate $d_D^2 w$,

$$\begin{aligned} d_D^2 w &= d_D(d_0 w + A \wedge w) \\ &= d_0(d_0 w + A \wedge w) + A \wedge (d_0 w + A \wedge w) \\ &= F_0 \wedge w + (d_0 A + A \wedge A) \wedge w. \end{aligned}$$

This, although not rigorously, says that,

$$F = F_0 + d_0 A + A \wedge A. \tag{C.5}$$

Now, let us take the variation of F,

$$\begin{aligned} \delta F &= \frac{d}{ds}(F_0 + d_0 A_s + A_s \wedge A_s)\big|_{s=0} \\ &= d_0 \left(\frac{d}{ds} A_s\right) + \frac{d}{ds} A_s \wedge A + A \wedge \frac{d}{ds} A_s\big|_{s=0} \\ &= d_0 \delta A + \delta A \wedge A + A \wedge \delta A \\ &= d_0 \delta A + [A, \delta A] \\ &= d_D \delta A. \end{aligned}$$

So the variation of the curvature equals the exterior covariant derivative of the variation of the vector potential.

Let us now prove some useful results about the trace of End(E)-valued forms that we will soon use to get the variation of the Yang-Mills action.

1- If w and μ are End(E)-valued p- and q-forms, respectively, then $\mathrm{tr}(w \wedge \mu) = (-1)^{pq}\mathrm{tr}(\mu \wedge w)$:

$$\begin{aligned} \mathrm{tr}(w \wedge \mu) &= \mathrm{tr}(w_I \mu_J) \otimes dx^I \wedge dx^J \\ &= \mathrm{tr}(\mu_J w_I) \otimes dx^I \wedge dx^J \\ &= (-1)^{pq}\mathrm{tr}(\mu_J w_I) \otimes dx^J \wedge dx^I \\ &= (-1)^{pq}\mathrm{tr}(\mu \wedge w). \end{aligned}$$

This property is called the graded cyclic property and in the second step we used the commutativity of elements of End(E) inside the trace.

2- With w and μ of the item before, it holds that $\mathrm{tr}([w, \mu]) = 0$: one just has to use item 1 and the linearity of the trace.

3- Let D be a connection on E. If w is an End(E)-valued form, then $\mathrm{tr}(d_D w) = d\mathrm{tr}(w)$:

$$\mathrm{tr}(d_D w) = \mathrm{tr}([D_\mu, w_I]) \otimes dx^\mu \wedge dx^I.$$

To continue, let us show how $[D_\mu, w_I]$ acts on a section s of E,

$$\begin{aligned}
[D_\mu, w_I](s) &= (D_\mu w^a_{Ib} e_a \otimes e^b - w^a_{Ib} e_a \otimes e^b D_\mu)(s) \\
&= D_\mu w^a_{Ib} s^b e_a - w^a_{Ib} e_a s^b_{,\mu} - w^a_{Ib} e_a \otimes e^b A^c_{\mu d} s^d e_c \\
&= w^a_{Ib,\mu} s^b e_a + w^a_{Ib} s^b_{,\mu} e_A + w^a_{Ib} s^b A^c_{\mu a} e_c - w^a_{Ib} s^b_{,\mu} e_a - w^a_{Ib} A^b_{\mu c} s^c e_a \\
&= \left(w^a_{Ib,\mu} + w^c_{Ib} A^a_{\mu c} - w^a_{Ic} A^c_{\mu b} \right) e_a \otimes e^b \, (s)
\end{aligned}$$

Now let us go on from where we left,

$$\begin{aligned}
\mathrm{tr}(d_D w) &= \left(w^a_{Ib,\mu} + w^c_{Ib} A^a_{\mu c} - w^a_{Ic} A^c_{\mu b} \right) \mathrm{tr}(e_a \otimes e^b) \otimes dx^\mu \wedge dx^I \\
&= w^a_{Ia,\mu} \otimes dx^\mu \wedge dx^I \\
&= d(\mathrm{tr}(w_I) \otimes dx^I) \\
&= d\mathrm{tr}(w),
\end{aligned}$$

where d is the ordinary exterior derivative. This identity allows to "commute" the trace and the exterior covariant derivative.

4- Let M be an n-dimensional oriented manifold, and w and μ as usual with $p + q + 1 = n$, then

$$\int_M \mathrm{tr}(d_D w \wedge \mu) = (-1)^{p+1} \int_M \mathrm{tr}(w \wedge d_D \mu). \tag{C.6}$$

One just has to use the last result and to integrate by parts.

5- As in the last point, but with $p = q$, then holds,

$$\int_M \mathrm{tr}(w \wedge \star\mu) = \int_M \mathrm{tr}(\mu \wedge \star w). \tag{C.7}$$

To prove it, just use the cyclic property of the trace and the identity $d\Omega = dx^I \wedge \star dx^J = dx^J \wedge \star dx^I$, where $d\Omega$ is the volume form.

We are now in position of computing the equations of motion of Yang-Mills theory,

$$\begin{aligned}
\delta S_{YM} &= \frac{1}{2}\delta \int_M \mathrm{tr}(F \wedge \star F) \\
&= \frac{1}{2} \int_M \mathrm{tr}(\delta F \wedge \star F + F \wedge \star \delta F) \\
&= \int_M \mathrm{tr}(d_D \delta A \wedge \star F) \\
&= \int_M \mathrm{tr}(\delta A \wedge d_D \star F).
\end{aligned}$$

Chern-Simons theory

As me do not want a theory with some fixed given metric, we would like to cook a lagrangian that only depends on the connection (or the vector potential). The easy guess would be,

$$S(A) = \int_M \mathrm{tr}(F^n), \tag{C.8}$$

but is easy to see that it gives a trivial equation of motion,

$$\mathrm{tr}(d_D F^{n-1}) = 0, \tag{C.9}$$

satisfied always by means of the Bianchi identity. Not useful as an action, it is a simple (or not) example of a bundle invariant, since its vanishing variation with respecto to A says that the integral of the *nth-Chern form*, $\int_M \mathrm{tr}(F^n)$, is independent of the connection! In other words, given some bundle E, you can take *any* connection on E to compute the integral of the nth-Chern form.

