Sistemas diferenciales singulares de segundo orden. Un enfoque topológico

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Sistemas diferenciales singulares de segundo orden.
Un enfoque topológico

Estudiamos el siguiente tipo de sistemas de segundo orden:

$$Lu + g(u) = f(x) \qquad x \in \Omega,$$

con $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$ y $\mathcal{S}$ un conjunto acotado de singularidades; la función $f \in C(\overline{\Omega}, \mathbb{R}^N)$ tal que $\overline{f} := \frac{1}{|\Omega|} \int_\Omega f(x)\, dx = 0$ y u que satisfaga alguna condición de borde.

Primero trabajamos con el problema Periódico: $d = 1$, $\Omega = (0, T)$, $Lu = u''$ con condiciones de borde periódicas:

$$u(0) = u(T), \qquad u'(0) = u'(T).$$

En segundo lugar estudiamos el problema elíptico: $Lu = \Delta u$, $d > 1$ con una condición de borde no local:

$$\begin{cases} u \equiv C & x \in \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, & \end{cases}$$

dónde C es un vector desconocido constante. Esta condición de borde puede verse como una generalización de la condición periódica cuando $d = 1$ y Ω es un intervalo abierto.

En ambos casos usamos la teoría de grado topológico para probar existencia de soluciones cuando g satisface una cierta condición geométrica tanto cerca del conjunto $\mathcal{S}$ como en infinito.

Estudiamos por separado el caso en el que $\mathcal{S} = \{0\}$, una singularidad aislada. Aquí buscamos soluciones de problemas no singulares aproximados. Finalmente buscamos algún tipo de convergencia de estas soluciones a un candidato de solución para el problema original.

Palabras clave: problemas resonantes; teoría de grado; sistemas elípticos; sistemas periódicos; singularidades repulsivas.

2010 MSC: 34B16, 34C25, 35D99, 35J66, 47H11.

Second Order Singular Differential Systems.
A Topological Approach

We study the following type of Second Order Systems:

$$Lu + g(u) = f(x) \qquad x \in \Omega,$$

with $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$, and $\mathcal{S}$ a bounded set of singularities; the function $f \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\overline{f} := \frac{1}{|\Omega|}\int_\Omega f(x)\,dx = 0$ and u satisfying some boundary condition.

We first work with the Periodic Problem: $d = 1$, $\Omega = (0, T)$, $Lu = u''$ with periodic boundary conditions:

$$u(0) = u(T), \qquad u'(0) = u'(T).$$

Secondly we study an Elliptic Problem: $Lu = \Delta u$, $d > 1$ with a nonlocal boundary condition:

$$\begin{cases} u \equiv C & x \in \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, & \end{cases}$$

with C an unknown constant vector. This boundary conditions can be seen as a generalization of a periodic condition when $d = 1$ and Ω is an open interval.

In both cases we apply topological degree theory to prove existence of solutions when g satisfies certain geometrical conditions both near the set $\mathcal{S}$ and at infinity.

We study separately the case when $\mathcal{S} = \{0\}$, an isolated singularity. Here we look for solutions of the nonsingular problem and study approximated problems. Finally, we look for some kind of convergence of the solutions.

Keywords: resonant problems; degree theory; elliptic systems; periodic systems; repulsive singularities.

2010 MSC: 34B16, 34C25, 35D99, 35J66, 47H11.

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Introducción

El Análisis No lineal es un área en la Matemática que tiene un gran número de aplicaciones. En este trabajo se estudian sistemas no lineales de ecuaciones diferenciales de segundo orden. En particular, problemas de contorno de la forma:

$$Lu = Nu \qquad \text{en } \Omega,$$

en dónde $\Omega \subset \mathbb{R}^d$ es un dominio acotado, L un operador diferencial lineal y N un operador no lineal. Trabajamos solamente con operadores de segundo orden y nuestros resultados principales son para el caso $L = \Delta$, el Lapalaciano. Trabajamos en su mayoría con no linearidades de la forma $Nu = f - g(u)$. Dependiendo el contexto, fueron estudiadas diferentes condiciones de borde.

Trabajaremos con $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$, con $\mathcal{S}$ un conjunto acotado de singularidades. El caso no singular ($\mathcal{S} = \emptyset$) tiene, por supuesto, mucha importancia y un capítulo entero está dedicado a él (Capítulo 3). Debido al tipo de condiciones de contorno que serán explicadas más adelante, asumiremos que el término forzante $f \in C(\overline{\Omega}, \mathbb{R}^N)$ tiene promedio cero en cada coordenada: $\overline{f} = \frac{1}{|\Omega|} \int_\Omega f(x)dx = 0$.

El objetivo principal de este trabajo fue, en primer lugar, generalizar y extender resultados previos para el caso no singular, descripto en el Capítulo 3. Primero trabajamos con un sistema diferencial no lineal de segundo orden:

$$u'' + g(u) = p(t), \qquad t \in (0, T),$$

con $p \in C([0, T], \mathbb{R}^N)$, y condiciones de borde periódicas:

$$\begin{cases} u(0) = u(T) \\ u'(0) = u'(T) \end{cases}.$$

El resultado seminal en este tema se debe a Nirenberg [29], quien generalizó el trabajo pionero en el caso escalar, de Landesman y Lazer [23], quienes llegaron a una condición que en pocas palabras pedía a la g tener límites en el infinito g_+ y g_- de diferente signo. Nirenberg pidió que la g tuviera límites radiales en infinito uniformes diferentes de cero, en todas las direcciones. Eso significa que para cada $v \in S^{N-1}$, exista $g_v = \lim_{s\to+\infty} g(sv)$ uniformemente y sea distinto de 0.

A partir de este punto hay diversas direcciones en las cuales se pueden generalizar estos resultados. La reestricción en [29] que la g no pueda anularse en infinito fue descartada por Ortega y Ward Jr en [32], en dónde permitieron a la g tener las llamadas *vanishing nonlinearities*, es decir, que tienda a cero en infinito.

Amster y De Nápoli, en [6], lidiaron con el problema de los límites radiales uniformes. A partir de resultados en el caso escalar en los que se pedían condiciones más débiles, los autores fueron motivados a intentar debilitar dicha condición. Alcanzaron una condición geométrica realmente interesante, bastante más débil que la condición de Nirenberg. Se trata de cubrir a la esfera S^{N-1} con un número finito de abiertos U_j y tomar direcciones $w_j \in S^{N-1}$ tales que el límite uniforme exista, pero en cada U_j :

$$\limsup_{r\to+\infty} \langle g(ru), w_j\rangle := S_j(u) < 0.$$

Esta tesis nace con la idea de juntar estas dos últimas generalizaciones en el caso periódico, para obtener nuevos resultados de existencia. La herramienta principal usada para este proposito son los métodos topológicos, en particular trabajamos con métodos de punto fijo, grado topológico y teoría de continuación de Mawhin.

Nuestro resultado principal para este problema es el Teorema 3.2.1 en el cual alcanzamos este objetivo. También pudimos probar otro resultado, con condiciones algo menos técnicas, realmente similares a aquellas de Landesman y Lazer [23]. Nos referimos al Teorema 3.2.5.

Luego de tener éxito y lograr resultados de existencia, tuvimos la grata visita en Buenos Aires del Profesor Rafael Ortega, de la Universidad de Granada. Nos sugirió considerar no linealidades con singularidades, teniendo como motivación el problema de Kepler y el de un potencial eléctrico, dado que ambos eran ejemplos de casos con *vanishing nonlinearities.*

Esta nueva perspectiva nos llevó a nuevos horizontes y comenzamos a estudiar problemas singulares, que en nuestro contexto es cuando el conjunto singular $\mathcal{S}$ consiste de un único punto, y tomamos 0 como ese punto, pero por supuesto podría ser cualquier otro en $\mathbb{R}^N$. Las principales referencias con las que trabajamos fueron Coti Zelati [15], Solimini [34] y Fonda y Toeder [18], en el cual encontramos las principales dificultades y problemas abiertos en el área. Vale la pena mencionar un trabajo de Zhang[40], en el cual son usados métodos topológicos. Decidimos trabajar con singularidades de tipo repulsivo, eso es, cuando $\langle g(u), u\rangle < 0$ cerca del origen.

