Funciones de correlación en la correspondencia Ads/CFT

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Resúmen

La dualidad entre teorías de gravedad y teorías de campos de gauge es una de las ideas que más ha influido en el desarrollo de la física teórica de altas energías en la última decada. Hoy en día, esta se ha convertido en una poderosa herramienta en el estudio de sistemas fuertemente interactuantes. Sin embargo, la correspondencia AdS/CFT no ha sido probada completamente y aún se encuentra al nivel de conjetura. En este trabajo pretendemos chequear la correspondencia en algunos casos simples. Concretamente, calculamos funciones de correlación de tres y cuatro puntos en teoría de supercuerdas sobre $AdS_3 \times S^3 \times T^4$ a nivel árbol para operadores primarios quirales y mostramos que estas coinciden perfectamente con los correladores esperados en la teoría de campos dual. Además sugerimos una interesante conexión entre funciones de correlación de n puntos de operadores con número de torcimiento (twist) igual a dos y espín muy grande y valores de expectacion de loops de Wilson poligonales con lados nulos en teoría de Super-Yang-Mills con cuatro super-simetrías. Justificamos esta conexión a través de la comparación de las divergencias ultravioletas de ambos objetos en acoplamiento fuerte y presentamos algunas consideraciones perturbativas a este respecto en teoría de campos.

Palabras Clave: Teoría de Cuerdas, AdS/CFT, Teorías de Campos Supersimétricas.

Correlation functions in AdS/CFT correspondence

Abstract

The duality between theories of gravity and gauge field theories has been one of the most influential ideas in theoretical high energy physics within the last decade. Nowadays, it has become a truly powerful tool to study strongly interacting systems. However, the AdS/CFT correspondence still has the status of a conjecture. In this work we would like to contribute to test the correspondence in some simple cases. Concretely, we compute correlation functions of three- and four-points in string theory on $AdS_3 \times S^3 \times T^4$ at tree-level for chiral primary operators and we find perfect agreement with the corresponding correlators in dual the field theory in all the cases considered. We also suggest a possible connection between $n-$point functions of twist-two large spin operators in $\mathcal{N} = 4$ super Yang-Mills and null-polygonal Wilson loops. We support that suggestion by a semi-classical computation in the string theory side and by some perturbative considerations in the dual field theory.

Keywords: String Theory, AdS/CFT, Supersymmetric Quantum Field Theory

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Contents

Introduction

Quantum field theory and general theory of relativity are the fundamental pillars of modern physics.

Nowadays the microscopic understanding of nature is described with a high degree of precision by quantum field theory. The fundamental objects are particles which correspond to excitations of the fields, and with the exception of gravity, all the content of the known particles in nature (almost all observed so far) is encoded in the Standard Model. Even though this theory explains very successfully most of the phenomena we can measure at the energy scales that we can access with the current technology, there exists strong theoretical evidence suggesting that new physics will show up at higher energies. This can be exemplified by looking closer at the theory of General Relativity at microscopic scales. It is well known that General Relativity can not be quantized by the usual methods of quantum field theory. Specifically, if one performs quantum perturbation theory around Minkowski space, the usual UV infinities appear. In order to subtract those infinities one has to introduce an infinite number of counter-terms to the Einstein-Hilbert action which render the theory non-renormalizable.

Another related puzzle concerning the high energy behaviour of General Relativity comes from the existence of singular classical solutions such as black-holes, where the geometrical description of space-time provided by the General Theory of Relativity stops working beyond a given length scale. A quantum theory of gravity must contain a complete understanding of these singularities. Because of this and other several reasons, it is widely believed that at very high-energies Nature will be described by a more general theory from which we should obtain all the known quantum field theories as well as the theory of General Relativity as low energy limits.

Indeed, string theory is a potential candidate for such a theory, in which all the content of fields arises as oscillations of a relativistic string. From the effective low energy description of string theory (i.e. at length scales much larger than the string length) the Einsten-Hilbert action emerges and, as we will discuss later, some gauge field theories sharing similarities with the Standard Model can be also obtained as low energy limits of systems of strings. But currently a deeper understanding of the theory is needed in order to know how exactly the Standard Model should arise.

There are many properties of string theory which lead us to believe it is the right theory describing all the known interactions in all the energy spectrum. The main framework of this thesis, namely the AdS/CFT correspondence, is among the most interesting reasons, but at this point it is worth mentioning a couple more. As a quantum theory of gravity, there are good reasons to believe that string theory solves the ultraviolet problems of General Relativity, mainly because the extended nature of string interactions provide an inherent ultraviolet cut-off given by the fundamental string length [1, 2]. Furthermore, string theory has been able to 'resolve' a class of singularities of black-hole type even at the classical level [3].

String theory has a richer structure than field theories. It contains several extended objects, other than strings, which arise naturally without adding any extra content to the theory. This multi-dimensional extended objects, called $D-$branes, arise by studying the dynamics of open strings on compactified backgrounds. They can also be obtained as classical solutions of the effective description of low energy gravity . From the open string point of view, the effective low energy description of systems of branes is generically given by gauge theories, some of them similar to those that standard quantum field theory is used to dealing with. On the other hand, solutions of classical gravity associated with $D-$brane systems, are generically described by black hole-like geometries with fluxes. The study of the relation between these two alternative points of view, unravelled the AdS/CFT duality [4]. We shall elaborate on this point in chapter 1.

The AdS/CFT correspondence is a concrete realization of an old idea due to G. 't Hooft and L. Susskind [5, 6], which suggests that in a quantum theory of gravity, the physics in some region can be described by a theory on the boundary of the given region. From the AdS/CFT perspective, string theory (which as we said above is an actual quantum theory of gravity) defined on an AdS space is dual to a conformal field theory living on the flat boundary of the AdS space.

The most interesting property of the duality is that it relates both theories in opposite coupling regimes, i.e, by performing perturbative computations on the string theory side we are exploring the strong coupling regime of the corresponding dual field theory and vice-versa. Although it is a very exciting property from the practical point of view, it makes the correspondence hard to prove.

In this work we are going to focus on the two most popular examples of the correspondence. We will be mainly interested in the duality between superstring theory on $AdS_3 \times S^3 \times T^4$ and the two-dimensional conformal field theory whose target space is given by the symmetric product of N four-torus, namely $(T^4)^N/S(N)$, $S(N)$ being the permutation group of $N-$indices (AdS_3/CFT_2). We will also work out an explicit computation in the context of the duality between superstring theory on $AdS_5 \times S^5$ and the four-dimensional $\mathcal{N} = 4$ superconformal-Yang-Mills theory (AdS_5/CFT_4). All the theories involved in these examples are $\mathcal{N} = 4$ supersymmetric invariant, and their shortest multiplets are built from chiral primary operators, whose $R-$charges equal their conformal weights Δ and do not receive quantum corrections, i.e, they can be evaluated exactly in a tree-level computation. This last property of the chiral spectrum provides a nice window to test the correspondence, since the perturbative computation of some observables built out of chiral primaries should agree with their corresponding strong coupling values.

Because of this, correlation functions involving the chiral spectrum become useful in testing the correspondence. For the AdS_3/CFT_2 case, the structure constants of single-cycle operators in the chiral spectrum of the symmetric product were computed originally in [7] and, for a subset of these operators, they were extended in [8, 9] to the full 1/2 BPS $SU(2)$ multiplet. These three-point functions were exactly reproduced in the supergravity approximation of string theory [10, 11, 12, 13] and also in the full string theory [14, 15, 17, 18]. This agreement between bulk and boundary correlators was at first surprising because the computations on the string and CFT sides are performed at very different points in the moduli space [19, 20]. It can be explained by a non-renormalization theorem proved in [21] and recently revisited in [22]. Some four-point functions have been also considered in the conformal field theory on the symmetric product in [23] and some of them were reproduced in the string theory dual in an operator product expansion limit [81].

For the case AdS_5/CFT_4 the analysis of three-point functions for chiral primaries in the supergravity approximation has been made in [25, 26] and the results match the expected ones from the field theory. On the other hand, it was argued that in the limit where the number of colors in $\mathcal{N} = 4$ SYM goes to infinity,[1] only planar diagrams contribute to the perturbative expansion, but even more, the theory is integrable [27, 28, 29]. Exploiting integrability, it was possible to compute the spectrum of anomalous dimensions for some single trace operators for all values of the 't Hooft coupling [30] (see also [28, 29, 31]). In this case, following the discovery of the pp wave correspondence [32], single trace operators in $\mathcal{N} = 4$ SYM at large N were mapped to "spin chains" [33], with the spin chain formed by the different fields inside the trace. Since the spin chains were known to be related to integrability via the Bethe ansatz, this in turn led to integrability of $\mathcal{N} = 4$ SYM as above. Unlike the case of strings on $AdS_3 \times S^3 \times T^4$, the quantum formulation of string theory on $AdS_5 \times S^5$ is not yet fully defined. However, there are some regimes where we still could see some quantum effects. Among those regimes we can mention the pp-wave limit [32] where a sort of quantization could be done which is similar to the flat space case, and the regime where the quantum numbers of the operators, such as their energy or angular momentum, are very large. In the last case, most developments have been made for single trace operators in $\mathcal{N} = 4$ SYM theory built out of large numbers of derivatives of a couple of scalar fields. In [34] it has been suggested that the above states should correspond to excited string states in $AdS_5 \times S^5$ with large angular momentum in the AdS_5 factor. In that case we can perform semi-classical computations in string theory.

This work is organized as follows. In chapter 1 we present a brief review of the formulation of the AdS/CFT correspondence as close as possible to its original formulation [4]. We will show the arguments relating string theory on AdS space to both, four dimensional $\mathcal{N} = 4$ super-Yang-Mills and two-dimensional conformal field theory on $(T^4)^N/S(N)$. In chapter 2 we review the construction of string theory on AdS_3 with a NS antisymmetric field turned on, and we will see that it is related to a Wess-Zumino-Novikov-Witten model. After studying some simple classical solutions we will proceed to the construction of the whole quantum spectrum and the associated vertex fields. In chapter 3 we generalize the construction of the quantum spectrum made in chapter 2 to the whole $AdS_3 \times S^3 \times T^4$ space-time and from there we build the spectrum of chiral vertex

[1] $SU(N)$ is the gauge group of $\mathcal{N} = 4$ SYM

operators in generic representations. In chapter 4 we make a quick review of the sigma model on the symmetric product orbifold $(T^4)^N/S(N)$, mainly focusing on the chiral spectrum. We also list a set of correlation functions we are going to use in chapters 5 and 6. Chapter 5 is devoted to the computation of three-point functions in superstring theory on $AdS_3 \times S^3 \times T^4$ of chiral primary operators and we compare them with the expected correlators in the dual field theory. In chapter 6 we perform a detailed computation of the leading contribution on the sphere to some extremal and non-extremal unflowed four-point functions of chiral primary operators in an operator product expansion (OPE). We show that in the short-distance OPE limit, these correlators agree with the single-particle contribution to the boundary correlators. We also show how in the OPE limit these four-point functions factorize as expected from a conformal field theory in space-time. In chapter 7 we go back to the classical string theory on AdS_3 but now as a sub-factor of a bigger AdS_5 space. We summarize the Pohlmeyer reduction of strings on AdS_3 and then use it in order to study worldsheet solutions which contain asymptotically rotating folded strings (GKP) at large spin, and from these we compute semi-classically the leading divergent factor of an $n-$point correlation function of large spin operators. At the end of the chapter, we give some hints from the field theory perspective supporting the relation between correlation functions of large spin operators and expectation values of null polygonal Wilson loops. Finally chapter 8 contains the conclusions and some suggestions for further work.

Chapter 1

Formulation of the AdS/CFT correspondence

In this chapter, we will present a very brief review of the AdS/CFT correspondence as originally formulated by Juan Maldacena [4] just to give the context which motivates this work. In the first section we will start reviewing the arguments relating type IIB superstrings on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) in four dimensions. In the second section we will show how these arguments extend to type IIB superstring theory on $AdS_3 \times S^3 \times T^4$ which turns out to be dual to the two-dimensional conformal field theory whose target space corresponds to the symmetric product of $N-$copies of the four torus T^4, namely, $(T^4)^N/S(N)$. The content of this chapter is mainly based on [35].

1.1 The AdS$_5$/CFT$_4$ correspondence

Consider a system of N parallel $D3$-branes that are sitting together and let us study the low energy limit of this system from two different points of view. At the string level, this system contains both closed and open strings. The closed strings are the excitations of empty space (outside of the brane system) and the open strings end on the $D-$branes and describe their excitations. Taking the low energy limit, i.e, considering the system at energies lower than the string scale $\alpha' \to 0$, only the massless string states can be excited. Considering that the closed string sector is given by type IIB string theory, the massless states are described by type IIB supergravity. The open string massless states give a low-energy effective Lagrangian which corresponds to $\mathcal{N} = 4$ SYM. The complete effective action is given by

$$S = S_{sugra} + S_{brane} + S_{int} \,. \tag{1.1}$$

Both S_{sugra} and S_{brane} contain higher derivative corrections, which are subleading at low energies. In the limit $\alpha' \to 0$, S_{sugra} reduces to the action of a free graviton plus other free fields such as the dilaton and the anti-symmetric field. A quick way to see this is by expanding the gravity piece of S_{sugra} around a flat metric η, i.e. taking $g = \eta + \kappa h$, κ being the gravitational coupling, which gives schematically,

$$S_{sugra} \sim \frac{1}{2\kappa^2} \int \sqrt{g}\mathcal{R} \sim \int (\partial h)^2 + \kappa(\partial h)^2 h + \cdots \quad . \tag{1.2}$$

On other hand, S_{int} depends explicitly on the coupling $\kappa \sim g_s \alpha'^2$, g_s being the string coupling which we keep fixed. So, equally as the interaction term in S_{sugra}, S_{int} is proportional to positive powers of κ and in the low energy limit $\alpha' \to 0$, all interaction terms drop out. In this low energy limit we end up with two decoupled systems. On the one hand we have free gravity in the bulk and, on the other hand, we have the four dimensional gauge theory.

Now, let us study the low energy regime of the system from the pure gravitational perspective.

$D-$branes are massive charged objects which act as sources for the various supergravity fields. We can find solutions to type IIB supergravity containing branes from the action,

$$S^{IIB}_{sugra} = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{-g} \left[e^{-2\phi}(R + 4(\nabla\phi)^4) - \frac{2}{(8-p)!} F^2_{p+2} \right] , \tag{1.3}$$

where F_{p+2} is the stress tensor associated to the p+1-form potential $F_{p+2} = dA_{p+1}$, which couples to the brane. We are particularly interested in the $D3-$brane solutions, which have the following form [36, 37],

$$\begin{aligned} ds^2 &= f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \\ F_5 &= (1 + *) dt dx_1 dx_2 dx_3 df^{-1} \\ f &= 1 + \frac{R^4}{r^4} \,, \end{aligned} \tag{1.4}$$

where $R^4 \equiv 4\pi g_s \alpha'^2 N$ and $*$ denotes the 10d Hodge$-*$ dual. Since g_{tt} is non-constant, the energy E_p of an object as measured by an observer at a constant position r and the energy E_∞ measured by an observer at infinity are related by the redshift factor,

$$E_\infty = f^{-\frac{1}{4}} E_p \,. \tag{1.5}$$

This implies that an object close to $r = 0$ would appear to have lower energy for an observer at infinity. From the point of view of an observer at infinity, there are two kinds of excitations. On one hand we can have massless particles propagating in the bulk with very large wavelengths, and therefore they are insensitive to the curvature (i.e. to the presence of the branes). They behave as free gravitons (or other fields) propagating in the bulk. On the other hand we can have excitations getting closer and closer to $r = 0$ which find it harder and harder to escape to the asymptotic region. In other words, the two kinds of excitations decouple in the low energy limit [38, 34], and we end up with free supergravity in the bulk and physical excitations in the near horizon geometry ($r << R$) of (1.4). In the near horizon region, we can approximate,

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2(\frac{dr^2}{r^2} + d\Omega_5^2)\,, \tag{1.6}$$

which is the geometry of $AdS_5 \times S^5$.

Comparing the above two approaches to the low-energy limit of a system of N $D3-$branes we see that they both have two decoupled regions in that limit. In both cases one of the regions is described by free supergravity. So, it is natural to identify the second system which appears in both descriptions. Thus, we are led to the conjecture that $\mathcal{N} = 4$ SYM in four dimensions is dual to type IIB superstring theory on $AdS_5 \times S^5$ [4].

In the gauge theory, the perturbative analysis is reliable when $g_{YM}^2 N << 1$, g_{YM} being the coupling constant of the Yang-Mills theory. From the physics of $D-$branes we know that g_{YM} is related to the string coupling through [39],

$$g_s = \frac{g_{YM}^2}{4\pi}\,. \tag{1.7}$$

By using this relation and

$$R^4 \equiv 4\pi g_s \alpha'^2 N \tag{1.8}$$

we have,

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{\alpha'^2} << 1\,. \tag{1.9}$$

Therefore, when the Yang-Mills theory is at weak coupling, the dual string theory is at large α' and the supergravity approximation is not reliable. Conversely, the classical gravity description becomes reliable when the radius of curvature of AdS and of S^5 become large compared to the string length,

$$\frac{R^4}{\alpha'^2} \sim g_{YM}^2 N >> 1\,. \tag{1.10}$$

The two theories are conjectured to be the same, but when one side is weakly coupled the other is strongly coupled and viceversa. This makes the correspondence both hard to prove and useful, as we can solve a strongly coupled gauge theory via classical supergravity.

1.2 The AdS$_3$/CFT$_2$ correspondence

All the arguments leading to the formulation of the AdS_5/CFT_4 correspondence in previous section, analogously apply to the case of the AdS_3/CFT_2 duality. The only novelty is that now we have to consider a system of Q_1 parallel $D1$-branes which at the same time are parallel to one of the directions of Q_5 parallel $D5$-branes. Additionally, we wrap (compactify) the four directions (x^6, x^7, x^8, x^9) which are non-parallel to the $D1$ on a T^4 torus. This is known as the $D1-D5$ system. Let us start considering the brane system from the purely gravitational point of view. The type IIB effective string action is again given by

$$\frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}(R+4(\nabla\phi)^4)-\frac{1}{12}F^2\right], \tag{1.11}$$

where $F = dC^{(2)}$ is a three-form, and $C^{(2)}$ is the 2-form RR gauge potential. The solution characterizing the $D1-D5$ system is given by

$$\begin{aligned}
ds^2_{10} &= f_1^{-1/2}f_5^{-1/2}(-dt^2+dx_5^2)+f_1^{1/2}f_5^{1/2}dx_idx^i+f_1^{1/2}f_5^{-1/2}dx_adx^a\,,\\
C^2_{05} &= -\frac{1}{2}(f_1^{-1}-1)\,,\\
F_{ijk} &= \epsilon_{ijkl}\partial_l f_5\,,\\
e^{-2\phi} &= f_5 f_1^{-1}\,,\\
f_{1,5} &= (1+r^2_{1,5}/r^2)\,,\\
r_1^2 &= \frac{g_s\alpha'}{\tilde{v}}Q_1\,,\\
r_5^2 &= g_s\alpha' Q_5\,,
\end{aligned} \tag{1.12}$$

where $\tilde{v}$ is associated to the volume of the T^4 by $V_{T^4}\equiv\alpha'^2(2\pi)^4\tilde{v}$ and $r^2 = x_ix^i$ denotes the distance measured in the transverse direction to all $D-$branes. As in the $D3-$brane system, the energy of an object measured by anyone at infinity is redshifted as

$$E_\infty = (f_1f_5)^{-1/4}E_r\,. \tag{1.13}$$

Once again, from the point of view of an observer at infinity, excitations sitting near the horizon geometry and bulk excitations decouple in the low energy limit and we end up with free supergravity and the physical excitations in the near horizon geometry. More precisely, the supergravity approximation is reliable in the limit $\alpha' \to 0$, therefore a more appropriate scaling is,

$$\begin{aligned}
\alpha' \to 0, \quad \frac{r}{\alpha'} &\equiv U = \text{fixed}\,, \\
\tilde{v} &= \frac{V_4}{\alpha'^2 (2\pi)^4} = \text{fixed}, \quad g_6 \equiv \frac{g_s}{\sqrt{\tilde{v}}} = \text{fixed}\,.
\end{aligned} \tag{1.14}$$

In this limit, the redshift is given by,

$$E_\infty = \frac{r}{R} E_r \tag{1.15}$$

with $R = \sqrt{\alpha'}(g_6^2 Q_1 Q_5)^{1/4}$ and the metric (1.12) becomes,

$$ds^2 = \alpha' \Big(ds_3^2 + ds^2[S^3] + ds^2[T^4] \Big)\,, \tag{1.16}$$

where

$$ds_3^2 = \Big[\frac{U^2}{l^2}(-dx_0^2 + dx_5^2) + l^2 \frac{dU^2}{U^2} \Big], \tag{1.17}$$

represents three-dimensional anti-de Sitter space AdS_3 and

$$\begin{aligned}
ds^2[S^3] &= l^2 d\Omega_3^2, \\
ds^2[T^4] &= \sqrt{\frac{Q_1}{v Q_5}} (dx_6^2 + \ldots dx_9^2),
\end{aligned} \tag{1.18}$$

represent a three-sphere and a four-torus. Thus the near horizon geometry is that of $AdS_3 \times S^3 \times T^4$. Now from the point of view of the open string theory describing the dynamics of the $D1 - D5$ system we will work in the following region of parameter space,

$$V_4 \sim \mathcal{O}(\alpha'^2), \quad R >> \sqrt{\alpha'}\,, \tag{1.19}$$

which by using (1.14) implies that $g_s \sim \frac{R^2}{\alpha'^2 \sqrt{N}}$, being $N = Q_1 Q_5$. In this region, the size of the torus T^4 is of the order of the string scale, the masses of the winding and momentum modes of the strings are of order $1/\sqrt{\alpha'}$. This implies that for energies $E << 1/\sqrt{\alpha'}$ we can neglect these modes. This means that from the point of view of the effective low energy theory on the brane system $(x^0, x^5, x^6, x^7, x^8, x^9)$, the elementary excitations correspond to the dimensional reduction to 1+1 dimensions (the $(t, x^5)-$space) of the spectrum of open strings ending on the branes. There are three classes of such strings. Those with both ends on a $D1-$brane ([1,1]-strings), those with both ends on a $D5-$brane ([5,5]-strings) and those with one end on a $D1-$brane and the other one on a $D5-$brane

([1,5]-strings). The part of the spectrum coming from [1,1] ([5,5]) strings is simply the dimensional reduction to $1+1$ dimensions of the $\mathcal{N}=1, U(Q_1)$ ($\mathcal{N}=1, U(Q_5)$) gauge theory in $9+1$ dimensions [40]. The bosonic fields of this theory can be organized into the vector multiplet and the hypermultiplet of $\mathcal{N}=2$ theory in four-dimensions as

$$\begin{aligned} \text{Vector multiplet:} \quad & A_0^{(1)}, A_5^{(1)}, Y_m^{(1)}, m=1,2,3,4 \qquad (1.20) \\ \text{Hypermultiplet:} \quad & Y_i^{(1)}, i=6,7,8,9 \\ \text{Vector multiplet:} \quad & A_0^{(5)}, A_5^{(5)}, Y_m^{(5)}, m=1,2,3,4 \qquad (1.21) \\ \text{Hypermultiplet:} \quad & Y_i^{(5)}, i=6,7,8,9 \end{aligned}$$

The $A_0^{(1)}, A_5^{(1)}$ ($A_0^{(5)}, A_5^{(5)}$) are the $U(Q_1)$ ($U(Q_5)$) gauge fields in the non-compact directions. The $Y_m^{(1)}$'s, $Y_i^{(1)}$'s, $Y_m^{(5)}$'s, $Y_i^{(5)}$ are gauge fields in the compact directions of the $\mathcal{N}=1$ super Yang-Mills in ten-dimensions. They are hermitian $Q_1 \times Q_1$ ($Q_5 \times Q_5$) matrices transforming as adjoints of $U(Q_1)$ (Q_5). The hypermultiplets of $\mathcal{N}=2$ supersymmetry are doublets of the $SU(2)_R$ symmetry of the theory. The adjoint matrices $Y_i^{(1)}$'s ($Y_i^{(5)}$'s) can be arranged as doublets under $SU(2)_R$ as

$$N^{(1)} = \begin{pmatrix} N_1^{(1)} \\ N_2^{(1)\dagger} \end{pmatrix} = \begin{pmatrix} Y_9^{(1)} + iY_8^{(1)} \\ Y_7^{(1)} - iY_6^{(1)} \end{pmatrix},$$

$$N^{(5)} = \begin{pmatrix} N_1^{(5)} \\ N_2^{(5)\dagger} \end{pmatrix} = \begin{pmatrix} Y_9^{(5)} + iY_8^{(5)} \\ Y_7^{(5)} - iY_6^{(5)} \end{pmatrix}.$$

Similarly, the part of the bosonic spectrum coming from [1,5] is given by

$$\begin{aligned} \text{Vector multiplet:} \quad & A_0^{(1,5)}, A_5^{(1,5)}, Y_m^{(1,5)}, m=1,2,3,4 \qquad (1.22) \\ \text{Hypermultiplet:} \quad & Y_i^{(1,5)}, i=6,7,8,9\,, \end{aligned}$$

but now all these fields transform as bi-fundamentals of $U(Q_1) \times \overline{U(Q_5)}$. Adding the fermionic superpartners we endup with an effective 1+1 dimensional (4,4) supersymmetric gauge theory with gauge group $U(Q_1) \times U(Q_5)$. The matter content of this theory consists of hypermultiplets $Y^{(1)}$'s, $Y^{(5)}$'s transforming in the adjoint representation of $U(Q_1)$, $U(Q_5)$ respectively, and hypermultiplets $Y^{(1,5)}$'s transforming in the bi-fundamental representation of $U(Q_1) \times \overline{U(Q_5)}$. It has been argued that in the Higgs branch this gauge theory flows in the infrared to an $\mathcal{N}=4$ superconformal field theory (SCFT) on the symmetric product of $N=Q_1Q_5$ four torus [41, 42], i.e, the target space of this theory is given by $(T^4)^{Q_1Q_5}/S(Q_1Q_5)$, S being the symmetric group of Q_1Q_5 indices.

Thus, we are led to the conjecture in this case that $\mathcal{N} = 4$ SCFT in two dimensions is dual to type IIB superstring theory on $AdS_3 \times S^3 \times T^4$.

Chapter 2

String theory on AdS$_3$

In order to explore the AdS$_3$/CFT$_2$ correspondence we need to understand string propagation in the bulk geometry. String theory on $AdS_3 \times S^3 \times T^4$ involves Ramond-Ramond fluxes F_{ijk} (see (1.12)). Although some progress has been made for strings propagating on Ramond-Ramond backgrounds [43, 44, 45, 46, 47, 48], the quantization of this models is still difficult and results are hard to obtain. For that reason, it is convenient to take advantage of the $S-$duality on the $D1-D5-$system. The near horizon geometry of the $S-$dual brane system is also $AdS_3 \times S^3 \times T^4$ but with Neveu-Schwarz $H-$flux which is the field strength of a two-form field B.

2.1 Classical theory.

Let us start considering strings propagating on AdS_3 spaces with $H-$flux [49, 50]. The isometry group of AdS_3 is $SO(2,2) \sim SL(2,R) \times SL(2,R)$. This is obvious from the embedding of the hyperboloid in $M^{2,2}$,

$$-X_0^2 - X_{-1}^2 + X_1^2 + X_2^2 = -1\,. \tag{2.1}$$

The above three-dimensional hyperboloid can be parametrized by the following element of $SL(2,R)$,

$$g = \begin{pmatrix} X_{-1} + X_1 & X_0 - X_2 \\ -X_0 - X_2 & -X_{-1} - X_1 \end{pmatrix}\,. \tag{2.2}$$

Another useful parametrization is given by

$$g = \,\mathrm{e}^{(i\frac{t+\phi}{2}\sigma_2)}\,\mathrm{e}^{\rho\sigma_3}\,\mathrm{e}^{(i\frac{t-\phi}{2}\sigma_2)}\,, \tag{2.3}$$

where σ_i are the Pauli matrices. We use the following generators for the $SL(2,R)$ Lie algebra

$$T^0 = -\frac{i}{2}\sigma^2, \qquad T^{\pm} = \frac{1}{2}(\sigma^3 \pm i\sigma_1). \tag{2.4}$$

Then the metric on the $SL(2,R)$ group manifold is given by

$$g_{\mu\nu} = \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g), \tag{2.5}$$

where μ,ν are indices referring to ρ, t, ϕ. Evaluating the metric using the parametrization given in (2.3) we get

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2, \tag{2.6}$$

which is the metric on AdS_3 expressed in the global coordinates (t,ϕ,ρ). Thus string propagation on AdS_3 can be expressed in terms of the WZW action given by,

$$\begin{aligned} S &= \frac{k}{4\pi\alpha'}\int_M dx^+ dx^- \mathrm{Tr}(g^{-1}\partial^\mu g g^{-1}\partial_\mu g) + \\ &\quad \frac{k}{24\pi\alpha'}\int_N d^3y \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(g^{-1}\partial^\alpha g\, g^{-1}\partial^\beta g\, g^{-1}\partial^\gamma g). \end{aligned} \tag{2.7}$$

The last term corresponds to the $H-$field, which can be written in terms of the Maurer-Cartan 1-form

$$w = g^{-1}dg, \quad H = w^3\,, \tag{2.8}$$

and N is a three dimensional manifold whose boundary is the world-sheet M. k is a parameter of the theory which can be interpreted as $1/\hbar$. The equations of motion for this action are given by,

$$\partial_-\left(\partial_+ g\, g^-\right) = 0\,. \tag{2.9}$$

Therefore, a general solution is,

$$g = g_+(x^+)g_-(x^-)\,. \tag{2.10}$$

Let us consider a simple solution given by,

$$g_+(x^+) = U\, e^{\frac{i\alpha}{2}\sigma_2 x^+}\,, \quad g_-(x^-) = e^{\frac{i\alpha}{2}\sigma_2 x^-}\, V\,, \tag{2.11}$$

where we have used $x^\pm = \tau \pm \sigma$, with (τ,σ) the world sheet coordinates and $U\,, V$ are constant elements of $SL(2,R)$. When $V = U^{-1}$, without loss of generality, we can set $U = V^{-1} = \mathrm{e}^{\rho_0\sigma_3/2}$, and then we have $g = \mathrm{e}^{\rho_0\sigma_3/2}\, e^{i\alpha\tau\sigma_2}\, \mathrm{e}^{-\rho_0\sigma_3/2}$. From

the parametrization in (2.3) we see that in the particular case $\rho_0 = 0$, the solution is a time-like geodesic with $\rho = \phi = 0$ and $t = \alpha\tau$, i.e. it represents a particle trajectory sitting at the center of AdS_3.
One can generate new solutions by the following transformations,

$$g_+ = e^{i\frac{1}{2}\omega x^+ \sigma_2} \tilde{g}_+ \,, \quad g_- = \tilde{g}_- e^{i\frac{1}{2}\omega x^- \sigma_2} \quad , \tag{2.12}$$

where $\tilde{g}_+$ and $\tilde{g}_-$ are the old solutions, i.e, $g(\omega) = e^{i\frac{1}{2}\omega x^+ \sigma_2}\, \tilde{g}\, e^{i\frac{1}{2}\omega x^- \sigma_2}$. From (2.3) we see that this acts on t and ϕ as

$$t \to t + \omega\tau \quad \phi \to \phi + \omega\sigma \tag{2.13}$$

This transformation is called spectral flow and will be of great importance in our discussion of the spectrum and correlation functions. It stretches the geodesics in the t-direction and gives them σ dependence. In fact, now the solution represents a string winding w times around the centre $\rho = 0$ of AdS_3; the resulting solution describes a circular string which repeats expansion and contraction.

As we argued above, the theory is invariant under the action of $SL(2,R) \times SL(2,R)$, $g \to \Omega(x^+) g \bar{\Omega}^{-1}(x^-)$, both Ω and $\bar{\Omega}$ being elements of $SL(2,R)$. Under infinitesimal transformations $\Omega \sim 1 + \epsilon(x^+)$, $\bar{\Omega} \sim 1 + \bar{\epsilon}(x^-)$, g transforms as

$$\delta g = \delta_\epsilon g + \delta_{\bar{\epsilon}} g \,, \tag{2.14}$$

where $\delta_\epsilon g = \epsilon g$ and $\delta_{\bar{\epsilon}} g = -g\bar{\epsilon}$. The variation of the action (2.7) is given by,

$$\begin{aligned} \delta S &= \frac{k}{2\pi\alpha'} \int_M d^2x \mathrm{Tr}(g^{-1}\delta g \partial_+(g^{-1}\partial_- g)) \,, \\ &= \frac{k}{2\pi\alpha'} \int_M d^2\, x \mathrm{Tr}(\epsilon \partial_-(\partial_+ g g^{-1}) - \bar{\epsilon}\partial_+(g^{-1}\partial_- g)) \,, \end{aligned} \tag{2.15}$$

from where we read the set of conserved currents,

$$j_+ = k\partial_+ g g^{-1}, \quad j_- = k g^{-1}\partial_- g \,, \tag{2.16}$$

or in components,

$$j_+^a = \mathrm{Tr}(T^a j_+), \quad j_-^a = \mathrm{Tr}(T^{a*} j_-), \quad a = 0, \pm \,, \tag{2.17}$$

which generate an infinite set of conserved charges associated to the affine $SL(2,R) \times SL(2,R)$ symmetry of the theory given by

$$j_n^a = \int_0^{2\pi} \frac{dx^+}{2\pi} e^{inx^+} j_+^a \,, \quad \bar{j}_n^a = \int_0^{2\pi} \frac{dx^-}{2\pi} e^{inx^-} j_-^a \,. \tag{2.18}$$

The zero modes of $j^0_\pm$ are related to the energy E and angular momentum l,

$$j^0_0 = \int_0^{2\pi} \frac{dx^+}{2\pi} j^0_+ = \frac{1}{2}(E+l)\,, \quad \bar{j}^0_0 = \int_0^{2\pi} \frac{dx^-}{2\pi} j^0_- = \frac{1}{2}(E-l)\,. \tag{2.19}$$

The energy-momentum tensor can be written in terms of the currents as,

$$T^{AdS}_{++} = \frac{1}{k} j^a_+ j^a_+\,, \tag{2.20}$$

with a similar expression for T^{AdS}_{--}.
Under the spectral flow transformation the currents change in the following way,

$$j^0_+ = \tilde{j}^0_+ + \frac{k}{2}\omega, \quad j^\pm_+ = \tilde{j}^{\mp i\omega x^+}\,, \tag{2.21}$$

and a similar expression for j^a_-. In terms of the charges (Fourier modes) (2.18),

$$j^0_n = \tilde{j}^0_n + \frac{k}{2}\omega\delta_{n,0}, \quad j^\pm_n = \tilde{j}^\pm_{n\mp\omega}\,, \tag{2.22}$$

with a similar expression for $\bar{j}^a_n$. The energy-momentum tensor transforms as,

$$T^{AdS}_{++} = \tilde{T}^{AdS}_{++} - \omega\tilde{j}^0_+ - \frac{k}{4}\omega^2\,. \tag{2.23}$$

2.2 Quantum theory

From now on we will use Euclidean world-sheet signature, i.e, we analytically continue to $z = \tau + i\sigma$ and work on the complex plane.