Let us talk a bit about the Chern forms. First of all, they are closed, which can be seen using the item 3 result in the previous section and the Bianchi identity. This means that the kth Chern form defines a cohomology class in $H^{2k}(M)$, $[\mathrm{tr}(F^k)]$. Now, although the Chern form depends on A, its cohomology class does not, it changes by an exact form:

$$\begin{aligned} \delta\mathrm{tr}(F^k) &= k\mathrm{tr}(\delta F \wedge F^{k-1}) \\ &= k\mathrm{tr}(d_D\delta A \wedge F^{k-1}) \\ &= k\mathrm{tr}(d_D(\delta A \wedge F^{k-1})) \\ &= kd\mathrm{tr}(\delta A \wedge F^{k-1}) \end{aligned}$$

We can compute the difference between two Chern forms given by different connections. Say A' is a vector potential with curvature F', $\delta A = A' - A$, $A_s = A + s\delta A$, and F_s the curvature of A_s, then,

$$\begin{aligned} \mathrm{tr}F'^k - \mathrm{tr}F^k &= \int_0^1 \frac{d}{ds}\mathrm{tr}(F_s^k)ds \\ &= k\int_0^1 ds\,\mathrm{tr}(\frac{d}{ds}F_s \wedge F_s^{k-1}) \\ &= k\int_0^1 ds\,\mathrm{tr}(d_D\delta A \wedge F_s^{k-1} + 2s\delta A^2 \wedge F_s^{k-1}) \\ &= k\int_0^1 ds\,d\mathrm{tr}(\delta A \wedge F_s^{k-1}) + k\int_0^1 ds\,\mathrm{tr}(\delta A \wedge d_D F_S^{k-1}) + 2ks\int_0^1 ds\,\mathrm{tr}(\delta A^2 \wedge F_s^{k-1}) \end{aligned}$$

Let us work on the second term of the RHS:

$$\begin{aligned} k\int_0^1 ds\,\mathrm{tr}(\delta A\wedge d_D F_S^{k-1}) &= k\int_0^1 ds\,\mathrm{tr}(\delta A\wedge(dF_s^{k-1}+[A,F_s^{k-1}])) \\ &= k\int_0^1 ds\,\mathrm{tr}(\delta A\wedge(-[A_s,F_s^{k-1}]+[A,F_s^{k-1}])) \\ &= k\int_0^1 ds\,\mathrm{tr}(\delta A\wedge[-s\delta A,F_s^{k-1}]) \\ &= -sk\int_0^1 ds\,\mathrm{tr}(\delta A\wedge(\delta A\wedge F_s^{k-1}-F_s^{k-1}\wedge\delta A)) \\ &= -sk\int_0^1 ds\,\mathrm{tr}(\delta A^2\wedge F_s^{k-1}-(-1)\delta A^2\wedge F_s^{k-1}) \\ &= -2sk\int_0^1 ds\,\mathrm{tr}(\delta A^2\wedge F_s^{k-1}), \end{aligned}$$

so we finally get,

$$\mathrm{tr}F'^k-\mathrm{tr}F^k = d\left(k\int_0^1 ds\,\mathrm{tr}(\delta A\wedge F_s^{k-1})\right). \tag{C.10}$$

Therefore, we can define the *kth-Chern class* $c_k(E)$ of the vector bundle E to be the cohomlogy class of $\mathrm{tr}(F^k)$, where F is the curvature of any connection on E. Actually, it can be shown that

$$\frac{(i/2\pi)^k}{k!}\mathrm{tr}(F^k) \tag{C.11}$$

is an integral class, meaning that its integral over a compact and orientable manifold M is an integer.

Now, the Chern-Simons form is a $(2k-1)$-form whose exterior derivative gives the kth-Chern form. This is well defined only locally, of course. For example, take (C.10), with $A=F=0$, and now call $A'=\delta A:=A$, then

$$\mathrm{tr}(F^k) = d\left(k\int_0^1 ds\,\mathrm{tr}(A\wedge F_s^{k-1})\right). \tag{C.12}$$

For a 3-dimensional theory ($\dim(M)=3$), we want the Chern-Simons form to be a 3-form, so $k=2$,

$$\mathrm{tr}(F^2) = d\left(2\int_0^1 ds\,\mathrm{tr}(\delta A\wedge F_s)\right) = d\mathrm{tr}(A\wedge dA+\frac{2}{3}A^3), \tag{C.13}$$

so the Chen-Simons form, in three dimensions, is

$$L_{CS}(A) = A\wedge dA+\frac{2}{3}A^3. \tag{C.14}$$

The three dimensional Chern-Simons theory is defined as

$$S_{CS}(A) = \int_M \mathrm{tr}(L_{CS}(A)). \tag{C.15}$$

The variation of this action is easily computed and gives,

$$\delta S_{CS} = 2\int_M \mathrm{tr}(\delta A \wedge F), \tag{C.16}$$

so flat connections are the ones which solve the equations of motion. For other odd dimensions, $\dim(M) = 2k-1$, the variation of the Chern-Simons action is,

$$\delta S_{CS} = k\int_M \mathrm{tr}(\delta A \wedge F^{k-1}). \tag{C.17}$$

The Chern-Simons action is not gauge invariant. If we compute the action for the gauge transformed vector potential $A' = g^{-1}dg + g^{-1}Ag$, we get,

$$L_{CS}(A') = L_{CS}(A) - \frac{1}{3}\mathrm{tr}((g^{-1}dg)^3) + d\mathrm{tr}(dgg^{-1}A). \tag{C.18}$$

Appendix D

Many AdS$_3$ black holes living in harmony

The discovery of Bañados, Teitelboim and Zanelli triggered the search for other black hole solutions in AdS$_3$ gravity. What it was found later is that one can construct spacetimes with many BTZ black holes coexisting, thanks to the absence of gravitational waves which would make them interact [59, 60, 61].