Atacamos las singularidades perturbando el problema con aproximaciones continuas de g. Para cada una de ellas hicimos uso de nuestros resultados para el caso no singular, que fueron descriptos anteriormente, y obtuvimos una sucesión de soluciones. Una tarea dificil fue la de hallar cotas uniformes para estas sucesiones con el fin de asegurar existencia de una función límite, candidata a ser solución del problema original.

Logramos esto con el Teorema 4.2.4. Este resultado nos dió una función límite y un candidato a solución del problema original. Con condiciones algo más fuertes, conseguimos probar en el Teorema 4.2.5 que este candidato era de hecho una *solución generalizada* (será explicada más adelante, en el Capítulo 4) del problema.

También a partir de este último teorema mencionado, obtuvimos un resultado fuerte para el caso en el que g sea un gradiente ($g = \nabla G$), con $\lim_{u\to} G(u) = +\infty$, que implica un tipo más fuerte de repulsividad. En este caso probamos que el límite de los problemas aproximados debía a su vez ser una solución clásica del problema original.

Estas ideas fueron plasmadas en [7] y serán discutidas en profundidad en el Capítulo 4.

Nuestro próximo paso fue trabajar con el problema elíptico:

$$\Delta u + g(u) = f(x), \qquad x \in \Omega \subset \mathbb{R}^d,$$

con g como antes y $f \in C(\overline{\Omega}, \mathbb{R}^N)$, con las siguientes condiciones de borde:

$$\begin{cases} u \equiv C & u \in \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, & \end{cases}$$

en dónde C es un vector constante desconocido. Estas condiciones pueden verse como una generalización de las condiciones periódicas pues, si $d = 1$ y $\Omega = (0, T)$, la primera condición resulta $u(0) = u(T)$ y la segunda indica que $u'\big|_0^T = 0$. Este tipo de condiciones fue estudiada por Berestycki y Brézis en [11] y por Ortega en [30] y proviene de un problema de la física del plasma, que fue estudiado exhaustivamente en un trabajo de Temam [37].

Las técnicas que usamos para probar resultados para el caso no singular en el problema elíptico fueron similares a aquellas que usamos para el problema periódico, obviamente teniendo en cuenta las dificultades que surgen en el contexto de los problemas elípticos. Aquí es prudente mencionar que esto fue posible dada la naturaleza del operador diferencial, sin importar el espacio en el que está definido. Tanto el operador u''

con condiciones de borde periódico, como el operador Δu con las condiciones de borde no locales previamente mencionadas tienen núcleo de una dimensión, las funciones constantes de $\mathbb{R}^N$. Este hecho hace que la extensión sea posible.

Sin embargo, cuando tratamos de extender los resultados obtenidos en [7] para el caso singular tuvimos serias dificultades. La perdida de compacidad hizo que no consigamos obtener el mismo tipo de resultados de convergencia de los problemas aproximados. No obstante, con condiciones más fuertes, pudimos obtener resultados importantes. Estos fueron probados en [8] y están disponibles el Capítulo 5.

Este contratiempo nos llevó a estudiar tipos más generales de singularidades. Comenzamos a considerar a $\mathcal{S}$ como un conjunto acotado arbitrario. En [8] obtuvimos resultados de existencia usando una condición geométrica introducida por Ruiz y Ward Jr en [33] y extendida por Amster y Clapp en [5]. Está basada en aplicar la teoría de continuación de Mawhin [27] en conjuntos convenientes provenientes de cotas a-priori de la solución del problema.

Primero, en el Teorema 3.3.1 probamos la versión no singular del resultado, que fue una adaptación de los resultados recién comentados para el sistema elíptico que estamos considerando, con condiciones no locales.

El Teorema 5.2.2 fue nuestro principal resultado en este contexto, debido a que trabajamos con un conjunto $\mathcal{S}$ general de singularidades y obtuvimos soluciones clásicas en conjuntos convenientes.

Conseguimos probar resultados interesantes de existencia e incluso encontramos una forma de detectar multiplicidad de soluciones, dependiendo del conjunto de singularidades. Desde ya, dado que el problema depende esencialmente de los aspectos topológicos del operador, los resultados que probamos también son válidos en el caso periódico.

Finalmente, también probamos un resultado similar al del caso periódico en el caso que el conjunto $\mathcal{S}$ fuera un punto aislado y la singularidad es de tipo repulsivo. Una noción distinta de solución generalizada tuvo que ser definida debido a que la falta de compacidad de las inmersiones de Sobolev no nos permitió obtener estimaciones uniformes fuertes para nuestros problemas aproximados. Probamos en el Teorema 5.3.4 que dadas ciertas condiciones, la existencia de este tipo de soluciones generalizadas puede ser asegurada.

Esta tesis está organizada de la siguiente manera:

En el próximo Capítulo, se presenta la matemática necesaria para entender por completo los resultados aquí presentados. Está dividido en una sección de preliminares analítcos y otra de preliminares topológicos. En la primera se enuncian resultados de inmersión de espacios de Sobolev junto algunos otros resultados relevantes. En la segunda, se repasan los teoremas de punto fijo y hay una introdcción autocontenida a la teoría de grado topológico hasta llegar a la teoría de continuación de Mawhin.

El Capítulo 2 es una breve historia de los dos principales problemas tratados en este trabajo: Los problemas resonantes y los problemas singulares. Aquí, las principales referencias son explicadas con más detalle y se presentan las dificultades principales de los problemas.

En el Capítulo 3 damos resultados para el problema cuando g es no singular. Consisten en generalizaciones y extensiones de los previamente enumerados resultados del Capítulo 2. Los resultados provienen tanto de [7] como de [8], ya que se trata el problema periódicos como el elíptico. Este capítulo será constantemente usado en los dos siguientes.

En el Capítulo 4 el problema periódico es estudiado. La mayoría del mismo está dedicado al caso en el que se trata de una singularidad aislada y repulsiva. El esquema de aproximación es explicado y los resultados principales de [7] son probados.

Por último, el Capítulo 5 trata el problema elíptico, tanto el caso de la singularidad aislada y repulsiva como el caso del conjunto de singularidades. Los resultados de este último capítulo fueron publicados en [8].

Introduction

Nonlinear Analysis is an area of Mathematics that has a great number of applications. The study of Second Order Nonlinear Differential Equations is the one treated in this work. In particular, our objects of study will be Boundary Value Problems (BVP) of this type:

$$Lu = Nu \qquad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^d$ will be a bounded domain, L a Linear Differential Operator and N a nonlinear one. We will work only with Second Order Operators and our main results will be for the case $L = \Delta$, the Laplacian. We will work mostly with nonlinearities of the form $Nu = f - g(u)$. Different boundary conditions are studied depending on the context.

We will work with $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$, with $\mathcal{S}$ a bounded set of singularities. The nonsingular case ($\mathcal{S} = \emptyset$) will of course have an important role and an entire chapter is dedicated to it (Chapter 3). Because of the type of boundary conditions that will be explained later, the forcing term $f \in C(\overline{\Omega}, \mathbb{R}^N)$ will have zero average in each coordinate: $\overline{f} = \frac{1}{|\Omega|} \int_\Omega f(x)dx = 0$.