The variation of the action (2.15) is given in $z-$complex coordinates by the following,

$$\delta S = -i\frac{k}{4\pi\alpha'}\int_M d^2\, z \mathrm{Tr}(\epsilon(z)\partial_{\bar{z}}(\partial_z g g^{-1}) - \bar{\epsilon}(\bar{z})\partial_z(g^{-1}\partial_{\bar{z}}g))\,. \tag{2.24}$$

We have used $d^2x = (-i/2)d^2z$. Now performing an integration by parts, we get,

$$\begin{aligned} \delta S &= -i\frac{1}{4\pi\alpha'}\int_M d^2\, z\, [\partial_{\bar{z}}\mathrm{Tr}(\epsilon(z)j(z)) + \partial_z \mathrm{Tr}(\bar{\epsilon}(\bar{z})\bar{j}(\bar{z}))]\,, \\ &= \frac{i}{4\pi\alpha'}\oint dz \mathrm{Tr}(\epsilon(z)j(z)) - \frac{i\,k}{4\pi\alpha'}\oint d\bar{z}\mathrm{Tr}(\bar{\epsilon}(\bar{z})\bar{j}(\bar{z}))\,, \end{aligned} \tag{2.25}$$

which in components looks like,

$$\delta S = \frac{i}{2\pi\alpha'}\oint dz \sum_a \epsilon^a j^a - \frac{i}{4\pi\alpha'}\oint d\bar{z}\sum_a \bar{\epsilon}^a \bar{j}^a\,. \tag{2.26}$$

For a generic correlation function of a number of operators, denoted here as $\langle X \rangle$, the Ward identity

$$\delta \langle X \rangle = \langle (\delta S) X \rangle \,, \tag{2.27}$$

should be satisfied, i.e,

$$\delta \langle X \rangle = \frac{i}{2\pi\alpha'} \oint dz \sum_a \epsilon^a \langle j^a X \rangle - \frac{i}{4\pi\alpha'} \oint d\bar{z} \sum_a \bar{\epsilon}^a \langle \bar{j}^a X \rangle \,. \tag{2.28}$$

On the other hand, we can compute directly the infinitesimal transformation law for the currents from expressions (2.16),

$$\delta j(z) = [\epsilon, j] - k \partial_z \epsilon \,, \tag{2.29}$$

or,

$$\delta j^a(z) = \sum_{b,c} i f_{abc} \epsilon^b j^c - k \partial_z \epsilon \,, \tag{2.30}$$

where f_{abc} are the structure constants of the $SL(2,R)$ algebra and the non-vanishing components are given by,

$$f_{0\pm\pm} = \pm 1, \quad f_{-+0} = -2 \tag{2.31}$$

By comparing (2.28) and (2.30) we deduce the following OPE between currents,

$$j^a(z) j^b(z') \sim \frac{k \delta_{ab}}{(z-z')^2} + \sum_c i f_{abc} \frac{j^c(w)}{(z-w)} \,. \tag{2.32}$$

This is known in the literature as current algebra. In terms of the Fourier modes, the above current algebra is equivalent to the following commutation relations,

$$[j^a_n, j^b_m] = \sum_c i f_{abc} j^c_{n+m} + \frac{k}{2} n \delta_{n+m,0}. \tag{2.33}$$

There is a similar set of commutation relations for the right movers $\bar{j}^a_n$. Using equation (2.20) one can find the Fourier modes of the energy-momentum tensor[1]. They are given by

$$\begin{aligned} L_0 &= \frac{1}{k-2} \Bigg[\frac{1}{2} (j^+_0 j^-_0 + j^-_0 j^+_0) - (j^0_0)^2 \\ &\qquad + \sum_{m=1}^{\infty} (j^+_{-m} j^-_m + j^-_{-m} j^+_m - 2 j^0_{-m} j^0_m) \Bigg] \\ L_{n\neq 0} &= \frac{1}{k-2} \sum_{m=1}^{\infty} (j^+_{n-m} j^-_m + j^-_{n-m} j^+_m - 2 j^0_{n-m} j^0_m) \,. \end{aligned} \tag{2.34}$$

[1] This is known in the literature as Sugawara construction [51]

It can be compactly written as,

$$L_n = \frac{1}{k-2}\sum_m : j^a_m j^a_{n-m} : , \tag{2.35}$$

where : ... : denotes normal ordering. These generators obey the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\, n(n-1)\delta_{n+m,0}\,, \tag{2.36}$$

with the central charge given by

$$c = \frac{3k}{k-2}\,. \tag{2.37}$$

The action of the Virasoro generators on the currents is given by,

$$[L_n, j^a_m] = -m j^a_{n+m}\,. \tag{2.38}$$

2.2.1 Spectrum

The physical spectrum of a string in AdS_3 must be in unitary ghost-free representations of the following algebra,

$$\begin{aligned}
[j^a_n, j^b_m] &= \sum_c i f_{abc} j^c_{n+m} + \frac{k}{2} n\delta_{n+m,0}, \quad f_{0\pm\pm} = \pm 1,\ f_{-+0} = -2\,,\\
[L_n, j^a_m] &= -m j^a_{n+m}\\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}\, n(n-1)\delta_{n+m,0}\,.
\end{aligned} \tag{2.39}$$

The unitary representations of the zero modes of the currents obeying $[j^a_0, j^b_0] = \sum_c i f_{abc} j^c_0$, are classified by the eigen-values of j^0_0 which we denote by m and take highest value $-h$ (lowest h) by repeated action of j^+_0 (j^-_0). They are divided in five classes[2]:

1. Identity:
   The trivial representation $|0\rangle$. This representation has $h = 0, m = 0$ and $j^\pm_0|0\rangle = 0$.

2. Principal discrete representations (lowest weight):
   These are representations of the form

$$\mathcal{D}^+_h = \{|h;m\rangle : m = h, h+1, h+2\,\cdots\}, \tag{2.40}$$

[2]See [52] for a review.

Here $|h;h\rangle$ is annihilated by j_0^-. The tower of states over $|h;h\rangle$ is built by the repeated action of j_0^+. The norm of these states is positive and the representation is unitary if h is real and $h > 0$. h is restricted to be half integer if we are considering representations of the group $SL(2,R)$, however for the universal cover of $SL(2,R)$ which is our interest, h can be any positive integer.

3. Principal discrete representations (highest weight):
   These are representations of the form

$$\mathcal{D}_h^- = \{|h;m\rangle : m = -h, -h-1, -h-2, \cdots\}, \tag{2.41}$$

   where $|h;-h\rangle$ is annihilated by j_0^+. The representation has positive norm and is unitary if h is real and $h > 0$. This representation is the charge conjugate of $\mathcal{D}_h^+$.

4. Principal continuous representations:
   These representations are of the form

$$\mathcal{C}_h^\alpha = \{|h,\alpha;m\rangle : m = \alpha, \alpha \pm 1, \alpha \pm 2, \cdots\} \tag{2.42}$$

   Without loss of generality, we can restrict $0 \leq \alpha < 1$. The representation has positive norm and is unitary if $h = 1/2 + is$, where s is real.

5. Complementary representations:
   These are of the form

$$\mathcal{E}_h^\alpha = \{|h,\alpha;m\rangle : m = \alpha, \alpha \pm 1, \alpha \pm 2, \cdots\}. \tag{2.43}$$

   Again without loss of generality we can restrict $0 \leq \alpha < 1$ The representation has positive norm and is unitary if h is real and $h(1-h) > \alpha(1-\alpha)$.

Among these representations, we restrict to those which admit square integrable wave functions in the point particle limit. As AdS_3 is non-compact, square-integrability refers to delta function normalizable wave functions. This imposes the restriction $h > 1/2$ [3]. It is known that $\mathcal{C}^\alpha_{h=1/2+is} \otimes \mathcal{C}^\alpha_{h=1/2+is}$ and $\mathcal{D}_h^\pm \otimes \mathcal{D}_h^\pm$ with $h > 1/2$ form the complete basis of square integrable wave functions on AdS_3. Along this work we are going to use $\mathcal{D}_h^\pm$ representations only, for reasons which will become clear in the next section.

[3] This condition is also the condition for the Breitenlohner-Freedman bound [53] on $\mathcal{D}_h^\pm$ which states that the mass of a scalar in AdS_3 is given by $m^2 = h(h-1) \geq -\frac{1}{4}$.

Given a unitary representation $\mathcal{H}$ of $SL(2,R)$, one can construct a representation of the affine algebra (2.39) or $\widehat{SL(2,R)}$ by regarding $\mathcal{H}$ as the primary states annihilated by $j^{0,\pm}_{n\geq 1}$, and the full representation space is generated by acting with $j^{0,\pm}_{n\leq 1}$ on $\mathcal{H}$. We denote the representations of the full current algebra built on the corresponding representations of the zero modes by $\widehat{\mathcal{D}}^{\pm}_{h}$.

The above spectrum in general contain states with negative norms or ghosts. As in strings on flat space, we should be able to remove them by imposing the Virasoro conditions,

$$L_n|\text{physical}\rangle = 0\,, n \geq 0\,. \tag{2.44}$$

It has been shown that this spectrum does not contain ghosts when $\widehat{\mathcal{D}}^{\pm}_{h}$ is restricted to $0 < h < k/2$ [54, 55, 56, 57, 58].

Finally, as we saw in the previous section, we can generate new classical solutions from the old ones by the spectral flow transformation. At the quantum level, this is the statement that we can generate new representations from the standard ones by application of the spectral flow transformation on the currents. Indeed, we can easily see that, for any integer w, the transformation given by,

$$\begin{aligned} j^0_n &= \tilde{j}^0_n + \frac{k}{2}\omega\delta_{n,0}, \quad j^{\pm}_n = \tilde{j}^{\pm}_{n\mp\omega}\,, \\ L_n &= \tilde{L}_n - wj^0_n + \frac{k}{4}w^2\delta_n, 0\,, \end{aligned} \tag{2.45}$$

preserves the commutation relations (2.39). The above transformation maps one representation into an inequivalent one for a given w. We are going to denote the spectral flow images of $\widehat{\mathcal{D}}^{\pm}_{h}\otimes\widehat{\mathcal{D}}^{\pm}_{h}$, $\widehat{\mathcal{C}}^{\alpha}_{h}\otimes\widehat{\mathcal{C}}^{\alpha}_{h}$ as $\widehat{\mathcal{D}}^{\pm,w}_{h}\otimes\widehat{\mathcal{D}}^{\pm,w}_{h}$, $\widehat{\mathcal{C}}^{\alpha,w}_{h}\otimes\widehat{\mathcal{C}}^{\alpha,w}_{h}$ respectively, with $w \in \mathbb{Z}$.

Moreover, it has been shown in [59] that in order to produce a modular invariant partition function for string theory on AdS_3, spectral flowed states should be taken into account.

2.2.2 Primary fields

In the WZW theory, a primary field is defined as a field that transforms covariantly with respect to the group in which the model is defined. Explicitly in the case discussed in this chapter, the group is $SL(2,R) \times SL(2,R)$. This implies that a primary field transforms infinitesimally as,

$$\delta_{\epsilon}\Phi_{\lambda}(z,\bar{z}) = \epsilon\Phi_{\lambda}(z,\bar{z}), \quad \delta_{\bar{\epsilon}}\Phi_{\bar{\lambda}}(z,\bar{z}) = -\Phi_{\bar{\lambda}}(z,\bar{z})\bar{\epsilon}\,, \tag{2.46}$$

where $(\lambda,\,\bar{\lambda})$ denotes a given representation of $SL(2,R)\times SL(2,R)$. Using (2.28), we deduce the following OPEs between the currents and the primary fields,

$$j^a(z)\Phi_\lambda(w) \sim \frac{-t^a_\lambda \Phi_\lambda(w,\bar{w})}{z-w}, \quad \bar{j}^a(z)\Phi_\lambda(w) \sim \frac{\Phi_\lambda(w,\bar{w})t^a_{\bar{\lambda}}}{z-w}\,, \tag{2.47}$$

where t^a_λ and $t^a_{\bar{\lambda}}$ are the generators of the $SL(2,R)\times SL(2,R)$ algebra in the corresponding representation.

In this work we will be mainly interested in the following representation,

$$-\,t^+_x \equiv D^+_x = \partial_x, \quad -t^0_x \equiv D^0_x = x\partial_x + h, \quad -t^-_x \equiv D^-_x = x^2\partial_x + 2hx\,. \tag{2.48}$$

This is known as iso-spin representation and can be interpreted as a usual scalar representation in coordinate x.

Chapter 3

String theory on AdS$_3$$\times$ S$_3$$\times$ T^4

In this chapter, we shall generalize the construction of the string theory made in chapter 2 to the full super-string theory on $AdS_3 \times S_3 \times T^4$.

3.1 Quantum theory

Type IIB superstring theory on $AdS_3 \times S^3 \times T^4$ was originally studied in [60, 61, 35, 62, 63, 64]. As in the AdS_3 case, we can build an $SU(2)-$WZW and a $U(1)^4-$WZW models [65], where $SU(2)$ and $U(1)^4$ are the corresponding isometry groups of the S^3 sphere and T^4 torus respectively. Therefore we have a bosonic action with an $\widehat{\mathrm{SL}(2)} \times \widehat{\mathrm{SU}(2)} \times \widehat{\mathrm{U}(1)}^4$ affine worldsheet symmetry. We would like to consider a world-sheet with $\mathcal{N} = 1$ supersymmetry. To do that, we introduce a set of fermions ψ^A, χ^A, λ^i, $A = 0, \pm$, $i = 1, 2, 3, 4$ corresponding to the super-partners of the bosonic $SL(2, R)$, $SU(2)$ and T^4 respectively.

The $\widehat{\mathrm{SL}(2)}$ and $\widehat{\mathrm{SU}(2)}$ supercurrents $\psi^A + \theta J^A$ and $\chi^A + \theta K^A$, respectively, satisfy the following OPE

$$\begin{aligned}
J^A(z)J^B(w) &\sim \frac{\frac{k}{2}\eta^{AB}}{(z-w)^2} + \frac{i\epsilon^{AB}{}_C J^C(w)}{z-w}\,, \\
J^A(z)\psi^B(w) &\sim \frac{i\epsilon^{AB}{}_C \psi^C(w)}{z-w}\,, \\
\psi^A(z)\psi^B(w) &\sim \frac{\frac{k}{2}\eta^{AB}}{z-w}\,,
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
K^A(z)K^B(w) &\sim \frac{\frac{k}{2}\delta^{AB}}{(z-w)^2} + \frac{i\epsilon^{AB}{}_C K^C(w)}{z-w}\,, \\
K^A(z)\chi^B(w) &\sim \frac{i\epsilon^{AB}{}_C\chi^C(w)}{z-w}\,, \\
\chi^A(z)\chi^B(w) &\sim \frac{\frac{k}{2}\delta^{AB}}{z-w}\,,
\end{aligned}
\tag{3.2}
$$

with $A = 0, 1, 2$, $\epsilon^{012} = 1$ and $\eta^{AB} = (-, +, +)$. It is convenient to introduce new currents as

$$
J^A(z) = j^A(z) + \hat{j}^A(z)\,, \qquad K^A(z) = k^A(z) + \hat{k}^A(z)\,, \tag{3.3}
$$

where j^A ($\hat{j}^A$) and k^A ($\hat{k}^A$) generate $\mathrm{SL}(2)_{k+2}$ ($\mathrm{SL}(2)_{-2}$) and $\mathrm{SU}(2)_{k-2}$ ($\mathrm{SU}(2)_2$) affine algebras, respectively, with

$$
\hat{j}^A(z) = -\frac{i}{k}\epsilon^A{}_{BC}\psi^B(z)\psi^C(z), \qquad \hat{k}^A(z) = -\frac{i}{k}\epsilon^A{}_{BC}\chi^B(z)\chi^C(z). \tag{3.4}
$$

The advantage of the above spliting is that the bosonic $j^A(z)$ and ferminonic $\hat{j}^A(z)$ currents are decoupled. The $\widehat{\mathrm{U}(1)}^4$ is realized in terms of free bosonic currents $i\partial Y^i$ and free fermions $\lambda^i, i = 1, 2, 3, 4$.

The stress tensor and supercurrent are given by

$$
\begin{aligned}
T(z) &= \frac{\eta_{AB}}{k}(j^A j^B - \psi^A\partial\psi^B) + \frac{\delta_{AB}}{k}(k^A k^B - \chi^A\partial\chi^B) + \frac{1}{2}(\partial Y^i\partial Y_i - \frac{1}{2}\lambda^i\partial\lambda_i), \\
G(z) &= \frac{2}{k}(\eta_{AB}\psi^A j^B + \frac{2i}{k}\psi^0\psi^1\psi^2) + \frac{2}{k}(\delta_{AB}\chi^A k^B - \frac{2i}{k}\chi^0\chi^1\chi^2) + \lambda^i\partial Y_i\,.
\end{aligned}
\tag{3.5}
$$

The spectrum of the theory is built from those of the SL(2,R) and SU(2) WZNW models. As we mentioned in the previous section, the Hilbert space of the former [49] is decomposed into unitary representations of the SL(2,R)$\times$ SL(2,R) current algebra [1], namely the discrete lowest- and highest-weight representations $\mathcal{D}_h^\pm \otimes \mathcal{D}_h^\pm$ with $h \in \mathbb{R}$, $\frac{1}{2} < h < \frac{k+2}{2}$ and $m = \pm h, \pm h \pm 1, \dots$, the continuous representations $\mathcal{C}_h^\alpha \otimes \mathcal{C}_h^\alpha$ with $h = \frac{1}{2} + i\mathbb{R}$, $m = \alpha + \mathbb{Z}$, $\alpha \in [0, 1)$, their current algebra descendants and spectral flow images, $\widehat{\mathcal{D}}_h^{\pm,w} \otimes \widehat{\mathcal{D}}_h^{\pm,w}$, $\widehat{\mathcal{C}}_h^{\alpha,w} \otimes \widehat{\mathcal{C}}_h^{\alpha,w}$ with $w \in \mathbb{Z}$ and the same spin and amount of spectral flow on the left- and right-moving sectors.

Primary operators of spin h and worldsheet conformal dimension $\Delta^{sl}(\Phi_h) = -\frac{h(h-1)}{k}$, are defined by (2.48),

$$
j^a(z)\Phi_h(x, \overline{x}; w, \overline{w}) \sim \frac{D_x^a\Phi_h(x, \overline{x}; w, \overline{w})}{z-w}\,, \quad a = 0, \pm\,, \tag{3.6}
$$

[1]Actually, the spectrum is built on representations of the universal cover of SL(2,R), to which we refer simply as SL(2,R) for short.

where $D_x^+ = \partial_x$, $D_x^0 = x\partial_x + h$ and $D_x^- = x^2\partial_x + 2hx$. Expanding in modes as

$$\Phi_h(x, \overline{x}) = \sum_{m,\overline{m}} \Phi_{h,m,\overline{m}} x^{-h+m} \overline{x}^{-h+\overline{m}} , \tag{3.7}$$

one can read the action of the zero modes of the currents on $\Phi_{h,m,\overline{m}}$, namely

$$j_0^0 \Phi_{h,m,\overline{m}} = m\Phi_{h,m,\overline{m}}, \quad j_0^{\pm}\Phi_{h,m,\overline{m}} = [m \mp (h-1)]\, \Phi_{h,m\pm 1,\overline{m}}, \quad (m \neq \pm h) , \tag{3.8}$$

and $j_0^- \Phi_{h,h,\overline{m}} = j_0^+ \Phi_{h,-h,\overline{m}} = 0$, which exhibit the usual state-field correspondence. We can go back from the $x-$basis to the $m-$basis by the inverse transform,

$$\Phi_{h,m,\overline{m}} = \int d^2x x^{h-m-1} \overline{x}^{h-\overline{h}-1} \Phi_h(x, \overline{x}) . \tag{3.9}$$

It is worth mentioning at this point that the $\widehat{SL(2,R)}$ current algebra on the string worldsheet induces a Virasoro algebra in coordinates $(x, \bar{x})$, under which the fields $\Phi_h(x, \overline{x})$ behave as conformal primaries with conformal weight equal to h [60]. That is the reason why we have made the usual CFT expansion in (3.7). This also lead us to think about the coordinates $(x, \bar{x})$ as the boundary coordinates where the dual field theory should be defined. In forthcoming chapters we are going to use this fact strongly.

Similarly, the primary fields of the SU(2)$_{k-2}$ WZNW model with conformal dimension $\Delta^{su}(V_j) = \frac{j(j+1)}{k}$ verify

$$k^a(z) V_j(y, \overline{y}; w, \overline{w}) \sim \frac{P_y^a V_j(y, \overline{y}; w, \overline{w})}{z - w} , \tag{3.10}$$

with $P_y^+ = \partial_y$, $P_y^0 = y\partial_y - j$, $P_y^- = -y^2\partial_y + 2jy$ and can be expanded in modes as

$$V_j(y, \overline{y}) = \sum_{m', \overline{m}'=-j}^{j} V_{j,m',\overline{m}'} y^{j+m'} \overline{y}^{j+\overline{m}'} . \tag{3.11}$$

The spin $j \in \mathbb{Z}/2$ is bounded by $0 \leq j \leq \frac{k-2}{2}$ and $k_0^+ V_{j,j,\overline{m}'} = k_0^- V_{j,-j,\overline{m}'} = 0$,

$$k_0^0 V_{j,m',\overline{m}'} = m' V_{j,m',\overline{m}'}, \quad k_0^{\pm} V_{j,m',\overline{m}'} = (\pm m' + 1 + j) V_{j,m'\pm 1,\overline{m}'}, \quad (m' \neq \pm j) . \tag{3.12}$$

In the fermionic sector, the fields ψ^a transform in the spin $\hat{h} = -1$ representation of the global SL(2,R) algebra and χ^a transform in the spin $\hat{j} = 1$ of the SU(2) global algebra.

Vertex operators creating unflowed physical states in the NS sector were constructed in [64]. The construction of the physical spectrum is very similar to that corresponding to super-strings in flat space. The ground state corresponds to the following vertex,

$$\mathcal{V}_{h,j,p} = \Phi_{h,m,\bar{m}} V_{j,m',\bar{m}'} \, e^{i\vec{p}\cdot\vec{Y} + i\vec{\bar{p}}\cdot\vec{\bar{Y}}} \,, \tag{3.13}$$

where $e^{i\vec{p}\cdot\vec{Y} + i\vec{\bar{p}}\cdot\vec{\bar{Y}}}$ is the plane wave on the torus, i.e, $\vec{Y}$ are coordinates on the torus and $\vec{p}$ is a vector in an even self-dual Narain lattice $\Gamma^{4,\bar{4}}$. The towers of string states are obtained by multiplying this zero mode wave function by a polynomial in the fermionic oscillators, the bosonic oscillators and their derivatives, and a similar polynomial in the antiholomorphic oscillators. More concretely, the most general state in the (-1, -1) picture of the NS-NS sector has the form

$$\mathcal{B}^{N,\bar{N}}_{h,j,p} = e^{-\varphi-\bar{\varphi}} P_N(\psi^A,\, \partial\psi^A, ..., j^a,\, \partial j^A, ..) \bar{P}_{\bar{N}}(...) \mathcal{V}_{h,j,p} \,, \tag{3.14}$$

where P_N is a polynomial in the bosonic and fermionic worldsheet fields and their derivatives with scaling dimension N, and similarly for $\bar{P}_{\bar{N}}$. $e^{-\varphi}$ is a bosonised super-ghost field ensuring that those states are in -1 picture. A physical state should satisfy the Virasoro conditions,

$$L_n|\mathrm{Phys}\rangle = \left(L_{n,AdS} + L_{n,SU(2)} + L_{n,T^4}\right)|\mathrm{Phys}\rangle = 0, \quad n \geq 0\,, \tag{3.15}$$

which implies,

$$-\frac{1}{2} + N - \frac{h(h-1)}{k} + \frac{j(j+1)}{k} + \frac{|p|^2}{2} = 0\,, \tag{3.16}$$

and a similar relation for $\bar{N}$, $\bar{p}^2$. From the above relation we can see that generically, a physical state has to satisfy,

$$h = j + 1\,. \tag{3.17}$$

Additionally, as we have discussed in chapter 1, the size of the T^4 (1.19) is of the order of the string length to the four $V_4 \sim \mathcal{O}(\alpha'^2)$. In this work we are going to compute only tree-level correlators, therefore we can neglect the massive contributions from T^4 since they will become visible in higher α' corrections. In other words, we set $|p|^2 = |\bar{p}|^2 = 0$.

As we mentioned in the introduction, there is a nice window to test the AdS/CFT correspondence if we restrict the spectrum to the chiral ring. The chiral (antichiral) fields, are primaries which satisfy the condition $\mathcal{H} = \mathcal{J}$ ($\mathcal{H} = -\mathcal{J}$) (see chapter 4), $\mathcal{H}$ being the total space-time conformal dimension[3] and $\mathcal{J}$ the

[2]From now on we display only the holomorphic indices.

[3]This is the quantum number we have denoted h in the bosonic sector, but now generalized to the super-symmetric case

total SU(2) charge. As we see from (3.16) the lowest physical primary (without covariant derivatives) should have $N = 1/2$. In the -1 picture, for generic h there are eight primary physical states satisfying the above conditions [4],

$$\begin{aligned} \mathcal{V}^i_{h,m,m'} &= \mathrm{e}^{-\varphi}\lambda^i \Phi_{h,m} V_{j,m'} \,, \\ \mathcal{W}^{\pm}_{h,m,m'} &= e^{-\varphi}(\psi\Phi_{h,m})_{h\pm1,m_T} V_{j,m'} \,, \\ \mathcal{Y}^{\pm}_{h,m,,m'} &= e^{-\varphi}\Phi_{h,m}(\chi V_{j,m'})_{h,m'_T} \,, \end{aligned} \tag{3.18}$$

where $(\psi\Phi_{h,m})_{h\pm1,m_T}$ denotes the SL(2,R) current algebra product representation of the representation h and the representation 1 with total 'angular momentum' $\mathcal{H} = h \pm 1$.

The four operators $\mathcal{V}^i_{h,m,m'}$ have $\mathcal{H} = h = j+1$ and $\mathcal{J} = j$; $\mathcal{W}^{\pm}_{h,m,m'}$ have $\mathcal{H} = h = j+1 \pm 1$ and $\mathcal{J} = j$; and $\mathcal{Y}^{\pm}_{h,m,,m'}$ have $\mathcal{H} = h = j+1$ and $\mathcal{J} = j \pm 1$. Therefore we have the following chiral vertex in the NS sector,

$$\mathcal{O}^{(0)}_{h,m,m'} \equiv \mathcal{W}^{-}_{h,m,m'} = e^{-\varphi}(\psi\Phi_{h,m})_{h-1,m_T} V_{j,m'} \,, \tag{3.19}$$

$$\mathcal{O}^{(2)}_{h,m,,m'} \equiv \mathcal{Y}^{+}_{h,m,,m'} = e^{-\varphi}\Phi_{h,m}(\chi V_{j,m'})_{j+1,m'_T} \,. \tag{3.20}$$

To study the Ramond sector one needs to construct the spin fields for ψ^a, χ^a, λ^i [64]. It is convenient to have a bosonised form of the fermions such as

$$\begin{aligned} \partial H_1 &= \frac{2}{k} i\psi^+\psi^-, \quad \partial H_2 = \frac{2}{k} i\chi^+\chi^-, \\ \partial H_3 &= -\frac{2}{k} i\psi^3\chi^3, \quad \partial H_4 = \lambda^1\lambda^2, \quad \partial H_5 = \lambda^3\lambda^4 \,. \end{aligned} \tag{3.21}$$

The spin fields take the form $S_{[\epsilon_1,\cdots,\epsilon_5]} = \exp\frac{i}{2}\sum_{i=1}^5 \epsilon_i H_i$, with $\epsilon_i = \pm 1$. They transform as two copies of $(\frac{1}{2},\frac{1}{2})$ under SL(2)$\times$ SU(2). GSO projection requires $\prod_{i=1}^5 \epsilon_i = +1$ and BRST invariance demands $\prod_{i=1}^3 \epsilon_i = -1$. Following [66] we define the spin fields associated with ψ^a, χ^a as $\tilde{S}_{[\epsilon_1,\epsilon_2,\epsilon_3]} = \exp\frac{i}{2}(\epsilon_1 H_1 + \epsilon_2 H_2 + \epsilon_3 H_3)$.

Decomposing the product $(\tilde{S}\Phi_{h,m}V_{j,m'})$ into representations of the total currents J^a, K^a, the chiral vertex operators in the $-\frac{1}{2}$ picture take the form

$$\mathcal{O}^{(a)}_{h,m,m'} = e^{-\frac{\varphi}{2}}(\tilde{S}\Phi_{h,m}V_{j,m'})_{h-\frac{1}{2},m_T+\frac{1}{2};j+\frac{1}{2},m'_T+\frac{1}{2}} e^{ai(\hat{H}_4-\hat{H}_5)} \,, \tag{3.22}$$

where H_i are redefined as $\hat{H}_i = H_i + \pi\sum_{j<i} N_j$, $N_j = i\oint \partial H_i$ and $a = \pm$.

- *Spectral flow*

[4] There are two remaining contributions which involve $(\psi\Phi_{h,m})_{h,m_T}$ and $(\chi V_{j,m'})_{j,m'_T}$, but they are not physical [64].

The algebras (3.1) are invariant under the following spectral flow automorphisms

$$\widetilde{J}^0_n = J^0_n - \frac{k}{2} w \delta_{n,0}, \quad \widetilde{J}^{\pm}_n = J^{\pm}_{n\pm w}, \qquad \widetilde{K}^0_n = K^0_n + \frac{k}{2} w' \delta_{n,0}, \quad \widetilde{K}^{\pm}_n = K^{\pm}_{n\pm w'} .$$

The currents j^a, $\hat{j}^a$, k^a and $\hat{k}^a$ transform under spectral flow as

$$j^0_n = \tilde{j}^0_n + \frac{k+2}{2} w \delta_{n,0} , \quad j^{\pm}_n = \tilde{j}^{\pm}_{n\mp w} , \qquad k^0_n = \tilde{k}^0_n - \frac{k-2}{2} w' \delta_{n,0} , \quad k^{\pm}_n = \tilde{k}^{\pm}_{n\mp w'} ,$$
$$\hat{j}^0_n = \tilde{\hat{j}}^0_n - w\delta_{n,0}, \quad \hat{j}^{\pm}_n = \tilde{\hat{j}}^{\pm}_{n\mp w} , \qquad \hat{k}^0_n = \tilde{\hat{k}}^0_n - w'\delta_{n,0} , \quad \hat{k}^{\pm}_n = \tilde{\hat{k}}^{\pm}_{n\mp w'} .$$

As we did with the currents, we split the Virasoro generators into their bosonic and fermionic pieces as, $L^{sl}_n = l^{sl}_n + \hat{l}^{sl}_n, L^{su}_n = l^{su}_n + \hat{l}^{su}_n$,

$$\begin{aligned}
\tilde{L}^{sl}_n &= L^{sl}_n + w\tilde{J}^0_n + \frac{k}{4} w^2 \delta_{n,0} , \qquad & \tilde{L}^{su}_n &= L^{su}_n + w'\tilde{K}^0_n - \frac{k}{4} w'^2 \delta_{n,0} \\
l^{sl}_n &= \tilde{l}^{sl}_n - w\tilde{j}^0_n - \frac{k+2}{4} w^2 \delta_{n,0} , \qquad & l^{su}_n &= \tilde{l}^{su}_n - w'\tilde{k}^0_n + \frac{k-2}{4} w'^2 , \\
\hat{l}^{sl}_n &= \tilde{\hat{l}}^{sl}_n - w\tilde{\hat{j}}^0_n + \frac{1}{2} w^2 \delta_{n,0} , \qquad & \hat{l}^{su}_n &= \tilde{\hat{l}}^{su}_n - w'\tilde{\hat{k}}^0_n + \frac{1}{2} w'^2 \delta_{n,0} .
\end{aligned} \tag{3.23}$$

The closure of the SL(2,R) and SU(2) algebras requires the same amount of spectral flow w (w') for j^a and $\hat{j}^a$ (k^a and $\hat{k}^a$). The spectral flow maps primaries to descendants of SU(2) and, unlike in SL(2,R), it does not generate new representations. For the sake of simplicity, we restrict to $w > 0$ in this section.

To construct spectral flow images of chiral primaries in generic frames, we consider the SL(2,R) sector first. A $w = 0$ affine primary is mapped by the spectral flow to a lowest-weight state of the global algebra $\Phi^{h,w}_{H,M}$ with $H = M$ satisfying [49]

$$j^0_0 \Phi^{h,w}_{H,M} = M\Phi^{h,w}_{H,M} = \left(m + \frac{k+2}{2} w \right) \Phi^{h,w}_{H,M} , \tag{3.24}$$

$$l_0 \Phi^{h,w}_{H,M} = \left(-\frac{h(h-1)}{k} - wm - \frac{k+2}{4} w^2 \right) \Phi^{h,w}_{H,M} . \tag{3.25}$$

Using (3.23) − (3.23), the fermions ψ^a in the spectral flow frame obey

$$\hat{j}^0_0 \psi^a = (a-w)\psi^a, \qquad \hat{j}^-_0 \psi^a = \tilde{\hat{j}}^-_w \psi^a = 0 , \tag{3.26}$$

$$\hat{l}_0 \psi^a = (\frac{1}{2} - wa + \frac{1}{2} w^2)\psi^a , \tag{3.27}$$

i.e. ψ^a is a lowest-weight field with angular momentum $\hat{h} = a - w$. Acting with $\hat{j}^+_0$, one obtains the global representation in the w sector as $\psi^{|\hat{h}|}_{\hat{m}} \sim (\hat{j}^+_0)^n \psi^a$ with $\hat{m} = -\hat{h}, \cdots, \hat{h}$ up to a normalization.