In order to briefly sketch how to construct these multi-black hole solutions, let us first present another way to obtain the BTZ black hole from initial data. Consider the surface $x^0 = t = 0$ on AdS$_3$ as the "initial data" surface (with $x^3 > 0$) in (1.6), which has zero extrinsic curvature and then also has negative constant curvature: it is the hyperbolic space H^2. This space can be projected on $x^3 = \ell$ by an stereographic projection with "North pole" at $x^3 = -\ell$, and one gets the Poincar disk,

$$ds^2 = \left(\frac{1}{1-r^2}\right)^2 (dx^2 + dy^2), \quad r^2 = \frac{x^2+y^2}{4\ell^2}, \tag{D.1}$$

which is an Euclidean geometry of constant negative curvature. Every geodesic in this space is an arc of circle that intersects the boundary at right angles. The curves $\phi = 0$ and $\phi = 2\pi$ are geodesics and the interior the interior region they define makes for the initial geometry of the BTZ black holes, once one has identified this two geodesics, The identification of the geodesics permits the idea of folding the diagram making it a three-dimensional image, like a cigar, where the horizon line in Figure D.1 becomes a circle.

Now, to describe in a nutshell how to construct a geometry with many BTZ black holes coexisting in harmony, we proceed pictorially based on Figure D.1. The idea is to have many disconnected boundaries each of them identical to the boundary of the BTZ black hole. In Figure D.2 the initial surface corresponding to three BTZ black holes is shown. If one would like to draw a three-dimensional picture when the identifications are performed, then the result would be a trinion or pant, as the one used in [128], which is a genus-zero Riemann surface with three punctures. Each of these punctures represents a horizon. The asymptotic regions cannot be embedded in flat space so this kind of visualization is only valid from the origin to the position of the horizons.

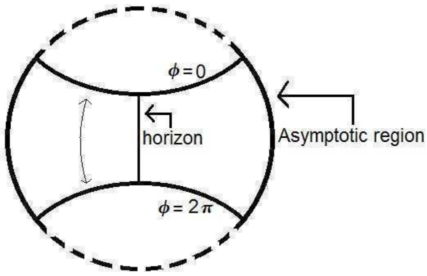

Figure D.1: Picture of the procedure to obtain the initial surface of the BTZ black hole. The disk is the Poincaré disk. The two arcs of a circle that intersect the boundary of the disk are the geodesics to be identified (as shows the double thin arrow). The region between the geodesics and the dotted lines are eliminated and remains the interior between the geodesics. The horizon and the asymptotic region are also shown.

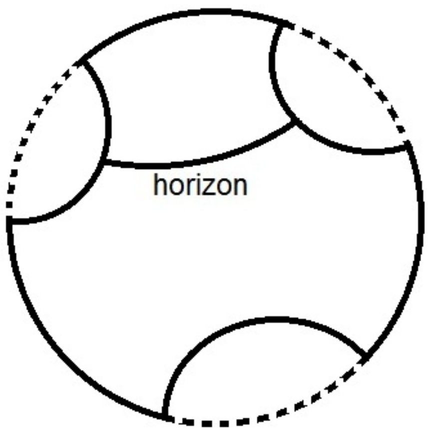

Figure D.2: Picture of the procedure to obtain the initial surface of the three-BTZ black holes-spacetime. The disk is the Poincaré disk. Now there are three asymptotic regions and three geodesics to be identified. Only one horizon is shown, although there are two other horizons joining the other geodesics.

Appendix E

3D curvature from parallel transport

The Riemann curvature of the three-dimensional 0-brane described in Section 4.2 can be obtained from parallel transport of a Lorentz vector V^a around the point $r = 0$. The equation of parallel transport along $B(r) = \text{Const.}$ reads,

$$(DV)^c = \left(\partial_\phi V^c + \omega_\phi^{cd}\, V_d\right) d\phi = 0\,, \tag{E.1}$$

or in components,

$$\begin{aligned} 0 &= \partial_\phi V^0 + \frac{bB}{\ell}\, V^1\,, \\ 0 &= \partial_\phi V^1 + \frac{bB}{\ell}\, V^0 - \frac{a}{\ell}\sqrt{B^2+\ell^2}\, V^2\,, \\ 0 &= \partial_\phi V^2 + \frac{a}{\ell}\sqrt{B^2+\ell^2}\, V^1\,. \end{aligned} \tag{E.2}$$

The most general solution to these equations has the form,

$$\begin{aligned} V^0(\phi) &= \frac{Bb}{\ell\Omega}\left(\beta\cos\Omega\phi - \alpha\sin\Omega\phi\right) + \frac{a}{bB}\sqrt{B^2+\ell^2}\gamma\,, \\ V^1(\phi) &= \alpha\cos\Omega\phi + \beta\sin\Omega\phi\,, \\ V^2(\phi) &= \frac{a}{\ell\Omega}\sqrt{B^2+\ell^2}\left(\beta\cos\Omega\phi - \alpha\sin\Omega\phi\right) + \gamma\,, \end{aligned} \tag{E.3}$$

where the angular frequency is,

$$\Omega^2(r) = \frac{a^2-b^2}{\ell^2}\, B^2 + a^2 = \frac{r^2}{\ell^2} + a^2 + b^2\,. \tag{E.4}$$