The main goal of this work was to, at first, generalize and extend previous results in the nonsingular case, described in Chapter 3. We first worked with a second order nonlinear ordinary differential system:

$$u'' + g(u) = p(t), \qquad t \in (0, T),$$

with $p \in C([0, T], \mathbb{R}^N)$ and Periodic Boundary Conditions:

$$\left\{ \begin{array}{l} u(0) = u(T) \\ u'(0) = u'(T) \end{array} \right. .$$

The seminal result in this area is due to Nirenberg [29], who generalized the pioneer work in the area done by Landesman and Lazer [23], who had worked on the scalar case, with the hypothesis that g had to have limits at infinity g_+ and g_- with different sign. Nirenberg asked g to have nonzero uniform radial limits at infinity in all directions. That means that for every $v \in S^{N-1}$, the limit $g_v = \lim_{s\to+\infty} g(sv)$ exists uniformly and it is not equal to zero.

From this point forth, there are several ways to generalize the results. The restriction in [29] that g can not vanish at infinity was discarded by Ortega and Ward Jr in [32], where they allowed g to have *vanishing nonlinearities* at infinity.

Amster and De Nápoli, in [6], dealt with the uniform radial limit problem. Due to results with much weaker conditions for the scalar case, the authors were motivated to try to weaken such condition. They reached an interesting geometrical condition, much weaker than the classical condition in [29]. It involves covering S^{N-1} with a finite number of open sets U_j and taking directions $w_j \in S^{N-1}$ such that the uniform limit exists for each $u \in U_j$:

$$\limsup_{r\to+\infty} \langle g(ru), w_j \rangle := S_j(u) < 0.$$

The genesis idea of this thesis was to mix these last two generalizations in the periodic problem, to obtain new existence results. Topological Methods are the main tools used for this purpose, in particular we worked with Fixed point Methods, Topological Degree and Mawhin's Continuation Theory.

Our main result for this problem is Theorem 3.2.1 in which we achieved this last goal. We were able to prove another result, with slightly less technical conditions, and really similar to those of Landesman and Lazer [23]. We are referring to Theorem 3.2.5.

After succeeding with this problem, we fortunately had the visit in Buenos Aires of Professor Rafael Ortega. He suggested us to consider nonlinearities with singularities, having as a motivation the Kepler problem and the electrical charges potential problem, as both were examples of *Vanishing Nonlinearities* cases.

This took us to a quite different framework and we started to study singular problems, that in our context is when the singular set consist of an isolated point, $\mathcal{S} = \{0\}$, and took 0 to be this point, but of course it could be any point $s \in \mathbb{R}^N$. The main references we worked with were Coti Zelati [15], Solimini [34] and Fonda and Toeder [18], in which we found the main difficulties and open problems in the area. It is also worth mentioning a work of Zhang [40], in which topological methods are used. We decided to work with repulsive type singularities, that is, when $\langle g(u), u \rangle < 0$ near the origin.

We attacked the singularities by perturbating the problem with continuous approximations of g. For each one of them we used the continuous results we had studied in the beginning, and got a sequence of solutions. A difficult task was to find uniform bounds to these sequences to ensure the existence of a limit function, candidate to be a solution of the original problem.

We accomplished this with Theorem 4.2.4. This result gave us the existence of a limit function, and a candidate for a solution for the original problem. With stronger conditions, we were able to prove in Theorem 4.2.5 that this candidate was in fact a *generalized solution* (this concept will be explained in Chapter 4) of the problem. Also as a part of this last theorem, we got a strong result for the periodic case: If the nonlinearity g was a gradient ($g = \nabla G$), with $\lim_{u \to} G(u) = +\infty$, which implies a stronger kind of repulsiveness, we proved that the limit function was indeed a classical solution of the problem. These ideas, along with the nonsingular results were done in [7] and are thoroughly discussed in Chapter 4.

Our next step was to work with the Elliptic Problem:

$$\Delta u + g(u) = f(x), \qquad x \in \Omega \subset \mathbb{R}^d,$$

with g as before and $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with the following *Nonlocal* Boundary Conditions:

$$\begin{cases} u \equiv C & u \in \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, & \end{cases}$$

where C is an unknown vector constant in $\mathbb{R}^N$. This conditions can be seen as a generalization of the periodic problem, because if $d = 1$ and $\Omega = (0, T)$, the first condition reads $u(0) = u(T)$ and the second one $u'\big|_0^T = 0$. This type of condition was studied by Berestycki and Brézis in [11] and by Ortega in [30] and comes from a Plasma Physics problem. This problem is thoroughly studied in a work by Temam [37].

The techniques we used to prove results for the nonsingular case in the Elliptic Problem were similar to those used for the periodic case, obviously taking into account the difficulties that arise in the elliptic framework. Here it is worth to mention that this was possible because of the nature of the operator, regardless the space it is defined in. Both the u'' with periodic boundary conditions, and Δu with the nonlocal boundary conditions described before have a one dimensional Kernel, the constant functions. This fact is the one that makes the extensions possible.

Nevertheless, when trying to extend the results obtained in [7] for the singular case, we had serious difficulties. The loss of compactness made it impossible to obtain the same results of convergence of the approximate solutions. Nevertheless, by strengthening some conditions we were able to get some important results. These results were proved in [8] and are available in Chapter 5

This setback led us to study more general type of singularities. We started to consider $\mathcal{S}$ as an arbitrary bounded set. In [8] we obtained existence results of solutions using a geometrical condition introduced by Ruiz and Ward Jr in [33] and extended by Amster and Clapp in [5]. It is based in applying Mawhin's Continuation Theory [27] in convenient sets given by a priori bounds of the solutions.

First, in Theorem 3.3.1 we proved the nonsingular version, that was an adaptation of the results just commented to the Elliptic System we are considering, with the Nonlocal Boundary Conditions.

Theorem 5.2.2 was our main result in this context, because we worked with a general set $\mathcal{S}$ of singularities and obtained classical solutions in convenient sets.

We obtained interesting existence results and even found some way of detecting multiple solutions, depending on the set of singularities. Of course, because it is a problem that essentially depends on the topological aspects of the spaces and operators, these results are valid for the periodic case.

Finally, we also proved a result similar to that of the periodic case when the set $\mathcal{S}$ is an isolated point and the singularity is of a repulsive kind. A different notion of generalized solution had to be defined because the lack of compactness of the Sobolev embeddings did not allow us to have such strong estimates for the approximated problems. We proved in Theorem 5.3.4 that given certain conditions, the existence of this type of generalized solution can be ensured.

This thesis is organized as follows:

In the next Chapter, we give the mathematics needed to fully understand the results here showed. It is divided in a topological section, in which fixed point theorems, degree theory and continuation theory are described; and an analytical section, where Sobolev spaces are revised and the main classical results are enumerated.

Chapter 2 is a brief but thorough history of the two main type of problems this thesis works with: Resonant Problems and Singular Problems. Here, the main references are described with more detail and the difficulties of the problems are presented.

In Chapter 3 we give results for the case when g is nonsingular. They consist on generalizations and extensions of the previous results enumerated in Chapter 2. The results come both from [7] and [8] as they are both on the periodic problem and the elliptic one. This chapter will be constantly used in the last two chapters.

In Chapter 4 the periodic problem with a repulsive singularity is studied. The main section deals with the case of the isolated singularity of a repulsive type.The approximation scheme is explained and the main results from [7] are stated.

Finally, Chapter 5 deals with the Elliptic problem and both the singular repulsive nonlinearity as well as the general set of singularities are studied. The results from this chapter were published in [8].

Chapter 1

Preliminaries

This section is meant to present the mathematical background needed to appreciate and understand the concepts that will be used throughout the work.