All these ingredients allow to construct the representations of J^a. We denote the fields of the product representation in the NS sector as $(\psi_{\hat{m}}^{|\hat{h}|}\Phi_{H,M}^{w,h})_{\mathcal{H},\mathcal{M}}(z,\overline{z})$, where $|H-\hat{h}| \leq \mathcal{H} \leq H+\hat{h}$, $\mathcal{M}=\mathcal{H},\mathcal{H}+1,...$ and their worldsheet conformal weight is given by

$$\Delta^{sl}\left[(\psi_{\hat{m}}^{|\hat{h}|}\Phi_{H,M}^{h,w})_{\mathcal{H},\mathcal{M}}\right] = -\frac{h(h-1)}{k} - w(m-a) + \frac{1}{2} - \frac{k}{4}w^2\,. \tag{3.28}$$

Repeating the analysis for SU(2), one obtains the product representation $(\chi_{\hat{m}'}^{\hat{j}}V_{J,M'}^{j,w'})_{\mathcal{J},\mathcal{M}'}$, with $|J-\hat{j}| \leq \mathcal{J} \leq J+\hat{j}$, $-\mathcal{J} \leq \mathcal{M}' \leq \mathcal{J}$, $J=m'-\frac{k-2}{2}w'$, $\hat{j}=|a-w'|$ and worldsheet conformal weight

$$\Delta^{su}\left[(\chi_{\hat{m}'}^{\hat{j}}V_{J,M'}^{j,w'})_{\mathcal{J},\mathcal{M}'}\right] = \frac{j(j+1)}{k} - w'(m'-a) + \frac{1}{2} + \frac{k}{4}w'^2\,. \tag{3.29}$$

In order to construct chiral states in generic representations, we should apply the spectral flow operation on the chiral primaries (3.19) and (3.20). A detailed derivation of these vertices is left until the next section. We notice that the physical and chiral state conditions require to simultaneously spectral flow the SL(2,R) and SU(2) product representations and we obtain [18],

$$\mathcal{O}^{(0),w}_{\mathcal{H},\mathcal{M}} \equiv e^{-\varphi}(\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M})_{\mathcal{H},\mathcal{M}}(\chi^{w'}_{\hat{m}'}V^{j,w'}_{J,M'})_{\mathcal{J},\mathcal{M}'}\,, \tag{3.30}$$

$$\mathcal{O}^{(2),w}_{\mathcal{H},\mathcal{M}} \equiv e^{-\varphi}(\psi^{w}_{\hat{m}}\Phi^{h,w}_{H,M})_{\mathcal{H},\mathcal{M}}(\chi^{w'+1}_{\hat{m}'}V^{j,w'}_{J,M'})_{\mathcal{J},\mathcal{M}'}\,, \tag{3.31}$$

where φ is the bosonization of the β,γ ghosts, $\mathcal{M}=\mathcal{H}$ and $\mathcal{M}'=-\mathcal{J}$. For generic level k, the physical state condition $(L_0-1)\mathcal{W}=0$ implies $h=j+1$, $w=w'$ and $m'_T=-m_T$ (see (3.19),(3.20)), and similarly for $\mathcal{Y}$. Analogously, $G_r\mathcal{W}=(\tilde{G}_r-w\tilde{\psi}^0_r-w\tilde{\chi}^0_r)\mathcal{W}=0$ ($G_r\mathcal{Y}=0$) for $r>0$ requires $m_T=h-1$ ($m_T=h$) [66]. Finally, chirality (or antichirality) demands, for both operators $\mathcal{O}^{(0)}$ and $\mathcal{O}^{(2)}$,

$$\mathcal{H}=m_T+\frac{k}{2}w=\pm\mathcal{J}. \tag{3.32}$$

To obtain the spectral flowed $\frac{1}{2}$ BPS operators in the Ramond sector we need the product representation $(S^{\hat{j}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w'}_{J,M'})$. The discussion about the fermions applies analogously to the spin fields, *i.e.* from the lowest-weight component of the $\hat{h}=-\hat{j}=-|w\pm\frac{1}{2}|$ spin representation, given by

$$S^{w+\frac{1}{2}}_{-w-\frac{1}{2},w+\frac{1}{2}} \equiv e^{-i(w+\frac{1}{2})(\hat{H}_1+\hat{H}_2)-\frac{i}{2}\hat{H}_3}\,, \tag{3.33}$$

one constructs the global representation acting with $\hat{j}^+_0,\hat{k}^+_0$.

The chiral fields in the w sector are [66]

$$\mathcal{O}^{(a),w}_{\mathcal{H},\mathcal{M}}=e^{-\frac{\varphi}{2}}(S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J,M'})_{\mathcal{H},\mathcal{M},\mathcal{J},\mathcal{M}'}e^{\pm\frac{i}{2}(\hat{H}_4-\hat{H}_5)}\,, \tag{3.34}$$

where $S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}$ has conformal weight $\frac{3}{8}+w^2+w$, $\hat{h}=-w-\frac{1}{2}=-\hat{j}$ and $\mathcal{H}=h-\frac{1}{2}+\frac{k}{2}w=\mathcal{J}$.

3.2 Vertex operators of chiral states

In this section we present a derivation of the vertex operators creating spectral flow images of chiral primary states. These operators were proposed and built in [18] and [66] .

3.1 NS sector

The Clebsch-Gordan coefficients expanding the product representation $(\psi\Phi)$ in (3.30) and (3.31) are computed in appendix A1. We find

$$(\psi^{|\hat{h}|}_{\hat{m}}\Phi^{h,w}_{H,M})_{\mathcal{H},\mathcal{M}} = \sum_{\hat{m}=-\hat{h}}^{\hat{h}} C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}\psi^{|\hat{h}|}_{\hat{m}}\Phi^{h,w}_{H,M}\,, \tag{3.35}$$

where only the holomorphic part has been written and [5]

$$\begin{aligned} C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} &= \frac{(\mathcal{M}+\mathcal{H})!}{(\hat{m}+|\hat{h}|)!(\mathcal{M}-\hat{m}+H)!}\sum_{s=0}^{\hat{m}+|\hat{h}|}(-1)^{s-|\hat{h}|}\times \\ &\binom{\hat{m}+|\hat{h}|}{s}\frac{(\mathcal{M}-s+|\hat{h}|+H)!}{(\mathcal{M}-s+\mathcal{H})!}\times \\ &\frac{(|\hat{h}|-H+\mathcal{M}-s-1)!}{(\mathcal{M}-s-\mathcal{H}-1)!(\mathcal{H}+|\hat{h}|-H)!}\,. \end{aligned} \tag{3.36}$$

This can be rewritten using the generalized hypergeometric function ${}_3F_2(a,b,c;e,f|1)$ as

$$\begin{aligned} C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} = \frac{(-1)^{\hat{m}-|\hat{h}|}\Gamma(-H+|\hat{h}|+\mathcal{M})\Gamma(H+|\hat{h}|+\mathcal{M}+1)}{\Gamma(\mathcal{H}-H+|\hat{h}|+1)\Gamma(\mathcal{M}-\mathcal{H})\Gamma(H+\mathcal{M}-\hat{m}+1)\Gamma(|\hat{h}|+\hat{m}+1)}\times \\ {}_3F_2(-\mathcal{H}-\mathcal{M},\mathcal{H}-\mathcal{M}+1,-|\hat{h}|-\hat{m};-H-|\hat{h}|-\mathcal{M},H-|\hat{h}|-\mathcal{M}+1;1)\,, \end{aligned}$$

with the advantage that it can be represented in terms of the Pochhammer double-loop contour integral, possessing a unique analytic continuation in the complex plane for all its indices [67, 68]. The analogous coefficients for SU(2) are related to these ones through analytic continuation.

For our purposes, it is convenient to write the vertex operators in the $x-$basis, where the isospin can be identified with the coordinates on the boundary [61]. This can be done using $J^+\sim\partial_x$ (3.6), i.e, every observable $\mathcal{O}(x)$ is conjugate to $\mathcal{O}(0)$,

$$e^{-xJ_0^+}\mathcal{O}(z)e^{xJ_0^+}\equiv\mathcal{O}(x,z). \tag{3.37}$$

[5]The coefficients $C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}$ are given by the dot product $\langle H,M;\hat{h},\hat{m}|H,\hat{h};\mathcal{H},\mathcal{M}\rangle$ as we show in the appendix.

Performing this operation on the fermion fields, in the unflowed frame one gets

$$e^{-xJ_0^+}\psi^+(z)e^{xJ_0^+} = \psi^+(x,z) \equiv \psi(x,z) \quad (3.38)$$

$$= -2x\psi^0(z) + \psi^+(z) + x^2\psi^-(z), \quad (3.39)$$

and in a generic w frame

$$e^{-xJ_0^+}\psi^{|\hat{h}|}_{\hat{m}=\hat{h}}(z)e^{xJ_0^+} \equiv \psi^{|\hat{h}|}(x,z) = \sum_{\hat{m}=-\hat{h}}^{\hat{h}} \frac{(-1)^{\hat{m}+\hat{h}}\,\Gamma(2|\hat{h}|+1)}{\Gamma(\hat{m}+|\hat{h}|+1)\Gamma(|\hat{h}|-\hat{m}+1)}\psi^{|\hat{h}|}_{\hat{m}}x^{-\hat{h}+\hat{m}}\,. \quad (3.40)$$

Inserting $H = m + \frac{k+2}{2}w$ and $\hat{h} = -w-1$ in (3.36) we get

$$C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} = (-1)^{\hat{m}+w+1}\frac{\Gamma(2w+3)}{\Gamma(\hat{m}+w+2)\Gamma(w-\hat{m}+2)}\,, \quad (3.41)$$

which coincide with the coefficients in (3.40). Therefore, the SL(2,R) part of the chiral vertex (3.30) may be written as

$$(\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M})_{\mathcal{H},\mathcal{M}} = \sum_{\hat{m}=-w-1}^{w+1}(-1)^{\hat{m}+w+1}\frac{\Gamma(2w+3)}{\Gamma(\hat{m}+w+2)\Gamma(w-\hat{m}+2)}\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M}\,. \quad (3.42)$$

Expanding in modes, it is easy to see that they may be expressed in the following factorized form

$$(\psi\Phi)^{h,w}_{\mathcal{H}}(x) \equiv \sum_{\mathcal{M}}(\psi^{w+1}_{\hat{m}}\Phi^{h,w}_{H,M})_{\mathcal{H},\mathcal{M}}\;x^{-\mathcal{H}+\mathcal{M}} = \psi^{w+1}(x)\Phi^{h,w}_{H}(x)\,. \quad (3.43)$$

Surprisingly, this factorization always occurs in (3.30) when H and $\hat{h}$ combine to produce a chiral state.

So far, we have restricted to the holomorphic SL(2,R) sector, but the same analysis applies to SU(2) [68] and to their antiholomorphic parts. Putting all together, we get the following vertex operators creating spectral flow images of chiral primary states in arbitrary spectral flow frames

$$\mathcal{O}^{(0),w}_{\mathcal{H}\overline{\mathcal{H}}}(x,y,\overline{x},\overline{y}) = e^{-\varphi}\;\Phi^{h,w}_{H\overline{H}}(x,\overline{x})\psi^{w+1}(x)\overline{\psi}^{w+1}(\overline{x})V^{h-1,w}_{J\overline{J}}(y,\overline{y})\chi^{w}(y)\overline{\chi}^{w}(\overline{y}), \quad (3.44)$$

$$\mathcal{O}^{(2),w}_{\mathcal{H}\overline{\mathcal{H}}}(x,y,\overline{x},\overline{y}) = e^{-\varphi}\Phi^{h,w}_{H\overline{H}}(x,\overline{x})\psi^{w}(x)\overline{\psi}^{w}(\overline{x})\chi^{w+1}(y)\overline{\chi}^{w+1}(\overline{y})V^{h-1,w}_{J,\overline{J}}(y,\overline{y})\;, \quad (3.45)$$

with $J = H-2w, \overline{J} = \overline{H}-2w$, $\mathcal{H} = -J-1, \overline{\mathcal{H}} = -\overline{J}-1$.

3.2 Ramond sector

The product representation needed to construct the vertex operators (3.34) in the Ramond sector can be expanded as

$$
\begin{aligned}
(S^{\hat{j}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J,M'})_{\mathcal{H},\mathcal{M},\mathcal{J},\mathcal{M}'} &= \sum_{\hat{m},\hat{m}'=-\hat{h}}^{\hat{h}} (S^{\hat{j}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J,M'})C^{(M,\hat{m},\mathcal{M}),(M',\hat{m}',\mathcal{M}')}_{(H,\hat{h},\mathcal{H}),(J,\hat{h},\mathcal{J})} \\
&\equiv \sum_{\hat{m}=-\hat{h}}^{\hat{h}} (S^{\hat{j}}_{\hat{m}}\Phi^{h,w}_{H,M})C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}} \otimes \sum_{\hat{m}'=-\hat{h}}^{\hat{h}} (S^{\hat{j}}_{\hat{m}'}V^{j,w}_{J,M'})C^{M',\hat{m}',\mathcal{M}'}_{J,\hat{h},\mathcal{J}}, \qquad (3.46)
\end{aligned}
$$

i.e. the SL(2) and SU(2) parts factorize. The Clebsch-Gordan coefficients $C^{M,\hat{m},\mathcal{M}}_{H,\hat{h},\mathcal{H}}$ can be computed from (3.36) taking $H = m + \frac{k+2}{2}w$ and $\hat{h} = -w - \frac{1}{2}$. Using (3.37), it is easy to see that the triple product factorizes in the $x-$basis as

$$
\begin{aligned}
(S\Phi V)^{h,w}_{\mathcal{H},\mathcal{J}}(x,y) &\equiv \sum_{\mathcal{M},\mathcal{M}'} (S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}\Phi^{h,w}_{H,M}V^{j,w}_{J;M'})_{\mathcal{H},\mathcal{M},\mathcal{J},\mathcal{M}'}\; x^{-\mathcal{H}+\mathcal{M}}y^{\mathcal{J}+\mathcal{M}'} \\
&= S^{w+\frac{1}{2}}(x,y)\Phi^{h,w}_{H}(x)V^{j,w}_{J}(y)\,, \qquad (3.47)
\end{aligned}
$$

where

$$
\begin{aligned}
&S^{w+\frac{1}{2}}(x,y) \equiv \\
&\sum_{\hat{m},\hat{m}'=-(w+\frac{1}{2})}^{w+\frac{1}{2}} \left[\frac{(-1)^{\hat{m}+w+\frac{1}{2}}\;\Gamma(2w+2)}{\Gamma(\hat{m}+w+\frac{3}{2})\Gamma(w-\hat{m}+\frac{3}{2})}\frac{(-1)^{\hat{m}'+w+\frac{1}{2}}\;\Gamma(2w+2)}{\Gamma(\hat{m}'+w+\frac{3}{2})\Gamma(w-\hat{m}'+\frac{3}{2})}\right] \\
&\qquad \times\; S^{w+\frac{1}{2}}_{\hat{m},\hat{m}'}\,x^{\hat{m}+w+\frac{1}{2}}y^{\hat{m}'+w+\frac{1}{2}}\,. \qquad (3.48)
\end{aligned}
$$

Taking into account the antiholomorphic part, the vertex operators creating spectral flow images of chiral primary states in the Ramond sector are given by

$$
\begin{aligned}
\mathcal{O}^{(a),w}_{\mathcal{H}\overline{\mathcal{H}}}(x,\overline{x},y,\overline{y}) &= e^{-\frac{\varphi}{2}}S^{w+\frac{1}{2}}(x,y)\overline{S}^{w+\frac{1}{2}}(\overline{x},\overline{y})\Phi^{h,w}_{H,\overline{H}}(x,\overline{x}) \\
&\qquad V^{j,w}_{J,\overline{J}}(y,\overline{y})e^{a\frac{i}{2}(\hat{H}_4-\hat{H}_5)}e^{\pm\frac{i}{2}(\hat{\overline{H}}_4-\hat{\overline{H}}_5)}\,. \qquad (3.49)
\end{aligned}
$$

Chapter 4

Sigma Model On The Symmetric Product Orbifold of $\mathbf{T}^4$

In this chapter we briefly review the dual field theory to type IIB super-string theory in $AdS_3 \times S^3 \times T^4$, which is given by a two-dimensional conformal field theory whose target space corresponds to the symmetric product of N T^4 torus, $(T^4)^N/S(N)$.

4.1 The Model

Type IIB superstring theory on $\mathrm{AdS}_3\times$ $\mathrm{S}^3\times$ T^4 with RR background is conjectured to be dual to the infrared fixed point theory living on a D1-D5 system compactified on T^4. As we mentioned at the end of chapter 1, it has been argued that in the Higgs branch this theory flows in the infrared to an $\mathcal{N}=4$ superconformal field theory (SCFT) on the symmetric product of $N=Q_1Q_5$ four-torus [41], i.e, the target space of this theory is given by $(T^4)^N/S(N)$, S being the symmetric group of N indices.

The $\mathcal{N}=(4,4)$ SCFT on T^4 is described by the free Lagrangian

$$S=\frac{1}{2}\int d^2z\ \left[\partial x^i_A\bar{\partial}x_{i,A}+\psi^i_A(z)\bar{\partial}\psi^i_A(z)+\widetilde{\psi}^i_A(\bar{z})\partial\widetilde{\psi}^i_A(\bar{z})\right] \tag{4.1}$$

Here i runs over the T^4 coordinates 1,2,3,4 and $A=1,2,\ldots,N$ labels various copies of the four-torus. The symmetric group $S(N)$ acts by permuting the A indices.

The $\mathcal{N}=4$ superconformal algebra with central charge $c=6N$ can be constructed out of N copies of four real fermions and bosons.

The generators are given by

$$
\begin{aligned}
T(z) &= \partial X_A(z)\partial X_A^{\dagger}(z) + \frac{1}{2}\Psi_A(z)\partial\Psi_A^{\dagger}(z) - \frac{1}{2}\partial\Psi_A(z)\Psi_A^{\dagger}(z)\,, \\
G^a(z) &= \begin{pmatrix} G^1(z) \\ G^2(z) \end{pmatrix} = \sqrt{2}\begin{pmatrix} \Psi_A^1(z) \\ \Psi_A^2(z) \end{pmatrix}\partial X_A^2(z) + \sqrt{2}\begin{pmatrix} -\Psi_A^{2\dagger}(z) \\ \Psi_A^{1\dagger}(z) \end{pmatrix}\partial X_A^1(z)\,, \\
J_R^i(z) &= \frac{1}{2}\Psi_A(z)\sigma^i\Psi_A^{\dagger}(z)\,.
\end{aligned}
\tag{4.2}
$$

The summation over A which runs from 1 to N is implied. By performing all the operator product expansions among the currents, we can realize that they generate an $\mathcal{N}=4$ super conformal algebra [69]. The bosons X and the fermions Ψ are given in terms of the torus coordinates and their fermionic super-partners as,

$$
\begin{aligned}
X_A(z) &= (X_A^1(z), X_A^2(z)) = \sqrt{1/2}(x_A^1(z)+ix_A^2(z), x_A^3(z)+ix_A^4(z)), \\
\Psi_A(z) &= (\Psi_A^1(z), \Psi_A^2(z)) = \sqrt{1/2}(\psi_A^1(z)+i\psi_A^2(z), \psi_A^3(z)+i\psi_A^4(z)) \\
X_A^{\dagger}(z) &= \begin{pmatrix} X_A^{1\dagger}(z) \\ X_A^{2\dagger}(z) \end{pmatrix} = \sqrt{\frac{1}{2}}\begin{pmatrix} x_A^1(z)-ix_A^2(z) \\ x_A^3(z)-ix_A^4(z) \end{pmatrix} \\
\Psi_A^{\dagger}(z) &= \begin{pmatrix} \Psi_A^{1\dagger}(z) \\ \Psi_A^{2\dagger}(z) \end{pmatrix} = \sqrt{\frac{1}{2}}\begin{pmatrix} \psi_A^1(z)-i\psi_A^2(z) \\ \psi_A^3(z)-i\psi_A^4(z) \end{pmatrix}
\end{aligned}
\tag{4.3}
$$

4.2 Short Multiplets

The representations of the supergroup are classified according to the conformal weight and $SU(2)_R$ quantum number. The highest weight states $|\mathrm{hw}\rangle = |h, \mathbf{j}_R, j_R^3 = j_R\rangle$ satisfy the following properties

$$
\begin{aligned}
L_1|\mathrm{hw}\rangle = 0 &\qquad L_0|\mathrm{hw}\rangle = h|\mathrm{hw}\rangle \\
J_R^{(+)}|\mathrm{hw}\rangle = 0 &\qquad J_R^{(3)}|\mathrm{hw}\rangle = j_R|\mathrm{hw}\rangle \\
G_{1/2}^a|\mathrm{hw}\rangle = 0 &\qquad G_{1/2}^{a\dagger}|\mathrm{hw}\rangle = 0
\end{aligned}
\tag{4.4}
$$

where $J_R^{\pm} = J_R^{(1)} \pm iJ_R^{(2)}$. $L_{\pm,0}, J_R^{(\pm),(3)}$ are the global charges of the currents $T(z)$ and $J_R^{(i)}(z)$. The charges $G^a_{1/2,-1/2}$ are the global charges of the supersymmetry currents $G^a(z)$.

Highest weight states which satisfy $G_{-1/2}^{2\dagger}|\mathrm{hw}\rangle = 0, \quad G_{-1/2}^1|\mathrm{hw}\rangle = 0$ are chiral primaries. Short multiplets are generated from the chiral primaries through the action of the raising operators $J_-, G^{1\dagger}_{-1/2}$ and $G^2_{-1/2}$. The structure of the short multiplet is given below

$$\begin{array}{cccc} \text{States} & j & L_0 & \text{Degeneracy} \\ |\text{hw}\rangle_S & h & h & 2h+1 \\ G^{1\dagger}_{-1/2}|\text{hw}\rangle_S, G^2_{-1/2}|\text{hw}\rangle_S & h-1/2 & h+1/2 & 2h+2h=4h \\ G^{1\dagger}_{-1/2}G^2_{-1/2}|\text{hw}\rangle_S & h-1 & h+1 & 2h-1 \end{array} \tag{4.5}$$

and $h = 1, 3/2, ..., (N+1)/2$.

The chiral primary operators on the symmetric product associated to the above states are given by the *single-cycle* twist operators

$$O^{(0)}_n(x,\bar{x})\,, \qquad O^{(a)}_n(x,\bar{x})\,, \qquad O^{(2)}_n(x,\bar{x})\,, \tag{4.6}$$

with $a = \pm$, and $n = 1, ..., N$ denotes the length of the cycle (For a precise definition see *e.g.* [23, 8, 9]). The corresponding conformal dimensions are

$$h^{(0)} = \frac{n-1}{2}\,, \qquad h^{(a)} = \frac{n}{2}\,, \qquad h^{(2)} = \frac{n+1}{2} \tag{4.7}$$

and similarly for the antiholomorphic sector. For the comparison with string theory computations, we will later use the label $h = (n+1)/2$ instead of n such that

$$h^{(0)} = h-1\,, \qquad h^{(a)} = h-1/2\,, \qquad h^{(2)} = h\,. \tag{4.8}$$

The (anti-)chiral operators $O^{(A)}_n$ $(O^{(A)\dagger}_n)$ $(A = 0, a, 2)$ form a (anti-)chiral ring under an $\mathcal{N} = 2$ subalgebra and satisfy $h^{(A)} = Q$ $(h^{(A)} = -Q)$, where q is the corresponding $U(1)$ charge.

Let us summarize the holomorphic chiral ring in the following table,

Field	$\Delta = Q$	Range of Δ
$O^{(0)}_h$	$h-1$	$0, \frac{1}{2} \dots \frac{N-1}{2}$
$O^{(a)}_h$	$h-1/2$	$\frac{1}{2}, 1 \dots \frac{N}{2}$
$O^{(2)}_h$	h	$1, \frac{3}{2} \dots \frac{N+1}{2}$

Table 1: Holomorphic chiral ring.

As the notation suggests, the operators displayed in table 1, are identified to be dual to (3.45), (3.44) and (3.49) respectively.

4.3 Some Correlation functions

Correlators of single-cycle twist operators are computed on covering surfaces of different genera. Quite generally, it can be shown from the Riemann-Hurwitz formula that if the cycle lengths of a p-point correlator satisfy

$$n_p = \sum_{i=1}^{p-1} n_i - p + 2\,, \tag{4.9}$$

the sphere is the only covering surface which contributes to the correlator [23].

Two- and three-point functions on the sphere for the operators in above table are given by [8, 9] [1]

$$\langle \mathcal{O}_n^{\epsilon_n}(x_1,\overline{x}_1)\mathcal{O}_{-n}^{-\epsilon_n}(x_2,\overline{x}_2)\rangle = |x_{12}|^{-4h}\ , \tag{4.10}$$

$$\langle \mathcal{O}_{n_1}^{\epsilon_{n_1}}\mathcal{O}_{n_2}^{\epsilon_{n_2}}\mathcal{O}_{n_3}^{\epsilon_{n_3}\dagger}\rangle = \sqrt{\frac{n_1n_2n_3}{\mathrm{N}}}\delta^2(\sum_{i=1}^{3}\mathcal{M}_{n_i})C_{n_1n_2n_3}\prod_{i<j}|x_{ij}|^{-2h_{ij}}\ , \tag{4.11}$$

where $(\epsilon_n;\overline{\epsilon}_{\overline{n}}) = \pm$, and with respect to the notation in Table 1 $(-)$ corresponds to the index (0) and $(+)$ corresponds to index (2), $h_{12} = h_1 + h_2 - h_3$, etc., $-h_i \le m_i \le h_i$ and the coefficients $C_{n_1n_2n_3}$ are defined in terms of the SU(2) 3j symbols as

$$\begin{aligned} C_{n_1n_2n_3} &= \frac{|\epsilon_{n_1}n_1 + \epsilon_{n_2}n_2 + \epsilon_{n_3}n_3 + 1|^2}{4n_1n_2n_3} \\ &\times \left|\begin{pmatrix} h_1 & h_2 & h_3 \\ m_1 & m_2 & m_3\end{pmatrix}\right|^2 \left|\frac{h_{12}!h_{23}!h_{31}!(\sum_{i=1}^{3}h_i+1)!}{(2h_1)!(2h_2)!(2h_3)!}\right|\ . \end{aligned}$$

Using (4.7) and $m_i = \pm h_i$, the delta function in (4.11) implies $h_{ij} = 0$ for certain i,j. Specifying $n_3 = n_1 + n_2 - 1$, the non-vanishing three-point functions are those with $(\epsilon_{n_1},\epsilon_{n_2},\epsilon_{n_3}) = (-,-,-)$ and $(+,-,+)$. In this case, the product in the second line reduces to one.

Two other correlators that will be important below have been evaluated in the particular case $n_3 = n_1 + n_2 - 1$ [7], namely (we omit the obvious coordinate dependence)

$$\left\langle \mathcal{O}_{n_1}^{a}\mathcal{O}_{n_2}^{-}\mathcal{O}_{n_3}^{a'\dagger}\right\rangle = \frac{1}{\sqrt{\mathrm{N}}}\left(\frac{n_1n_3}{n_2}\right)^{1/2}\delta^{a,a'}\delta^{\overline{a},\overline{a}'}\ , \tag{4.12}$$

$$\left\langle \mathcal{O}_{n_1}^{a}\mathcal{O}_{n_2}^{a'}\mathcal{O}_{n_3}^{+\dagger}\right\rangle = \frac{1}{\sqrt{\mathrm{N}}}\left(\frac{n_1n_2}{n_3}\right)^{1/2}\xi^{a,a'}\xi^{\overline{a},\overline{a}'}\ ,\quad \xi^{a,a'} = \xi^{\overline{a},\overline{a}'} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\ . \tag{4.13}$$

[1]Contributions from surfaces with higher genus are suppressed in the large N limit.

Several four-point correlators satisfying (4.9) on the sphere have been computed in [23]. Among others, the authors found the extremal four-point functions

$$\langle O^{(0)\dagger}_{n_4}(\infty) O^{(0)}_{n_3}(1) O^{(0)}_{n_2}(x,\bar{x}) O^{(0)}_{n_1}(0)\rangle = F_4(n_i)\frac{n_4^{5/2}}{(n_1 n_2 n_3)^{1/2}}\,, \tag{4.14}$$

$$\langle O^{(2)\dagger}_{n_4}(\infty) O^{(2)}_{n_3}(1) O^{(0)}_{n_2}(x,\bar{x}) O^{(0)}_{n_1}(0)\rangle = F_4(n_i)\frac{n_3^{3/2} n_4^{1/2}}{(n_1 n_2)^{1/2}}\,, \tag{4.15}$$

$$\langle O^{(b)\dagger}_{n_4}(\infty) O^{(a)}_{n_3}(1) O^{(0)}_{n_2}(x,\bar{x}) O^{(0)}_{n_1}(0)\rangle = \delta^{ab} F_4(n_i)\frac{n_4^{3/2} n_3^{1/2}}{(n_1 n_2)^{1/2}}\,, \tag{4.16}$$

$$\langle O^{(2)\dagger}_{n_4}(\infty) O^{(a)}_{n_3}(1) O^{(b)}_{n_2}(x,\bar{x}) O^{(0)}_{n_1}(0)\rangle = \epsilon^{ab} F_4(n_i)\frac{(n_4 n_3 n_2)^{1/2}}{n_1^{1/2}}\,, \tag{4.17}$$

where the function $F_4(n_i)$ is given by

$$F_4(n_i) = \left[\frac{(N-n_1)!(N-n_2)!(N-n_3)!}{(N-n_4)!(N!)^2}\right]^{1/2}\,. \tag{4.18}$$

Note that $F_4 \approx 1/N$ at large N and that the correlators are independent of the cross-ratio $x \equiv \frac{x_{12}x_{34}}{x_{13}x_{24}}$. The extremality conditions

$$h_4 = h_1 + h_2 + h_3\,, \quad \textit{etc.} \tag{4.19}$$

imposed on these correlators imply the condition (4.9), $n_4 = n_1 + n_2 + n_3 - 2$.[2]

There are also some *non-extremal* correlators satisfying (4.9). An example is given by the correlator [23]

$$\langle O^{(0)\dagger}_{n+2}(\infty) O^{(0)}_{2}(1) O^{(0)\dagger}_{2}(x,\bar{x}) O^{(2)}_{n}(0)\rangle = G(x,\bar{x})\,, \tag{4.20}$$

where for small x

$$G(x,\bar{x}) \approx \frac{(n+2)^{3/2}}{2(n+1)n^{1/2}}\sqrt{\frac{(N-n)(N-n-1)}{N^2(N-1)^2}}|x|^{-2}\,. \tag{4.21}$$

The correlator scales as $1/N$ at large N. The conformal dimensions are $h_1 = h_4 = \frac{n+1}{2}$ and $h_2 = h_3 = \frac{1}{2}$ and similarly for the anti-holomorphic sector. The correlator is clearly non-extremal since

$$h_4 = h_1 + h_2 + h_3 - 1\,, \tag{4.22}$$

[2] There is also a fifth extremal correlator, $\langle O^{(2,2)\dagger}_{n_4}(\infty) O^{(0,0)}_{n_3}(1) O^{(0,0)}_{n_2}(x,\bar{x}) O^{(0,0)}_{n_1}(0)\rangle$ with $n_4 = n_1 + n_2 + n_3 - 4$ [23], which does not satisfy (4.9).

but nevertheless satisfies (4.9). The appearance of two anti-chiral operators in (4.20) ensures charge conservation since

$$\sum_{i=1}^{4} q_i = h_1 - h_2 + h_3 - h_4 = 0\,. \tag{4.23}$$

Extremal correlators satisfy a non-renormalization theorem [21] and are thus protected along the entire moduli space. They should therefore be reproducible by a string or supergravity computation. The non-extremal correlator (4.20) is not a priori protected by a non-renormalization theorem.

Chapter 5

Three-point functions in AdS$_3\times$ S$^3\times$ T^4

In this chapter we shall compute three-point functions of physical vertex operators in the full WZNW model on AdS$_3\times$ S$^3\times$ T^4. Since our main interest is to compare these correlators with their corresponding duals by AdS/CFT, we will restrict to three-point functions involving chiral vertex operators only. This is based upon [18].

5.1 Three-point functions of chiral states

As we saw in chapter 2 (see also chapter 3), the usual unitary "unflowed" representations of the $SL(2,R)$ current algebra have the 'spin' bounded as $0 < h \leq k/2$. On the other hand, the symmetric product orbifold contains cycles of lengh $n \leq N$ (see Table 1 in chapter 4). We will see below in this chapter that perturbative worldsheet theory is valid when N is large. In that limit, it appears that infinitely many chiral operators are missing in the bulk [70]. This puzzle is solved once we recall that spectral flow generates inequivalent $SL(2,R)$ current algebra representations with 'spin' given by

$$\mathcal{H}_n = h - 1 + \frac{k}{2}w\,. \tag{5.1}$$

Thus we see that spectral flowed representations can take any value of $\mathcal{H}$ without violating the unitarity bound imposed on h. By comparing $h^{(0)}$ in (4.7) with (5.1) we see that the 'missing' string states should correspond to chiral primaries in the boundary with $n = 2h - 1 + kw$.

In order to explore to whole chiral spectrum and verify this correspondence, we compute correlation functions involving spectral flowed fields.

In this section we compute $w-$conserving three-point functions involving spectral flow images of chiral primary states [18, 66]. We restrict to the so called extremal correlators, satisfying $j_n = j_m + j_l$. Our results reduce to the unflowed case when taking all the winding numbers w inside the correlator to zero [14, 15, 17] (see also [16]).

NS-NS-NS three-point functions

Let us start by evaluating the following amplitudes

$$\mathcal{A}_3 = g_s^{-2} \left\langle \mathcal{O}^{(0),w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,y_1,\overline{x}_1,\overline{y}_1)\mathcal{O}^{(0),w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,y_2,\overline{x}_2,\overline{y}_2)\mathcal{O}^{(0),w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,y_3,\overline{x}_3,\overline{y}_3)\right\rangle_{S^2}, \tag{5.2}$$

$$\mathcal{A}'_3 = g_s^{-2} \left\langle \mathcal{O}^{(2),w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,y_1,\overline{x}_1,\overline{y}_1)\mathcal{O}^{(0),w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,y_2,\overline{x}_2,\overline{y}_2)\mathcal{O}^{(2),w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,y_3,\overline{x}_3,\overline{y}_3)\right\rangle_{S^2}. \tag{5.3}$$

The vertices $\mathcal{O}^{(0),w}_{\mathcal{H},\overline{\mathcal{H}}}$, $\mathcal{O}^{(2),w}_{\mathcal{H},\overline{\mathcal{H}}}$ were defined in the -1 ghost picture. Note that the total ghost number of a correlator on a genus-g surface must be $-\chi = -(2-2g)$. In order to have total ghost number -2, as required on the sphere, the above three-point correlator should contain one operator in 0 ghost picture. We change the picture of an operator by applying to it the picture changing operator [14, 15]. We change the picture of an unflowed operator for simplicity, *i.e.*

$$\begin{aligned}&\tilde{\mathcal{O}}^{(0)}_h(x,y,\overline{x},\overline{y})\\ &= \left[\left((1-h)\hat{\jmath}(x) + j(x) + \frac{2}{k}\psi(x)\chi_a(y)P^a_y\right)\times c.c.\right]\Phi_h(x,\overline{x})V_{h-1}(y,\overline{y})\,,\end{aligned} \tag{5.4}$$

$$\begin{aligned}&\tilde{\mathcal{O}}^{(2)}_h(x,y,\overline{x},\overline{y})\\ &= \left[\left(h\hat{k}(y) + k(y) + \frac{2}{k}\chi(y)\psi_a(x)D^a_x\right)\times c.c.\right]\Phi_h(x,\overline{x})V_{h-1}(y,\overline{y})\,.\end{aligned} \tag{5.5}$$

As discussed in detail below, this restriction is not strictly necessary to evaluate (5.3), but further knowledge on spectral flowed affine representations than is currently available is needed to compute (5.2) in a more general situation. In any case, we shall see that including an unflowed operator does not imply any loss of generality for correlators involving spectral flow images of chiral primary states in the SL(2,R) sector.