If the transport starts in the point $\phi = 0$ with the vector $\bar{V}^a$, the integration constants are,

$$\begin{aligned} \alpha &= \bar{V}^1\,, \\ \beta &= \frac{1}{\ell\Omega}\left(-bB\bar{V}^0 + a\sqrt{B^2+\ell^2}\bar{V}^2\right)\,, \\ \gamma &= \frac{bB}{\ell^2\Omega^2}\left(a\sqrt{B^2+\ell^2}\bar{V}^0 - bB\bar{V}^2\right)\,. \end{aligned} \tag{E.5}$$

After moving along the circle and returning to the initial point, the vector becomes,

$$V^c(2\pi) = S^c{}_d\,\bar{V}^d\,, \tag{E.6}$$

where the transformation matrix, in the limit $r \to 0$, reads,

$$S = \begin{pmatrix} 1+\frac{b^4}{a^4-b^4}\left(1-\cos\theta\right) & -\frac{b^2}{\sqrt{a^4-b^4}}\,\sin\theta & -\frac{a^2b^2}{a^4-b^4}\left(1-\cos\theta\right) \\ -\frac{b^2}{\sqrt{a^4-b^4}}\,\sin\theta & \cos\theta & \frac{a^2}{\sqrt{a^4-b^4}}\,\sin\theta \\ \frac{a^2b^2}{a^4-b^4}\left(1-\cos\theta\right) & -\frac{a^2}{\sqrt{a^4-b^4}}\,\sin\theta & \frac{1}{a^4-b^4}\left(-b^4+a^4\cos\theta\right) \end{pmatrix}. \tag{E.7}$$

We have introduced the angle,

$$\theta := \lim_{r\to 0} 2\pi\Omega = 2\pi\sqrt{a^2+b^2}\,. \tag{E.8}$$

In the static case ($b=0$), we have $\theta_0 = 2\pi a_0$ and,

$$S_{\text{static}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_0 & \sin\theta_0 \\ 0 & -\sin\theta_0 & \cos\theta_0 \end{pmatrix} = e^{\theta_0 J_{12}}\,. \tag{E.9}$$

A non-trivial curvature appears due to the static current,

$$S_{\text{static}} = e^{-\int j_{\text{static}}}\,, \qquad \int j_{\text{static}} = -2\pi a_0\,J_{12}\,. \tag{E.10}$$

In the rotating case ($b \neq 0$), in analogy to the static case, we can rewrite the matrix S as a composition of the 12-rotation and 02-boost,

$$S = e^{\eta J_{02}}\,e^{\theta J_{12}}\,e^{-\eta J_{02}}\,, \tag{E.11}$$

with the angles,

$$\theta \;:=\; \lim_{r\to 0} 2\pi\Omega = 2\pi\sqrt{a^2+b^2}\,, \tag{E.12}$$

$$\tanh\eta \;:=\; \lim_{r\to 0}\frac{bB}{a\sqrt{B^2+\ell^2}} = \frac{b^2}{a^2}\,. \tag{E.13}$$

Again, a spinning brane is obtained from the static one after applying the Lorentz boost,

$$e^{\eta J_{02}} = \begin{pmatrix} \cosh\eta & 0 & \sinh\eta \\ 0 & 1 & 0 \\ \sinh\eta & 0 & \cosh\eta \end{pmatrix}. \tag{E.14}$$

The non-trivial curvature ($S \neq 1$) generated by the spinning source becomes explicit now as,

$$S = e^{-\int j_{\text{curvature}}}\,. \tag{E.15}$$

Because the generators J_{12} and J_{02} do not commute, $[J_{12}, J_{02}] = J_{01}$, we apply the Baker-Campbell-Hausdorff formula,

$$e^{\eta J_{02}} e^{\theta J_{12}} e^{-\eta J_{02}} = e^{\theta\left(J_{12}\,\cosh\eta - J_{01}\,\sinh\eta\right)}\,, \tag{E.16}$$

and we obtain the source as,

$$\begin{aligned}\int j_{\text{curvature}} &= \theta\left(-J_{12}\cosh\eta + J_{01}\sinh\eta\right) \\ &= \frac{2\pi}{\sqrt{a^2-b^2}}\left(-a^2 J_{12} + b^2 J_{01}\right)\ . \end{aligned} \tag{E.17}$$

We have then reproduced the Riemannian piece of the curvature (4.35). Anyway, this analysis is incomplete because the full source contains a torsional piece as well, $j = j_{\text{curvature}} + j_{\text{torsion}}$.

Appendix F

BPS branes in 2+1 dimensions

The supersymmetric extension of the AdS group in three dimensions, with $\mathcal{N} = p+q$ supersymmetries, is $OSp(p|2) \times OSp(q|2)$ [1, 16] with the corresponding algebra generators $G_K = \left\{G_K^+, G_K^-\right\}$. The connection 1-form can be written as,

$$A = A^K G_K = A^+ + A^- \ , \tag{F.1}$$

where

$$A^\pm = \left(\omega^{\,a} \pm \frac{1}{\ell}\, e^{\,a}\right) J_a^\pm + \frac{1}{2}\, b_\pm^{IJ}\, T_{IJ}^\pm + \psi_{\pm\alpha}^I\, Q_I^{\pm\alpha} \ . \tag{F.2}$$

Here $\{T_{IJ}^+, T_{I'J'}^-\}$ generate the $O(p) \times O(q)$ subgroup. The corresponding field strength also splits as $F = F^+ + F^-$, with

$$F^\pm = \left(R^a \pm \frac{1}{\ell}\, T^a + \frac{1}{2\ell^2}\, \epsilon^a{}_{bc}\, e^b \wedge e^c\right) J_a^\pm + \frac{1}{2}\, \mathcal{F}_\pm^{IJ}\, T_{IJ}^\pm + \text{spinors} \ , \tag{F.3}$$

the curvature being given by $R^a = \frac{1}{2}\, \epsilon^{abc}\, R_{bc} = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c$.