We divide the preliminaries in two parts: An Analytical one with classical results in Sobolev spaces and Differential Equations, and a Topological one, where we give more than just the definitions and ideas from the following areas: Fixed Point Theorems, Topological Degree Theory, Mawhin's Continuation Theory and some Nonlinear Functional Analysis.

1.1 Analytical Preliminaries

1.1.1 Sobolev Embeddings

Here we enumerate the main results in the classical theory. Let us recall some notation and definitions:

Definition 1.1.1.

$$W^{k,p}(U) := \{u \in L^1_{loc}(U) : D^\alpha u \in L^p(U) \quad \forall \alpha : |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index.

If $p = 2$ we write $H^k(U) := W^{k,2}(U)$.

In these spaces we define the following norms:

Definition 1.1.2.

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|=0}^k \int_U |D^\alpha u|^p dx\right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha|=0}^k ess\sup_U |D^\alpha u| & p = \infty. \end{cases}$$

We recall that with these norms, Sobolev spaces are Banach spaces, while the H^k are also Hilbert spaces with the natural inner product:

$$\langle f, g \rangle := \sum_{0 \leq |\alpha| \leq k} \int_\Omega D^\alpha f D^\alpha g dx.$$

We have the classical Sobolev inequalities that give an answer to the embedding problems. The three big results depend on the relationship between p and n, the dimension. Another important fact for the theory is the Sobolev conjugate, also known as the Sobolev critical exponent:

Definition 1.1.3. If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* := \frac{np}{n-p}.$$

Note that we have the following relations:

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \qquad p^* > p.$$

Theorem 1.1.4 (Gagliardo-Nirenberg-Sobolev)**.** *Assume $1 \leq p < n$, then there exists a constant $C = C(n, p)$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall u \in C_0^1(\mathbb{R}^n).$$

This last results gives us estimates for bounded domains $U \subset \mathbb{R}^n$ for the Sobolev spaces:

Theorem 1.1.5. *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

with $C = C(n, p, U)$.

And for the $W_0^{1,p}$ spaces we have the following important result:

Theorem 1.1.6. *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$, and $u \in W_0^{1,p}(U)$. Then $u \in L^q(U)$, for each $q \in [1, p^*]$, and we have the estimate:*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}.$$

A particular case of this is the well-known Poincaré inequality:

Theorem 1.1.7 (Poincaré). *Assume* $1 \leq p \leq \infty$, *and* $u \in W_0^{1,p}(U)$. *Then there exists a constant* $C = C(p,n)$ *such that we have the estimate*

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

The case $n < p < \infty$ is due to Morrey:

Theorem 1.1.8 (Morrey). *There exists a constant* $C = C(p,n)$ *such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n, \mathbb{R})$, *where* $\gamma := 1 - n/p$.

We now give the famous general Sobolev Inequalities (only for the case $k < n/p$):

Theorem 1.1.9. *Let* U *be a bounded open subset of* $\mathbb{R}^n$ *with a* C^1 *boundary. Assume that* $u \in W^{k,p}(U)$: *If* $k < n/p$, *then* $u \in L^q(U)$, *where* $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ *and the following estimate holds:*

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

and $C = C(k,p,n,U)$.

Next, we focus on the compact embeddings. We first recall the definition:

Definition 1.1.10. Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, written $X \subset\subset Y$ if there exist a constant C such that:

- $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$.
- each bounded sequence in X is precompact in Y, that is that it has a convergent subsequence in Y.

We can now state the Embedding Theorem:

Theorem 1.1.11 (Rellich-Kondrachov). *Assume* U *is a bounded open subset of* $\mathbb{R}^n$ *and* ∂U *is* C^1. *Suppose* $1 \leq p < n$. *Then*

$$W^{1,p}(U) \subset\subset L^q(U) \qquad \forall q \in [1, p^*).$$

Remark 1.1.12. By letting $p \to n$, we have that $p^* \to \infty$ since $p^* > p$, so we have in particular:

$$W^{1,p}(U) \subset\subset L^p(U) \qquad \forall p \in [1, \infty].$$

We already knew this result if $p \in [n, +\infty]$ using Arzela-Ascoli's Theorem. Finally, note that

$$W_0^{1,p}(U) \subset\subset L^p(U) \qquad \forall p \in [1, \infty],$$

even without assuming ∂U to be C^1.

Let us end this section of the preliminaries with an inequality we will use throughout this work: Poincaré Inequality, a generalization of Theorem 1.1.7. For the case $n = 1$ it is also known as Wirtinger Inequality. First we recall the definition of the average:

Definition 1.1.13. We define the average of a function as

$$\overline{u} := \frac{1}{|U|} \int_U u(x)dx.$$

If $n = 1$, $U = (0, T)$, then it becomes

$$\overline{u} := \frac{1}{T} \int_0^T u(t)dt.$$

Note that if the function is periodic, i.e. $u(t + T) = u(t)$ for all $t \in \mathbb{R}$, then the average is also defined as before.

Remark 1.1.14. An important remark is that the average will be also a projection to the Kernel for the operators we are going to work with, for example, when $L = u''$, $n = 1$ and we work with Periodic Boundary Conditions.

Theorem 1.1.15. *Let U be a bounded, connected, open subset of $\mathbb{R}^n$, $n > 1$ with a C^1 boundary ∂U. Assume $p \in [1, \infty]$, then there exists a constant $C = C(n, p, U)$ such that*

$$\|u - \overline{u}\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}, \qquad \forall u \in W^{1,p}(U).$$

If $n = 1$ and $U = (0, T)$ we have the so called Wirtinger Inequality:

$$\|u - \overline{u}\|_{L^p(0,T)} \leq C\|u'\|_{L^p(0,T)}.$$

Finally, we recall the Dual Space $H^{-1}(U)$:

Definition 1.1.16. We denote by $H^{-1}(U)$ the dual space to $H_0^1(U)$ and we write $\langle , \rangle$ the pairing between $H^{-1}(U)$ and $H_0^1(U)$ as if $\langle f, v\rangle = f[v]$ In other words, if $f \in H^{-1}(U)$ there exist functions f^0 and $\tilde{f} = (f^1, \cdots, f^n)$ in $L^2(U)$ such that

$$\langle f, v\rangle = \int_U f^0 v + \sum_{i=1}^{n} \int_U f^i v_{x_i} dx \quad \forall v \in H_0^1(U).$$

For more on this, see Evans [17].

1.1.2 Elliptic Equations

In Chapter 5 we will deal with Elliptic equations of the form:

$$\Delta u + g(u) = f(x), \qquad x \in \Omega \subset R^d, \tag{1.1}$$

with some kind of Boundary Conditions. We will work only with the Laplacian in this work, although most of the results can be extended to a broader type of operators, the so called $p-$Laplacian type Operator. For example in the ordinary differential equation framework can be deffined as:

Definition 1.1.17. $Lu = \phi(u')'$ is called a $p-$Laplacian if $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

- For every $x_1 \neq x_2 \in \mathbb{R}^N$, we have that
$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2\rangle > 0.$$
- There exists a funcion $\alpha : (0, +\infty) \to (0, +\infty)$ such that it verifies $\alpha(s) \to +\infty$ as $s \to +\infty$ and
$$\langle \phi(x), x\rangle \geq \alpha(|x|)|x| \qquad \forall\, x \in \mathbb{R}^N.$$

Both conditions imply that ϕ is an homeomorphism onto $\mathbb{R}^N$. The most standard example are the $N-$dimensional $p-$Laplacian given by

$$\phi(x) = |x|^{p-2}x \qquad p > 1.$$

or a system of one-dimensional $p-$Laplacians, namely:

$$\phi(x) = \left(|x_1|^{p_1-2}x_1, \cdots, |x_N|^{p_N-2}x_N\right) \qquad p_j > 1.$$

Finally, we enumerate a series of results that we use freely in the rest of this work, we begin by giving an important resut regarding the Strong Maximum Principle, The Hopf's Lemma. In these results, we consider $Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$, with a^{ij}, b^i, c continuous and L uniformly elliptic.