Replacing (5.4) in (5.2), $\mathcal{A}_3$ explicitly reads

$$\begin{aligned}
\mathcal{A}_3 &= g_s^{-2}\left\langle e^{-\varphi(z_1,\bar{z}_1)}e^{-\varphi(z_2,\bar{z}_2)}\right\rangle \left\langle V^{h_1-1,w}_{J_1,\overline{J}_1}(y_1,\overline{y}_1)V^{h_2-1,w}_{J_2,\overline{J}_2}(y_2,\overline{y}_2)V_{h_3-1}(y_3,\overline{y}_3)\right\rangle \\
&\times \ \langle\overline{\chi}^w(\overline{y}_1)\overline{\chi}^w(\overline{y}_2)\rangle\,\langle\chi^w(y_1)\chi^w(y_2)\rangle \\
&\times \ \Big\langle \Phi^{h_1,w}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\psi^{w+1}(x_1)\overline{\psi}^{w+1}(\overline{x}_1)\Phi^{h_2,w}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\psi^{w+1}(x_2)\overline{\psi}^{w+1}(\overline{x}_2) \\
&\times \ \{(1-h_3)\hat{\jmath}(x_3)+j(x_3)\}\,\{(1-h_3)\hat{\jmath}(\overline{x}_3)+j(\overline{x}_3)\}\,\Phi_{h_3}(x_3,\overline{x}_3)\Big\rangle\ ,
\end{aligned}$$

and inserting (5.5) into (5.3) and using $\psi_a(x)D^a_x = \frac{1}{2}(\psi(x)\partial_x + h\partial_x\psi(x))$ we get

$$\begin{aligned}
\mathcal{A}'_3 &= g_s^{-2}\left\langle e^{-\varphi(z_1,\bar{z}_1)}e^{-\varphi(z_2,\bar{z}_2)}\right\rangle \left\langle V^{h_1-1,w}_{J_1,\overline{J}_1}(y_1,\overline{y}_1)V^{h_2-1,w}_{J_2,\overline{J}_2}(y_2,\overline{y}_2)V_{h_3-1}(y_3,\overline{y}_3)\right\rangle \\
&\times \ \Big\{\Big[\left\langle\psi^w(x_1)\psi^{w+1}(x_2)\psi(x_3)\right\rangle\partial_{x_3}\left\langle\Phi^w_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^w_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle + \\
&\qquad h_3\,\left\langle\psi^w(x_1)\psi^{w+1}(x_2)\partial_{x_3}\psi(x_3)\right\rangle\left\langle\Phi^w_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^w_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle\Big] \\
&\times \ \left\langle\chi^{w+1}(y_1)\chi^w(y_2)\chi(y_3)\right\rangle\times c.c.\Big\}\ ,
\end{aligned} \tag{5.6}$$

where $w = w_1 = w_2$.

The SU(2) and fermionic expectation values involved in the above expression were discussed in [66] and we will recall them below. We now compute the SL(2,R) correlators, applying the technique developed in [71].

As we argued in chapter 3, the $(x,\bar{x})$ can be interpreted as the boundary coordinates, hence we would like to compute the correlation functions in that basis in order to compare the results in this chapter with those expected from the dual field theory. Since we do not know a prescription to perform generic spectral flow transformations in the $x-$basis[1] we first turn the expressions into the $m-$basis, perform the spectral flow, and then transform them back into the $x-$basis.

Recalling that the fields $\Phi^{h,w}_{H,\overline{H}}(x,\overline{x})$ behave as primary fields in $(x,\bar{x})-$coordinates, a generic three-point function in the $x-$basis can be written as, (we omit the z dependence for short)

$$\begin{aligned}
&\left\langle\Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi^{h_3,w_3}_{H_3,\overline{H}_3}(x_3,\overline{x}_3)\right\rangle \\
&\qquad = D(H_i,\overline{H}_i)\left(x_{12}^{-H_{12}}x_{23}^{-H_{23}}x_{13}^{-H_{13}}\times c.c.\right),
\end{aligned} \tag{5.7}$$

($c.c.$ stands for the antiholomorphic dependence). The goal is thus to determine the unknown function $D(H_i,\overline{H}_i)$.

From the inverse transform (3.9),

$$\Phi^{h,w}_{H,M,\overline{H},\overline{M}} = \int d^2x\,x^{H-M-1}\overline{x}^{\overline{H}-\overline{M}-1}\Phi^{h,w}_{H,\overline{H}}(x,\overline{x})\ , \tag{5.8}$$

[1] [71] gave a prescription to perform one unit of spectral flow in the $x-$basis.

(5.7) can be transformed to the $m-$basis as

$$\left\langle \prod_{i=1}^{3} \Phi^{h_i, w'_i}_{H_i, M_i, \overline{H}_i, \overline{M}_i} \right\rangle = (2\pi)^2 D(H_i, \overline{H}_i) W(H_i, M_i, \overline{H}_i, \overline{M}_i) \delta^2(M_1 + M_2 + M_3), \tag{5.9}$$

where

$$\begin{aligned} W(H_i, M_i, \overline{H}_i, \overline{M}_i) &= \int d^2x_1 d^2x_2 x_1^{H_1-M_1-1} x_2^{H_2-M_2-1} \overline{x}_1^{H_1-\overline{M}_1-1} \overline{x}_2^{H_2-\overline{M}_2-1} \\ &\times |x_{12}|^{-2H_{12}} |1-x_1|^{-2H_{13}} |1-x_2|^{-2H_{23}} . \end{aligned} \tag{5.10}$$

In principle, in order to obtain a general expression, we should integrate (5.10) for arbitrary H_i, M_i, $i = 1, 2, 3$. However, recall that the spectral flow with $w > 0$ ($w < 0$) turns primary states of the current algebra into lowest- (highest-) weight states of a global representation with $H = M = m + \frac{k+2}{2}w$, $\overline{H} = \overline{M} = \overline{m} + \frac{k+2}{2}w$ ($H = -M = -m + \frac{k+2}{2}|w|$, $\overline{H} = -\overline{M} = -\overline{m} + \frac{k+2}{2}|w|$) [2]. We are particularly interested in the case, $H_1 = M_1, \overline{H}_1 = \overline{M}_1$ and $H_2 = -M_2, \overline{H}_2 = -\overline{M}_2$, where the function (5.10) develops two single poles. The corresponding divergences are related to the regions $x_1, \overline{x}_1 \to 0$ and $x_2, \overline{x}_2 \to \infty$ in the integrand of $W(H_i, \overline{H}_i, M_i, \overline{M}_i)$. Near these two regions we simply get,

$$\left\langle \prod_{i=1}^{3} \Phi^{h_i, w'_i}_{H_i, M_i, \overline{H}_i, \overline{M}_i} \right\rangle = (2\pi)^2 V^2_{conf} \delta^2(M_1 + M_2 + M_3) D(H_i, \overline{H}_i), \tag{5.11}$$

where $V_{conf} = \int dx^2/|x|^2$.

On the other hand, it is known that spectral flow preserving $n-$point functions in the $m-$basis are related to correlators involving only unflowed operators as [73]

$$\begin{aligned} &\left\langle \prod_{i=1}^{n} \Phi^{h_i, w_i}_{H_i, M_i, \overline{H}_i, \overline{M}_i}(z_i, \overline{z}_i) \right\rangle_{\sum_i w_i = 0} = \\ &\prod_{j<i} (z_{ij})^{-w_j m_i - w_i m_j - \frac{k+2}{2} w_i w_j} \times c.c. \left\langle \prod_{i=1}^{n} \Phi^{w_i=0}_{h_i, m_i, \overline{m}_i}(z_i, \overline{z}_i) \right\rangle , \end{aligned} \tag{5.12}$$

and three-point functions of $w = 0$ primary states have the following form in $m-$basis [74, 75]:

$$\begin{aligned} &\left\langle \prod_{i=1}^{3} \Phi^{w_i=0}_{h_i, m_i, \overline{m}_i}(z_i, \overline{z}_i) \right\rangle = \\ &(2\pi)^2 \delta^2(\sum_i m_i) W(h_i, m_i, \overline{m}_i) C(h_i) |z_{12}|^{-2\Delta_{12}} |z_{13}|^{-2\Delta_{13}} |z_{23}|^{-2\Delta_{23}}, \end{aligned} \tag{5.13}$$

[2] The spectral flow labels w and w' for highest/lowest weight states of global representations in the $x-$ and $m-$basis, respectively, may be related as $w' = \frac{M}{H} w$.

with

$$C(h_1,h_2,h_3) = -\frac{G(1-h_1-h_2-h_3)G(-h_{12})G(-h_{13})G(-h_{23})}{2\pi^2\nu^{h_1+h_2+h_3-1}\gamma\left(\frac{k+1}{k}\right)G(-1)G(1-2h_1)G(1-2h_2)G(1-2h_3)}\,, \tag{5.14}$$

where $G(h) = k^{\frac{j(k+1-h)}{2k}}\Gamma_2(-h|1,k)\Gamma_2(k+1+h|1,k)$, Γ_2 being the Barnes double gamma function and $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$, $h_{12} = h_1 + h_2 - h_3$, etc.

Comparing with (5.11), one finds that the three-point functions involving spectral flow images of primary operators in arbitrary $w-$sectors in the $x-$basis corresponding to $w-$preserving amplitudes in the $m-$basis are given by [3]

$$\begin{aligned}&\left\langle \Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi^{h_3,w_3}_{H_3,\overline{H}_3}(x_3,\overline{x}_3)\right\rangle \\ &\qquad = \frac{1}{V^2_{conf}}W(h_i,m_i,\overline{m}_i)C(h_i)x_{12}^{-H_{12}}x_{13}^{-H_{13}}x_{23}^{-H_{23}}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}}\,.\end{aligned} \tag{5.15}$$

Recall that this result holds for operators satisfying $m_1 + m_2 + m_3 = 0$.

As discussed above, for highest/lowest weight states the function $W(h_i,m_i,\overline{m}_i)$ develops poles which cancel the factor V^{-2}_{conf}. Taking, for instance, a chiral field at $x_1,\overline{x}_1$ and an antichiral one at $x_2,\overline{x}_2$, *i.e.* $m_1 = \overline{m}_1 = h_1, m_2 = \overline{m}_2 = -h_2$, the residue of the double pole is just one, and we obtain [4]

$$\begin{aligned}\mathcal{A}^1_3 &\equiv \langle \Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)\Phi^{h_3,w_3}_{H_3,\overline{H}_3}(x_3,\overline{x}_3)\rangle \\ &= C(h_i)x_{12}^{-H_{12}}x_{13}^{-H_{13}}x_{23}^{-H_{23}}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}}\,.\end{aligned}$$

The following expectation value is also needed to evaluate $\mathcal{A}_3$:

$$\mathcal{A}^2_3 \equiv \langle \Phi^{h_1,w_1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w_2}_{H_2,\overline{H}_2}(x_2,\overline{x}_2)j(x_3)\Phi_{h_3}(x_3,\overline{x}_3)\rangle\,.$$

The OPE $j(x)\Phi^{h,w}_{H,\overline{H}}(x',\overline{x}')$ is only known so far for $w=1$ fields [71], namely

$$\begin{aligned}j(x',z')\Phi^{h,w=1}_{H,\overline{H}}(x,\overline{x};z,\overline{z}) &= (m-h+1)\frac{(x-x')^2}{(z-z')^2}\Phi^{h,w=1}_{H+1,\overline{H}}(x,\overline{x};z,\overline{z}) \\ &\quad + \frac{1}{z'-z}\left[2H(x-x') + (x-x')^2\partial_x\right]\Phi^{h,w=1}_{H,\overline{H}}(x,\overline{x};z,\overline{z})\,.\end{aligned} \tag{5.16}$$

[3]This correlation function was directly computed in the $x-$basis in [72] in the particular case $w_1 = w_2 = 1$, $w_3 = 0$ using the definition of $w=1$ vertex operators given in [71]. Here we have used a different technique which is useful to evaluate correlators involving fields in arbitrary w sectors and, specially, expectation values including currents.

[4]Normalizing the two-point functions of these operators to the identity, this result agrees with the prediction formulated in [66] when the correlator involves one unflowed state. Three flowed chiral primary operators obeying $m_1+m_2+m_3 = 0$ cannot meet the condition $h_3 = h_1+h_2-1$ under which the prediction of [66] holds.

Therefore, we restrict to this case. Inserting (5.16) into $\mathcal{A}_3^2$, one gets

$$\begin{aligned}\mathcal{A}_3^2 &= (1-h_1+m_1)\frac{(x_1-x_3)^2}{(z_1-z_3)^2} < \Phi^{h_1,w=1}_{H_1+1,\overline{H}_1}(x_1,\bar{x}_1)\Phi^{h_2,w=1}_{H_2,\overline{H}_2}(x_2,\bar{x}_2)\Phi_{h_3}(x_3,\bar{x}_3) > \\ &+ (1-h_2-m_2)\frac{(x_2-x_3)^2}{(z_2-z_3)^2} < \Phi^{h_1,w=1}_{H_1,\overline{H}_1}(x_1,\bar{x}_1)\Phi^{w=1,h_2}_{H_2+1,\overline{H}_2}(x_2,\bar{x}_2)\Phi_{h_3}(x_3,\bar{x}_3) > \\ &+ \frac{1}{z_3-z_1}\left[2H_1(x_1-x_3)+(x_1-x_3)^2\partial_{x_1}\right]\mathcal{A}_3^1 \\ &+ \frac{1}{z_3-z_2}\left[2H_2(x_2-x_3)+(x_2-x_3)^2\partial_{x_2}\right]\mathcal{A}_3^1 .\end{aligned}$$

The first two terms are easily evaluated using the procedure discussed above and we get

$$\begin{aligned}\left\langle \Phi^{h_1,w=1}_{H_1+1,\overline{H}_1}(x_1,\bar{x}_1)\Phi^{h_2,w=1}_{H_2,\overline{H}_2}(x_2,\bar{x}_2)\Phi_{h_3}(x_3,\bar{x}_3)\right\rangle &= W(h_i,m_1=h_1+1,m_2=-h_2,m_3) \\ \times V_{conf}^{-2}\, C(h_i)\, x_{12}^{-H_{12}-1}x_{13}^{-H_{13}-1}x_{23}^{-H_{23}+1}&\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} ,\end{aligned} \tag{5.17}$$

where

$$W(h_i,m_1=h_1+1,m_2=-h_2,m_3) = V_{conf}^2\frac{h_{13}}{m_1-h_1+1} , \tag{5.18}$$

and similarly,

$$\begin{aligned}\left\langle \Phi^{h_1,w=1}_{H_1,\overline{H}_1}(x_1,\overline{x}_1)\Phi^{h_2,w=1}_{H_2+1,\overline{H}_2}(x_2,\overline{x}_2)\Phi_{h_3}(x_3,\overline{x}_3)\right\rangle &= C(h_i)\frac{h_{23}}{1-h_2-m_2} \\ \times x_{12}^{-H_{12}-1}x_{13}^{-H_{13}+1}x_{23}^{-H_{23}-1}&\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} .\end{aligned} \tag{5.19}$$

Putting all together, we obtain

$$\begin{aligned}\mathcal{A}_3^2 = (3h_3-H_1-H_2)C(h_i)\, &x_{12}^{-H_{12}-1}x_{13}^{-H_{13}+1}x_{23}^{-H_{23}+1}\overline{x}_{12}^{-\overline{H}_{12}}\overline{x}_{13}^{-\overline{H}_{13}}\overline{x}_{23}^{-\overline{H}_{23}} \\ &\times z_{12}^{-\Delta_{12}+1}z_{13}^{-\Delta_{13}-1}z_{23}^{-\Delta_{23}-1}\overline{z}_{12}^{-\overline{\Delta}_{12}}z_{13}^{-\overline{\Delta}_{13}}\overline{z}_{23}^{-\overline{\Delta}_{23}} ,\end{aligned} \tag{5.20}$$

and analogously for the term containing the antiholomorphic current $\bar{j}(x)$ in $\mathcal{A}_3$.

To write down the final result, let us recall the fermionic and SU(2) correlators (see [66] for details).

$$< \psi^{w+1}(x_1)\psi^{w+1}(x_2) >= \frac{k}{2}\frac{x_{12}^{2(w+1)}}{z_{12}^{(w+1)^2}} , \tag{5.21}$$

$$\begin{aligned}&< \psi^{w+1}(x_1)\psi^{w+1}(x_2)\hat{\jmath}(x_3) > \\ &= \sum_{i=1}^{2}\frac{1}{z_{3i}}\left[2(w+1)x_{3i}+(x_{3i})^2\partial_{x_i}\right] < \psi^{w+1}(x_1)\psi^{w+1}(x_2) > \\ &= k(w+1)\frac{x_{13}x_{23}}{x_{12}}\frac{z_{12}}{z_{13}z_{23}}\frac{x_{12}^{2(w+1)}}{z_{12}^{(w+1)^2}} ,\end{aligned} \tag{5.22}$$

$$< \psi^w(x_1)\psi^{w+1}(x_2)\psi(x_3) >= k\frac{x_{12}^{2w}x_{23}^2 z_{13}^w}{z_{12}^{w2+w} z_{23}^{w+1}}\,. \tag{5.23}$$

Similar expressions are obtained for χ^w.

In the SU(2) WZNW model, normalizing the two−point functions as

$$\langle V_{j_1}(y_1,\overline{y}_1;z_1,\overline{z}_1)V_{j_2}(y_2,\overline{y}_2;z_2,\overline{z}_2) = \delta_{j_1j_2}\frac{|y_{12}|^{2j_1}}{|z_{12}|^{4\Delta_{j_1}}}\,, \tag{5.24}$$

the three−point functions are given by [76]

$$\langle V_{j_1}(y_1,\bar{y}_1;z_1,\bar{z}_1)V_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)V_{j_3}(y_3,\bar{y}_3;z_3,\bar{z}_3)\rangle = C'(j_1,j_2,j_3)\prod_{i<j}\frac{|y_{ij}|^{2j_{ij}}}{|z_{ij}|^{2\Delta_{ij}}}\,, \tag{5.25}$$

for $j_n \leq j_m + j_l$, where

$$C'(j_1,j_2,j_3) = \sqrt{\frac{\gamma(\frac{1}{k})}{\gamma(\frac{2j_1+1}{k})\gamma(\frac{2j_2+1}{k})\gamma(\frac{2j_3+1}{k})}}\;\frac{P(j_1+j_2+j_3+1)\,P(j_{12})\,P(j_{23})\,P(j_{31})}{P(2j_1)\,P(2j_2)\,P(2j_3)}$$

with $P(j) = \prod_{m=1}^{j}\gamma(\frac{m}{k}), P(0) = 1$.

As argued in [66], the structure constants for spectral flowed chiral fields in SU(2) are also given by $C'(j_i)$ for $j_n = j_m + j_l$. Therefore, collecting all the contributions and suppressing the $x-$ and $z-$dependence for short, we get

$$\mathcal{A}_3 = g_s^{-2}\frac{k^2}{4}\left|\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+1\right|^2 C'(j_i)C(h_i)\,, \tag{5.26}$$

$$\mathcal{A'}_3 = g_s^{-2}\frac{k^2}{4}\left|\mathcal{H}_1-\mathcal{H}_2+\mathcal{H}_3-1\right|^2 C'(j_i)C(h_i)\,. \tag{5.27}$$

As shown in [14, 15],

$$C(h_1,h_2,h_3)C'(j_1,j_2,j_3) = \frac{c_\nu^{1/2}}{2\pi}\prod_{i=1}^{3}\sqrt{B(h_i)}\,, \tag{5.28}$$

where

$$B(h_i) = \frac{k}{4\pi^3}\frac{\nu^{1-2h_i}}{\gamma\left(\frac{2h_i-1}{k}\right)}\,, \qquad \nu = \pi\frac{\Gamma\left(1-\frac{1}{k}\right)}{\Gamma\left(1+\frac{1}{k}\right)}\,, \tag{5.29}$$

with $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, and c_ν a free parameter.

In order to compare these results with the conjectured dual counterparts, the two-point functions must be normalized to the identity.

In the NS sector the two-point function is ($h = j+1$) [14, 15]

$$\langle \mathcal{O}^{(0),w}_{\mathcal{H},\bar{\mathcal{H}}}(x_1,\bar{x}_1,y_2,\bar{y}_2)\mathcal{O}^{(0),w}_{\mathcal{H},\bar{\mathcal{H}}}(x_2,\bar{x}_2,y_2,\bar{y}_2)\rangle = \frac{k^2}{g_s^2}\frac{B(h)\delta(0)\,|y_{12}|^{4\mathcal{J}}}{|z_{12}|^4|x_{12}|^{4(\mathcal{H})}}\,, \tag{5.30}$$

where we defined $\phi_i = \phi(z_i)$ and used (5.22), (5.24) and

$$\langle e^{-\phi(z_1)} e^{-\phi(z_2)} \rangle = \frac{1}{z_{12}} ,$$
$$\langle \Phi^{h,w}_{H,\overline{H}}(x_1,\overline{x}_1)\Phi^{h,w}_{H,\overline{H}}(x_2,\overline{x}_2)\rangle = g_s^{-2}\delta(0) x_{12}^{-2H}\overline{x}_{12}^{-2\overline{H}} , \tag{5.31}$$

The two-point function scales as $|x_{12}|^{-4h^{(0)}}$ with $h^{(0)}$ as in (4.8), which agrees with the scaling of the dual boundary operator.

In the Ramond sector we get the two-point function $(h = j+1)$

$$\langle \tilde{\mathcal{O}}^{a,w}_{\mathcal{H},\bar{\mathcal{H}}}(x_1,\bar{x}_1,y_2,\bar{y}_2)\mathcal{O}^{b,w}_{\mathcal{H},\bar{\mathcal{H}}}(x_2,\bar{x}_2,y_2,\bar{y}_2) \;=\; \frac{1}{g_s^2}\frac{k}{(2h-1+kw)^2}\frac{B(h)\delta(0)\,|y_{12}|^{4\mathcal{J}}}{|z_{12}|^4|x_{12}|^{4\mathcal{H}}}\delta^{ab} , \tag{5.32}$$

where we used

$$\langle s^a_+(x_1,y_1)s^b_-(x_2,y_2)\rangle = \delta^{ab}\frac{ix_{12}y_{12}}{(z_{12})^{5/4}} , \qquad \langle e^{-3\phi(z_1)/2}e^{-\phi(z_2)/2}\rangle = \frac{1}{(z_{12})^{3/4}} . \tag{5.33}$$

Note that one primary is in the $-1/2$ picture while the other one is in the $-3/2$ picture such that the total ghost number is -2, as required on the sphere. The two-point function scales as $|x_{12}|^{-4h^{(a)}}$ with $h^{(a)}$ as in (4.8).

In order to obtain the corresponding boundary correlators we need to integrate the above two-point functions over the worldsheet coordinates z_1 and z_2. Equivalently, we may fix $z_1 = 1$ and $z_2 = 0$ and divide the correlator by the volume of the conformal group V_{conf} which keeps the two points fixed. As shown in appendix A in [71], this removes the divergence coming from $\delta(0)$ and introduces the factor[5]

$$-\frac{2h-1}{2\pi\nu k^2\gamma(\frac{k+1}{k})c_\nu} = \frac{2h-1}{2\pi k} \quad \text{for} \quad \nu = \frac{\pi}{c_\nu}\frac{\Gamma(1-\frac{1}{k})}{\Gamma(1+\frac{1}{k})} . \tag{5.34}$$

We observe that other than the operators in the boundary conformal field theory, the chiral primaries are not normalized to unity. We therefore rescale the operators as

$$\mathbb{O}^{(0),w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x}) = \frac{\sqrt{2\pi}}{\sqrt{k\,B(h)(2h-1)}} g_s\,\mathcal{O}^{(0),w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x}) ,$$
$$\mathbb{O}^{a,w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x}) = \sqrt{\frac{2\pi(2h-1)}{B(h)}} g_s \mathcal{O}^{a,w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x}) . \tag{5.35}$$

The operator $\mathcal{O}^{(2),w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x})$ is rescaled as $\mathcal{O}^{(0),w}_{\mathcal{H},\overline{\mathcal{H}}}(x,\bar{x})$.

[5] $\nu = \nu(k)$ is a free parameter. As in [15], we leave c_ν (and thus ν) undetermined for the moment. c_ν will later be fixed, when we compare the bulk and boundary correlators. Note that $c_\nu = 1$ in [71].

Omitting the standard dependence on the coordinates, we thus get

$$\langle \mathbb{O}^{(0),w}_{\mathcal{H}_1,\overline{\mathcal{H}}_1} \mathbb{O}^{(0),w}_{\mathcal{H}_2,\overline{\mathcal{H}}_2} \mathbb{O}^{(0)}_{h_3} \rangle \;= 4 g_s (2\pi k\, c_\nu)^{1/2} \frac{|\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+1|^2}{\sqrt{(2h_1-1+kw)(2h_2-1+kw)(2h_3-1)}} \tag{5.36}$$

$$\langle \mathbb{O}^{(2),w}_{\mathcal{H}_1,\overline{\mathcal{H}}_1} \mathbb{O}^{(0),w}_{\mathcal{H}_2,\overline{\mathcal{H}}_2} \mathbb{O}^{(2)}_{h_3} \rangle \;= 4 g_s (2\pi k\, c_\nu)^{1/2} \frac{|\mathcal{H}_1-\mathcal{H}_2+\mathcal{H}_3-1|^2}{\sqrt{(2h_1-1+kw)(2h_2-1+kw)(2h_3-1)}}\,. \tag{5.37}$$

While (5.36) was obtained for $w=1$, (5.37) holds for arbitrary w.

These three-point functions involve one unflowed operator. We restricted to this case for simplicity. However, notice that when the three operators are spectral flow images of chiral primaries of SL(2,R) or the unflowed operator creates a highest/lowest weight primary state, the condition $h_i = \pm m_i$ together with the requirement $m_1+m_2+m_3=0$ imply, for example, $h_2 = h_1 + h_3$. Combined with the chirality condition $j_i = h_i - 1$, this gives $j_2 = j_1 + j_3 + 1$ which violates the triangular inequality $j_2 \leq j_1 + j_3$ of the SU(2) WZNW model. Therefore the SU(2) factor gives a zero for the whole three-point function. This conclusion does not apply when the unflowed operator obeys $h_3 \neq \pm m_3$. Therefore, the results (5.36) and (5.37) hold for amplitudes containing two flowed and one unflowed chiral primary operators as long as the latter does not create a highest/lowest weight state in the SL(2,R) sector.

Let us now compare these results with the correlators in the dual theory. Recall from chapter 1 that $g_s \sim \frac{R^2}{\alpha'^2\sqrt{N}}$ when the volume of the T^4 can be neglected. Besides, we are considering now the regime where stringy corrections are important, i.e $R \sim \sqrt{\alpha'}$. Therefore, if we choose $c_\nu = 1/2\pi k$, we can see the correlation functions (5.36) and (5.37) scale as $N^{-1/2}$ and the contribution to these from the sphere is reliable if N is large. Recall that the chiral string states $\mathcal{O}^{(0),w}_{\mathcal{H}\overline{\mathcal{H}}}$, $\mathcal{O}^{(2),w}_{\mathcal{H}\overline{\mathcal{H}}}$ have been identified with the chiral operators $\mathcal{O}^{(0)}_n$, $\mathcal{O}^{(2)}_n$ of the SCFT, respectively [14, 15, 66]. Moreover, the proposed identification between the quantum numbers of $\mathbb{O}^{(0),w}_{\mathcal{H}\overline{\mathcal{H}},h}$ and those of $\mathcal{O}^{(0)}_n(x,\bar{x})$ is the following [14, 15, 66]

$$\mathcal{H}_n = \frac{n-1}{2} = h - 1 + \frac{k}{2}w \qquad \Rightarrow \qquad n = 2h - 1 + kw\,, \tag{5.38}$$

and for $\mathbb{Y}^{h,w}_{\mathcal{H}\overline{\mathcal{H}}}$ and $\mathcal{O}^{(2)}_n(x,\bar{x})$ it is

$$\mathcal{H}_n = \frac{n+1}{2} = h + \frac{k}{2}w \qquad \Rightarrow \qquad n = 2h - 1 + kw\,. \tag{5.39}$$

Replacing these values of n in the boundary three-point functions (4.11), one gets at leading order

$$\langle \mathcal{O}^{(0)}_{n_1}\mathcal{O}^{(0)}_{n_2}\mathcal{O}^{(0)\dagger}_{n_3}\rangle = \frac{1}{\sqrt{\mathrm{N}}}\frac{|\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+1|^2}{\sqrt{(2h_1-1+kw_1)(2h_2-1+kw_2)(2h_3-1)}}, \quad (5.40)$$

$$\langle \mathcal{O}^{(2)}_{n_1}\mathcal{O}^{(0)}_{n_2}\mathcal{O}^{(2)\dagger}_{n_3}\rangle = \frac{1}{\sqrt{\mathrm{N}}}\frac{|\mathcal{H}_1-\mathcal{H}_2+\mathcal{H}_3-1|^2}{\sqrt{(2h_1-1+kw_1)(2h_2-1+kw_2)(2h_3-1)}}, \quad (5.41)$$

in perfect agreement with (5.36) and (5.37), respectively. Furthermore, using the bulk-to-boundary dictionary, one can verify that the boundary correlators corresponding to three spectral flow images of chiral primary operators is zero because in both cases (5.40) and (5.41) the relation $h_2 = h_1 + h_3$ implies $n_2 = n_1 + n_3$, which violates the U(1) charge by one unit.

Two other correlators can be considered in the string theory corresponding to the vanishing correlators $((2),(0),(2))$ and $((0),(0),(2))$ in the boundary CFT, namely $\langle\prod_{i=1}^{3}\mathcal{O}^{(2),w_i}_{\mathcal{H}_i,\overline{\mathcal{H}}_i}\rangle$ and $\langle\prod_{i=1}^{2}\mathcal{O}^{(0),w_i}_{\mathcal{H}_i,\overline{\mathcal{H}}_i}\mathcal{O}^{(2),w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}\rangle$. It is easy to see that they violate the SU(2) charge conservation in the case $j_2 = j_1 + j_3$ that we are considering, and therefore they also vanish.

R-R-NS three-point functions

The chiral states $\mathcal{O}^{a,w}_{\mathcal{H},\overline{\mathcal{H}}}$ were identified with the operators $\mathcal{O}^{a}_{n}$ in [17, 66]. To compare the corresponding three-point functions in the dual theories, the two R-R-NS correlators needed are

$$\mathbb{A}_3 = g_s^{-2}\left\langle \mathcal{O}^{a,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,\overline{x}_3)\mathcal{O}^{a,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,\overline{x}_2)\mathcal{O}^{(o),w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,\overline{x}_1)\right\rangle_{S^2}, \quad (5.42)$$

$$\mathbb{A}'_3 = g_s^{-2}\left\langle \mathcal{O}^{(2),w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}(x_3,\overline{x}_3)\mathcal{O}^{a,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}(x_2,\overline{x}_2)\mathcal{O}^{a,w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}(x_1,\overline{x}_1)\right\rangle_{S^2}. \quad (5.43)$$

The R vertices (3.49) were obtained in the $-\frac{1}{2}$ picture, so it is not necessary to insert a picture changing operator and we can compute this amplitude for states in arbitrary w sectors, as long as $w_n = w_m + w_l$.

The SU(2) part of the three-point functions is given by $C'(j_i)$ for $j_n = j_m + j_l$ and the fermionic contributions are the following [66] [6]

$$\begin{aligned}\langle S^{w_3+\frac{1}{2}}(x_3,y_3)S^{w_2+\frac{1}{2}}(x_2,y_2)\psi^{w_1+1}(x_1)\chi^{w_1}(y_1)\rangle &= \frac{(w_1+w_2)!}{w_1!w_2!},\\ \langle \psi^{w_3}(x_3)\chi^{w_3+1}(y_3)S^{w_2+\frac{1}{2}}(x_2,y_2)S^{w_1+\frac{1}{2}}(x_1,y_1)\rangle &= \frac{(w_1+w_2)!}{w_1!w_2!}. \quad (5.44)\end{aligned}$$

As shown in the previous section, the SL(2,R) contribution is simply $C(h_i)$ for two or three flowed chiral primary states satisfying $m_1 + m_2 + m_3 = 0$. If the

[6] We get the inverse of the result reported in [66].

three operators are flowed, the SU(2) spins violate the triangular inequality and the correlator vanishes, analogously to the NS-NS-NS case. When one operator is unflowed, the factor $\frac{(w_1+w_2)!}{w_1!w_2!}$ reduces to unity and we have

$$\mathbb{A}_3 = \mathbb{A}_3' = g_s^{-2}\sqrt{B(h_1)B(h_2)B(h_3)}\,. \tag{5.45}$$

Normalizing the R operators as (5.35) we get

$$\begin{aligned}\left\langle \mathbb{O}^{a,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}\mathbb{O}^{a,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}\mathbb{O}^{(0),w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}\right\rangle &= \left\langle \mathbb{O}^{a,w_3}_{\mathcal{H}_3,\overline{\mathcal{H}}_3}\mathbb{O}^{a,w_2}_{\mathcal{H}_2,\overline{\mathcal{H}}_2}\mathbb{O}^{(2),w_1}_{\mathcal{H}_1,\overline{\mathcal{H}}_1}\right\rangle \\ &= 2g_s\left[\frac{(2h_3+kw_3-1)(2h_2+kw_2-1)}{(2h_1+kw_1-1)}\right]^{1/2}\,,\end{aligned} \tag{5.46}$$

for $w_1 = 0$ or $w_2 = 0$, again in agreement with the boundary correlators (4.12) and (4.13). These correlators as well as (5.36), (5.37) were also predicted in [66] by using the AdS/CFT correspondence.

Summarizing, in this chapter we have compute three-point functions of chiral primary operators in spectral flowed representations and have found that they agree with the corresponding correlation functions in the conformal field theory on the symmetric product.

On one hand, this result confirms the non-renormalization theorem for three-point functions of chiral operators in the field theory side [21]. On the other hand, we can notice from (5.36), (5.37) and (5.46) that three-point functions involving spectral flowed operators in the full superstring theory are just copies of the ones involving only unflowed states. This is a very non-trivial fact since in the full supersymmetric theory we have to use picture-changing currents which introduce additional dependence on the winding number w, as well as correlation functions of fermions in higher spin representations, such as (5.21), which also contain a non-trivial dependence on the winding number w. Thus it is quite surprising that at the end all the dependence on w factorizes in a way which reproduces a copy of the unflowed result.

Chapter 6

Four-point functions in AdS$_3\times$ S$^3\times$ T^4

In this chapter we shall compute four-point functions of chiral vertex operators in string theory on $AdS_3 \times S^3 \times T^4$, based upon [24].