We seek for a bosonic configuration ($\psi^\pm = 0$) that possesses non-trivial supersymmetries $\varepsilon = \varepsilon^+ + \varepsilon^- = \varepsilon_I^{+\alpha}\, Q_\alpha^{+I} + \varepsilon_{I'}^{-\alpha}\, Q_\alpha^{-I'}$, so that the spinor ε is a solution of the Killing spinor equation,

$$D\varepsilon := \left(D_+\varepsilon^+\right)_I^{\ \alpha}\, Q_\alpha^{+I} + \left(D_-\varepsilon^-\right)_{I'}^{\ \alpha}\, Q_\alpha^{-I'} = 0 \ .$$

Each term must be zero independently, so we have

$$D_\pm\varepsilon^\pm = \left[d - \frac{1}{2}(\omega^a \pm \frac{1}{\ell}\, e^a)\Gamma_a + b_\pm\right] \varepsilon^\pm = 0 \ , \tag{F.4}$$

where $b_\pm$ is a square matrix with components $(b_\pm)^K_{\ L}$ (the components of the $o(p)$ or $o(q)$ gauge fields), and Γ_a are Dirac matrices. The term $b_\pm\varepsilon^\pm$ means $(b_\pm\varepsilon^\pm)^I{}_\alpha = (b_\pm)^I{}_J\varepsilon^{\pm J}_{\ \alpha}$.

The AdS connection in the region around the brane is locally flat, $F^\pm = 0$. This means that the torsion must vanish and the metric is that of a locally AdS spacetime. The only effect of the presence of the brane is in the topology of the region around it. Next, the conditions for the geometry to admit a global Killing spinor in the presence of the defect will be analyzed.

$\mathcal{N}=1$ supersymmetry

The minimal supersymmetry, $\mathcal{N}=1$, is described by the super AdS algebra $osp(1|2)$ with $(p,q)=(1,0)$ or $(0,1)$, so the $b^{\pm}$ are absent, and only one gravitino, either ψ^{+} or ψ^{-}, is present, and consequently, either Q^{+} or Q^{-} is included. Suppose the supersymmetry generator is Q^{+}; then, the Killing spinor is $\varepsilon=\varepsilon^{+}Q^{+}$ and equation (F.4) must be solved for the + choice only.

The ansatz for the metric of a static three-dimensional 0-brane displayed in (4.3,4.4) can be described by the vielbein e^a and the spin connection $\omega^a \equiv \frac{1}{2}\varepsilon^{a}{}_{bc}\,\omega^{bc}$, at $r\neq 0$, as

$$e^0 = A\,d\phi_{03}\ ,\qquad e^1=\frac{\ell}{A}\,dB\ ,\qquad e^2=B\,d\phi_{12}\ ,$$

$$\omega^0=-\frac{A}{\ell}\,d\phi_{12}\ ,\qquad \omega^1=0\ ,\qquad \omega^2=\frac{B}{\ell}\,d\phi_{03}\ ,$$

where $A^2-B^2=\ell^2$, $B=r/a_0$, $\phi_{12}=a_0\phi$, and $\phi_{03}=a_0\,t/\ell$. We want to solve the Killing spinor equation (F.4) for $\varepsilon^{+}=\varepsilon_I^{+\alpha}\,Q_\alpha^{+I}\equiv\varepsilon^{+}Q^{+}$. In Chern-Simons AdS$_3$ supergravity, considering the gauge connection A describing only the 0-brane (*i.e.*, without additional $O(p)\times O(q)$ gauge fields switched on),

$$D\varepsilon^{+}=\left[d-\frac{1}{2}\left(\omega^a+\frac{1}{\ell}\,e^a\right)\Gamma_a\right]\varepsilon^{+}=0\ . \tag{F.5}$$

Here, Γ_a are three-dimensional Γ-matrices and, for simplicity, we choose only one of the two inequivalent representation of Γ-matrices, with $c=1$. In our notation, $\epsilon^{012}=1$.

The radial component of the Killing spinor equation,

$$D_r\varepsilon^{+}=\left(\partial_r-\frac{1}{2a_0A}\,\Gamma_1\right)\varepsilon^{+}=0\ , \tag{F.6}$$

has the general solution,

$$\varepsilon^{+}=e^{f(r)\Gamma_1}\,\xi^{+}(t,\phi)\ , \tag{F.7}$$

where ξ^{+} is a spinor and,

$$f(r)=\frac{1}{2}\int_0^{r/a_0}\frac{dr'}{\sqrt{r'^2+a_0^2\ell^2}}=\frac{1}{2}\,\sinh^{-1}\!\left(\frac{r}{a_0\ell}\right)\ . \tag{F.8}$$

The two remaining components of the Killing equation (F.5) are,

$$\left[\partial_\phi-\frac{a_0}{2\ell}\,e^{-2f\Gamma_1}(A+B\,\Gamma_1)\,\Gamma_0\right]\xi^{+}=0\ , \tag{F.9}$$

$$\left[\partial_t-\frac{a_0}{2\ell^2}\,e^{-2f\Gamma_1}(A+B\,\Gamma_1)\,\Gamma_0\right]\xi^{+}=0\ . \tag{F.10}$$