Theorem 1.1.18 (Hopf). *Assume $u \in C^2(U) \cap C^1(\overline{U})$. Suppose further that*

$$Lu \leq 0 \qquad \text{in } U,$$

and there exists a point $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \qquad \text{for all } x \in U.$$

Assume finally that U satisfies the interior ball condition at x^0, that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

i) If $c \equiv 0$ in U, then

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

with ν the outer unit normal to B at x^0.

ii) Moreover, if $c \geq 0$ in U, the same holds provided $u(x^0) \geq u(x)$.

Mean-Value Theorem for Vector-Valued integrals:

Theorem 1.1.19. *If $\gamma \in C([0,T],\Omega)$, with $\Omega \subset \mathbb{R}^n$, then*

$$\overline{\gamma} = \frac{1}{T}\int_0^T \gamma(t)dt \in \overline{co(\Omega)},$$

where $co(\Omega)$ is the convex hull of Ω.

Definition 1.1.20. Given $A \in \mathbb{R}^n$, we define the *Convex Hull* of A as the smallest convex set that contains A. Formally, the convex hull may be defined as the intersection of all convex sets containing A or as the set of all convex combinations of points in A.

Here we also recall Fredholm's alternative Theorem:

Theorem 1.1.21. *Let E be a Banach space and $T : E \to E$ a linear compact operator. Then for any $\lambda \neq 0$, we have*

1) *The equation $(T - \lambda I)v = 0$ has a nonzero solution.*

or

2) *The equation $(T - \lambda I)v = f$ has a unique solution v for any function f.*

In the second case, the solution v depends continuously on f.

The Fredholm alternative can be restated as follows: any $\lambda \neq 0$ which is not an eigenvalue of a compact operator is in the resolvent, i.e., $(T - \lambda I)^{-1}$, is continuous.

Next, let us define the Green's Function for ordinary differential equations:

We will assume that the operator is in divergence form now, that is: $Lu = (-pu')' + qu$, with $p \in C^1([a,b],\mathbb{R})$, $p > 0$ and $q \in C([a,b],\mathbb{R}), q \geq 0$.

The problem is, given $\varphi \in C([a,b],\mathbb{R})$ find u such that:

$$\begin{cases} L[u](t) = \varphi(t) & t \in (a,b) \\ \mathcal{B}[u] = 0, & \end{cases}$$

with $\mathcal{B}$ an operator indicating the boundary conditions, for example:

$$\mathcal{B}[u] = \begin{cases} u(a) \\ u(b) \end{cases}, \quad \mathcal{B}[u] = \begin{cases} u(a) - u(b) \\ u'(a) - u'(b) \end{cases}, \quad \mathcal{B}[u] = \begin{cases} \alpha u(a) + \beta u(b) \\ \gamma u'(a) + \delta u'(b) \end{cases}.$$

It is worth remarking that not for all boundary conditions there will be a solution.

We state that u is given by:

$$u(t) = \int_a^b G(t,s)\varphi(s)ds,$$

with $G : [a,b] \times [a,b] \to \mathbb{R}$ the so called Green's Function. This function G has the following properies:

1) $L_t[G](t,s) = 0$ for $a < t < s$ and for $s < t < b$.

2) It satisfy the boundary conditions.

3) $G \in C([a,b] \times [a,b],\mathbb{R})$. In particular, in $t = s$, which implies $G(s_-,s) = G(s_+,s)$.

4) $G \in C^1([a,b] \times [a,b] \backslash \{t = s\},\mathbb{R})$, and it has a jump:

$$\frac{\partial G(s_-,s)}{\partial t} - \frac{\partial G(s_+,s)}{\partial t} = \frac{1}{p(t)}.$$

1.1.3 Resonant Problems

Finally, we give a short introduction to resonant problems. Let us consider the general nonlinear problem:

$$Lu = Nu,$$

where L is a differential operator and N is a nonlinear operator, which might involve also derivatives of less degree of those of L. Boundary conditions are also present, and they define the space where the operator is defined. For example the scalar problem

$$u'' = f(t, u, u') \qquad t \in (0, T),$$

with $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ a bounded function. If L is invertible in a suitable space then the problem is called *non resonant*. A simple example of a non resonant problem is the previous equation under Dirichlet Boundary Conditions, $u(0) = u(T) = 0$. In this case, $Lu = u''$ and $ker(L) = 0$. The problem reduces to a fixed point problem:

$$u = L^{-1}Nu$$

and fixed point theory can be applied directly.

If on the other hand L is not invertible, then the problem is called *resonant*. This is the case if in the previous example we consider Neumann, or periodic conditions, where $ker(L)$ is non trivial. If $L = u''$ as in the example, and $L : D \subset C([0,T], \mathbb{R}) \to D$, with D the subspace of the constant functions, the Kernel is in fact D. This is a case of resonance in the first eigenvalue (in this case 0). This denomination comes from the following:

If we consider the eigenvalue problem

$$-u'' = \lambda u$$

with periodic conditions, then it is not hard to see that the eigenvalues are:

$$\lambda_k = \left(\frac{2k\pi}{T}\right)^2, \qquad k = 0, 1, \cdots.$$

The first eigenvalue is 0, and the associated eigenspace is the space of constant functions. More on this type of problems can be found below, where an example of the Mawhin's Continuation Theory is explained. For more of this see Amster [3].

1.2 Topological Preliminaries

1.2.1 Fixed Point Theorems

We here give a brief enumeration of the most important fixed point theorems, which are the cornerstones of the Topological Methods for solving nonlinear problems.

The classical proof of existence and uniqueness of solution for an ordinary differential equation with initial conditions relies in the Piccard method of successive approximation. In his PhD thesis (1917) Banach proved that Piccard's method was in fact a particular case of a much more general result. First we recall the definition of a contraction:

Definition 1.2.1. Let X, Y be two metric spaces, we say that $T : X \to Y$ is a contraction if there exists $\alpha < 1$ such that:

$$\forall\, x, y \in X, \quad d_Y(Tx, Ty) \leq \alpha d_X(x, y).$$

We state here the famous Banach's Fixed Point Theorem:

Theorem 1.2.2 (Banach)**.** *Let X be a complete metric space and let $T : X \to X$ a contraction. Then, T has a unique fixed point $\hat{x}$. Moreover, $\hat{x}$ can be calculated in an iterative way from the sequence $x_{n+1} = T(x_n)$, starting from any $x_0 \in X$.*

Other important Fixed Point Theorem is due to Brouwer:

Theorem 1.2.3 (Brouwer)**.** *Let $B = B_1(0) \subset \mathbb{R}^N$ and $f \in C(\overline{B}, \overline{B})$. Then there exists $x \in \overline{B}$ such that $f(x) = x$.*

The Brouwer Fixed Point Theorem was one of the early achievements of algebraic topology, and is the basis of more general fixed point theorems which are important in functional analysis. The case $N = 3$ first was proved by Piers Bohl in 1904. It was later proved by L. E. J. Brouwer in 1909. Jacques Hadamard proved the general case in 1910, and Brouwer found a different proof in 1912. Since these early proofs were all non-constructive and indirect, they ran contrary to Brouwer's intuitionist ideals. However, methods to construct (approximations to) fixed points guaranteed by Brouwer's Theorem are now known. It can also be proven that it is equivalent to the axiom of completeness.