6.1 Four-point function of chiral states

In this section we compute a four-point correlator which involves only unflowed chiral primary operators of the NS sector [1]. In particular we are interested in computing the correlator

$$G_4^{NS}(x,\bar{x}) = g_s^{-2}\int d^2z \left\langle \tilde{\mathcal{O}}^{(0)}_{j_4,m_4}(\infty)\mathcal{O}^{(0)}_{j_3,m_3}(1)\tilde{\mathcal{O}}^{(0)}_{j_2,m_2}(x,\bar{x};z,\bar{z})\mathcal{O}^{(0)}_{j_1,m_1}(0)\right\rangle\,, \quad (6.1)$$

where we choose m-labels as ($d \in \mathbb{Z}$)

$$\begin{aligned} m_1 &= \bar{m}_1 = j_1\,, \\ m_2 &= \bar{m}_2 = j_2 - d\,, \\ m_3 &= \bar{m}_3 = j_3\,, \\ m_4 &= \bar{m}_4 = -j_4 = -(j_1+j_2+j_3-d)\,. \end{aligned} \quad (6.2)$$

By conformal invariance the worldsheet coordinates are fixed as $z_{1,2,3,4} = 0, z, 1, \infty$, where z is the cross-ratio $z = z_{12}z_{34}/(z_{13}z_{24})$ on the worldsheet. Similarly, in the $x-$basis we choose $x_{1,2,3,4} = 0, x, 1, \infty$. Thus, the complex x corresponds to the

[1] Recall the general vertex $\mathcal{O}^{(0),w}_{\mathcal{H},\mathcal{M}}$ is denoted by $\mathcal{O}^{(0)}_{h,m}$ in the unflowed $w = 0$ sector. We also use h or j indistinguishably, keeping in mind that they are simply related by $h = j + 1$.

spacetime (boundary) cross-ratio. The correlator $G_4^{NS}(x,\bar{x})$ involves two ghost number zero and two ghost number -1 operators, $\tilde{\mathcal{O}}_j^{(0)}$ and $\mathcal{O}_j^{(0)}$, respectively.

The correlator (6.1) is extremal, if the spacetime scalings of the operators in $G_4^{NS}(x,\bar{x})$ satisfy (4.19), $\mathcal{H}_4 = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$. Using (4.8) and $h_i = j_i + 1$ $(i = 1, ..., 4)$, for (6.1) this translates into the condition

$$j_4 = j_1 + j_2 + j_3 \tag{6.3}$$

or $d = 0$. We will first consider the non-extremal case $d > 0$ and come back to the extremal case $d = 0$ in section 6.1.4.

Substituting the explicit expressions for these operators, as given by (5.4), (5.5) and (3.44) with $w = 0$, we get[2]

$$\begin{aligned}
G_4^{NS}(x,\bar{x}) = g_s^{-2} \int d^2 z \Bigg[&(1-h_2)(1-h_4) \left\langle \psi(0)\hat{\jmath}(x)\psi(1)\hat{\jmath}(\infty)\right\rangle \left\langle \prod_{i=1}^{4} \Phi_{h_i} \right\rangle \\
&+(1-h_2) \left\langle \psi(0)\hat{\jmath}(x)\psi(1)\right\rangle \left\langle j(\infty) \prod_{i=1}^{4} \Phi_{h_i} \right\rangle \\
&+(1-h_4) \left\langle \psi(0)\psi(1)\hat{\jmath}(\infty)\right\rangle \left\langle j(x) \prod_{i=1}^{4} \Phi_{h_i} \right\rangle \\
&+ \left\langle \psi(0)\psi(1)\right\rangle \left\langle j(x) j(\infty) \prod_{i=1}^{4} \Phi_{h_i} \right\rangle \Bigg] \left\langle \prod_{i=1}^{4} \Phi'_{j_i,m_i} \right\rangle \left\langle e^{-\phi(0)} e^{-\phi(1)} \right\rangle \times c.c\,.
\end{aligned} \tag{6.4}$$

The actual computation of $G_4^{NS}(x,\bar{x})$ will be done along the lines of [71].

6.1.1 Some correlators inside $G_4^{NS}(x,\bar{x})$

Following [71], we write the $SL(2)$ four-point function (see appendix (B))

$$\begin{aligned}
\left\langle \prod_{i=1}^{4} \Phi_{h_i} \right\rangle = &|x_{24}|^{-4h_2} |x_{14}|^{2(h_2+h_3-h_1-h_4)} |x_{34}|^{2(h_1+h_2-h_3-h_4)} |x_{13}|^{2(h_4-h_1-h_2-h_3)} \\
&\times |z_{24}|^{-4\Delta_2} |z_{14}|^{2\nu_1} |z_{34}|^{2\nu_2} |z_{13}|^{2\nu_3} \mathcal{F}_{SL(2)}(x,\bar{x};z,\bar{z})\,,
\end{aligned} \tag{6.5}$$

in terms of the factorization Ansatz [78]

$$\mathcal{F}_{SL(2)}(x,\bar{x};z,\bar{z}) = \int_{\frac{1}{2}+iR} dh\, \mathcal{C}(h) |\mathcal{F}_h(x;z)|^2\,, \tag{6.6}$$

[2] There is also a non-vanishing term involving the correlator $\langle(\chi_a P^a_{y_4})(\chi_b P^b_{y_2}) \prod_{i=1}^4 \Phi_{h_i} \Phi'_{j_i}\rangle$. This term turns out to be subleading in x and may be neglected in the small x region, see the discussion below.

where the normalization $\mathcal{C}(h)$ is given by $\mathcal{C}(h) = \frac{C(h_1,h_2,h)C(h,h_3,h_4)}{B(h)}$. The functions $B(h)$ and $C(h_1, h_2, h_3)$ are the scaling of the $SL(2)$ two-point function and the $SL(2)$ structure constants, respectively. They are given by (5.29) and (5.14) in chapter 5. As in [71], we change variables from z to u by defining $u = z/x$ and consider the case $|x| < 1$. We may then perform an expansion of $\mathcal{F}_h(x; u)$ in powers of x as

$$\mathcal{F}_h(x; u) = x^{\Delta(h)-\Delta(h_1)-\Delta(h_2)+h-h_1-h_2} u^{\Delta(h)-\Delta(h_1)-\Delta(h_2)} \sum_{m=0}^{\infty} g_m(u) x^m \,. \tag{6.7}$$

Substituting this expansion into the KZ equation for $SL(2)$ [78], one finds that the first term obeys the hypergeometric equation in u, *i.e.*

$$g_0(u) = F(a, b, c|u)\,, \tag{6.8}$$

with $a = h_1 + h_2 - h$, $b = h_3 + h_4 - h$, $c = k - 2h$. We will sometimes use the shorthand notation $F_h(u) \equiv F(a, b, c|u)$. In what follows we will focus on the leading term in the x expansion,

$$\mathcal{F}_h(x; u) = x^{\Delta(h)-\Delta(h_1)-\Delta(h_2)+h-h_1-h_2} u^{\Delta(h)-\Delta(h_1)-\Delta(h_2)} F_h(u) + \ldots\,, \tag{6.9}$$

where the ellipsis represents higher order terms in x. Such terms correspond to descendants under the global $SL(2)$ algebra [71], which do not play a role in the small x region. It is convenient to write $F_h(u)$ as a power series in u,

$$F_h(u) = \sum_{n=0}^{\infty} \mathcal{H}(a, b, c, n) u^n\,, \tag{6.10}$$

with coefficients

$$\mathcal{H}(a, b, c, n) = \frac{\Gamma(a + n)\Gamma(b + n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c + n)\Gamma(n + 1)}\,. \tag{6.11}$$

A similar factorization ansatz can be found for the $SU(2)$ four-point function. As shown in appendix D, at small z the $SU(2)$ four-point function with m-values as in (6.2) can be expanded as[3]

$$\left\langle \prod_{i=1}^{4} \Phi'_{j_i,m_i} \right\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{C}'(j)\, |\mathcal{G}_j(z)|^2\,, \tag{6.12}$$

[3]We assume that the level k is large enough. For small k, the upper bound of summation is changed [76].

with

$$
\begin{aligned}
|\mathcal{G}_j(z)|^2 &= \sum_{n'=0}^{\infty} G_{j,n'}\, |z|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}\,, \\
G_{j,n'} &= \delta^2_{j_1+j_2+j_3-j_4,d}\, c_{2j_2}^{j_2+m_2} \mathcal{D}(j_1,j_2,J)\mathcal{D}(J,j_3,j_4) \\
&\quad \times \frac{\Gamma(0)^2}{\Gamma(j+n'-j_1-j_2+1+d)^2\Gamma(j_4-j-n'-j_3)^2}\,. \qquad (6.13)
\end{aligned}
$$

c_{2j}^{j+m} are the inverse of the binomial coefficients,

$$
c_{2j}^{j+m} = \frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(2j+1)}\,. \qquad (6.14)
$$

The δ-function reflects the charge conservation $m_1+m_2+m_3+m_4=0$. The normalization $\mathcal{C}'(j)$ is given by $\mathcal{C}'(j) = C'_{j,j_1,j_2}C'_{j,j_3,j_4}$ (no summation over j). The $SU(2)$ structure constants C'_{j_1,j_2,j_3} and the functions $\mathcal{D}(j_1,j_2,J)$ are given by (B-20) and (D-8) in the appendix, respectively.

We will also need some other four-point correlators for $G_4^{NS}(x,\bar{x})$. For the following, it is useful to define the n-point correlators

$$
d_k^{(n)} = \left\langle j(x_k)\prod_{i=1}^{n}\Phi_{h_i}(x_i)\right\rangle\,, \qquad d_{km}^{(n)} = \left\langle j(x_k)j(x_m)\prod_{i=1}^{n}\Phi_{h_i}(x_i)\right\rangle\,, \qquad (6.15)
$$

with $k,m=1,...,n$, in which one or two bosonic currents $j(x)$ act on the product of n $SL(2)$ functions $\Phi_h(x)$. As shown in appendix C, such correlators can entirely be expressed in terms of derivatives of the $SL(2)$ n-point function. In particular, the functions $d_2^{(4)}$, $d_4^{(4)}$ and $d_{24}^{(4)}$ appearing in (6.4) can be computed by means of (C-6) and (C-7). Using only the first term in the small x expansion (6.9) of the $SL(2)$ four-point function (6.5) (and $x = x_{12}x_{34}/(x_{13}x_{24})$), we find

$$
d_k^{(4)} = \langle j(x_k)\prod_{i=1}^{4}\Phi_{h_i}(x_i)\rangle = \int dh\, \mathcal{C}(h)\left|\sum_{n=0}^{\infty}\hat{d}_{k,n}^{(4)}\,\mathbb{S}_n\right|^2\,, \qquad (6.16)
$$

with

$$
\begin{aligned}
\mathbb{S}_n &= (x_{24})^{-2h_2}(x_{14})^{h_2+h_3-h_1-h_4}(x_{34})^{h_1+h_2-h_3-h_4}(x_{13})^{h_4-h_1-h_2-h_3} \\
&\quad \times (z_{24})^{-2\Delta_2}(z_{14})^{\nu_1}(z_{34})^{\nu_2}(z_{13})^{\nu_3} \\
&\quad \times x^{h-h_1-h_2-n} z^{\Delta(h)-\Delta(h_1)-\Delta(h_2)+n}\mathcal{H}(a,b,c,n) \qquad (6.17)
\end{aligned}
$$

and $\mathcal{H}(a,b,c,n)$ as in (6.11). For $k=4,2,1$, the coefficients are given by[4]

$$
\begin{aligned}
\hat{d}^{(4)}_{4,n} &= -\frac{z_{13}}{z_{34}z_{14}}\frac{x_{34}x_{14}}{x_{13}}(h+h_3-h_4-n) \\
&\quad + \frac{z_{12}}{z_{24}z_{14}}\frac{x_{24}x_{14}}{x_{12}}(h-h_1-h_2-n)\,, && (6.18)\\
\hat{d}^{(4)}_{2,n} &= \frac{z_{34}}{z_{24}z_{23}}\frac{x_{24}x_{23}}{x_{34}}(h-h_3-h_4-n) \\
&\quad + \frac{z_{14}}{z_{24}z_{12}}\frac{x_{24}x_{12}}{x_{14}}(h_1-h_2-h_3+h_4-n) \\
&\quad - \frac{z_{13}}{z_{23}z_{12}}\frac{x_{23}x_{12}}{x_{13}}(h+h_3-h_4-n)\,, && (6.19)\\
\hat{d}^{(4)}_{1,n} &= \frac{z_{34}}{z_{14}z_{13}}\frac{x_{14}x_{13}}{x_{34}}(h-h_3-h_4-n) \\
&\quad - \frac{z_{24}}{z_{14}z_{12}}\frac{x_{14}x_{12}}{x_{24}}(h-h_1+h_2-n)\,. && (6.20)
\end{aligned}
$$

Finally, the correlator $d^{(4)}_{24}$ is given by

$$
d^{(4)}_{24} = \int dh\, \mathcal{C}(h) \left| \sum_{n=0}^{\infty} \hat{d}^{(4)}_{24,n}\, \mathbb{S}_n \right|^2 , \tag{6.21}
$$

$$
\hat{d}^{(4)}_{24,n} = -\frac{(h+h_3-h_4-n)(-h-h_1+h_2+n)x}{z} + \ldots\,, \tag{6.22}
$$

which, for brevity, is expanded around $z=0$ (the ellipses denote further terms subleading in z). Also the x- and z-dependence is already fixed as above. Note that the above expressions for the $d^{(4)}$ correlators are only valid for small x.

We will also need the fermionic correlators

$$
\begin{aligned}
\langle \psi(x_1)\psi(x_2)\rangle &= k\frac{(x_{12})^2}{z_{12}}\,, \\
\langle \psi(x_1)\psi(x_2)\jmath(x_3)\rangle &= 2k\frac{x_{12}x_{23}x_{31}}{z_{31}z_{23}}\,, \\
\langle \psi(x_1)\jmath(x_2)\psi(x_3)\jmath(x_4)\rangle &= 2k\left[\frac{z_{13}x_{23}x_{14}}{z_{34}z_{23}z_{14}x_{13}^2}(x_{13}x_{24}+x_{12}x_{34})\right. \\
&\qquad \left. -\frac{z_{13}x_{34}x_{12}}{z_{34}z_{14}z_{12}x_{13}^2}(x_{14}x_{32}+x_{13}x_{42})\right]\,, && (6.23)
\end{aligned}
$$

which have been computed using (C-8) in appendix C.

Substituting now the correlators (6.18), (6.19), (6.21) and (6.23) as well as the expansions (6.5) (with (6.9)) and (6.12) for the $SL(2)$ and $SU(2)$ four-point

[4]Here we also list the coefficient $\hat{d}^{(4)}_{1,n}$ for later use.

functions into (6.4), for small x we find

$$
\begin{aligned}
G_4^{NS}(x,\bar{x}) = g_s^{-2}k^2 \int d^2u \sum_{j,n'} \mathcal{C}'(j) \int dh\, \mathcal{C}(h)\, |x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1+n')} \\
\times |u|^{2(\Delta(h)+\Delta(j)+n')} \times \Bigg| \sum_{n=0}^{\infty} \Bigg[(1-h_2)(1-h_4) 2 \frac{(2x-1)z + x(x-2)}{z(z-1)} + \\
\hat{d}^{(4)}_{4,n} \quad + \; (1-h_2) 2 \frac{(x-1)x}{(z-1)z} + (1-h_4) 2\, \hat{d}^{(4)}_{2,n} + \hat{d}^{(4)}_{24,n} \Bigg] \mathcal{H}(a,b,c,n) u^n \Bigg|^2 G_{j,n'}\,,
\end{aligned}
\tag{6.24}
$$

where it is understood that z needs to be replaced by $z = ux$. Note also $\Delta(h_i) + \Delta(j_i) = 0$ for the external fields.

6.1.2 Moduli integration and integral over h

We now perform the integrals over the worldsheet cross-ratio u and the $SL(2)$ representation label h. We wish to do the u-integral before the integral over h but need to be careful about the occurrence of divergences. Following [71, 79], we therefore regularize the u-integral by introducing a cut-off parameter ε and divide the range of u into two regions:

$$
\begin{aligned}
\text{region I:} \quad & |u| < \varepsilon \\
\text{region II:} \quad & |u| > \varepsilon\,.
\end{aligned}
$$

In region I there are only operators in the intermediate channel whose $SL(2)$ part is associated with short strings with winding number $w = 0$ [71]. In region II there can be long strings with $w = 1$ and two-particle states [71]. The representation theory of $SL(2)$ does not allow any other spectrally-flowed states in the intermediate channel.

An important observation is that "single-cycle" operators in the spacetime CFT arise locally on the worldsheet, *i.e.* in the small u region, while "multi-cycle" operators correspond to non-local contributions coming from the large u region [71, 79].[5] Since at large N multi-particle contributions are suppressed in non-extremal correlators [23], we may restrict to the one-particle contributions to the four-point correlator. We therefore consider only region I and ignore possible two-particle contributions coming from region II.

[5]The "single-cycle" operators (or "single-trace" operators in higher-dimensional CFTs) correspond to one-particle states in the worldsheet theory. Similarly, "multi-cycle" operators correspond to multi-particle states.

Formally, the one-particle contributions are taken into account by first integrating over the small u region, $|u| < \varepsilon$, and then taking the limit $\varepsilon \to 0$. This is the limit where the operators approach each other in their worldsheet coordinates. For $|u| < \varepsilon$, we may then expand $G_4^{NS}(x, \bar{x})$ in powers of u as

$$
\begin{aligned}
&G_4^{NS}(x, \bar{x}) \qquad (6.25)\\
&= g_s^{-2} k^2 \int d^2 u \int dh \sum_{j,n'} \mathcal{C}(h)\mathcal{C}'(j) G_{j,n'}\, |x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1+n')} |u|^{2(\Delta(h)+\Delta(j)+n')} \\
&\quad \times \left| \sum_{n=0}^{\infty} \left[-\frac{(h+h_1+h_2-2-n)(h+h_3+h_4-2-n)}{u} + O(u^0) \right] \mathcal{H}(a,b,c,n) u^n \right|^2 ,
\end{aligned}
$$

where we display only the most singular term in the square brackets. Subleading terms are summarized in $O(u^0)$.

The relevant u-integral inside $G_4^{NS}(x, \bar{x})$ is

$$
\sum_{n,\bar{n}=0}^{\infty} \int_{|u|<\varepsilon} d^2 u\, |u|^{2(\lambda-1)} u^n \bar{u}^{\bar{n}} = \sum_{n,\bar{n}=0}^{\infty} \frac{\pi}{\lambda+n} \varepsilon^{2(\lambda+n)} \delta_{n,\bar{n}} \qquad (6.26)
$$

with $\lambda = \Delta(h) + \Delta(j) + n'$.

We now turn to the integration over h. The h-integral is defined along the line $h = \frac{k-1}{2} + is$ ($s \in \mathbb{R}$), away from the locus of the continuous representation of $SL(2)$, $h = \frac{1}{2} + is$. The reason for the deformation is that only there the integrand is equivalent to a monodromy invariant solution, cf. (4.34) in [71]. It is possible to shift the integration contour back to $h = \frac{1}{2} + is$. However, in general, the integral picks up pole residues when the poles cross the integration contour. At small u there are altogether four types of poles of the h-integral which may contribute to the integral. These are [71]:

$$
\begin{aligned}
\text{type I:} &\quad \lambda + n = 0\,, \\
\text{type II:} &\quad h = h_1 + h_2 + n\,, \\
\text{type III:} &\quad h = k - h_1 - h_2 + n\,, \\
\text{type IV:} &\quad h = |h_1 - h_2| - n\,, \qquad n \in \{0, 1, 2, ...\}\,.
\end{aligned}
$$

The poles of type II-IV are poles in the structure constants $C(h, h_1, h_2)$. As discussed extensively in [79], none of these poles contributes to the integral. Even though naively one might interpret the contributions from the poles of type II as "double-cycle" operators in the spacetime CFT, such contributions go to zero in the $\varepsilon \to 0$ limit [79]. Type III poles do not appear if $h_1 + h_2 < \frac{k+1}{2}$ [71].

The contribution coming from poles of type IV was found to be canceled by the same contribution from crossing the integration contour [79].

We are left with poles of type I. These poles correspond to short string representations (with zero winding number) in the $SL(2)$ WZW model [71]. The condition

$$\lambda + n = \Delta(h) + \Delta(j) + n + n' = 0 \qquad (n, n' \geq 0) \tag{6.27}$$

is solved by $(h > 0)$

$$h = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4k(n + n') + 4j(j+1)}\,. \tag{6.28}$$

A particular solution is $n + n' = 0$ and $h = j + 1$. Since n and n' are both positive, $n = n' = 0$ and we recover the on-shell condition for chiral primaries in the intermediate channel. As such they map to single-cycle chiral primary operators in the spacetime CFT.

For $n + n' \neq 0$, we generically do not get a rational conformal weight h. Substituting the condition (6.27) into (6.25), we find that the correlator depends on x as $x^{h-n-h_1-h_2}$. This should be compared with the x dependence of the corresponding boundary four-point function, which is $x^{H-H_1-H_2}$ (see e.g. (4.2) in [71]), where H denotes the corresponding spacetime conformal weights. Since $H = h - n$ with h as in (6.28), one therefore identifies this contribution as coming from $SL(2)$ short string descendants (of the type $(J^-_{-1})^n(J^-_{-1})^{\bar{n}}|h, m = \bar{m} = h\rangle$) in the intermediate channel [71]. These states have a continuous spectrum for $h > 0$, if one chooses the universal cover of $SL(2)$ as the target space. Since $H = h - n$ is generically irrational, it is not clear to us which boundary states can be identified with the current algebra descendants. In the following we therefore restrict to the case $n = n' = 0$ $(h = j + 1)$, for which there are only chiral primary operators in the intermediate channel, and ignore possible contributions from current algebra descendants.

This leads to some simplification of the product $\mathcal{C}(h)\mathcal{C}'(j)$. Recalling the relation between the structure constants of $SL(2)$ and $SU(2)$ (5.28), which holds for $h_i = j_i + 1$ $(i = 1, 2, 3)$, we find the identity

$$\mathcal{C}(h)\mathcal{C}'(j) = \frac{c_\nu}{(2\pi)^2}\prod_{i=1}^{4}\sqrt{B(h_i)} \tag{6.29}$$

since $h = j + 1$. In other words, the poles of the $SL(2)$ structure constants cancel against the zeros of the $SU(2)$ structure constants.

With these identities, we may now return to $G_4^{NS}(x,\bar{x})$. Applying the residue theorem[6] and taking the limit $\varepsilon \to 0$, we get

$$G_4^{NS}(x,\bar{x}) = g_s^{-2}k^2 \sum_j \prod_{i=1}^{4} \sqrt{B(j_i+1)} G_{j,0} \frac{c_\nu}{(2\pi)^2} \frac{2\pi^2}{\partial_h(\Delta(h))|_{h=j+1}} |x|^{2(j-j_1-j_2)} \\ \times \left((j+j_1+j_2+1)(j+j_3+j_4+1)\right)^2 . \tag{6.30}$$

Where the factor $\partial_h(\Delta(h))|_{h=j+1}/(2\pi) = (2j+1)/(2\pi k)$ in the denominator is related to the fact that we need to integrate over the conformal group on the worldsheet when comparing two-point functions on the worldsheet to two-point functions in spacetime. Recall that spacetime four-point functions can be considered as a sum over the product of two three-point functions divided by the two-point function.

We must still normalize the four-point function with respect to the scaling of the two-point functions. For the four-point function of the corresponding normalized operators (5.35), we then find

$$\mathbb{G}_4^{NS}(x,\bar{x}) = s(k) \sum_j \frac{(j+j_1+j_2+1)^2(j+j_3+j_4+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}} \frac{G_{j,0}}{2j+1} |x|^{2(j-j_1-j_2)} , \tag{6.31}$$

where we introduced the factor

$$s(k) = g_s^{-2}k^2 \left(g_s \sqrt{\frac{2\pi}{k}} \right)^4 \frac{c_\nu}{(2\pi)^2} 2\pi\, k \, . \tag{6.32}$$

If we choose $c_\nu = 1/(2\pi k)$ as in chapter (5), then $s(k) = g_s^2$, which scales as $1/N$.

[6] Let us denote the r.h.s. of (6.26) by $f(h)$ such that for $n=0$ we have $f(h) \equiv \frac{\pi\varepsilon^{2\lambda(h)}}{\lambda(h)}$. Define also h_0 by $\lambda(h_0)=0$. Then $\oint dh f(h) = 2\pi i \,\mathrm{Res}(f;h_0)$ with $\mathrm{Res}(f;h_0) = \frac{\pi\varepsilon^{2\lambda(h_0)}}{\lambda'(h_0)}$ such that

$$\int dh \frac{\pi\varepsilon^{2\lambda(h)}}{\lambda(h)} \propto \frac{2\pi^2}{\partial_h \Delta(h_0)}$$

with $h_0 = j+1$.

6.1.3 Factorization into three-point functions

It is possible to rewrite $\mathbb{G}_4^{NS}(x,\bar{x})$ as the product of two three-point functions. For that, we label the state in the intermediate channel by j and set its m quantum number as $m = j$.[7] Then, the charge conservation $m = m_1 + m_2$ selects the term with

$$j = j_1 + j_2 - d \tag{6.33}$$

in the sum over j. For this particular value of j, or $d = j_1 + j_2 - j$, $G_{j,0}$ reduces to

$$\begin{aligned} G_{j,0} &= c_{2j_2}^{j_2+m_2}\delta^2_{j_1+j_2+j_3-j_4,d} \\ &= \frac{\Gamma(j_2+j-j_1+1)\Gamma(j_1+j_2-j+1)}{\Gamma(2j_2+1)}\,\delta^2_{j_1+j_2+j_3-j_4,d} \end{aligned} \tag{6.34}$$

and $\mathbb{G}_4^{NS}(x,\bar{x})$ becomes

$$\begin{aligned} \mathbb{G}_4^{NS}(x,\bar{x}) = g_s^2 &\frac{\Gamma(j_2+j-j_1+1)\Gamma(j_1+j_2-j+1)}{\Gamma(2j_2+1)}\frac{(j+j_1+j_2+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j+1)}} \\ &\times \frac{(j+j_3+j_4+1)^2}{\sqrt{(2j+1)(2j_3+1)(2j_4+1)}}\,|x|^{-2d} + \dots . \end{aligned} \tag{6.35}$$

However, this is nothing but the expected factorization in terms of three-point functions,

$$\mathbb{G}_4^{NS}(x,\bar{x}) = \frac{\left\langle \mathbb{O}_j^{(0)}(\infty)\tilde{\mathbb{O}}_{j_2}^{(0)}(x,\bar{x})\mathbb{O}_{j_1}^{(0)}(0)\right\rangle \left\langle \tilde{\mathbb{O}}_{j_4}^{(0)}(\infty)\mathbb{O}_{j_3}^{(0)}(1)\mathbb{O}_{j}^{(0)}(0)\right\rangle}{\left\langle \mathbb{O}_j^{(0)}(\infty)\mathbb{O}_j^{(0)}(0)\right\rangle} + \dots \tag{6.36}$$

with [14]

$$\left\langle \mathbb{O}_{j_1}^{(0)}(\infty)\mathbb{O}_{j_2}^{(0)}(1)\tilde{\mathbb{O}}_{j_3}^{(0)}(0)\right\rangle = g_s\frac{(j_1+j_2+j_3+1)^2}{\prod_i(2j_i+1)^{\frac{1}{2}}}\frac{\Gamma(j_{13}+1)\Gamma(j_{12}+1)}{\Gamma(2j_1+1)} . \tag{6.37}$$

The ellipsis indicates terms subleading in x. The x-dependence $|x|^{-2d}$ is now contained in the left three-point function.

[7] More generally, one could have set $m = j - \tilde{d}$ with $\tilde{d} \geq 0$. Each term in $\mathbb{G}_4^{NS}(x,\bar{x})$ would then scale as $|x|^{2(j-j_1-j_2)} = |x|^{2(-d+\tilde{d})}$. Since at small x the leading term in the sum over j is that for $\tilde{d} = 0$, we may neglect global $SU(2)$ descendants. Note that we have already ignored global $SL(2)$ descendants in (6.9).

6.1.4 The extremal case and comparison with the boundary theory

So far, general non-extremal four-point functions have not been considered in the dual symmetric orbifold theory. For comparison with the results in the boundary conformal field theory, we therefore specialize now to the extremal case $j_4 = j_1 + j_2 + j_3$, for which the dual boundary correlator is known [23].

As we can see from (6.34) for $d = 0$ (*i.e.* $j = j_1 + j_2$), $G_{j,0} = \delta^2_{j_1+j_2+j_3,j_4}$, and hence

$$\mathbb{G}_4^{NS}(x,\bar{x}) = g_s^2 \, \frac{(2j+1)(2j_4+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}} \, . \tag{6.38}$$

The result is independent of the cross-ratio x, as expected for extremal correlators. Changing variables from j to n by setting $n_i = 2j_i + 1$ $(i = 1,2,3,4)$, we get

$$\mathbb{G}_4^{NS}(x,\bar{x}) = \frac{1}{N} \frac{n_4^{5/2}}{(n_1 n_2 n_3)^{1/2}} \frac{\tilde{n}}{n_4} \tag{6.39}$$

with $\tilde{n} = n_1 + n_2 - 1$. In the large N limit, this is in agreement with the single-cycle contribution to the boundary correlator (4.14), which is given by (4.14) times the factor $\tilde{n}/n_4$ [23]. This is the contribution coming from single-cycle operators in the intermediate channel.

As argued in [23], in the extremal case contributions coming from double-cycle operators in the intermediate channel are not suppressed at large N. It was found that the combined effect of single- and double-cycle operators is given by the single-cycle contribution times the factor $n_4/\tilde{n}$, symbolically:

$$\begin{aligned} \text{full extremal correlator} &= \text{single-} + \text{double-cycle contribution} \\ &= \frac{n_4}{\tilde{n}} \cdot (\text{single-cycle contribution})\, . \end{aligned}$$

Clearly, it would be desirable to reproduce this factor in the worldsheet theory. Double-cycle terms in the spacetime OPE arise nonlocally on the worldsheet and are presently not very-well understood.

6.1.5 Crossing symmetry

We conclude this section with some comments on the crossing symmetry of $\mathbb{G}_4^{NS}(x, \bar{x})$.

An essential part of the correlator is the $SL(2)$ four-point function, which may be denoted by

$$\mathcal{G}_{34}^{12}(x, z) \equiv \left\langle \prod_{i=1}^{4} \Phi_{h_i}(x_i, z_i) \right\rangle . \tag{6.40}$$

On the right hand side we set again $z_{1,2,3,4} = 0, z, 1, \infty$ and $x_{1,2,3,4} = 0, x, 1, \infty$. As shown by Teschner in [80], the $SL(2)$ four-point function is invariant under crossing symmetry, i.e it satisfies the following identity:

$$\mathcal{G}_{34}^{12}(x, z) = \mathcal{G}_{14}^{32}(1 - x, 1 - z) . \tag{6.41}$$

This corresponds to the simultaneous exchanges

$$x_1 \leftrightarrow x_3 \, , \qquad z_1 \leftrightarrow z_3 \, , \qquad h_1 \leftrightarrow h_3 \, , \tag{6.42}$$

which map the cross-ratios as $x \leftrightarrow 1 - x$ and $z \leftrightarrow 1 - z$. The operators $\mathcal{O}_{j,m}^{(0)}$ are basically $SL(2)$ primaries dressed by some spinors ψ and $e^{-\phi}$ (and currents in case of $\tilde{\mathcal{O}}_{j,m}^{(0)}$). We need to show that this dressing does not violate crossing symmetry.

Let us investigate the crossing symmetry of (6.1) (or, equivalently, (6.4)), which follows if each term in (6.4) is invariant under (6.42). For instance, consider the four-point function

$$\begin{aligned} d_2^{(4)} = \left\langle j(x_2) \prod_{i=1}^{4} \Phi_{h_i}(x_i, z_i) \right\rangle = & \left[\frac{x_{21}}{z_{21}} (x_{21}\partial_{x_1} - 2h_1) + \frac{x_{23}}{z_{23}} (x_{23}\partial_{x_3} - 2h_3) \right. \\ & \left. + \frac{x_{24}}{z_{24}} (x_{24}\partial_{x_4} - 2h_4) \right] \left\langle \prod_{i=1}^{4} \Phi_{h_i}(x_i, z_i) \right\rangle . \end{aligned} \tag{6.43}$$

Here we used the explicit expression (C-6) in Appendix B. Clearly, due to (6.41), this expression is invariant under the exchange (6.42), and similarly $d_4^{(4)}$ and $d_{24}^{(4)}$ appearing in (6.4). The action of the currents $j(x)$ on the $SL(2)$ four-point function therefore remains crossing invariant. Similarly, we can verify the crossing symmetry of correlators in (6.4) which involve only $SL(2)$ fermions by checking the explicit expressions (6.23).

In summary, assuming the crossing invariance of the $SL(2)$ four-point function $\mathcal{G}_{34}^{12}(x, z)$ (proven in [80]), we find that (6.1) is also invariant under this symmetry.

Note however that in the computation of the one-particle contribution we used an approximation for the $SL(2)$ four-point function (Eq. (6.9)), valid at small x and u, which is not crossing invariant. The one-particle contribution computed here is therefore not crossing invariant by itself. The above analysis shows however that it can in principle be made invariant by including the two-particle contributions in the intermediate channel.

6.2 Mixed NS and R four-point function

The computation of the previous section can easily be adapted to other four-point functions. As a further example, we next compute a four-point function which involves two chiral primaries in the NS sector and two in the R sector. Such a four-point function is given by

$$\begin{aligned}G_4^R(x,\bar{x}) &= g_s^{-2}\int d^2z\left\langle\mathcal{O}^{(b)}_{j_4,m_4}(\infty)\mathcal{O}^{(a)}_{j_3,m_3}(1)\mathcal{O}^{(0)}_{j_2,m_2}(x,\bar{x};z,\bar{z})\tilde{\mathcal{O}}^{(0)}_{j_1,m_1}(0)\right\rangle\\ &= g_s^{-2}\int d^2z\,\langle e^{-\frac{\phi(\infty)}{2}}e^{-\frac{\phi(1)}{2}}e^{-\phi(z)}\rangle\left[(1-h_1)\left\langle s^a_-(1)s^b_-(\infty)\psi(x)\jmath(0)\right\rangle\right.\\ &\quad\times\left.\left\langle\prod_{i=1}^4\Phi_{h_i}\right\rangle+\left\langle s^a_-(1)s^b_-(\infty)\psi(x)\right\rangle\left\langle\prod_{i=1}^4 \jmath(0)\Phi_{h_i}\right\rangle\right]\left\langle\prod_{i=1}^4\Phi'_{j_i,m_i}\right\rangle\times c.c.\end{aligned}\tag{6.44}$$

with m-values as in (6.2). The first two operators are Ramond chiral primaries with ghost number $-1/2$. The third and fourth operators are NS chiral primaries with ghost number -1 and 0. The total ghost number is therefore again -2, as required on the sphere.