It turns out that $f(r)$ satisfies the identities,

$$\ell e^{\pm 2f\Gamma_1}=A\pm B\,\Gamma_1\ , \tag{F.11}$$

and therefore,

$$e^{\mp 2f\Gamma_1}\,(A\pm B\,\Gamma_1)=\ell\ . \tag{F.12}$$

Thus, the general solution of (F.9)-(F.10) reads $\xi^+ = e^{\frac{1}{2}\,a_0\,\Gamma_0\left(\phi+\frac{t}{\ell}\right)}\,\eta^+$, and,

$$\varepsilon^+ = e^{f(r)\Gamma_1}\,e^{\frac{1}{2}\,a_0\,\Gamma_0\left(\phi+\frac{t}{\ell}\right)}\,\eta^+ \ . \tag{F.13}$$

Here, η^+ is a constant spinor that can always be chosen as an eigenvector of the matrix Γ_0, for instance,

$$\Gamma_0\,\eta^+ = i\eta^+ \ . \tag{F.14}$$

The Killing spinor ε^+ has to be globally single-valued, that is, it must be either periodic or antiperiodic under rotations by 2π: $\varepsilon^+(\phi+2\pi) = \pm\varepsilon^+(\phi)$. This is satisfied by (F.13) provided the topological defect is quantized,

$$a_0 = n \in \mathbb{Z} \ . \tag{F.15}$$

Because $\alpha_0 \in (0,1]$, one must have $n = 1$, that corresponds to global AdS$_3$ (remember that a_0 is defined modulo the sum of integers, so $n = 1$ is equivalent to $a_0 = 0$). We conclude that purely gravitational static 0-branes in three-dimensional $\mathcal{N} = 1$ CS supergravity with all additional matter fields switched off do not admit Killing spinors (they are not BPS states). This means that there are no globally defined Killing spinors except in the known cases ($M = 0, -1$), as reported in [129].

$\mathcal{N} = 2$ supersymmetries

$\mathcal{N} = 2$ supersymmetries occur for $(p,q) = (1,1)$, $(2,0)$, and its symmetric reflection, $(0,2)$. The case $(p,q) = (1,1)$ admits the Killing spinor $\varepsilon = \varepsilon^+\,Q^+ + \varepsilon^-\,Q^-$, where ε^+ is,

$$\varepsilon^+ = e^{f(r)\Gamma_1}\,e^{\frac{1}{2}\,ia_0\left(\phi+\frac{t}{\ell}\right)}\,\eta^+ \ , \tag{F.16}$$

and similarly, for the $(0,1)$ spinor $\varepsilon = \varepsilon^-\,Q^-$, one obtains

$$\varepsilon^- = e^{-f(r)\Gamma_1}e^{-\frac{1}{2}\,ia_0\left(\phi+\frac{t}{\ell}\right)}\eta^- \ , \tag{F.17}$$

as is seen from the case $\mathcal{N} = 1$ previously discussed. Again, this implies $a_0 = 1$.

In the case $(p,q) = (2,0)$, the algebra contains a generator of $o(2)$ that (modulo reflections) acts as $u(1)$. The corresponding Abelian field, b, introduces an additional charge in one of the two copies (say, ε^+). In this representation, $(b_+)^I{}_J := -b\,\sigma^I{}_J$, where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{F.18}$$

The CS field equations around the source are $F^{\pm} = 0$, where the curvatures read

$$\left(F^+\right)_I{}^J \;=\; \delta_I{}^J\left(R^a + \frac{1}{\ell}\,T^a + \frac{1}{2\ell^2}\,\varepsilon^a{}_{bc}e^b\wedge e^c\right)J_a^+ - \frac{1}{2}\,db\,\sigma_I{}^J \ , \tag{F.19}$$

$$\left(F^-\right)_I{}^J \;=\; \delta_I{}^J\left(R^a - \frac{1}{\ell}\,T^a + \frac{1}{2\ell^2}\,\varepsilon^a{}_{bc}e^b\wedge e^c\right)J_a^- \ . \tag{F.20}$$

Therefore, the geometry is locally AdS and torsion-free as in the previous case, and $db = 0$. The last condition enables us to write the 1-form b locally as $b = d\Omega$. Globally, this is much more interesting than being a trivial connection, since Ω could be multivalued (like the angle ϕ_{12} itself), allowing for different topological sectors for b, labelled by the winding number. This provides the basics to find a non-trivial Killing spinor charged with respect to b, producing a Aharonov-Bohm phase that cancels the contribution of the spin connection [130, 131]. Thus, a Killing spinor $\varepsilon = \varepsilon_I^+ Q^{+I}$ satisfies

$$d\varepsilon_I^+ - \frac{1}{2}\left(\omega^a + \frac{1}{\ell}\, e^a\right)\Gamma_a\, \varepsilon_I^+ - d\Omega\ {\sigma_I}^J\, \varepsilon_J^+ = 0\ . \tag{F.21}$$

Choosing $\Omega = q\,\phi_{12}$, only one component of the Killing equation receives a correction when compared with its form for $b = 0$,

$$\left(\partial_{\phi_{12}} - \frac{1}{2}\,\Gamma_0 - q\,\sigma\right)\varepsilon^+ = 0\ . \tag{F.22}$$