Although Theorem 1.2.3 is valid for any set homeomorphic to the unit ball $\overline{B} \subset \mathbb{R}^N$, Kakutani (1943) showed that it is not true for infinite dimensional spaces. Some additional hypothesis is needed for the operator T.

J. Schauder, around 1930, proved another Fixed Point Theorem, this time for infinite dimensional spaces:

Theorem 1.2.4 (Schauder)**.** *Let $(E, \|\cdot\|)$ be a normed space and let C be a closed convex and bounded subset of E. If $T : C \to C$ is a continuous function such that $T(C)$ is relatively compact ($\overline{T(C)}$ is compact), then T has at least a fixed point.*

The last fixed point theorem in this enumeration is an extension of the previous one, and has important applications in nonlinear problems, in particular it is the starting point of the Continuation Theory which will be explained later in this section. It was stated and proved by Leray and Schauder in 1934. We give here a particular case, due to Schauder:

Theorem 1.2.5 (Leray-Schauder)**.** *Let E be a Banach space and the operator $T : E \to E$ is compact. If there exists $R > 0$ such that the following property holds:*

$$\text{If } x = \lambda Tx \qquad \text{for some } \lambda \in [0,1] \quad \text{then,} \quad \|x\| < R.$$

Then T has at least a fixed point in X.

1.2.2 The Topological Mapping Degree

Introduction

Let us first of all recall the definition of two maps being Homotopic. This property will be the key point in the definition of the Degree.

Definition 1.2.6. Two maps $f_1 : E \to F$ and $f_2 : E \to F$ are homotopic if there is a continuous map $h : E \times [0,1] \to F$ such that $h(x,0) = f_1(x)$ and $h(x,1) = f_2(x)$.

Given two topological spaces E and F, one can define an equivalence relation on the continuous maps $f : F \to E$ using homotopies, by saying that $f_1 \sim f_2$ if f_1 is homotopic to f_2. Roughly speaking, two maps are homotopic if one can be deformed into the other. This equivalence relation is transitive because these homotopy deformations can be composed (i.e., one can follow the other). Thus, this relationship defines a class of homotopy.

A simple example is the case of continuous maps from S^1 to S^1. Consider the number of ways an infinitely stretchable string can be tied around a tree trunk. The string forms the first circle, and the tree trunk's surface forms the second circle. For any integer n, the string can be wrapped around the tree n times, for positive n clockwise, and negative n counterclockwise. Each integer n corresponds to a homotopy class of maps from S^1 to S^1.

After the string is wrapped around the tree n times, it could be deformed a little bit to get another continuous map, but it would still be in the same homotopy class, since it is homotopic to the original map. Conversely, any map wrapped around n times can be deformed to any other.

Let us start with a well known situation that will let us define the degree for $n = 2$. Let $\Omega \subset \mathbb{C}$ be a bounded domain, and for simplicity, let Ω be simply connected and that it's boundary $\gamma := \partial\Omega$ is a continuous curve, with positive orientation. Given an analytic function $f : \overline{\Omega} \to \mathbb{C}$, such that $f \neq 0$ in γ, we recall the following formula (a particular case of the theorem of zeros and poles):

$$d(f,\Omega) := \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = \#\{\text{zeros of } f \text{ in } \Omega\}.$$

We can make the following remarks concerning $d(f,\Omega)$:

- If $f = Id$ and $0 \notin \gamma$ then $d(f,\Omega) = 1$ if $0 \in \Omega$ and $d(f,\Omega) = 0$ if $0 \notin \Omega$.
- If $d(f,\Omega) \neq 0$, then f has at least a zero in Ω. This trivial fact for analytic functions will be the fundamental property and application of the extension of this definition for a continuous f.
- Homotopy Invariance: If $f \sim g$ then $d(f,\Omega) = d(g,\Omega)$.
- $d(f,\Omega)$ only depends on $f|_\gamma$. This can be seen as a direct consequence of the previous item. Because if $f|_\gamma = g|_\gamma$, the homotopy $h(z,\lambda) = \lambda f(z) + (1-\lambda)g(z)$ is such that for every $z \in \gamma$, $h(z,\lambda) = f(z) = g(z) \neq 0$, then $f \sim g$.

Recalling the *Index function* from Complex Analysis we can remark:

$$d(f,\Omega) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{1}{z}dz = I(f\circ\gamma, 0).$$

This Index is defined for continuous curves, as long as the function is not zero along this curve. This tells us that $h_\lambda := h(\lambda,\cdot)$ would not need to be analytical. Therefore, we could be able to extend our definition for a function $f \in C(\overline{\Omega},\mathbb{R})$ such that $f \neq 0$ in γ, just by defining this *degree* as $d(f,\Omega) := I(f\circ\gamma,0)$. It is not hard to show that the previous properties are still valid.

In the following section we will try to extend this definition for any continuous function $f : \overline{\Omega} \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. For convenience, we will define for every $y \in \mathbb{R}^n - f(\gamma)$, the degree $d(f,\Omega,y) \in \mathbb{Z}$ that will *count* the number of solutions in Ω of the equation $f(x) = y$. In $\mathbb{C}$, $d(f,\Omega,y) = I(f\circ\gamma,y)$, but this index is equal to that of the function $f - y$ with respect to 0. So we'll define in general:

$$d(f,\Omega,y) = d(f-y,\Omega,0).$$

Finally, knowing that this is the case when $n = 2$, we will need the degree to have the additivity property: If $\Omega_1 \cap \Omega_2 = \emptyset$, $f : \overline{\Omega_1 \cup \Omega_2} \to \mathbb{R}^n$ and $f \neq y$ in $\partial\Omega_1 \cup \partial\Omega_2$, then:

$$d(f,\Omega_1\cup\Omega_2,y) = d(f,\Omega_1,y) + d(f,\Omega_2,y).$$

The Brouwer Degree The goal is to extend the last definition to an arbitrary continuous function in an arbitrary finite dimensional space. First let us define:

$$\mathcal{A}(y) = \{f \in C(\overline{\Omega}, \mathbb{R}^n) : g \neq y \text{ in } \partial\Omega\}$$

be the set of *admissible functions*. First of all one can prove that the set is open:

Lemma 1.2.7. *If $f \in \mathcal{A}(y)$ and $g \in C(\overline{\Omega}, \mathbb{R}^n)$ satisfies the inequality $\|g - f\|_{L^\infty} < d(y, f(\partial\Omega))$ where $d(\cdot, \cdot)$ is the distance, then $g \in \mathcal{A}(y)$.*

Now, let us define the concepts of Critical and Regular values. Our fist definition of Degree will be only possible on Regular values.

Definition 1.2.8. Let $m \leq n$, $f \in C^\infty(\overline{\Omega}, \mathbb{R}^m)$. The *regular values* and *critical values* of f are defined as follow:

$$RV(f) = \{y \in \mathbb{R}^m : \forall x \in f^{-1}(y), Df(x) : \mathbb{R}^n \to \mathbb{R}^m \text{ is onto}\}.$$

$$CV(f) = \mathbb{R}^m \backslash RV(f)$$

We also note that if $y \in RV(f)$, then the set $f^{-1}(y)$ is finite. With this fact we give the definition of the degree function on regular points of a function $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$:

Definition 1.2.9. Let $y \in RV(f)$, the Brouwer Degree is defined as:

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(J_f(x)),$$

where $J_f(x) = det(Df(x))$.

For example, if $f \in C^1(\mathbb{R}, \mathbb{R})$ and $0 \in RV(f)$, the degree over an open interval (a, b) at 0 is equal to the times the function f crosses the axis with positive slope minus the times it does it with negative slope.