For the computation, we will need the fermionic correlators

$$\left\langle s^b_-(x_4)\psi(x_2)s^a_-(x_3)\right\rangle = k^{1/2}\frac{x_{23}x_{24}}{z_{23}^{1/2}z_{24}^{1/2}z_{34}^{3/4}}\,\delta^{ab}\,,\tag{6.45}$$

$$\begin{aligned}&\left\langle s^a_-(x_4)s^b_-(x_3)\psi(x_2)\jmath(x_1)\right\rangle =\\ &\quad-\left[\frac{x_{14}x_{12}}{x_{24}}\frac{z_{42}}{z_{14}z_{12}}+\frac{x_{13}x_{12}}{x_{23}}\frac{z_{23}}{z_{13}z_{12}}\right]\left\langle s^b_-(x_4)\psi(x_2)s^a_-(x_3)\right\rangle\,.\end{aligned}\tag{6.46}$$

For simplicity, we neglected the dependence on the y-labels here. The contribution from the ghosts is $\langle e^{-\phi(z_4)/2}e^{-\phi(z_3)/2}e^{-\phi(z_2)}\rangle = z_{23}^{-1/2}z_{24}^{-1/2}z_{34}^{-1/4}$.

Proceeding as before, we use again the factorization ansatz (6.6) and get

$$G_4^R(x,\bar{x}) = g_s^{-2}k \int d^2u \int dh \sum_j \mathcal{C}(h)\mathcal{C}'(j)\,|x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1)}|u|^{2(\Delta(h)+\Delta(j))}$$
$$\times\, \delta^{ab}\left|(1-h_1)\left(\frac{1}{u}+\frac{1}{u}\frac{xu-1}{x-1}\right)+\hat{d}_{1,0}^{(4)}\right|^2 G_{j,0}\,, \tag{6.47}$$

where the first term in the four-point function $d_1^{(4)}$, denoted by $\hat{d}_{1,n}^{(4)}$ with $n=0$, is given by (6.20). As in the previous section, we keep only the terms with $n=n'=0$ (and $F_h(u)\approx 1$). Notice that in the small-u, small-x region, we have

$$\frac{1}{u}+\frac{1}{u}\frac{xu-1}{x-1}\approx\frac{2}{u}\,,\qquad \hat{d}_{1,0}^{(4)}(h,h_i,x,z)\approx-\frac{(h-h_1+h_2)x}{z}\,, \tag{6.48}$$

with $z=ux$, as before. The structure of $G_4^R(x,\bar{x})$ is similar to that of $G_4^{NS}(x,\bar{x})$ as given, for instance, by (6.24). The only change is the terms in the second line.

We now perform the u- and h-integrals. In the region $|u|<\varepsilon$ we expand (6.47) as

$$G_4^R(x,\bar{x}) = g_s^{-2}k \int d^2u \int dh \sum_j \mathcal{C}(h)\mathcal{C}'(j)G_{j,0}\,|x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1)}|u|^{2(\Delta(h)+\Delta(j))}$$
$$\times\,\delta^{ab}\left|-\frac{(h+h_1+h_2-2)}{u}+O(u^0)\right|^2 \tag{6.49}$$

and do the u-integral as in (6.26). Performing also the h-integral and taking the $\varepsilon\to 0$ limit we get

$$G_4^R(x,\bar{x}) = g_s^{-2}k\,\delta^{ab}\sum_j\prod_{i=1}^4\sqrt{B(j_i+1)}G_{j,0}\frac{c_\nu}{(2\pi)^2}|x|^{2(j-j_1-j_2)}\frac{2\pi^2(j+j_1+j_2+1)^2}{\partial_h(\Delta(h))|_{h=j+1}}\,. \tag{6.50}$$

As argued above, there are only chiral primary states in the intermediate channel (with $h=j+1$), which allows us to use (6.29).

With the above value for $c_\nu = 1/(2\pi k)$, the corresponding rescaled correlator is

$$\mathbb{G}_4^R(x,\bar{x}) = g_s^2\,\delta^{ab}\sum_j\frac{G_{j,0}}{2j+1}(j+j_1+j_2+1)^2\left[\frac{(2j_3+1)(2j_4+1)}{(2j_1+1)(2j_2+1)}\right]^{1/2}|x|^{2(j-j_1-j_2)}\,. \tag{6.51}$$

As argued in the previous section, at small x the leading term in the sum over j is that for $j = j_1 + j_2 - d$. Recalling now (6.34), $\mathbb{G}_4^R(x, \bar{x})$ can be rewritten as

$$
\begin{aligned}
\mathbb{G}_4^R(x,\bar{x}) &= g_s^2\, \delta^{ab} \\
&\times \frac{\Gamma(j_2+j-j_1+1)\Gamma(j_1+j_2-j+1)}{\Gamma(2j_2+1)} \frac{(j+j_1+j_2+1)^2}{[(2j+1)(2j_1+1)(2j_2+1)]^{1/2}} \\
&\times \left[\frac{(2j_3+1)(2j_4+1)}{(2j+1)}\right]^{1/2} |x|^{-2d} + \dots\,,
\end{aligned} \tag{6.52}
$$

with $j = j_1 + j_2 - d = j_4 - j_3$. Ellipses represent again subleading terms in x. After comparing with the three-point functions, we get the factorization

$$
\mathbb{G}_4^R(x,\bar{x}) = \frac{\left\langle \mathbb{O}_j^{(0,0)}(\infty)\mathbb{O}_{j_2}^{(0,0)}(x,\bar{x})\tilde{\mathbb{O}}_{j_1}^{(0,0)}(0)\right\rangle \left\langle \mathbb{O}_{j_4}^{(b,\bar{b})}(\infty)\mathbb{O}_{j_3}^{(a,\bar{a})}(1)\mathbb{O}_{j}^{(0,0)}(0)\right\rangle}{\left\langle \mathbb{O}_j^{(0,0)}(\infty)\mathbb{O}_j^{(0,0)}(0)\right\rangle} + \dots\,, \tag{6.53}
$$

with the left three-point function as in (6.37) and the right one given by [15]

$$
\left\langle \mathbb{O}_{j_3}^{(b,\bar{b})}(\infty)\mathbb{O}_{j_2}^{(a,\bar{a})}(1)\mathbb{O}_{j_1}^{(0,0)}(0)\right\rangle = g_s \delta^{ab}\delta^{\bar{a}\bar{b}} \left[\frac{(2j_2+1)(2j_3+1)}{(2j_1+1)}\right]^{1/2}, \quad j_3 = j_1 + j_2\,. \tag{6.54}
$$

For comparison with the corresponding boundary correlator, we restrict again to the extremal case, $d = 0$ or $j_4 = j_1+j_2+j_3$. Then, the only non-vanishing term in the sum over j is that for $j = j_1 + j_2$ (with $G_{j,0} = \delta^2_{j_1+j_2+j_3,j_4}$) and $G_4^R(x, \bar{x})$ as given by (6.51) becomes independent of x,

$$
\mathbb{G}_4^R(x,\bar{x}) = \delta^{ab}\, g_s^2 \left[\frac{(2j_3+1)(2j_4+1)}{(2j_1+1)(2j_2+1)}\right]^{1/2} (2j+1)\,. \tag{6.55}
$$

The result precisely coincides with the one-particle contribution to (4.16) upon identifying $n_i = 2j_i + 1$. At large N it is given by[8]

$$
\mathbb{G}_4^R(x,\bar{x}) = \delta^{ab} \frac{1}{N} \frac{(n_4 n_3)^{1/2}}{(n_1 n_2)^{1/2}}\, \tilde{n} \tag{6.56}
$$

with $\tilde{n} = n_1 + n_2 - 1$. The result does not include possible contributions from the exchange of two-particle states.

We expect that the remaining extremal spacetime four-point correlators (4.15) and (4.17) can be reproduced by a similar worldsheet computation.

[8] This is the contribution from single-cycle operators in the intermediate channel. It is given by (4.16) times the factor $\tilde{n}/n_4$ [23].

6.3 A particular non-extremal four-point function

In this section we consider the non-extremal four-point function

$$G_4(x,\bar{x}) = g_s^{-2}\int d^2z\,\left\langle \tilde{\mathcal{O}}^{(0)}_{j_4}(\infty)\mathcal{O}^{(0)}_{j_3}(1)\tilde{\mathcal{O}}^{(0)}_{j_2}(x,\bar{x};z,\bar{z})\mathcal{O}^{(2)}_{j_1}(0)\right\rangle\,, \tag{6.57}$$

for, at first, arbitrary j-values. Later we will fix the j-labels in order to compare the correlator with the corresponding boundary correlator (4.20).

We begin by substituting the explicit expressions for the chiral primary operators,

$$\begin{aligned} G_4(x,\bar{x}) = g_s^{-2}\int d^2z\,\big\langle &\big((1-h_4)\hat{\jmath}(\infty)+j(\infty)+\tfrac{2}{k}\psi(\infty)\chi_a P^a_{y_4}\big)\,\mathcal{O}_{j_4}\\ &\times e^{-\phi(1)}\psi(1)\mathcal{O}_{j_3}\\ &\times \big((1-h_2)\hat{\jmath}(x)+j(x)+\tfrac{2}{k}\psi(x)\chi_a P^a_{y_2}\big)\,\mathcal{O}_{j_2}\\ &\times e^{-\phi(0)}\chi(0)\mathcal{O}_{j_1}\big\rangle\times c.c\,. \end{aligned} \tag{6.58}$$

Keeping only the nonvanishing terms, we get

$$\begin{aligned} G_4(x,\bar{x}) = g_s^{-2}k^{-2}\int d^2z\,\Bigg[&(1-h_4)\langle\hat{\jmath}(\infty)\psi(1)\psi(x)\rangle\langle 2\chi_a P^a_{y_2}\chi(0)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle\\ &+\langle\psi(1)\psi(x)\rangle\langle 2\chi_a P^a_{y_2}\chi(0)j(\infty)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle\\ &+(1-h_2)\langle\hat{\jmath}(x)\psi(\infty)\psi(1)\rangle\langle 2\chi_a P^a_{y_4}\chi(0)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle\\ &+\langle\psi(\infty)\psi(1)\rangle\langle 2\chi_a P^a_{y_4}\chi(0)j(x)\prod_{i=1}^{4}\mathcal{O}_{j_i}\rangle\Bigg]\times c.c\,. \end{aligned} \tag{6.59}$$

This can be simplified by means of the identity

$$2\chi_a P^a_y = \chi(y)\partial_y - j\partial_y\chi(y)\,. \tag{6.60}$$

We will also need the correlators

$$d_2^{(4)} = \langle j(x)\prod_{i=1}^{4}\mathcal{O}_{j_i}(x_i,y_i)\rangle\,,\qquad d_4^{(4)} = \langle j(\infty)\prod_{i=1}^{4}\mathcal{O}_{j_i}(x_i,y_i)\rangle\,, \tag{6.61}$$

given by (6.16) with (6.19) and (6.18), and the relations

$$\begin{aligned}
\langle \hat{\jmath}(\infty)\psi(1)\psi(x)\rangle &= 2\frac{z-1}{x-1}\langle\psi(1)\psi(x)\rangle\,, && (6.62)\\
\langle \hat{\jmath}(x)\psi(1)\psi(\infty)\rangle &= 2\frac{x-1}{z-1}\langle\psi(1)\psi(\infty)\rangle\,, && (6.63)\\
\partial_y\langle\chi(y)\chi(0)\rangle &= \frac{2}{y}\langle\chi(y)\chi(0)\rangle\,, && (6.64)\\
\lim_{y_4\to\infty}\partial_{y_4}\langle\chi(y_4)\chi(0)\rangle &= \lim_{y_4\to\infty}\frac{2}{y_4}\langle\chi(y_4)\chi(0)\rangle\,. && (6.65)
\end{aligned}$$

Substituting everything back into (6.59), we get

$$G_4(x,\bar{x}) = g_s^{-2}k^{-2}\int d^2z\,\left[\left((1-h_4)2\frac{z-1}{x-1}+d_4^{(4)}\right)\frac{2j_2-(j_1+j_2-j)}{y}\right.$$
$$\left.\times\,\langle\psi(1)\psi(x)\rangle\langle\chi(y)\chi(0)\rangle+\ldots\right]\langle\prod_{i=1}^{4}\mathcal{O}_{j_i}(x_i,y_i)\rangle\times c.c\,, \tag{6.66}$$

where the ellipsis indicates terms subleading in x. As before, we use the factorization Ansatz (6.6) and change variables, $z = ux$. At small u and small x, we obtain

$$G_4(x,\bar{x}) = g_s^{-2}k^2\int d^2u\int dh\sum_j \mathcal{C}(h)\mathcal{C}'(j)\,|x|^{2(\Delta(h)+\Delta(j)+h-h_1-h_2+1)}|u|^{2(\Delta(h)+\Delta(j))}$$
$$\times\left|\frac{(2-h-h_3-h_4)(j-j_1+j_2)}{u\,x}\,y\right|^2\,. \tag{6.67}$$

At this point we need to specify the chirality of the operators in the dual boundary correlator. For this, we assign labels $a_{1,2,3,4}\in\{0,1\}$ to the boundary operators. The label a_i is zero (one), if the dual operator is chiral (antichiral). Then, $U(1)$ charge conservation,

$$\textstyle\sum_{i=1}^4 q_i = (-1)^{a_1}h_1^{(2)}+\sum_{i=2}^4(-1)^{a_i}h_i^{(0)} = 0\,, \tag{6.68}$$

yields the following relation among the j-values,

$$(-1)^{a_1}(j_1+1)+(-1)^{a_2}j_2+(-1)^{a_2}j_3+(-1)^{a_4}j_4 = 0\,. \tag{6.69}$$

In view of the boundary correlator (4.20) let us consider the case $a_1 = a_3 = 0$ (chirals) and $a_2 = a_4 = 1$ (antichirals) and fix the j-labels as $j_1 = \frac{n-1}{2}$, $j_2 = j_3 = \frac{1}{2}$ and $j_4 = \frac{n+1}{2}$. These values have been chosen to agree with the conformal dimensions of the dual chiral operators appearing in the correlator (4.20). For instance, the spacetime conformal dimensions of the operators dual to $\mathcal{O}^{(2)}_{j_1}$ and $\tilde{\mathcal{O}}^{(0)}_{j_4}$ are

$$h^{(2)}_1 = h_1 = j_1 + 1 = \tfrac{n+1}{2} \qquad \text{and} \qquad h^{(0)}_4 = h_4 - 1 = j_4 = \tfrac{n+1}{2}\,, \tag{6.70}$$

as required in (4.20). Using the relations (4.8) and $h_i = j_i + 1$, we find that the non-extremality condition (4.22) translates into $j_4 = j_1 + j_2 + j_3$. Since $j_2 = 1/2$, this relation is equivalent to the $U(1)$ charge conservation relation $j_4 = j_1 - j_2 + j_3 + 1$.

For the above values of j_i and a_i $(i = 1, 2, 3, 4)$, it was found in [23] that in the boundary theory $O^{(0)}_{n+1}$ is the only operator running in the intermediate channel. In the worldsheet theory this operator is dual to $\mathcal{O}^{(0)}_j$ with $j = j_1 + 1 - j_2 = j_1 + 1/2$. If we assume that the one-to-one correspondence between worldsheet and boundary operators also holds in the intermediate channel, then the sum over j reduces to a single term for which $j = j_1 + 1/2$.

Proceeding as before, we get

$$G_4(x,\bar{x}) = g_s^{-2} k^2 \prod_{i=1}^{4} \sqrt{B(j_i+1)}\, \frac{c_\nu}{(2\pi)^2} (2j_4+1)^2 \frac{2\pi^2}{2j+1} \frac{|y|^2}{|x|^2}\,. \tag{6.71}$$

The corresponding rescaled correlator is[9]

$$\mathbb{G}_4(x,\bar{x}) = g_s^2 \frac{(2j_4+1)^2}{\prod_{i=1}^{4}\sqrt{2j_i+1}} \frac{1}{2(j_1+j_2)+1} |x|^{-2} \tag{6.72}$$

or

$$\mathbb{G}_4(x,\bar{x}) = \frac{1}{N} \frac{(n+2)^{3/2}}{2n^{1/2}} \frac{1}{n+1} |x|^{-2}\,. \tag{6.73}$$

At large N this agrees with the non-extremal correlator (4.20).

Summarizing, we started this chapter computing a four point-function involving only NS chiral primaries in the unflowed sector by means of the factorization ansatz for the $SL(2,R)$ four-point function [78] and an analogous factorization for the $SU(2)$ four-point function. We were able to compute the whole four-point function at small cross-ratio x where only 'single-particle' chiral primaries propagate into the intermediate channel. The result agrees with the single-particle contribution of the corresponding four-point function in the conformal field theory on the boundary of AdS_3. We also showed that in the small $x-$limit, the

[9] The operator $\mathcal{O}^{(0,0)}_{j_2=1/2}$ is dual to the *anti*-chiral operator $O^{(0,0)\dagger}_2$. As compared to the corresponding chiral operator, it is rescaled by an additional factor $|y|^{-4j_2}$, which cancels $|y|^2$ in the numerator.

four-point function factorizes in space-time as expected for a conformal field theory. Finally, we have applied the same procedure to compute other extremal and non-extremal four point functions and we have obtained agreement with the corresponding correlators in the dual field theory in all the cases considered.

As in the case of the three-point function in chapter 5, it is natural to expect that a four-point function involving spectral flowed states will be just a copy of the four-point function made out only of unflowed chirals, at least in the OPE limit.

The results in this chapter confirm that the non-renormalization theorem [21] is still valid for extremal four-point functions of chiral primary operators. Furthermore, since some non-extremal correlators also agrees with their dual correlators in the dual field theory, it suggests that the non-renormalization theorem can be extended to the non-extremal case.

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Chapter 7

Semi-classical correlation functions on AdS_5

Unlike the case of strings on $AdS_3 \times S^3 \times T^4$, the quantum formulation of string theory on $AdS_5 \times S^5$ is not yet fully defined. However, there are regimes or subsectors where we still can make some computations in the context of AdS/CFT. Among those regimes, there are protected operators, namely BPS states (such as chiral states), whose correlation functions could be computed in the dual field theory from the supergravity approximation and the pp-wave limit of AdS_5 (in which case we can do a worldsheet quantization of the string [32]).

A nice regime that generalizes the BMN limit [32] (dual to the pp wave limit) corresponds to operators with large quantum numbers, which allow us to perform semiclassical computations on the string theory side of the correspondence.

In this chapter we are going to review some classical and semi-classical aspects of string theory on AdS_5 [1], focusing on operators with large spin which correspond to strings rotating on AdS_5, known as GKP strings [34]. We shall compute semiclassically the leading divergence for a $n-$point function of GKP strings and show how null Wilson loops could be related to these correlators [81].

7.1 Polhmeyer reduction of classical strings in $AdS_3 \subset AdS_5$

In this section we briefly review the Pohlmeyer reduction of strings on AdS_3 proposed in [83, 84] (see also [85, 82] and references therein). We closely follow

[1] Actually, we are going to work in AdS_3 but now thought as a subspace of AdS_5.

the notation of [82]. It is worth mentioning that along this chapter we will consider only an AdS_3 factor of the total AdS_5 space. However, the results could be straightforwardly generalized to AdS_5.

AdS_d spaces can be written as the following hyperboloid in $R^{2,d-1}$

$$\vec{Y}\cdot\vec{Y} = -Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_{d-1}^2 = -1\,. \tag{7.1}$$

In terms of these embedding coordinates, the equations of motion for the bosonic string in an AdS$_d$ space are given by

$$\partial\bar{\partial}\vec{Y} - (\partial\vec{Y}\cdot\bar{\partial}\vec{Y})\vec{Y} = 0\ , \tag{7.2}$$

and the Virasoro constraints read

$$\partial\vec{Y}\cdot\partial\vec{Y} = \bar{\partial}\vec{Y}\cdot\bar{\partial}\vec{Y} = 0\,. \tag{7.3}$$

Let us start defining the reduced fields α and p in AdS_3 as,

$$\begin{aligned} e^{2\alpha(z,\bar{z})} &= \frac{1}{2}\partial\vec{Y}\cdot\bar{\partial}\vec{Y}\ , \\ p = -\frac{1}{2}\vec{N}\cdot\partial^2\vec{Y}\ , &\qquad \bar{p} = \frac{1}{2}\vec{N}\cdot\bar{\partial}^2\vec{Y}\ , \\ N_a &= \frac{e^{-2\alpha}}{2}\epsilon_{abcd}Y^b\partial Y^c\bar{\partial}Y^d\,. \end{aligned} \tag{7.4}$$

From (7.4) and (7.2) it can be shown that $p = p(z)$ is a holomorphic function [2], where we have parametrized the world-sheet in terms of complex variables z and $\bar{z}$. Let us introduce the following basis of four-vectors in $R^{2,2}$

$$\vec{q}_1 = \vec{Y}\ , \qquad \vec{q}_2 = e^{-\alpha}\bar{\partial}\vec{Y}\ , \qquad \vec{q}_3 = e^{-\alpha}\partial\vec{Y}\ , \qquad \vec{q}_4 = \vec{N}\ , \tag{7.5}$$

which satisfies

$$\vec{q}_1^{\,2} = -1\ , \qquad \vec{q}_2.\vec{q}_3 = 2, \quad \vec{q}_4^{\,2} = 1\,, \tag{7.6}$$

with the remaining $\vec{q}_i.\vec{q}_j = 0$. The last property along with the equivalence between $SO(2,2)$ and $SL(2)\times SL(2)$ allows to write the basis vectors in the following matrix representation

$$W = \frac{1}{2}\begin{pmatrix} \vec{q}_1 + \vec{q}_4 & \vec{q}_2 \\ \vec{q}_3 & \vec{q}_1 - \vec{q}_4 \end{pmatrix}\ , \tag{7.7}$$

[2]For the real solutions considered here, $p(z)$ and $\bar{p}(\bar{z})$ are complex conjugates. This condition could in principle be relaxed.

W being an $SL(2)\times SL(2)$ element. The components of this matrix have indices $W_{\alpha\dot\alpha,a\dot a}$. The first two indices denote rows and columns in the above matrix, while the other two are associated with the space-time bispinor representation of each $\vec{q}_i$, i.e,

$$(q_i)_{a\dot a} = q_{i,\mu}\sigma^{\mu}_{a\dot a}, \quad \sigma^{\mu} = (1, i\sigma_3, \sigma_1, -\sigma_2)\,. \tag{7.8}$$

The two $SL(2)$ symmetries of the AdS_3 target space correspond to the $SL(2)$ group acting on the index a and the $SL(2)$ that acts on the index $\dot a$. The basis vectors (7.5) define the $SL(2)$ connections $B^{L,R}$ given by [84]

$$B^L_z = \begin{pmatrix} \frac{1}{2}\partial\alpha & -e^{\alpha} \\ -e^{-\alpha}p(z) & -\frac{1}{2}\partial\alpha \end{pmatrix}, \qquad B^L_{\bar z} = \begin{pmatrix} -\frac{1}{2}\bar\partial\alpha & -e^{-\alpha}\bar p(\bar z) \\ -e^{\alpha} & \frac{1}{2}\bar\partial\alpha \end{pmatrix} \tag{7.9}$$

$$B^R_z = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha}p(z) \\ -e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \qquad B^R_{\bar z} = \begin{pmatrix} \frac{1}{2}\bar\partial\alpha & -e^{\alpha} \\ e^{-\alpha}\bar p(\bar z) & -\frac{1}{2}\bar\partial\alpha \end{pmatrix}. \tag{7.10}$$

The consistency conditions of the equations (7.2) imply that these connections are flat

$$\partial B^L_{\bar z} - \bar\partial B^L_z + [B^L_z, B^L_{\bar z}] = 0\ , \quad \partial B^R_{\bar z} - \bar\partial B^R_z + [B^R_z, B^R_{\bar z}] = 0\,. \tag{7.11}$$

These imply that p is a holomorphic function and that α satisfies the generalized sinh-Gordon equation

$$\partial\bar\partial\alpha(z,\bar z) - e^{2\alpha(z,\bar z)} + |p(z)|^2 e^{-2\alpha(z,\bar z)} = 0\,. \tag{7.12}$$

We are especially interested in the area of the worldsheet, which is given in terms of the reduced fields by,

$$A = 4\int d^2z e^{2\alpha}\,. \tag{7.13}$$

Given a solution of the generalized sinh-Gordon model, one can find the associated classical string worldsheet.

In order to do it, consider the auxiliary linear problem [82]

$$\begin{aligned} \partial\psi^L_\alpha + (B^L_z)_\alpha^{\ \beta}\psi^L_\beta = 0, \qquad \bar\partial\psi^L_\alpha + (B^L_{\bar z})_\alpha^{\ \beta}\psi^L_\beta = 0\,, \\ \partial\psi^R_{\dot\alpha} + (B^R_z)_{\dot\alpha}^{\ \dot\beta}\psi^R_{\dot\beta} = 0, \qquad \bar\partial\psi^R_{\dot\alpha} + (B^R_z)_{\dot\alpha}^{\ \dot\beta}\psi^R_{\dot\beta} = 0\,. \end{aligned} \tag{7.14}$$

We denote by $\psi^L_{\alpha,a}$, $a = 1,2$, and $\psi^R_{\dot\alpha,\dot a}$, $\dot a = 1,2$, the two independent solutions for the left and right linear equations respectively. Since the connections are in $SL(2)$, we can define an $SL(2)$ invariant product and use it to normalize the pair of solutions as

$$\langle\psi^L_a,\,\psi^L_b\rangle \equiv \epsilon^{\beta\alpha}\psi^L_{\alpha,a}\psi^L_{\beta,b} = \epsilon_{ab}, \qquad \langle\psi^R_{\dot a},\,\psi^R_{\dot b}\rangle \equiv \epsilon^{\dot\beta\dot\alpha}\psi^R_{\dot\alpha,\dot a}\psi^R_{\dot\beta,\dot b} = \epsilon_{\dot a\dot b}\,. \tag{7.15}$$

It is easy to see from (7.14) that the normalizations (7.15) are both constant and then they can be evaluated at any point. Now we can reconstruct the space-time worlsheet coordinates from these solutions through the matrix $W_{\alpha\dot\alpha,a\dot a}$, as has been explained in [82]

$$W_{\alpha\dot\alpha,a\dot a} = \psi^L_{\alpha,a}\psi^R_{\dot\alpha,\dot a}\,. \tag{7.16}$$

This could be justified noticing that each component of W in (7.7) is a null vector in $R^{2,2}$, so they can be written as a product of spinors. The explicit form of the target coordinates is given by the element q_1 in (7.7), which can be written as

$$Y_{a\dot a} = \begin{pmatrix} Y_{-1}+Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix}_{a,\dot a} = \psi^L_{\alpha,a}\psi^R_{\dot\beta,\dot a}\ , \tag{7.17}$$

and similarly for the other q_i.
Hence, the problem is reduced to finding an $\alpha(z,\bar z)$ and the pair $p(z)$, $\bar p(\bar z)$ which satisfy (7.12) and use them to compute the solutions to the linear equations (7.14).

7.2 Large spin limit of the GKP String in reduced fields

In global coordinates the GKP string at large spin S is given in worldsheet coordinates $(\tau,\,\sigma)$ by [88, 86]

$$t = \kappa\tau, \quad \theta = \omega\tau, \quad \rho = \rho(\sigma) \sim \kappa\sigma, \quad \omega \to \kappa \sim \frac{1}{\pi}\ln S\,. \tag{7.18}$$

In embedding coordinates its looks like

$$Y_{-1} + iY_0 = \,\mathrm{e}^{\kappa\tau}\cosh\rho(\sigma), \quad Y_1 + iY_2 = \,\mathrm{e}^{-\kappa\tau}\sinh\rho(\sigma)\,. \tag{7.19}$$

As we can see from (7.18), in the limit S going to infinity the parameter κ also goes to infinity and therefore it is convenient to make a reparametrization of the cylinder coordinates as,

$$\tilde{\tau} = \kappa\tau \quad \tilde{\sigma} = \kappa\sigma\,. \tag{7.20}$$

Now the coordinates $\tilde{\tau} \in (-\infty, \infty)$ and $\tilde{\sigma} \in (-\infty, \infty)$. Inserting the last coordinates in (7.4), we get that the GKP strings are described by the following reduced fields in cylindrical coordinates $w = \tilde{\tau} + i\tilde{\sigma}$,

$$p(w) = -\frac{1}{4}, \quad \mathrm{e}^{2\alpha(w,\bar{w})} = \frac{1}{4}\,. \tag{7.21}$$

Going to the plane through the conformal transformation $z = \,\mathrm{e}^{w}$, we have

$$p(z) = -\frac{1}{4\,z^2}, \quad \mathrm{e}^{2\alpha(z,\bar{z})} = \sqrt{p\bar{p}}\,. \tag{7.22}$$

As we can see, $p(z)$ encodes the information on the vertex positions, which in this case appear at $z = 0, \infty$. Now we would like to solve the linear problem (7.14) associated to the GKP string. It is easier to solve it first in w-coordinates, since the reduced fields do not depend on the coordinates there. Doing so we obtain (see [87])

$$\tilde{\psi} = \mathcal{A}\psi = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\pm\frac{i}{2}(\zeta^{-1}w - \zeta\bar{w})}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \tag{7.23}$$

where $\mathcal{A} = \mathrm{diag}(p^{-1/4}\,\mathrm{e}^{\alpha/2}, p^{1/4}\,\mathrm{e}^{-\alpha/2})$ and we have introduced a spectral parameter ζ in order to write the solutions in a compact way. The actual space time solutions are given by $\zeta = 1$ for ψ^L and $\zeta = i$ for ψ^R. Taking $z = r\,\mathrm{e}^{i\phi}$, on the $z-$plane, the above solutions behave as

$$\eta^{L\pm} \sim \,\mathrm{e}^{\pm\phi}\,v_{L\pm}, \quad \eta^{R\pm} \sim r^{\pm}\,v_{R\pm}\,, \tag{7.24}$$

where the $v's$ are given by

$$v_{L+} = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad v_{L-} = \begin{pmatrix} -1 \\ i \end{pmatrix}, \quad v_{R+} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_{R-} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{7.25}$$

Every solution can be written as a linear combination of the above η solutions. Hence, we choose the following arbitrary combinations as a basis of solutions

$$\psi^L_a = c_{l,a}\eta^{L,l}, \quad \psi^R_{\dot{a}} = c_{l,\dot{a}}\eta^{R,l}, \quad a, \dot{a} = 1, 2\,. \tag{7.26}$$

Our main interest in this chapter is the computation of correlation functions of large spin GKP strings semi-classically. In order to do it, we should generalize the GKP solution to solutions which asymptotically look like GKP. More explicitly, we would like to find solutions with the topology of a sphere with $n-$insertions, each insertion corresponding to a GKP on the cylinder. Unfortunately, we can not find that kind of solutions analytically, although we believe they can be studied by using integrability instead, but that is beyond scope of this work. However, we can still use the known information in order to approximately compute the correlator. Namely, generically we assume the function $p(z)$ has n-singularities at points z_i, describing the insertion of vertex operators in the $z-$plane. More explicitly, we assume the function $p(z)$ behaves near singularities in the following way

$$p(z) \sim -\sum_{i=1}^{n} \frac{1}{4(z-z_i)^2} \,. \tag{7.27}$$

Hence, near each singular point, the basis of solutions behaves as

$$\psi^L_{a\,i} \sim c_{l,a\,i}\,\eta^{L,l}, \quad \psi^R_{\dot{a}\,i} \sim c_{l,\dot{a}\,i}\,\eta^{R,l}, \quad a,\dot{a}=1,2, \quad i=1,...,n-1\,. \tag{7.28}$$

In order to see how the solutions approach each vertex insertion in the generic case, it is convenient to write them in $(z,\bar{z})$ coordinates

$$\eta^{R\pm} \sim \prod_{i=1}^{n}[(z-z_i)(\bar{z}-\bar{z}_i)]^{\pm 1/2}, \quad \eta^{L\pm} \sim \prod_{i=1}^{n}\left(\frac{z-z_i}{\bar{z}-\bar{z}_i}\right)^{\pm i/2} \,. \tag{7.29}$$

The target space coordinates can be recovered from the above solutions using (7.17), such that in Poincare coordinates (Z,x_μ) we have

$$\frac{1}{Z}=Y_{1\dot{1}}, \quad x^+=\frac{Y_{1\dot{2}}}{Y_{1\dot{1}}}, \quad x^-=\frac{Y_{2\dot{1}}}{Y_{1\dot{1}}}, \quad x^\pm = x_0 \pm x_1\,. \tag{7.30}$$

As we see from (7.29), some solutions get bigger and some get smaller when we approach the vertex. Therefore the target coordinates will be dominated by the big solutions when evaluated near the insertions. Then, from (7.17) and (7.28), they are schematically approximated near the singularities by

$$Y_{a\dot{a}} = c^{L\,\text{big}}_{a\,i}c^{R\,\text{big}}_{\dot{a}\,i}f_{\text{big}}(z-z_i,\bar{z}-\bar{z}_i)\,, \tag{7.31}$$

where the label 'big' on the coefficients means we keep only those multiplying the $\eta^{R,L}$ which become bigger near the given z_i and $f_{\text{big}}(z-z_i,\bar{z}-\bar{z}_i)$ is the large contribution coming from the $\eta^{R,L}$ which grows up the most.

$$\frac{1}{Z}=c^{L\,\text{big}}_{1\,i}c^{R\,\text{big}}_{\dot{1}\,i}f_{\text{big}}(z-z_i,\bar{z}-\bar{z}_i)\,, \quad x^+=\frac{c^{R\,\text{big}}_{\dot{2}\,i}}{c^{R\,\text{big}}_{\dot{1}\,i}}\,, \quad x^-=\frac{c^{L\,\text{big}}_{2\,i}}{c^{L\,\text{big}}_{1\,i}}\,. \tag{7.32}$$

7.3 Semi-classical correlation functions of large spin operators at strong coupling

The calculation of n-point functions at strong coupling in the leading semiclassical approximation is intrinsically related to finding an appropriate classical solution of a string which ends on the boundary of AdS[93, 90, 91]. Let $V_{H_i}(z_i)$, $i = 1, ..., n$ be the $n-$vertex operators inserted at points z_i on the worldsheet corresponding to a folded string.[3] For large string tension, the n-point function should be dominated by its semiclassical limit $i.e.$ by the action evaluated at its stationary point

$$\langle \prod_{i=1}^{n} V_{H_i}(z_i) \rangle \sim e^{-\frac{R^2}{2\pi\alpha'} A} \,, \tag{7.33}$$

where A is the string action on $AdS_5 \times S^5$ in conformal gauge, which in the semiclassical limit is reduced to the area of the worldsheet. The information provided by the vertex operators on the left hand side is implicitly contained in the boundary conditions for the classical solution. For the case of the two-point function, the appropriate classical solution corresponds to some analytical continuation of the GKP string [92, 93], which reaches the boundary of AdS when the spin of the string goes to infinity. This implies that, in the limit we are considering, the area of the worldsheet diverges and should be somehow regularized, in the same way as for expectation values of Wilson lines at strong coupling. This should also happen in the classical solutions corresponding to higher point functions. Hence, the n-point functions split into a divergent contribution and a finite (regularized) part,

$$\langle \prod_{i=1}^{n} V_{H_i}(z_i) \rangle \sim e^{-\frac{R^2}{2\pi\alpha'} (A_{\rm div} + A_{\rm reg})} \,. \tag{7.34}$$

In this section we will study the divergent part of the correlator and we will show that it scales in the same way as the null polygonal Wilson loop associated with it[4]. It is worth mentioning here that our approach to compute the divergent factor of the correlation function is very much in the spirit of [94].