The solution is,

$$\varepsilon_I^+ = e^{f(r)}\ e^{\frac{i}{2\ell}\,a_0 t + \frac{i}{2}\,a_0(1+2q)\phi}\ \eta_I^+\ , \tag{F.23}$$

where we have used $\phi_{12} = a_0\phi$ and that η_I^+ is a constant simultaneous eigenspintor of σ and Γ_0,

$${\sigma_I}^J\,\eta_J^+ = i\eta_I^+\ , \qquad (\Gamma_0)^\alpha{}_\beta\,(\eta_I^+)^\beta = i(\eta_I^+)^\alpha\ . \tag{F.24}$$

The (anti-)periodic boundary condition $\epsilon^+(\phi + 2\pi) = \pm\epsilon^+(\phi)$ requires the $U(1)$ charge to be quantized,

$$a_0\,(1 + 2q) \in \mathbb{Z}\ . \tag{F.25}$$

Note that this extremality condition perfectly matches that obtained by Izquierdo and Townsend (after replacing $a_0 \to \beta$ and $a_0\, q \to Q$ in their eq.(3.5)) [78]. Therefore, for a given topological defect $a_0 \in (0, 1]$, all charges given by $q = \frac{k}{2a_0} - \frac{1}{2}$, $k \in \mathbb{Z}$ satisfy the BPS condition. Conversely, if the $U(1)$ charge is fixed, there are several possible values for angular defect given by

$$0 < a_0(q, k) = \frac{k}{2q + 1} < 1\ , \qquad k \in \mathbb{Z}\ . \tag{F.26}$$

Note that for a given value of q, the number of allowed values for a_0 increase with $|q|$.

We conclude that non-trivial Killing spinors exist for these choices of q and a_0, and the corresponding 0-branes should be stable BPS configurations. Each matrix condition in (F.24) projects out 1/2 of the spinor components, so the final solution preserves 1/4 of the original supersymmetries; a 1/4-BPS state. There is a single unbroken supercharge in the solution. Obviously, the same is true for $(p, q) = (0, 2)$, just replacing $+$ by $-$ in the preceding discussion.

The current that describes this 0-brane couples to the geometry and to the $U(1)$ field. The gravitational part of the current is given by Eq.(4.1). Additionally, the $U(1)$ charge of the brane couples to b. The form of this contribution can be found from the Abelian gauge field, $b = q\,d\phi_{12}$ (that carries an electromagnetic flux given by the integral of $q\,dd\phi_{12} = 2\pi a_0\, q\,\delta(\Sigma_{12})$), so the total current is,

$$j_{[0]} = -2\pi a_0\left(J_{12} - qT_+^{12}\right)\,\delta(\Sigma_{12})\ . \tag{F.27}$$

The presence of the $o(2)$ R-symmetry field b is responsible for stabilizing the 0-brane: the conical defect in the spatial section is compensated by the $U(1)$ charge in the internal gauge space [130, 131].

$\mathcal{N} = p + q$ supersymmetries

For the $osp(p|2) \times osp(q|2)$ superalgebra, the brane solution is again locally flat, $F = 0$, namely, locally AdS geometry, torsion-free, and has a flat R-connection $db^I_{(\pm)J} + b^I_{(\pm)K} \wedge b^K_{(\pm)J} = 0$,

$$A_{\mathrm{AdS}} = 0\text{-brane} , \qquad b^{IJ}_{\pm} : \text{locally flat} , \tag{F.28}$$

where the 0-brane is given by Eqs. (4.3, 4.4). The connection b has the general form $b = g^{-1}dg$ (where g belongs to $O(p) \times O(q)$), but here we consider a particular Abelian choice of this form in the Cartan subalgebra of $o(p) \times o(q)$, such that $db^I_{(\pm)J} = 0$, and $b^I_{(\pm)K} \wedge b^K_{(\pm)J} = 0$. The Cartan subalgebra is spanned by

$$\left\{ T^+_{12}, T^+_{34}, \ldots, T^+_{2[\frac{p}{2}]-1,2[\frac{p}{2}]}; T^-_{12}, T^-_{34}, \ldots, T^-_{2[\frac{q}{2}]-1,2[\frac{q}{2}]} \right\} .$$

This means that we can take the matter connection as $b = -T\, d\phi_{12}$, with T a linear combination of some Cartan generators, say $k_+ + k_-$ of them, with $k_+ \leq [p/2]$ and $k_- \leq [q/2]$, and the coefficients represent the corresponding charges $q^{\pm}_k$. Explicitly,

$$T = T_+ + T_- , \tag{F.29}$$

where

$$T_{\pm} = \sum_{k=1}^{k_{\pm}} q^{\pm}_k\, T^{\pm}_{2k-1,2k} . \tag{F.30}$$

Thus, the connection and the source for this configuration read

$$A = A_{\mathrm{AdS}} + T\ d\phi_{12} , \tag{F.31}$$

$$j_{[0]} = -2\pi a_0 \left(J_{12} - T \right) \delta(\Sigma_{12}) . \tag{F.32}$$

For the Killing spinors, we already know that when $k_+ = k_- = 0$, there is no solution apart from the global AdS space, whereas for k_+ or $k_- \neq 0$ (say, $k_+ = 1$), the system resembles the $\mathcal{N} = 2$ case, and a Killing spinor of the type (F.23) exists for $a_0 \left(1 + 2q^+_1\right) \in \mathbb{Z}$.