We now give the tools that will allow us to have a good definition of the degree not only on regular points. The first step is to state a version of Sard's Theorem:

Lemma 1.2.10. *Let $m \leq n$ and $f \in C^\infty(\overline{\Omega}, \mathbb{R}^m)$. Then, the set of critical values $CV(f)$ has measure 0. In particular the set of regular values $RV(f)$ is dense in $\mathbb{R}^m$.*

Now, calling $C^\infty_{reg}(\overline{\Omega}, \mathbb{R}^m)$ the set of functions in $C^\infty(\overline{\Omega}, \mathbb{R}^m)$ for which $0 \in RV(f)$, as a consequence of Sard's Theorem, we have the density of the functions that have 0 as a regular point:

Lemma 1.2.11. *$C^{\infty}_{reg}(\overline{\Omega}, \mathbb{R}^m)$ is dense in $C(\overline{\Omega}, \mathbb{R}^m)$.*

Now we state a result that says that for a function f and a set Ω, the degree is constant in a ball sufficiently small around 0 (we take 0 without loss of generality):

Lemma 1.2.12. *Let $f \in C^1(\overline{\Omega}, R^n)$ such that $0 \in RV(f)$ and $f \neq 0$ in $\partial\Omega$. Then, there exists a neighborhood V of 0 such that if $y \in V$, then $y \in RV(f)$, $f \neq y$ in $\partial\Omega$ and $\deg(f, \Omega, y) = \deg(f, \Omega, 0)$.*

The next Lemma shows that $\deg(f, \Omega, 0)$ is constant in the connected components of $\mathcal{A}(0) \cap C^{\infty}_{reg}(\overline{\Omega}, \mathbb{R}^n)$.

Lemma 1.2.13. *Let $f \in C^{\infty}_{reg}(\overline{\Omega}, R^n)$, then there exists $\varepsilon > 0$ such that if $g \in C^{\infty}(\partial\Omega, \mathbb{R}^n)$ is such that $\|g - f\|_{L^{\infty}} < \varepsilon$ then, $0 \in RV(g)$, $g \neq 0$ in $\partial\Omega$ and $\deg(g, \Omega, 0) = \deg(f, \Omega, 0)$.*

With all the previous results and remarks, is is possible to prove the good definition of the topological degree.

Definition 1.2.14. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, and let $y \in \mathbb{R}^n$. Then there exists one, and only one continuous function

$$\deg(\cdot, \Omega, y) : \mathcal{A}(y) \to \mathbb{Z}$$

called the *Brouwer's degree* with the following properties:

1. Normalization: If $y \in \Omega$, then $\deg(id, \Omega, y) = 1$.

2. Translation invariance: $\deg(f, \Omega, y) = \deg(f - y, \Omega, 0)$.

3. Additivity: If Ω_1, Ω_2 are two open disjoint subsets of Ω, then the following is true:

   If $y \notin f(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$, then:

   $$\deg(f, \Omega, y) = \deg(f|_{\overline{\Omega_1}}, \Omega_1, y) + \deg(f|_{\overline{\Omega_2}}, \Omega_2, y).$$

4. Excision: If Ω_1 is an open subset of Ω, $y \notin f(\overline{\Omega} - \Omega_1)$, then

   $$\deg(f, \Omega, y) = \deg(f, \Omega_1, y).$$

5. Solution: If $\deg(f, \Omega, y) \neq 0$, then $y \in f(\Omega)$, moreover, $f(\Omega)$ is a neighborhood of y.

6. Homotopy invariance: If $h : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ is continuous and $h(x,\lambda) \neq y$ for all $x \in \partial\Omega, \lambda \in [0,1]$, then $\deg(h(\cdot,\lambda),\Omega,y)$ does not depend on $\lambda \in [0,1]$. Moreover, y can be replaced by a continuous function $y : [0,1] \to \mathbb{R}^n$ such that the previous condition is valid.

Theorem 1.2.15. *There exist function as the one defined before and it is unique.*

For a proof of this and all the Lemmas stated in this section refer to the books of Amster [3] or Teschl [38], where they give a more detailed analysis of this subject. The first appearence of this notion was in 1911 in a work from Brower [13]

The Leray-Schauder Mapping Degree

The objective of this section is to extend the mapping degree form $\mathbb{R}^n$ to general Banach spaces E. It is not possible to define a general degree for continuous functions from closed domains $\Omega \subset E$.

We first remark that the Brouwer degree can be trivially generalized to finite dimensional Banach spaces, simply by identifying the space E with $\mathbb{R}^n$, where $n = dim(E)$. This degree can also be defined for functions $f \in C(\Omega,\mathbb{R}^m)$, with $\Omega \subset \mathbb{R}^n$, with $m \leq n$:

Lemma 1.2.16. *Let $\Omega \subset \mathbb{R}^n$ a bounded domain, $f \in C(\overline{\Omega},\mathbb{R}^m)$ and let $m < n$. Let also be $g : \overline{\Omega} \to \mathbb{R}^n$ in which we think $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ by the following identification $(x_1,\cdots,x_m) = (x_1,\cdots,x_m,0,\cdots,0)$. Then, for every $y \in \mathbb{R}^m \backslash g(\partial\Omega)$ we have*

$$\deg(g,\Omega,y) = \deg(g|_{\overline{\Omega}\cap\mathbb{R}^m},\Omega\cap\mathbb{R}^m,y).$$

For infinite dimensional spaces we will limit ourselves to consider operators T of the form $T = I - K$, where $K : \overline{\Omega} \to E$ is a compact operator. This kind of operators are called Fredholm Operators and can be approximated by finite range operators:

Lemma 1.2.17. *Let $K : \overline{\Omega} \to E$ be a compact operator, and let $T = I - K$. Given $\varepsilon > 0$ there is an operator $T_\varepsilon : \overline{\Omega} \to E$ continuous such that $Rg(T_\varepsilon) \subset V_\varepsilon$, with $dim(V_\varepsilon) < \infty$ and such that $\|T(x) - T_\varepsilon(x)\| < \varepsilon$, for all $x \in \overline{\Omega}$.*

The proof of this Lemma is a consequence of the proof of Schauder's fixed point Theorem (Theorem 1.2.4). An important fact is that this does not depend on the approximation K_ε chosen.

From now on, E will be a Banach space, $\Omega \subset E$ a bounded domain and $K : \overline{\Omega} \to E$ a compact operator. The following result is immediate from the compactness.

Lemma 1.2.18. *If $Kx \neq x$, for all $x \in \partial\Omega$, then*

$$\inf_{x\in\partial\Omega} \|x - Kx\| > 0.$$

Having stated all the results and remarks, we are now able to define the Leray-Schauder degree:

Definition 1.2.19. Let Ω, E and K as before such that $(I-K)x \neq 0$, for all $x \in \partial\Omega$ and let

$$\epsilon < \frac{1}{2} \inf_{x\in\partial\omega} \|x - Kx\|.$$

We can define the *Leray-Schauder's degree* as

$$\deg_{LS}(I - K, \Omega, 0) := \deg((I - K_\varepsilon)|_{V_\epsilon}, \Omega \cap V_\varepsilon, 0)$$

where K_ε is such that $Rg(K_\varepsilon) \subset V_\varepsilon$ and that $\|K(x) - K_\varepsilon(x)\| < \varepsilon$, for all $x \in \overline{\Omega}$.

Finally we state that the definition does not depend on the approximation we take.

The properties of the Leray-Schauder mapping degree are analogous to the ones of the Brouwer degree. It is interesting to note that the homotopy invariance requires the additional hypothesis that the homotopy h is of the form $h(\cdot, \lambda) = I - K_\lambda$ with K_λ compact.