As we already mentioned, the source of divergences for the area comes from the regions close to the insertions, which are precisely the regions where the string

[3] We consider planar AdS/CFT duality, i.e. tree-level string theory.

[4] In a recent paper [87] the finite contribution $A_{\rm reg}$ of the 3-point correlator has been computed by exploiting the integrability of string theory on AdS_3 through the use of the Pohlmeyer reduced system.

approaches the boundary of AdS. Therefore, it is obvious that putting a worldsheet cutoff near each singularity is completely equivalent to putting a cutoff in target space for the coordinate Z close to the boundary, which at the same time corresponds, through AdS/CFT, to putting an ultraviolet cutoff on the energy of the process. In this section we will show how the divergent contribution to the correlator (7.34) at strong coupling, $A_{\rm div}$, is related to the space-time cutoff and how it depends on the dynamics of the string at the boundary.

From equations (7.24) and (7.28) we can see the large spin solutions diverge near $\phi = \pm\infty$, $r = \pm\infty$, and therefore, the worldsheet is approaching the boundary at those points. Moreover, as we will see, the worldsheet approaches the boundary on light-like trajectories. This can be seen from equation (7.32), realizing that some coefficients $c^{\rm big}_{a,i}$ are equal for consecutive i, i.e, the points i and $i+1$ have the same, lets say x^+_i coordinate and therefore they are joined by a null line. The area is given by (7.13), and the divergent part of it comes from the regions near singularities, and as we said, near each singularity $\mathrm{e}^{2\alpha} \sim \sqrt{p\bar{p}}$. We will regularize the area by using a radial cut-off around each singularity $|z - z_i| > \epsilon_i$. Moreover, as we can see from the mapping $\mathrm{e}^w = z$, the $z-$plane is actually an infinite covering of the complex plane, and the integral over ϕ will introduce another source of divergence for the area, and then we should regularize that by putting a cut-off Λ_ϕ in ϕ. Since the leading contribution to the divergent area comes from the regions very near the singularities of $p(z)$, we are going to isolate the contributions of each z_i and approximate $A_{\rm div}$ as

$$A_{\rm div} = 4\sum_{i=1}^{n}\int_{|z-z_i|>\epsilon_i} d^2\, z\,\frac{1}{|z-z_i|^2} \sim -4\sum_{i=1}^{n}\Lambda_{\phi_i}\,\ln\epsilon_i\,, \tag{7.35}$$

As we mentioned, when we approach the singular points z_i, the worldsheet gets closer to the boundary, and hence the worldsheet cutoff ϵ should be related to some physical cutoff $1/Z = 1/\mu$, $\mu << 1$, which corresponds to putting a brane very close to the boundary where the tips of the folded string will end in the limit of infinite spin. In (7.35) we have left a label i for the cut-off in order to track the corresponding singularity which will be associated to a given space-time coordinate at the boundary. It will become clearer below.

Without loss of generality, let us start considering only three operator insertions, taking κ_1, $\kappa_3 > 0$ and $\kappa_2 < 0$. As we see from equation (7.31), the behaviour of the target coordinates near the boundary is well approximated by the big solutions near each vertex. In order to visualize the behavior of the target space coordinates near the insertions, we display the figures 2 and 3.

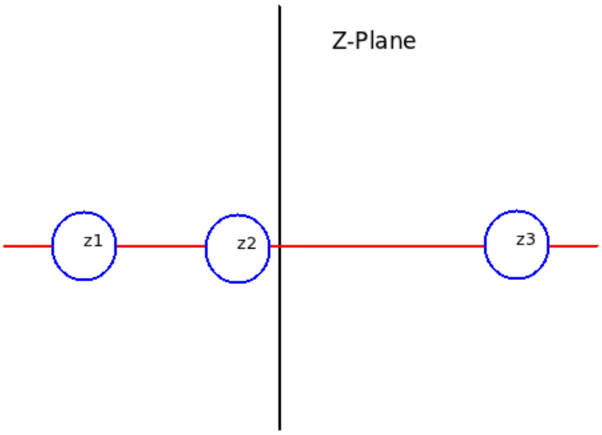

fig.2: $z-$plane with three holes of size ϵ representing the position of the insertions.

As we will see, red lines and green lines map to the null directions defining the Wilson loop. Therefore, the points $1, 2, ..., 6$ correspond in space-time to the cusp of the null Wilson loop. Let us denote the positions of those cusps as $(Z_l,\, x_l^{\pm})$, and see how we approach the boundary around each cusp. Since $\kappa_1,\, \kappa_3\, > 0$ and $\kappa_2\, < 0$, we have according to (7.31) and (7.32),

$$
\begin{aligned}
\frac{1}{Z_1} &= c^{L+}_{1\,1} c^{R-}_{\dot{1}\,1} \left(\frac{z-z_1}{\bar{z}-\bar{z}_1}\right)^{i/2} |z-z_1|^{-1}, \quad x_1^+ = \frac{c^{R-}_{\dot{2}\,1}}{c^{R-}_{\dot{1}\,1}}, \quad x_1^- = \frac{c^{L+}_{2\,1}}{c^{L+}_{1\,1}}, \\
\frac{1}{Z_2} &= c^{L-}_{1\,1} c^{R-}_{\dot{1}\,1} \left(\frac{z-z_1}{\bar{z}-\bar{z}_1}\right)^{-i/2} |z-z_1|^{-1}, \quad x_2^+ = \frac{c^{R-}_{\dot{2}\,1}}{c^{R-}_{\dot{1}\,1}}, \quad x_2^- = \frac{c^{L-}_{2\,1}}{c^{L-}_{1\,1}}, \\
\frac{1}{Z_3} &= c^{L+}_{1\,2} c^{R+}_{\dot{1}\,2} \left(\frac{z-z_2}{\bar{z}-\bar{z}_2}\right)^{i/2} |z-z_2|^{-1}, \quad x_3^+ = \frac{c^{R+}_{\dot{2}\,2}}{c^{R+}_{\dot{1}\,2}}, \quad x_3^- = \frac{c^{L+}_{2\,2}}{c^{L+}_{1\,2}}, \\
\frac{1}{Z_4} &= c^{L-}_{1\,2} c^{R+}_{\dot{1}\,2} \left(\frac{z-z_2}{\bar{z}-\bar{z}_2}\right)^{-i/2} |z-z_2|^{-1}, \quad x_4^+ = \frac{c^{R+}_{\dot{2}\,2}}{c^{R+}_{\dot{1}\,2}}, \quad x_4^- = \frac{c^{L-}_{2\,2}}{c^{L-}_{1\,2}}, \\
\frac{1}{Z_5} &= c^{L+}_{1\,3} c^{R-}_{\dot{1}\,3} \left(\frac{z-z_3}{\bar{z}-\bar{z}_3}\right)^{i/2} |z-z_3|^{-1}, \quad x_5^+ = \frac{c^{R-}_{\dot{2}\,3}}{c^{R-}_{\dot{1}\,3}}, \quad x_5^- = \frac{c^{L+}_{2\,3}}{c^{L+}_{1\,3}}, \\
\frac{1}{Z_6} &= c^{L-}_{1\,3} c^{R-}_{\dot{1}\,3} \left(\frac{z-z_3}{\bar{z}-\bar{z}_3}\right)^{-i/2} |z-z_3|^{-1}, \quad x_6^+ = \frac{c^{R-}_{\dot{2}\,3}}{c^{R-}_{\dot{1}\,3}}, \quad x_6^- = \frac{c^{L-}_{2\,3}}{c^{L-}_{1\,3}}.
\end{aligned}
\tag{7.36}
$$

As we mentioned in section 4, we can see from the equations above that the target positions of points connecting red lines on the worldsheet (see fig.2) live on the same null lines in space-time, $i.e$ $x_1^+ = x_2^+$, $x_3^+ = x_4^+$, $x_5^+ = x_6^+$.

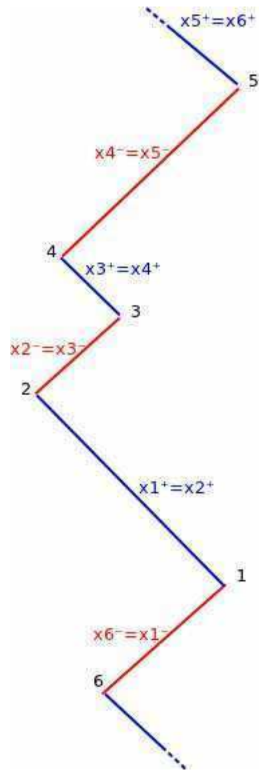

Fig. 2: Mapping of the boundaries of the half-worlsheet into space-time light-like lines.

The coordinates displayed in (7.36) are good approximations to the exact string solution only in the neighborhood of each insertion (think about each insertion as an asymptotic state in target space). Each asymptotic state is connected to each other through the worldsheet, but since we do not know the exact solution, we cannot determine how these states are exactly correlated. One way to relate each insertion to each other is by using the so-called monodromy matrix. As the auxiliary linear problem has two independent solutions, with a basis given by (ψ_1, ψ_2), we expect that as one analytically continues them around a singularity, they get linearly transformed as

$$\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = S \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{7.37}$$

It is worth mentioning that the monodromy matrix should belong to $SL(2, R)$,[5] and therefore it satisfies $\det|S| = 1$. Applying (7.37) to (7.28), we see that we can reinterpret (7.37) as

$$c^{L\alpha}_{a,i} = S^{L\alpha}_{\beta} c^{L\beta}_{a,i} \, , \tag{7.38}$$

[5]We have written $\vec{Y} \in$SO(2,2) as an element $Y_{a\dot{a}} \sim \psi^L_a \psi^R_{\dot{a}}$ of SL(2,R)×SL(2,R), or in other words, the eigenvectors of S should live on a representation of SL(2,R).

with a similar relation for the c^R coefficients and $\alpha,\, \beta = \pm$.

According to the discussion in section 2, we expect that the leading contribution to the correlator comes from the particles propagating between the operators through nearly null trajectories. Let us start imposing a null trajectory condition to the path connecting cusp two to cusp three

$$x_2^- = \frac{c_{2,1}^{L-}}{c_{1,1}^{L-}} \equiv x_3^- = \frac{c_{2,2}^{L+}}{c_{1,2}^{L+}} = \frac{S_{\beta,\,2\to 1}^{L+} c_{2,1}^{L\beta}}{S_{\beta,\,2\to 1}^{L+} c_{1,1}^{L-}}\,, \tag{7.39}$$

therefore, the S^L-matrix should satisfy $S_{+,\,2\to 1}^{L+} = 0$, and from the determinant we get $S_{-,\,2\to 1}^{L+} S_{+,\,2\to 1}^{L-} = -1$, i.e

$$S_{2\to 1}^L = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & \gamma_{2\to 1}^L \end{pmatrix}\,, \tag{7.40}$$

where we have defined $S_{-,\,2\to 1}^{L-} = \gamma_{2\to 1}^L$. Doing the same for the points connecting cusps 4 and 5 and cusps 6 and 1, we end up with[6]

$$S_{3\to 2}^L = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & \gamma_{3\to 2}^L \end{pmatrix}\,, \quad S_{1\to 3}^L = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & \gamma_{1\to 3}^L \end{pmatrix}\,. \tag{7.41}$$

Now we would like to compute the distances between consecutive points, or the separation length of points living on a null line. Let us start considering

$$x_1^- - x_2^- = \frac{c_{2,1}^{L+} c_{1,1}^{L-} - c_{1,1}^{L+} c_{2,1}^{L-}}{c_{1,1}^{L+} c_{1,1}^{L-}} = -\frac{\langle \psi_{1,1}^L \,,\, \psi_{2,1}^L \rangle}{c_{1,1}^{L+} c_{1,1}^{L-}}\,. \tag{7.42}$$

Recalling we have normalized the basis of solutions in each sector as $\langle \psi_{1,i}^L \,,\, \psi_{2,i}^L \rangle = 1$, we get[7]

$$x_1^- - x_2^- = -\frac{1}{c_{1,1}^{L+} c_{1,1}^{L-}} \equiv x_{12}^-\,. \tag{7.43}$$

On the other hand, we have $x_2^- \equiv x_3^-$, and then

$$x_2^+ - x_3^+ = \frac{c_{\dot{2},1}^{R-}}{c_{\dot{1},1}^{R-}} - \frac{c_{\dot{2},2}^{R+}}{c_{\dot{1},2}^{R+}} \equiv \frac{a_{\dot{2},\dot{1}}^R}{c_{\dot{1},1}^{R-} c_{\dot{1},2}^{R+}}\,, \tag{7.44}$$

where we have defined

$$\begin{aligned} a_{\dot{2},\dot{1}}^R &= c_{\dot{2},1}^{R-} (S_{+\,2\to 1}^{R+} c_{\dot{1},1}^{R+} + S_{-\,2\to 1}^{R+} c_{\dot{1},1}^{R-}) - c_{\dot{1},1}^{R-} (S_{+\,2\to 1}^{R+} c_{\dot{2},1}^{R+} + S_{-\,2\to 1}^{R+} c_{\dot{2},1}^{R-}) \\ &= \langle \psi_{1,1}^R \,,\, \psi_{2,1}^R \rangle \, S_{+\,2\to 1}^{R+} = \gamma_{2\to 1}^R\,, \end{aligned} \tag{7.45}$$

[6]These monodromy matrices take the form of the Stokes matrices considered in [82] by using a different argument.

[7]We define $x_{i,j}^{\pm} = x_i^{\pm} - x_j^{\pm}$.

with $S^{R+}_{+\,2\to 1} = \gamma^R_{2\to 1}$. The parameters $\gamma^{R,L}_{i+1\to i}$ parametrize our ignorance on the exact solution and we cannot compute them only using the approximated solutions around the insertions. Notice that in the notation we have used we get $x^-_{12} = x^-_{13} = x^-_{62} = x^-_{63}$ and similar relations for the other cusps (see Fig. 2).
Introducing a cutoff μ_i given by the value of Z_i closest to the boundary and recalling that the worldsheet radial cutoff is defined by $|z - z_i| > \epsilon_i$ we have from (7.36)

$$
\begin{aligned}
-\ \ln\mu_1 &= \ln(c^{L+}_{1,1} c^{R-}_{1,1}) - \ln\epsilon_1 \,, & (7.46)\\
-\ln\mu_3 &= \ln(c^{L-}_{1,2} c^{R-}_{1,2}) - \ln\epsilon_2 \,, & (7.47)
\end{aligned}
$$

where we have taken an arbitrary finite ϕ. Summing up (7.46) and (7.47) and using (7.43), we get

$$
-\ \ln\epsilon_1\epsilon_3 = -\ln\mu_1\mu_3 + \ln\left(x^-_{13}x^+_{13}\right) + \ln\gamma^R_{2\to 1} = -\ln\mu_1\mu_3 + \ln\left(x^-_{12}x^+_{23}\right) + \ln\gamma^R_{2\to 1}\,. \tag{7.48}
$$

In order to disentangle the divergence coming from small radius ϵ from the one coming from large ϕ, we re-evaluate (7.46) and (7.47) at arbitrary but not small $|z - z_i|$. Then, we have,

$$
-\Lambda_{\phi_1} - \Lambda_{\phi_3} = -\ln\mu_1\mu_3 + \ln\left(x^-_{13}x^+_{13}\right) + \ln\gamma^R_{2\to 1} = -\ln\mu_1\mu_3 + \ln\left(x^-_{12}x^+_{23}\right) + \ln\gamma^R_{2\to 1}\,. \tag{7.49}
$$

Doing the same for the other lines connecting consecutive cusps, we can see that,

$$
\begin{aligned}
-\ \ln\epsilon_3\epsilon_5 \sim -\Lambda_{\phi_3} - \Lambda_{\phi_5} &= -\ln\mu_3\mu_5 + \ln\left(x^-_{34}x^+_{45}\right) + \ln\gamma^R_{3\to 2}\\
-\ln\epsilon_5\epsilon_1 \sim -\Lambda_{\phi_5} - \Lambda_{\phi_1} &= -\ln\mu_5\mu_1 + \ln\left(x^-_{56}x^+_{61}\right) + \ln\gamma^R_{1\to 3}\,.
\end{aligned}
$$

We should take the same cut-off in all directions, i.e. $\mu_i = \mu$, $\epsilon_i = \epsilon$, $\Lambda_{\phi_i} = \Lambda_\phi$ and putting this in (7.35) we get,

$$
\begin{aligned}
A_{\rm div} \sim\ & \left[\ln\left(\frac{x^-_{12}x^+_{23}}{\mu^2}\right) + \ln\gamma^R_{2\to 1}\right]^2 + \left[\ln\left(\frac{x^-_{34}x^+_{45}}{\mu^2}\right) + \ln\gamma^R_{3\to 2}\right]^2\\
& + \left[\ln\left(\frac{x^-_{56}x^+_{61}}{\mu^2}\right) + \ln\gamma^R_{1\to 3}\right]^2 .
\end{aligned} \tag{7.50}
$$

For the sake of simplicity, let us change the cusp indexing by line indexing in the following way,

$$
x^-_{12} \equiv \mathbf{x}^-_1\,,\ \ x^+_{23} \equiv \mathbf{x}^+_2\,,\ \ x^-_{34} \equiv \mathbf{x}^-_3\,,\ \ x^+_{45} \equiv \mathbf{x}^+_4\,,\ \ x^-_{56} \equiv \mathbf{x}^-_5\,,\ \ x^+_{61} \equiv \mathbf{x}^+_6\,. \tag{7.51}
$$

Additionally, we are going to assume the contributions from the monodromy matrices $\ln\gamma^R_{i\to j}$ are finite and can be neglected with respect to the factors containing μ^2. Doing the above and generalizing for $n-$points or $2n-$cusp (lines) we have.

$$A_{\text{div}} \sim \sum_{i=1}^{n} \left[\ln[\frac{\mathbf{x}_i^- \mathbf{x}_{i+1}^+}{\mu^2}] \right]^2 , \tag{7.52}$$

where $n+1=1$.

The most remarkable outcome of the above computation is that the leading divergent factor $\left[\ln\frac{1}{\mu^2}\right]^2$ has the same scaling in terms of the cut-off as the corresponding divergence of a null polygonal Wilson loop [82]. This type of log square divergences are also usual in the collinear limit of deep inelastic scattering amplitudes in QCD. It is also worth mentioning that although the vertices are inserted at $n-$points on the worldsheet, they map to $n-$lines on the boundary of AdS (red lines, fig. 2) very much like space-time points map to lines in twistor space. However, those lines on the boundary do not touch each other, but are connected by other light-like lines (blue lines, fig. 2) which come from the region $\phi\to\pm\infty$ on the worldsheet. Based on this, we suggest that an $n-$point function of single trace twist-two large spin operators can be computed from the expectation value of a polygonal Wilson loop when the spins of the operators go to infinity.

Surely the correlator depends on the quantum numbers of the operators, such as the spin and energy. In the limit considered in (7.52) these quantities are encoded in the length of, lets say, red lines in fig. 2 as well as in the position of the operators. It would be interesting to study the exact mapping from spins and positions of the operators to adjoint Wilson lines.

7.4 A heuristic weak coupling analysis

Twist two operators of the form[8]

$$\mathcal{O}_S = \text{Tr}(\Phi \bigtriangledown_{\{\mu_1} \dots \bigtriangledown_{\mu_S\}} \Phi) V^{\mu_1} \dots V^{\mu_S} , \tag{7.53}$$

with spin S and anomalous dimension for large spin given by

$$\gamma_S = \Delta - S - 2 \sim -f(\lambda) \ln S, \tag{7.54}$$

are described at strong coupling by a macroscopic rotating GKP string on AdS_5.

[8]In this equation $\{\cdots\}$ means totally symmetrized and V^μ is a priori an arbitrary vector.

In this section we will give some field theoretical evidence suggesting the relation between the large spin limit of correlators involving the fields (7.53) and expectation values of null polygonal Wilson loops. We are considering operators of the schematic form $\mathcal{O}_S = \mathrm{Tr}[\Phi(V^\mu \nabla_\mu)^S \Phi]$, which are characterized by the spin number S and the conformal weight Δ. In the large spin limit, we will compute the correlation functions in the region in which the emitted particles tend to follow light-like directions. In order to make direct contact with Wilson loops, we recall that the operator O_S arises in a power series expansion of the following gauge-invariant bi-local operator

$$W(V) = \mathrm{Tr}\left(\Phi(0)\, \mathrm{e}^{\int_0^V A_\mu dV^\mu} \Phi(V^\mu)\, \mathrm{e}^{-\int_0^V A_\mu dV^\mu} \Phi(V^\mu)\right) = \sum_{S=0}^{\infty} \frac{1}{S!} \mathcal{O}_S(V^\mu)\,. \quad (7.55)$$

On one hand, we can compute the leading divergent factor of the following expectation value [95],

$$\langle p|\mathcal{O}_S|p\rangle \sim (p \cdot V)^S \left(\frac{\Lambda}{\mu}\right)^{\gamma_S}, \quad (7.56)$$

where Λ and μ are ultraviolet and infra-red cut-offs respectively. (7.56) means we can compute the expectation value of W_V as,

$$\langle p|W_V|p\rangle = \sum_0^S \frac{1}{S!}(p \cdot V)^S \left(\frac{\Lambda}{\mu}\right)^{-f(\lambda)\ln S} = \sum_S \frac{1}{S!}(p \cdot V)^S S^{-f(\lambda)\ln(\Lambda/\mu)}\,. \quad (7.57)$$

By using the following identity,

$$\sum_n \frac{a_n}{n!}(x)^n = x^\alpha \, \mathrm{e}^x\,, \qquad (x \to \infty) \quad \text{if} \quad a^n \sim n^\alpha\,, (n \to \infty) \quad (7.58)$$

we can conclude that

$$\langle p|W_V|p\rangle \,=\, \mathrm{e}^{P \cdot V} \left(\frac{\Lambda}{\mu}\right)^{-f(\lambda)\ln(p \cdot V)}. \quad (7.59)$$

On the other hand, we can compute $\langle p|W_V|p\rangle$ directly in perturbation theory as explained in [96, 97] and from there we expect the leading divergent contribution to be given by,

$$\langle p|W_V|p\rangle \,=\, \mathrm{e}^{p \cdot V} \left(\frac{L}{\epsilon}\right)^{-2\Gamma \ln(p \cdot V)}, \quad (7.60)$$

where L and ϵ are ultraviolet and infra-red cut-offs respectively. By doing the identification $2\Gamma = f(\lambda)$, we can see that (7.59) and (7.60) agree, which means that the leading contribution to the divergence of the expectation value of W_V is coming from the operators in the power expansion with the largest spin.

We will choose the vector V^μ along a null direction x^+ which means we are taking the Wilson line along the given null direction. That will not necessarily happen for finite spin S, but it is certainly the case for infinite spin limit. In that limit, correlators of Wilson lines of the type above should be dominated by the operators with large spin. Hence we will consider correlators of Wilson lines along null directions. Alternatively, as it has been argued in [98] and can be deduced through AdS/CFT considerations, the insertion of an operator $\mathcal{O}_S$ produces a displacement between fields Φ along the x^+ direction joined by an adjoint Wilson line connecting the two-operators. Moreover, the larger the spin the larger the distance between fields and they become displaced along the null direction as the spin goes to infinity.

In this section we will follow the lines of [96] in order to compute the correlator between Wilson lines (7.55) along null directions. Let us suppose one of the line operators emits a particle of momentum p as shown in figure 3[9], and compute its propagator at one loop. Consider the absorption of a gluon with momentum k and gauge potential $A^\mu(k)$ emitted by the Wilson line. In the momentum representation, this process contributes to the correlator the following vertex[10]

$$g \int \frac{d^4k}{(2\pi)^4} \frac{(2\,p' - k)_\mu}{((p'-k)^2 + i\epsilon)} A^\mu(k)\,. \tag{7.61}$$

In the massless limit or light-like trajectories, this vertex has singularities when k is collinear to p and for soft gluons $k \sim 0$. It has been argued in [96] that when the momentum k is collinear to p, the components of $A^\mu(k)$ transverse to k contribute to higher twist and can be neglected in the limit we are considering.

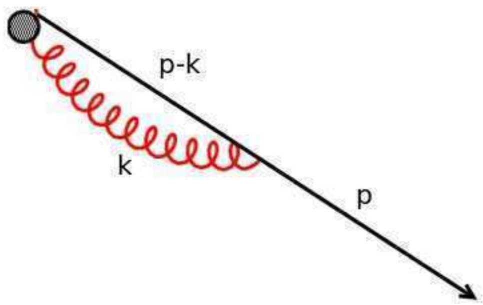

Fig. 3: One loop correction to the propagator of the emitted particle.

[9]The particle p is a single particle state of the scalar Φ.

[10]Here we have used the scalar-vector vertex, see appendix in [99] .

Hence only longitudinal polarization survives and A^μ becomes pure gauge,

$$A^\mu(k) = k^\mu \left(\frac{y \cdot A(k)}{y \cdot k + i\epsilon} \right) , \tag{7.62}$$

y^μ being a vector along the path followed by the particle. Since the on-shell emitted particle is supposed to be massless, the vector y^μ should be null, i.e, it should be along some x^- direction. Inserting (7.62) into (7.61), the absorption vertex is now given by,

$$g \int \frac{d^4k}{(2\pi)^4} \left(\frac{y \cdot A(k)}{y \cdot k + i\epsilon} \right) = g \int_0^\infty d\tau \, y_\mu A^\mu(y\tau) = g \int_0^\infty dx^- \cdot A(x^-) \,. \tag{7.63}$$

Summing all contributions at any number of loops of $p'-$collinear gluons, we should finally obtain that the propagator corresponding to the emitted particle in the light-cone limit due to the interaction with gluons turns out to be

$$\mathcal{G}(\Delta x^-) = G_{\text{free}}(\Delta x^-) \mathcal{P}\text{exp} \left(ig \int_{x_1^-}^{x_2^-} dx^- \cdot A(x^-) \right) , \tag{7.64}$$

where G_{free} corresponds to the propagator for the free particle propagating between points x_1^- and x_2^- separated by a light-like distance Δx^-. This can be justified in the following way. Since the Wilson line (7.55) is dominated by the contributions of local operators at large spin S, it could be thought of as a very fast particle propagating in the x^- direction which after a given threshold starts to emit particles and gluons. When the gluons and particles emitted are collinear, a divergence in the propagator of the emitted particle occurs, dominating the whole propagator. Since collinear gluons are pure gauge, they cannot change the state of the particle but only its phase. That pure phase is given by the Wilson line (7.64) along the light-like direction x^-.

Another way to see this is based on the arguments given in [101]. The propagator of a scalar particle propagating between points x_i^- and x_{i+1}^- interacting with a gauge field is given by replacing the free scalar propagator by the propagator in the background gluon field $S(x_i^-, x_{i+1}^-; A)$, which satisfies the Green equation

$$iD^\mu D_\mu S(x, y; A) = \delta^4(x - y) \,, \tag{7.65}$$

with $D^\mu = \partial_x^\mu - ig[A^\mu(x),]$. In the light-cone limit, it is convenient to look for a solution of the above equation of the form (7.64) times some function of x which goes to one when x tends to $x^\pm$.

Concerning the divergences coming from soft gluons, we expect they cancel against virtual gluon corrections as in deep inelastic scatterings in QCD.

In principle, besides the gluon absorption vertex, we should consider scalar absorption vertices also. However it has been claimed in [100, 101], using OPE arguments, that such kind of vertices only contribute at higher twist and consequently their contributions are suppressed in the light-like limit.

Finally, putting all the Wilson lines together, *i.e*, replacing particle propagators by Wilson lines of the type (7.64), the entire correlator becomes a polygonal Wilson loop with light-like edges, which in the very large spin limit we expect to be dominated by the contributions of local operators in the expansion (7.55) with large S, in a similar way like in the expectation value of a Wilson line with a single cusp [102].

Summarizing, we have computed semi-classically the leading divergence of an $n-$point function in string theory on $AdS_3 \subset AdS_5$ involving rotating strings with large angular momentum which corresponds to the strong coupling regime of twist-two large spin operators in $\mathcal{N} = 4$ super-Yang-Mills theory. We have seen that this divergence scales as the leading divergent factor of the expectation value of a null-polygonal Wilson loop, which suggests a possible relation between $n-$point functions of large spin operators and the mentioned expected values of Wilson loops. In order to support the given suggestion, we have presented a heuristic perturbative analysis which shows how the Wilson loop could arise from the correlator.

Chapter 8

Conclusions

In this work, we have studied correlation functions of strings propagating on AdS spaces in the context of the AdS/CFT correspondence. We have mainly focused on three- and four-point correlation functions of type IIB superstring theory on $AdS_3 \times S^3 \times T^4$ at tree level for chiral primary operators and we have found agreement with the corresponding correlators in the dual two-dimensional conformal field theory living on the boundary of AdS_3.

More specifically, we have evaluated spectral flow conserving three-point functions containing spectral flow images of chiral primary states in type IIB superstring theory and showed that they agree with the corresponding correlators in the dual boundary CFT. These results provide an additional verification of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, widening similar conclusions of previous works [14, 15, 17] to the non-trivial spectral flow sectors of the theory and also confirm the non-renormalization theorem for three-point functions of chiral operators in the field theory side [21]. Moreover, the agreement between three-point functions of spectral flowed chiral vertex operators with the correlator of their proposed dual fields, confirms the bulk-boundary dictionary relating both sides of the correspondence which also solves the problem of missing states [70] (see section 5.1).

We discussed extremal and non-extremal four-point correlators of unflowed chiral primary states in the worldsheet theory at small cross-ratios x where we were allowed to ignore subleading contributions from global $SL(2)$ and $SU(2)$ descendants in the intermediate channel (in the boundary theory this corresponds to neglecting spacetime descendants.) For simplicity, we also ignored possible contributions from current algebra descendants. This is certainly allowed for extremal correlators, for which the $N = 2$ chiral ring structure ensures that there are only chiral primary operators in the intermediate channel. For non-extremal correlators, however, there are in principle further contributions coming from current algebra descendants, which we have not computed. We found that the integrated *non-extremal* correlators $G_4^{NS}(x,\bar{x})$ and $G_4^{R}(x,\bar{x})$ factorize into the product of two spacetime three-point functions involving chiral primaries, see (6.36) and (6.53). Other than in the spacetime CFT, the factorization is non-trivial in the worldsheet theory because of the integration over the moduli space. If there were only chiral primary operators running in the intermediate channel, the factorization property would imply the non-renormalization of the correlator, at least at small x. However, as just stated, there can be additional terms coming from current algebra descendants, which would renormalize the four-point function.

We also evaluated $G_4^{NS}(x,\bar{x})$ and $G_4^{R}(x,\bar{x})$ for the *extremal* case and found agreement with the single-particle contribution to the corresponding extremal boundary correlators computed in [23]. This had been expected from the non-renormalization theorem of [21, 22]. Note that in contrast to their non-extremal cousins, extremal four-point correlators also have two-particle states in the intermediate channel, whose contribution to the correlator is not suppressed at large N. Clearly, it would be desirable to also derive these two-particle additional terms by taking into account nonlocal contributions on the worldsheet. We think such contributions are presently not very well understood.

We also computed a particular non-extremal four-point correlator, defined in (6.57). This correlator is not covered by the non-renormalization theorem of [21] and therefore need not necessarily agree with its boundary counterpart. Nevertheless, we find exact agreement, cf. our result (6.72) or (6.73) agrees with (4.20), again under the premise that we may ignore possible contributions from current algebra descendants in the intermediate channel, suggesting and extension of the non-renormalization theorem to non-extremal correlators.

Concerning the AdS_5/CFT_4 duality, a full quantum formulation of string theory is still missing in order to compute correlation functions similar to those in the AdS_3/CFT_2 correspondence. Nevertheless, we can perform some computations where the semi-classical approximation to the correlators is still reliable. One such situation occurs when operators get large quantum numbers. In this work we have considered large spin operators in $\mathcal{N}=4$ super-Yang-Mills which correspond to classical rotating strings in AdS_5. On the string theory side we have performed a semi-classical computation of the leading ultra-violet divergence of an $n-$point function of rotating GKP strings and we have found the same scaling as the leading divergence of the expectation value of a null polygonal Wilson loop [82].

We also have presented a heuristic perturbative analysis which shows how the expectation value of a null-polygonal Wilson loop could arise from the large spin limit of the correlator of twist two composite operators.

In order to confirm the correspondence between Wilson loops and large spin correlation functions, at least at semi-classical level, we should find an exact classical solution of a string in AdS which contains GKP classical states and has the topology of a sphere with n-vertex insertions. As far as we know, that solution is unknown. However, the correlator could be computed semi-classically by using integrability, as has been done at strong coupling for the finite part in [87] and at weak coupling in [104]. The next interesting step could be to use the results in [87] in order to extract the finite part of the Wilson loops from the correlators.