The ϕ_{12}-component of the Killing spinor $\varepsilon^{\pm}$ equation (F.4) reads

$$\left(\partial_{\phi_{12}} - \frac{1}{2}\,\Gamma_0 - \sum_{k=1}^{k_{\pm}} q^{\pm}_k\, \tau^{\pm}_{2k-1,2k} \right) \varepsilon^{\pm} = 0 , \tag{F.33}$$

and has the general lowest supersymmetry preserving solution

$$\varepsilon^{\pm} = \exp \left\{ \pm f(r) \pm \frac{i}{2\ell}\, a_0 t \pm \frac{i}{2}\, a_0 \left(1 + \sum_{k=1}^{k_{\pm}} q^{\pm}_k \right) \phi \right\} \eta^{\pm} . \tag{F.34}$$

The constant spinors $\eta^{\pm}$ are chosen such that

$$\left(\Gamma_0\right)^{\alpha}{}_{\beta}\, \eta^{\pm\beta}_I = i\eta^{\pm\alpha}_I , \tag{F.35}$$

$$\left(\tau^{\pm}_{2k-1,2k} \right)^J{}_I\, \eta^{\pm}_J = i\eta^{\pm}_I , \quad k = 1, \ldots, k_{\pm} . \tag{F.36}$$

This gives raise to $p-k_+$ $(q-k_-)$ independent components. The boundary condition $\varepsilon(\phi+2\pi) = \pm\varepsilon(\phi)$ leads to the condition on the charges

$$a_0\left(1+q^+_{12}+\cdots+q^+_{k_+-1,k_+}\right) \quad \in \quad \mathbb{Z}\ , \tag{F.37}$$

$$a_0\left(1+q^-_{12}+\cdots+q^-_{k_--1,k_-}\right) \quad \in \quad \mathbb{Z}\ . \tag{F.38}$$

Notice that each projection in (F.36) effectively acts on a two-dimensional subspace because it corresponds to an Abelian rotation inside the Cartan subgroup of $osp(p|2)\times osp(q|2)$. Thus, at the beginning, there were $\mathcal{N}=p+q$ (real two-component) spinors, and $k_+ + k_-$ Abelian projections leave $\mathcal{N}-(k_-+k_+)$ vectorial components unchanged. Furthermore, the spinorial projection (F.35) breaks a half of supersymmetries, that finally gives $\left[\mathcal{N}-(k_-+k_+)\right]/2$ supercharges.

So far, we have shown that the three-dimensional spacetime containing a 0-brane admits a globally defined Killing spinor, by explicitly constructing it. This should be sufficient to guarantee this geometry to be a stable vacuum for supersymmetric theories with different values of $\mathcal{N}$. The supersymmetry algebra establishes a lower bound for the energy, which is saturated by the vacuum configuration. Thus, it is possible to assert the stability of the purely bosonic configuration by just checking that a Killing spinor can exist in that background.

The only missing link in this proof of stability is that we have not shown the charges that satisfy the supersymmetry algebra to be defined for this configuration. In fact, the charges that generate the symmetry (super) group should be finite and satisfy the right Poisson algebra in the phase space of the theory.

In $2+1$ dimensions, it is rather straightforward to check that the canonical charges satisfy the algebra of the supersymmetric extension of the AdS group, that is the super Virasoro algebra [129, 132, 133]. Since there is not much difference in the construction for naked singularities and for the standard black holes, we will not devote more lines to this discussion here. However, in the five-dimensional case, where BPS co-dimension two-branes where constructed in [13], the construction of the charges and the establishment of the energy lower bound need to be explicitly carried out. This was done in [13].

Appendix G

A new massive gravity

In [31], Bergshoeff, Hohm and Townsend proposed a new theory of gravity in three dimensions which has certain similarities with TMG. In this thesis focus was not put on the theory of [31], so here we shall review its most important features for completeness.

This new massive gravity (NMG) is defined by the action,

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[R - 2\lambda - \frac{1}{m^2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right) \right]. \tag{G.1}$$

The associated field equations read,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \tag{G.2}$$

where,

$$\begin{aligned} K_{\mu\nu} \quad = \quad & 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - \frac{1}{2} \Box R g_{\mu\nu} + 4 R_{\mu\alpha\nu\beta} R^{\alpha\beta} \\ & - \frac{3}{2} R R_{\mu\nu} - R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}. \end{aligned} \tag{G.3}$$

This tensor $K_{\mu\nu}$ satisfies the property that,

$$g^{\mu\nu} K_{\mu\nu} = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2, \tag{G.4}$$

saying that its trace is equal to the Lagrangian where it comes from, after variation with respect to the metric field.

NMG is a fourth-order theory of gravity in three dimensions, with the important feature that it is unitary at the linear level, since its linearized equations of motion reduce to those of Fierz-Pauli for a spin-two massive particle. As TMG, it admits any locally AdS_3 solution provided its radius l satisfies,

$$l^2 = -\frac{1}{2\lambda} \left(1 \pm \sqrt{1 - \lambda m^{-2}} \right). \tag{G.5}$$

The central charge computed from the canonical realization of asymptotic symmetries for Brown-Henneaux boundary conditions is given by,

$$c = \frac{3l}{2G} \left(1 - \frac{1}{2m^2 l^2} \right), \tag{G.6}$$

which certainly reduces to the Brown-Henneaux central charge $c = 3l/2G$ in the limit $m \to \infty$. Since there is no parity-violating term in the action, the left and right central charges coincide.

Similar to the case of TMG, the space of solutions of NMG is much reacher than that of pure gravity, admitting solutions such as BTZ black holes, warped AdS geometries [32] and AdS-waves [33, 34]. Also, there are black hole geometries [35, 36] that satisfy the equations of motion but are not asymptotically AdS$_3$ in any sense. Their asymptotic behavior is that of a 'Lifshitz metric' [37],

$$ds^2 = -\frac{r^{2z}}{l^{2z}}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}dx^2. \tag{G.7}$$

These kind of geometries can be found in different theories of arbitrary dimension and are believed to be good candidates to holographically describe the dynamics of Lifshitz fixed points of condensed matter theories in one less dimension with dynamical exponent z .

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