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Appendix A

Clebsch-Gordan Coefficients

In this appendix we compute the Clebsch-Gordan coefficients (CG) expanding the product representation $(H \otimes \hat{h})$ of the SL(2,$\mathbb{R}$) algebra [1]. We consider the case $H \in \mathcal{D}_H^{+,w}$, $\hat{h} \in \mathcal{D}_{\hat{h}}$, where

$$\mathcal{D}_H^+ : \quad \{|H, M\rangle\,; \quad H \in \mathbb{R}, \quad M = H + n, \quad n = 1, 2, 3.....\}\,, \tag{A-1}$$

is an infinite discrete representation and

$$\mathcal{D}_{\hat{h}}^+ : \quad \{|\hat{h}, \hat{m}\rangle\,; \quad -\hat{h} \leq \hat{m} \leq \hat{h}\,, \quad \hat{m} \in \mathbb{Z}\}\,, \tag{A-2}$$

is a finite representation of the SL(2,$\mathbb{R}$)$_{-2}$ algebra. We use the following normalization

$$\mathbf{j}^{\pm}|H, M >= (M \mp H)|H, M \pm 1 > \tag{A-3}$$

and similarly for $|\hat{h}, \hat{m} >$. A state living in the product representation may be expanded as

$$|H \otimes \hat{h}\rangle \equiv |H, \hat{h}; \mathcal{H}, \mathcal{M}\rangle = \sum_{M,\hat{m}} |H, M; \hat{h}, \hat{m}\rangle\langle H, M; \hat{h}, \hat{m}|H, \hat{h}; \mathcal{H}, \mathcal{M}\rangle\delta_{\mathcal{M},M+\hat{m}}\,. \tag{A-4}$$

Applying the raising operator $\mathbf{H}^+ = \mathbf{j_1}^+ + \mathbf{j_2}^+$ and equating the coefficients on both sides of (A-4), the following recursion relation is obtained

$$\begin{aligned}(\mathcal{M} - \mathcal{H})\langle \mathcal{M} + 1 - \hat{m}, \hat{m}|\mathcal{H}, \mathcal{M} + 1\rangle = (\mathcal{M} - \hat{m} - H)\langle \mathcal{M} - \hat{m}, \hat{m}|\mathcal{H}, \mathcal{M}\rangle \\ + (\hat{m} - 1 - \hat{h})\langle \mathcal{M} - \hat{m} + 1, \hat{m} - 1|\mathcal{H}, \mathcal{M}\rangle\,,\end{aligned} \tag{A-5}$$

[1] For a review of the techniques used here see e.g [106]

where the indices $H,\ \hat{h}$ have been dropped for short. A similar recursion relation is obtained applying the lowering operator $\mathbf{H}^- = \mathbf{j_1^-} + \mathbf{j_2^-}$, namely

$$\begin{aligned}&(\mathcal{M}+\mathcal{H})\langle\mathcal{M}-1-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}-1\rangle = \\ &(\mathcal{M}-\hat{m}+H)\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle + (\hat{m}+\hat{h}+1)\langle\mathcal{M}-\hat{m}-1,\hat{m}+1|\mathcal{H},\mathcal{M}\rangle\,.\end{aligned} \tag{A-6}$$

The last term in (A-5) vanishes for $\hat{m} = -\hat{h}$, *i.e.*

$$\langle\mathcal{M}+\hat{h}+1,-\hat{h}|\mathcal{H},\mathcal{M}+1\rangle = \frac{\mathcal{M}+\hat{h}-H}{\mathcal{M}-\mathcal{H}}\langle\mathcal{M}+\hat{h},-\hat{h}|\mathcal{H},\mathcal{M}\rangle\,, \tag{A-7}$$

and for $\mathcal{M} = \mathcal{H}+1$, this reads

$$\langle\mathcal{H}+\hat{h}+2,-\hat{h}|\mathcal{H},\mathcal{H}+2\rangle = (\mathcal{H}+1+\hat{h}-H)\langle M',-\hat{h}|\mathcal{H},\mathcal{H}+1\rangle\,. \tag{A-8}$$

Then, taking successively $\mathcal{M} = \mathcal{H}+2,\cdots,\mathcal{H}+n$, one finds

$$\langle M,-\hat{h}|\mathcal{H},\mathcal{M}\rangle = \frac{(\hat{h}-H+\mathcal{M}-1)!}{(\mathcal{M}-\mathcal{H}-1)!(\mathcal{H}+\hat{h}-H)!}\langle M'_1,-\hat{h}|\mathcal{H},\mathcal{H}+1\rangle. \tag{A-9}$$

Defining $q(\hat{m},\mathcal{M}) \equiv \frac{(-1)^{\hat{m}}(\hat{m}+\hat{h})!(\mathcal{M}-\hat{m}+H)!}{(\mathcal{M}+\mathcal{H})!}$, (A-6) may be recast as

$$\begin{aligned}q(\hat{m}+1,\mathcal{M})\langle\mathcal{M}-(\hat{m}+1),\hat{m}+1|\mathcal{H},\mathcal{M}\rangle &= q(\hat{m},\mathcal{M})\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle \\ &\quad -q(\hat{m},\mathcal{M}-1)\langle\mathcal{M}-1-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}-1\rangle \\ &\equiv \Delta_{\mathcal{M}}[q(\hat{m},\mathcal{M})\langle M',\hat{m}|\mathcal{H},\mathcal{M}\rangle]\,.\end{aligned} \tag{A-10}$$

Applying this successively for $\hat{m}-1,\cdots,\hat{m}-n$ and using $\Delta_x^n[f](x) = \sum_{s=0}^n(-1)^s\begin{pmatrix}n\\s\end{pmatrix}f(x-s)$, we get

$$\begin{aligned}\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle &= \frac{1}{q(\hat{m},\mathcal{M})}\Delta_{\mathcal{M}}^{\hat{m}+\hat{h}}[q(-\hat{h},\mathcal{M})\langle M',-\hat{h}|\mathcal{H},\mathcal{M}\rangle] \\ &= \frac{1}{q(\hat{m},\mathcal{M})}\sum_{s=0}^{\hat{m}+\hat{h}}(-1)^s\begin{pmatrix}\hat{m}+\hat{h}\\s\end{pmatrix}q(-\hat{h},\mathcal{M}-s)\langle\mathcal{M}-s+\hat{h},-\hat{h}|\mathcal{H},\mathcal{M}-s\rangle\,.\end{aligned}$$

Substituting $q(\hat{m},\mathcal{M})$ and $\langle\mathcal{M}-s+\hat{h},-\hat{h}|\mathcal{H},\mathcal{M}-s\rangle$ in this equation, we obtain

$$\begin{aligned}\langle\mathcal{M}-\hat{m},\hat{m}|\mathcal{H},\mathcal{M}\rangle &= \frac{(\mathcal{M}+\mathcal{H})!}{(\hat{m}+\hat{h})!(\mathcal{M}-\hat{m}+H)!} \\ &\quad\times\sum_{s=0}^{\hat{m}+\hat{h}}(-1)^{s-\hat{h}}\begin{pmatrix}\hat{m}+\hat{h}\\s\end{pmatrix}\frac{(\mathcal{M}-s+\hat{h}+H)!}{(\mathcal{M}-s+\mathcal{H})!} \\ &\quad\times\frac{(\hat{h}-H+\mathcal{M}-s-1)!}{(\mathcal{M}-s-\mathcal{H}-1)!(\mathcal{H}+\hat{h}-H)!}\langle\mathcal{H}+1+\hat{h},-\hat{h}|\mathcal{H},\mathcal{H}+1\rangle\,.\end{aligned}$$

Therefore, all the CG coefficients in the expansion (A-4) are expressed in terms of just one coefficient, which can be set to one [2]. As a consistency check, we compute some known cases.

In the unflowed sector, we need to decompose the product representation with $\hat{h}=1$. In this case, there are three possible combinations of $\mathcal{H}$, according to the angular momentum selection rules, namely $\mathcal{H}=H+1,\ \mathcal{H}=H,\ \mathcal{H}=H-1$. In the first case, one gets

$$\begin{aligned}
(\psi\Phi)^{\omega=0}_{H+1,M} &= \sum_{M,\hat{m}}(\Phi^{\omega=0}_{H,M}\psi^{\hat{m}})\langle M-\hat{m},\hat{m}|H+1,M\rangle \\
&= \frac{1}{2}(H-M)(1+H-M)\Phi^{\omega=0}_{H,M+1}\psi^{-} \\
&\quad +(H+1-M)(1+H+M)\Phi^{\omega=0}_{H,M}\psi^{3} \\
&+ \frac{1}{2}(H+M)(1+H+M)\Phi^{\omega=0}_{H,M-1}\psi^{+}\,. \qquad \text{(A-11)}
\end{aligned}$$

For $\mathcal{H}=H$, the following field expansion is obtained

$$(\psi\Phi)^{\omega=0}_{H,M} = (M-H)\Phi^{\omega=0}_{H,M+1}\psi^{-} - 2M\Phi^{\omega=0}_{H,M}\psi^{3} + (H+M)\Phi^{\omega=0}_{H,M-1}\psi^{+}\,.$$

And finally, for $\mathcal{H}=H-1$, which satisfies the chirality condition in the unflowed sector,

$$(\psi\Phi)^{\omega=0}_{H-1,M} = \Phi^{\omega=0}_{H,M+1}\psi^{-} - 2\Phi^{\omega=0}_{H,M}\psi^{3} + \Phi^{\omega=0}_{H,M-1}\psi^{+}\,, \qquad \text{(A-12)}$$

in agreement with the decomposition given in [60], up to the global phase factor mentioned above.

[2] Recall that the CG are determined up to a global phase factor (which is a global multiplicative factor for all remaining CG).

Appendix B

Correlators in $SL(2)_k$ and $SU(2)_{k'}$ WZW models

B.1 Two- and three-point functions in the $SL(2)_k$ WZW model

The chiral primaries of the $SL(2)$ WZW model are denoted by[1]

$$\Phi_h(z,\bar{z};x,\bar{x}) = \Phi_h(z,x)\,\bar{\Phi}_h(\bar{z},\bar{x}) \qquad \text{with} \qquad \Delta(h) = \bar{\Delta}(h) = -\frac{h(h-1)}{k-2}\ , \tag{B-1}$$

where k is the level of the affine Lie algebra. In the current context only half-integer h will be relevant.

The two- and three-point functions of $\Phi_h(z,\bar{z};x,\bar{x})$ were computed in [**?**, **?**]. The two-point function is given by

$$\begin{aligned}&\langle\Phi_{h_1}(z_1,\bar{z}_1;x_1,\bar{x}_1)\Phi_{h_2}(z_2,\bar{z}_2;x_2,\bar{x}_2)\rangle\\ &\quad= \frac{1}{|z_{12}|^{4\Delta(h_1)}}\left[\frac{1}{(2\pi)^2}\,\delta(x_{12})\,\delta(\bar{x}_{12})\,\delta(h_1+h_2-1) + \frac{B(h_1)}{|x_{12}|^{4h_1}}\delta(h_1-h_2)\right]\ ,\end{aligned} \tag{B-2}$$

with coefficient

$$B(h) = \frac{k-2}{\pi}\,\frac{\nu^{1-2h}}{\gamma(\frac{2h-1}{k-2})} \qquad \text{and} \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\ , \qquad \nu = \frac{\pi}{c_\nu}\frac{\Gamma(1-\frac{1}{k-2})}{\Gamma(1+\frac{1}{k-2})}\ . \tag{B-3}$$

[1]In this appendix we only deal with the bosonic currents; k and k' therefore refer to the bosonic levels.

The parameter c_ν is free.

The three-point function is

$$\langle \Phi_{h_1}(z_1,\bar{z}_1;x_1,\bar{x}_1)\,\Phi_{h_2}(z_2,\bar{z}_2;x_2,\bar{x}_2)\,\Phi_{h_3}(z_3,\bar{z}_3;x_3,\bar{x}_3)\rangle = C(h_1,h_2,h_3)\prod_{i<j}\frac{1}{|x_{ij}|^{2h_{ij}}|z_{ij}|^{2\Delta_{ij}}}\ , \tag{B-4}$$

with $\Delta_{12}=\Delta(h_1)+\Delta(h_2)-\Delta(h_3)$, $h_{12}=h_1+h_2-h_3$, *etc.* and coefficients

$$C(h_1,h_2,h_3)=\frac{k-2}{2\pi^3}\,\frac{G(1-h_1-h_2-h_3)G(-h_{12})G(-h_{23})G(-h_{31})}{\nu^{h_1+h_2+h_3-2}G(-1)G(1-2h_1)G(1-2h_2)G(1-2h_3)}\ , \tag{B-5}$$

where

$$G(h)=(k-2)^{\frac{h(k-1-h)}{2(k-2)}}\,\Gamma_2(-h|1,k-2)\,\Gamma_2(k-1+h|1,k-2)\ , \tag{B-6}$$

and $\Gamma_2(x|1,\omega)$ is the Barnes double Gamma function. $G(h)$ has poles at $h=n+m(k-2)$ and $h=-n-1-(m+1)(k-2)$ with $n,m=0,1,....$ In C_{h_1,h_2,h_3} the poles $h_1+h_2+h_3=n+k$, $n=0,1,...$ are excluded by the condition

$$h_1+h_2+h_3\leq k-1\,. \tag{B-7}$$

The function $G(h)$ satisfies the recursion relation

$$G(h+1)=\gamma(-\tfrac{h+1}{k-2})\,G(h)\ . \tag{B-8}$$

B.2 Four-point function in the $SL(2)_k$ WZW model

The four-point function of the $SL(2)$ chiral primary $\Phi_{h_i}(z,\bar{z};x,\bar{x})$ is given by [105]

$$\begin{aligned}\langle\prod_{i=1}^{4}\Phi_{h_i}(z_i,\bar{z}_i;x_i,\bar{x}_i)\rangle &= |x_{24}|^{-4h_2}|x_{14}|^{2(h_2+h_3-h_1-h_4)}|x_{34}|^{2(h_1+h_2-h_3-h_4)}|x_{13}|^{2(h_4-h_1-h_2-h_3)}\\ &\times|z_{24}|^{-4\Delta_2}|z_{14}|^{2\nu_1}|z_{34}|^{2\nu_2}|z_{13}|^{2\nu_3}\,\mathcal{F}_{SL(2)}(z,\bar{z};x,\bar{x})\end{aligned} \tag{B-9}$$

with

$$\nu_1=\Delta_2+\Delta_3-\Delta_1-\Delta_4\,,\quad \nu_2=\Delta_1+\Delta_2-\Delta_3-\Delta_4\,,\quad \nu_3=\Delta_4-\Delta_1-\Delta_2-\Delta_3\ ,$$

and

$$z=\frac{z_{12}z_{34}}{z_{14}z_{32}}\ ,\qquad x=\frac{x_{12}x_{34}}{x_{14}x_{32}}\ . \tag{B-10}$$

The function $\mathcal{F}_{SL(2)}(z,\bar{z};x,\bar{x})$ is given by

$$\mathcal{F}_{SL(2)}(z,\bar{z},x,\bar{x}) = \mathcal{M}(h_1,h_2,h_3,h_4)\ |z|^{-\frac{4h_1h_2}{k-2}}\ |1-z|^{-\frac{4h_1h_3}{k-2}}\Gamma(2h_1)b^{-1}\mu^{-2h_1}\times \tag{B-11}$$

$$\times\int\prod_{i=1}\frac{dt_i d\bar{t}_i}{(2\pi i)}\ |t_i-z|^{-\frac{2\beta_1}{k-2}}\ |t_i|^{-\frac{2\beta_2}{k-2}}\ |t_i-1|^{-\frac{2\beta_3}{k-2}}\ |x-t_i|^2\ |D(t)|^{\frac{-4}{k-2}}\ ,$$

where

$$D(t) = \prod_{i<j}(t_i-t_j)\ , \tag{B-12}$$

and

$$\begin{aligned}\beta_1 &= h_1+h_2+h_3+h_4-1\,,\\ \beta_2 &= h_1+h_2-h_3-h_4-1+k\,,\\ \beta_3 &= h_1+h_3-h_2-h_4-1+k\ .\end{aligned} \tag{B-13}$$

The normalization is

$$\begin{aligned}\mathcal{M} = &\frac{\pi C_W^2(b)}{b^{5+4b^2}\Upsilon_0^2}\frac{(\nu(b))^s}{(\pi\mu\gamma(b^2)b^4)^{-2h_1}}\frac{G(1-h_1-h_2-h_3-h_4)}{G(1-2h_1)}\\ &\times\prod_{i=2}^{4}\frac{G(-h_2-h_3-h_4+h_1+2h_i)}{G(1-2h_i)}\,,\end{aligned} \tag{B-14}$$

where $s = 1-\sum_{i=1}^4 h_i$, $b^2 = \frac{1}{k-2}$, $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and

$$\nu(b) = -b^2\gamma(-b^2) = \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\ . \tag{B-15}$$

B.3 Two- and three-point functions in the $SU(2)_{k'}$ WZW model

The chiral primaries of the $SU(2)_{k'}$ WZW model are denoted by

$$\Phi'_j(z,\bar{z};y,\bar{y}) = \Phi'_j(z,y)\,\bar{\Phi}'_j(\bar{z},\bar{y})\ , \tag{B-16}$$

and have conformal dimension

$$\Delta(j) = \bar{\Delta}(j) = \frac{j(j+1)}{k'+2}\ , \qquad 0\le j\le\frac{k'}{2}\ , \tag{B-17}$$

where j is the $SU(2)$ representation label and k' the level of the affine Lie algebra.

The two- and three-point functions of $\Phi'_j(z,\bar{z};y,\bar{y})$ are then [103, **?**]

$$\langle \Phi'_{j_1}(z_1,\bar{z}_1;y_1,\bar{y}_1)\Phi'_{j_2}(z_2,\bar{z}_2;y_2,\bar{y}_2)\rangle = \delta_{j_1,j_2}\,\frac{|y_{12}|^{4j_1}}{|z_{12}|^{4\Delta(j_1)}}\ , \tag{B-18}$$

and

$$\langle \Phi'_{j_1}(z_1,\bar{z}_1;y_1,\bar{y}_1)\Phi'_{j_2}(z_2,\bar{z}_2;y_2,\bar{y}_2)\Phi'_{j_3}(z_3,\bar{z}_3;y_3\bar{y}_3)\rangle = C'_{j_1,j_2,j_3}\prod_{i<j}\frac{|y_{ij}|^{2j_{ij}}}{|z_{ij}|^{2\Delta_{ij}}}\ , \tag{B-19}$$

with $\Delta_{12} = \Delta(j_1) + \Delta(j_2) - \Delta(j_3)$, *etc.* The relevant coefficients are

$$C'_{j_1,j_2,j_3} = \sqrt{\frac{\gamma(\frac{1}{k'+2})}{\gamma(\frac{2j_1+1}{k'+2})\gamma(\frac{2j_2+1}{k'+2})\gamma(\frac{2j_3+1}{k'+2})}}\,\frac{P(j_1+j_2+j_3+1)\,P(j_{12})\,P(j_{23})\,P(j_{31})}{P(2j_1)\,P(2j_2)\,P(2j_3)} \tag{B-20}$$

and

$$P(j) = \prod_{m=1}^{j}\gamma(\tfrac{m}{k'+2})\ , \qquad P(0) = 1\ , \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\ . \tag{B-21}$$

The functions $P(j)$ are nonvanishing for $0 \le j \le k'+1$. Therefore, $C'_{j_1,j_2,j_3} \neq 0$, if

$$j_1 + j_2 + j_3 \le k'\ . \tag{B-22}$$

B.4 Four-point function in the $SU(2)_{k'}$ WZW model

The four-point function of the $SU(2)$ chiral primary $\Phi'_{j_i}(z,\bar{z};y,\bar{y})$ is given by [76]

$$\begin{aligned}\langle \prod_{i=1}^{4}\Phi'_{j_i}(z_i,\bar{z}_i;y_i,\bar{y}_i)\ \rangle &= |y_{24}|^{4j_2}|y_{14}|^{2(j_1+j_4-j_2-j_3)}|y_{34}|^{2(j_3+j_4-j_1-j_2)}|y_{13}|^{2(j_1+j_2+j_3-j_4)}\\ &\quad\times|z_{24}|^{-4\Delta'_2}|z_{14}|^{2\nu'_1}|z_{34}|^{2\nu'_2}|z_{13}|^{2\nu'_3}\mathcal{F}_{SU(2)}(z,\bar{z},y,\bar{y})\ ,\end{aligned} \tag{B-23}$$

with

$$\nu'_1 = \Delta'_1+\Delta'_3-\Delta'_2-\Delta'_4\,,\quad \nu'_2 = \Delta'_1+\Delta'_2-\Delta'_3-\Delta'_4\,,\quad \nu'_3 = \Delta'_4-\Delta'_1-\Delta'_2-\Delta'_3\ ,$$

and

$$z = \frac{z_{12}z_{34}}{z_{14}z_{32}}\ , \qquad y = \frac{y_{12}y_{34}}{y_{14}y_{32}}\ . \tag{B-24}$$

The function $\mathcal{F}_{SU(2)}(z,\bar{z},y,\bar{y})$ is given in terms of the Dotsenko-Fateev integral

$$\mathcal{F}_{SU(2)}(z,\bar{z},y,\bar{y}) = \mathcal{N}(j_1,j_2,j_3,j_4)\ |z|^{\frac{4j_1j_2}{k'+2}}\ |1-z|^{\frac{4j_1j_3}{k'+2}} \times \tag{B-25}$$

$$\times \int \prod_{i=1}^{2j_1} \frac{dt'_i d\bar{t}'_i}{(2\pi i)}\ |t'_i - z|^{-\frac{2\beta'_1}{k'+2}}\ |t'_i|^{-\frac{2\beta'_2}{k'+2}}\ |t'_i - 1|^{-\frac{2\beta'_3}{k'+2}}\ |y - t'_i|^2\ |D(t')|^{\frac{4}{k'+2}}\ ,$$

where

$$D(t') = \prod_{i<j} (t'_i - t'_j)\ , \tag{B-26}$$

and

$$\begin{aligned}
\beta'_1 &= j_1 + j_2 + j_3 + j_4 + 1\,, \\
\beta'_2 &= j_1 + j_2 - j_3 - j_4 + 1 + k'\,, \\
\beta'_3 &= j_1 + j_3 - j_2 - j_4 + 1 + k'\ .
\end{aligned} \tag{B-27}$$

The normalization is

$$\begin{aligned}
\mathcal{N}(j_1,j_2,j_3,j_4) &= \left[\gamma\left(\tfrac{1}{k'+2}\right)\right]^{2j_1+1} \frac{P(j_1+j_2+j_3+j_4+1)}{\gamma\left(\frac{2j_1+1}{k'+2}\right)^{1/2} P(2j_1)} \\
&\times \prod_{i=2}^{4} \frac{P(j_2+j_3+j_4-j_1-2j_i)}{\gamma\left(\frac{2j_i+1}{k'+2}\right)^{1/2} P(2j_i)}\ ,
\end{aligned} \tag{B-28}$$

with

$$P(n) = \prod_{m=1}^{n} \gamma\left(\tfrac{m}{k'+2}\right), \qquad P(0) = 1\ . \tag{B-29}$$

Appendix C

Some correlators

In this appendix we give some more details on the computation of some correlators used in the main text.

For the computation of these correlators we will need the following OPEs (the dependence of the fields on z is suppressed):

$$
\begin{aligned}
j(x_k)\Phi_{h_i}(x_i) &= (-j^+ + 2x_k j^3 - x_k^2 j^-)\Phi_{h_i}(x_i) \\
&\sim \frac{1}{z_{ik}}\left(-D^+_{x_i} + 2x_k D^3_{x_i} - x_k^2 D^-_{x_i}\right)\Phi_{h_i}(x_i) \\
&= \frac{1}{z_{ik}}\left(-x_i^2\partial_{x_i} - 2h_i x_i + 2x_k(x_i\partial_{x_i} + h_i) - x_k^2\partial_{x_i}\right)\Phi_{h_i}(x_i) \\
&= \mathcal{D}^{(h_i)}_{ki}\Phi_{h_i}(x_i)\,, \qquad \text{(C-1)}
\end{aligned}
$$

$$j(x_1)j(x_2) \sim (k+2)\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}^{(-1)}_{12}j(x_2)\,, \qquad \text{(C-2)}$$

$$\hat{\jmath}(x_1)\hat{\jmath}(x_2) \sim -2\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}^{(-1)}_{12}\hat{\jmath}(x_2)\,, \qquad \text{(C-3)}$$

$$\hat{\jmath}(x_1)\psi(x_2) \sim \mathcal{D}^{(-1)}_{12}\psi(x_2)\,, \qquad \text{(C-4)}$$

where we defined the operator $\mathcal{D}^{(h)}_{ki}$ as

$$\mathcal{D}^{(h)}_{ki} \equiv \frac{1}{z_{ki}}\left(x_{ki}^2\,\partial_{x_i} - 2h\,x_{ki}\right)\,. \qquad \text{(C-5)}$$

Recall that $j(x)$ generates a bosonic $SL(2)$ affine algebra at level $k_b = k+2$ (k is the supersymmetric level), while $\hat{\jmath}(x)$ forms a supersymmetric $SL(2)$ model at level -2.

We first show that an n-point correlator involving $j(x_k)$ ($k \in \{1, ..., n\}$) and

n $SL(2)$ primaries $\Phi_{h_i}(x_i)$ $(i = 1, ..., n)$ satisfies

$$d_k^{(n)} = \langle j(x_k) \prod_{i=1}^{n} \Phi_{h_i}(x_i) \rangle = \sum_{\substack{i=1 \\ i \neq k}}^{n} \mathcal{D}_{ki}^{(h_i)} \langle \prod_{i=1}^{n} \Phi_{h_i}(x_i) \rangle \,. \tag{C-6}$$

This follows directly from (C-1).

Acting with $j(x_m)$ $(m \in \{1, ..., n\})$ on (C-6), we find the n-point correlator

$$d_{k,m}^{(n)} = \langle j(x_k) j(x_m) \prod_{i=1}^{n} \Phi_{h_i}(x_i) \rangle = \big(\mathcal{D}_{km}^{(-1)} + \sum_{\substack{i=1 \\ i \neq k}}^{n} \mathcal{D}_{ki}^{(h_i)} \big) d_m^{(n)} \,. \tag{C-7}$$

Similarly, we may compute the fermionic correlators using[1]

$$\begin{aligned} \langle \psi(x_1) \psi(x_2) \rangle &= k \frac{(x_{12})^2}{z_{12}} \,, \\ \langle \hat{\jmath}(x_3) \psi(x_1) \psi(x_2) \rangle &= \sum_{i=1}^{2} \mathcal{D}_{3i}^{(-1)} \langle \psi(x_1) \psi(x_2) \rangle \,, \\ \langle \hat{\jmath}(x_4) \hat{\jmath}(x_3) \psi(x_1) \psi(x_2) \rangle &= \sum_{j=1}^{3} \mathcal{D}_{4j}^{(-1)} \langle \hat{\jmath}(x_3) \psi(x_1) \psi(x_2) \rangle \,. \end{aligned} \tag{C-8}$$

[1]In the third equation we ignore a term of the type $\langle \hat{\jmath}(x_4) \hat{\jmath}(x_3) \rangle \langle \psi(x_1) \psi(x_2) \rangle$. It turns out to be subleading at small u.

Appendix D

Comments on $SU(2)$ four-point function

In this appendix we derive the factorization (6.12) of the $SU(2)$ four-point function.

We start from the $SU(2)$ four-point function in y-space,

$$\left\langle \Phi'_{j_1}(0)\Phi'_{j_2}(y_2,\bar{y}_2)\Phi'_{j_3}(y_3,\bar{y}_3)\Phi'_{j_4}(\infty)\right\rangle \,, \tag{D-1}$$

in which we fixed $y_1 = 0$, $y_4 = \infty$. This corresponds to choosing states with $m_1 = j_1$ and $m_4 = -j_4$. This will now be expanded by means of the general OPE [103]

$$\Phi'_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)\Phi'_{j_1}(0) = \sum_j \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2))}}{|y_2|^{2(j-j_1-j_2)}} C'^{j}_{j_1,j_2}[\Phi'_j](y_2,\bar{y}_2;z_2,\bar{z}_2)\,, \tag{D-2}$$

where $C'^{j}_{j_1,j_2}$ are the $SU(2)$ structure constants given by (B-20) and the square brackets $[\Phi'_j]$ denote the contributions to the OPE from the primary field Φ'_j and all its descendants. This quantity can be presented in the form

$$[\Phi'_j](y_2,\bar{y}_2;z_2,\bar{z}_2) = R^j_{j_1,j_2}(y_2,z_2)\bar{R}^j_{j_1,j_2}(\bar{y}_2,\bar{z}_2)\Phi'_j(y_2,\bar{y}_2;z_2,\bar{z}_2)\,, \tag{D-3}$$

where the operator R is given by

$$R^j_{j_1,j_2}(y_2,z_2) = \sum_{n'=0}^{\infty}\frac{z_2^{n'}}{y_2^{n'}}\prod_{\alpha_i=1}^{3}\sum_{\{n^i p_i\}=n'} R_{n'}(n_i,p_i,j)(J^{\alpha_i}_{-n_i}(y_2,z_2))^{p_i} \tag{D-4}$$

where $i=1=+$, $i=2=-$, $i=3=3$ and $\{n^i p_i\}=n'$ means all combinations of $n_i p_i$ (partitions of n') such that $n_+p_+ + n_-p_- + n_3p_3 = n'$. In order to determine the coefficient $R_{n'}(n_i,p_i,j)$, let us take without loss of generality, a single combination of n_ip_i for each given n' (*i.e.* let us look at the contribution to the OPE from a single descendant for each level n'). In that case

$$\Phi'_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)\Phi'_{j_1}(0) = \sum_{j,n',\bar{n}'} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2))} z_2^{n'} \bar{z}_2^{\bar{n}'}}{|y_2|^{2(j-j_1-j_2)} y_2^{n'} \bar{y}_2^{\bar{n}'}} C'^{j}_{j_1,j_2} R_{n'}(n_i,p_i,j) \bar{R}_{\bar{n}'}(\bar{n}_i,\bar{p}_i,j) \Phi'^{jn'\bar{n}'}_{J,\bar{J}}(y_2,\bar{y}_2;z_2,\bar{z}_2)\,, \tag{D-5}$$

where $\Phi'^{jn'\bar{n}'}_{J,\bar{J}}$ defined by

$$\Phi'^{jn'\bar{n}'}_{J,\bar{J}} = \left[\prod_{\alpha_i=1}^{3} \sum_{\{n^ip_i\}=n'} (J^{\alpha_i}_{-n_i}(y_2,z_2))^{p_i} \sum_{\{\bar{n}^i\bar{p}_i\}=\bar{n}'} (\bar{J}^{\alpha_i}_{-\bar{n}_i}(\bar{y}_2,\bar{z}_2))^{\bar{p}_i}\right] \Phi'_j \tag{D-6}$$

is the descendant of Φ'_j at level $(n',\bar{n'})$. Let us now consider a three-point function with a descendant inside. Such a three-point function has the general form

$$\langle \Phi'_{j_1}(y_1,z_1)\Phi'_{j_2}(y_2,z_2)\Phi'^{j_3n'_3}_{J_3}(y_3,z_3)\rangle = C'_{j_1,j_2,j_3}\mathcal{D}(j_1,j_2,J_3)\prod_{i<j}\frac{|y_{ij}|^{2J_{ij}}}{|z_{ij}|^{2\tilde{\Delta}_{ij}}}\,, \tag{D-7}$$

with $J_{12}=j_1+j_2-J_3$, $\tilde{\Delta}_{12}=\Delta_{12}-n'$, *etc.*, where we have taken $n'=\bar{n}'$ for the sake of simplicity. Using the OPE (D-5) on the left hand side of (D-7) and putting $y_1=z_1=0\,, y_2=z_2=1\,, y_3=z_3=\infty$, we find

$$\mathcal{D}(j_1,j_2,J_3) = R_{n'_3}(n_i,p_i,j_3)\bar{R}_{n'_3}(n_i,p_i,j_3)\,. \tag{D-8}$$

This allows us to write

$$\begin{aligned}&\Phi'_{j_2}(y_2,\bar{y}_2;z_2,\bar{z}_2)\Phi'_{j_1}(0)\\ &\quad= \sum_{j,n'} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|y_2|^{2(j-j_1-j_2+n')}} C'^{j}_{j_2,j_3}\mathcal{D}(j_1,j_2,J)\,\Phi'^{jn'}_{J}(y_2,\bar{y}_2;z_2,\bar{z}_2)\,.\end{aligned} \tag{D-9}$$

Inserting this into the $SU(2)$ four-point function, we get

$$\begin{aligned}&\left\langle \Phi'_{j_1}(0)\Phi'_{j_2}(y_2,\bar{y}_2)\Phi'_{j_3}(y_3,\bar{y}_3)\Phi'_{j_4}(\infty)\right\rangle\\ &= \sum_{j,n'} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|y_2|^{2(j-j_1-j_2+n')}} C'^{j}_{j_1,j_2}\mathcal{D}(j_1,j_2,J)\,\langle\Phi'_{j_4}(\infty)\Phi'_{j_3}(y_3,\bar{y}_3)\Phi'^{jn'}_{J}(y_2,\bar{y}_2)\rangle\\ &= \sum_{j,n'} \mathcal{C}'(j)\frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}}\frac{|y_{23}|^{2(j+n'+j_3-j_4)}}{|y_2|^{2(j+n'-j_1-j_2)}}\mathcal{D}(j_1,j_2,J)\mathcal{D}(j_3,j_4,J)\,,\end{aligned} \tag{D-10}$$

with $\mathcal{C}'(j) = C'^{j}_{j_1,j_2} C'_{j,j_3,j_4}$.

We now convert the $SU(2)$ four-point function to the $m-$basis. This will be accomplished by the field transformation [103]

$$\Phi'_{j,m,\bar{m}} = \frac{1}{2\pi i} \oint d^2y |y|^{2(m-j-1)} c^{j+m}_{2j} \Phi'_j(y,\bar{y}) \,, \tag{D-11}$$

where c are the inverse of the binomial coefficients,

$$c^{j+m}_{2j} = \frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(2j+1)} \,. \tag{D-12}$$

We have restricted the quantum numbers to $m = \bar{m}$. We then get

$$\langle \Phi'_{j_1,j_1} \Phi'_{j_2,m_2} \Phi'_{j_3,m_3} \Phi'_{j_4,-j_4} \rangle = \frac{1}{(2\pi i)^2} \sum_{j,n'} \Big[\mathcal{C}'(j) \mathcal{D}(j_1,j_2,J) \mathcal{D}(j_3,j_4,J) c^{j_2+m_2}_{2j_2} c^{j_3+m_3}_{2j_3}$$
$$\frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}} \oint d^2y_2 d^2y_3 |y_2|^{2(j_1+m_2-j-n'-1)} |y_3|^{2(m_3-j_3-1)} |y_2-y_3|^{2(j+n'+j_3-j_4)} \Big]$$

or, after changing variables from y_2 to $y = y_2/y_3$,

$$\frac{1}{(2\pi i)^2} \sum_{j,n'} \Big[\mathcal{C}'(j) \mathcal{D}(j_1,j_2,J) \mathcal{D}(j_3,j_4,J) c^{j_2+m_2}_{2j_2} c^{j_3+m_3}_{2j_3} \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}}$$
$$\times \oint d^2y |y|^{2(j_1+m_2-j-n'-1)} |1-y|^{2(j+n'+j_3-j_4)} \oint d^2y_3 |y_3|^{2(j_1+m_2+m_3-j_4-1)} \Big] \,. \tag{D-13}$$

Both integrals can be carried out using the formula

$$\frac{1}{2\pi i} \oint \frac{dy}{y^n} \frac{1}{(1-y)^m} = \frac{\Gamma(n+m-1)}{\Gamma(n)\Gamma(m)} \tag{D-14}$$

such that the $SU(2)$ four-point function in the $m-$basis becomes

$$\langle \Phi'_{j_1,j_1} \Phi'_{j_2,m_2} \Phi'_{j_3,m_3} \Phi'_{j_4,-j_4} \rangle = \sum_{j,n'} \Big[\mathcal{C}'(j) \mathcal{D}(j_1,j_2,J) \mathcal{D}(j_3,j_4,J) c^{j_2+m_2}_{2j_2} c^{j_3+m_3}_{2j_3}$$
$$\times \frac{|z_2|^{2(\Delta(j)-\Delta(j_1)-\Delta(j_2)+n')}}{|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}} \frac{\Gamma(j_4-j_1-m_2-j_3)^2}{\Gamma(j+n'-j_1-m_2+1)^2 \Gamma(j_4-j-n'-j_3)^2} \delta^2_{j_1+m_2+m_3-j_4,0} \Big] \,. \tag{D-15}$$

We may eventually take $m_2 = j_2 - d$, $m_3 = j_3$ with $d \geq 0$ and set $z_{1,2,3,4} = 0, z, 1, \infty$. Then, $c^{j_3+m_3}_{2j_3} = 1$ (since $m_3 = j_3$) and (D-15) reduces to (6.12). Note that in the small z limit, the factor $|z_{23}|^{2(\Delta(j)+\Delta(j_3)-\Delta(j_4)+n')}$ with $z_{23} = z - 1$ is just one.

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