Another Definition of the Degree

Another way to define the Topological Degree is by means of Algebraic Topology. For more on this refer to Dold [16]. Every endomorphism ϕ of a free cyclic group is given by an integer. Applying this remark to homology groups defines the notion of degree in algebraic topology:

Definition 1.2.20. If $f : S^{n-1} \to S^{n-1}$ is a map, then the induced endomorphism f_* of $\tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ is given by $f_*(x) = deg(f) \cdot x$, where $deg(f) \in \mathbb{Z}$ is a uniquely determined integer. This integer is called the degree of f.

In this context, we can enumerate the main properties:

Proposition 1.2.21. *This definition of degree has the following properties:*

1. $deg(Id) = 1$.
2. $deg(f \circ g) = deg(f)deg(g)$.
3. $f \simeq g \Rightarrow deg(f) = deg(g)$.

4. *The degree of a homotopy equivalence is* ± 1.

We give this last definition because throughout this thesis we will work in both environments. Here, a connection between the two settings:

Proposition 1.2.22. *Let* $g : \mathbb{R}^N \to \mathbb{R}^N$ *a continuous function and let* $R > 0$ *and* $\Phi_R : S^{N-1} \to S^{N-1}$ *defined by:*

$$\Phi_R(v) = \frac{g(Rv)}{|g(Rv)|},$$

and suppose this limits exist. If $B_R(0) \subset \mathbb{R}^N$ *is the open ball of radius* R *and center in the origin, the following equivalence holds:*

$$\deg(g, B_R(0), 0) = deg(\Phi_R),$$

with the expresion on the left being the Brouwer degree and the the one on the right being the degree just defined.

1.2.3 Mawhin's Continuation Theory

Let us give a formal overview of the subject, mainly following Mawhin's classical book [27]. The objective is to have existence results for the following problem:

$$Lu = Nu.$$

We consider X, Z two normed spaces, $U \subset X$ a bounded set. The operator $L : Dom(L) \to Z$, $N : \overline{U} \to Z$ such that L is Fredholm of index 0. That is:

i) L is linear and $Im(L)$ is closed.

ii) $dim(ker(L)) = dim(coker(L)) = n < \infty$.

Let us recall the definition of co-dimension (the dimension of the co-Kernel):

Definition 1.2.23. *Co-dimension* is a term used in a number of algebraic and geometric contexts to indicate the difference between the dimension of certain objects and the dimension of a smaller object contained in it. For example

$$codim(W) = dim(V) - dim(W)$$

gives the co-dimension of a subspace W of a finite-dimensional abstract vector space V.

For infinite-dimensional spaces, the co-dimension is the dimension of the quotient space:

$$codim(W) = dim(V/W),$$

that agrees with the definition in the finite case.

Note that (i)-(ii) imply that there exist $P : X \to X$ and $Q : Z \to Z$ continuous projectors such that the following sequence is exact.

$$X \to Dom(L) \to Z \to Q.$$

Remember that being an exact sequence means that $Im(P) = ker(L)$ and $Im(L) = ker(Q)$. Moreover, $\Pi : Z \to coker(L)$ with $\Pi z = z + Im(L)$ is continuous. We then need N to be $L-$compact. That is:

iii) N continuous and bounded.

iv) $K_{P,Q}N : \overline{U} \to X$ is compact in $\overline{U}$.

with $K_{P,Q} := K_P(Id - Q)$ and K_P being the local inverse operator of L_P, with $L_P : Dom(L) \cap ker(P) \to Im(L)$. In this context we have the following

Proposition 1.2.24. *If i)-iv) hold and $\wedge : coker(L) \to ker(L)$ exists, then if $u \in Dom(L) \cap \overline{U}$ the following are equivalent:*

a) u is a solution of $Lu = Nu$.

b) u is a solution of $(I - P)\,u = (\wedge\Pi + K_{P,Q})Nu$.

c) u is a fixed point of $M = P + (\wedge\Pi + K_{P,Q})N$. Moreover, M is compact.

d) u is a zero of $I - M$ i.e.

$$0 = u - Pu + (\wedge\Pi + K_{P,Q})Nu.$$

Finally, if the following holds:

v) $0 \notin (L - N)(Dom(L) \cap \partial U)$

or equivalently, there is no $u \in Dom(L) \cap \partial U$ such that $Lu = Nu$, then the Leray-Schauder degree (1.2.19) $\deg_{LS}(I - M, U, 0)$ is well defined. It is also important that this degree is independent of the choice of P and Q:

Proposition 1.2.25. *If i)-v) hold:*

- $\deg_{LS}(I-M,U,0)$ *only depends on* L,N,U *and the homotopy class of* $\wedge$ *in* $\mathcal{L}=\{\wedge : coker(L)\to ker(L) : \wedge \text{ is an isomorphism}\}$.
- $|\deg_{LS}(I-M,U,0)|$ *only depends on* L,N *and* U.

The idea of this theory is to give something of a recipe to prove existence of solutions of nonlinear problems:

Given, L and N as before (i)-iv)) we consider the following family of operators $\tilde{N}:\overline{U}\times[0,1]\to Z$ such that $N=\tilde{N}(\cdot,1)$.

For $\lambda\in[0,1]$, we have the following family of problems:

$$(P_\lambda) \qquad\qquad Lu=\tilde{N}(u,\lambda).$$

We now state Mawhin's famous Continuation Theorem:

Theorem 1.2.26. *Let* $L,\tilde{N}$ *as before and* U *a bounded domain. Suppose that the following two conditions hold:*

- $\forall\lambda\in[0,1],\ u\in\partial U Dom(L)\Rightarrow\ Lu\neq Nu.$
- $d_{LS}(I-M,U,0)\neq 0.$

with $M=P+(\wedge\Pi+K_{P,Q})N$ *and* P *and* K *as before. Then, for all* $\lambda\in[0,1]$ (P_λ) *has a solution.*

An Example of the use of the C.T.

Let us show how all this technology is used. Let us consider the following scalar periodic problem:

$$\begin{cases} u''+g(u)=p(t) & t\in(0,T)\\ u(0)=u(T) & \\ u'(0)=u'(T), & \end{cases} \tag{1.2}$$

where $g\in C(\mathbb{R},\mathbb{R})$, $p\in C([0,T],\mathbb{R})$. We shall assume that the average of p, denoted by $\overline{p}=\frac{1}{T}\int_0^T p(t)dt$ is zero. Let $\varphi\in C([0,T],\mathbb{R})$ with $\overline{\varphi}=0$, Linear Theory assures us that there exist a unique u solution of problem

$$\begin{cases} u''=\varphi & t\in(0,T)\\ u(0)=u(T) & \\ u'(0)=u'(T) & \\ \overline{u}=0. & \end{cases} \tag{1.3}$$

With this construction in mind, it is possible to define an operator K that given φ as before, $K\varphi=u$. It is not hard to prove that this operator K is in fact compact.

If we call $Nu = p - g(u)$, and take $\lambda \in [0,1]$ we consider the following problems:

$$u'' = \lambda Nu.$$

We have the following particular case of Proposition 1.2.24 and we show a proof to it.

Proposition 1.2.27. *For $\lambda \in (0,1]$, u is a solution of (P_λ) if and only if u is a solution of*

$$u = \overline{u} + \overline{Nu} + \lambda K\left(Nu - \overline{Nu}\right) := T_\lambda u,$$

Proof:
On one hand, if u is a solution of $u'' = \lambda Nu$ taking average it holds that $\overline{Nu} = 0$, so $\lambda K(Nu - \overline{Nu}) = \lambda KNu = Ku'' = u - \overline{u}$ because K is a left inverse of u'' and $\overline{K\varphi} = 0$. So the second equation holds.

On the other hand, if $u = \overline{u} - \overline{Nu} + \lambda K\left(Nu - \overline{Nu}\right)$, also taking average, we have that $\overline{u} = \overline{\overline{u}} + \overline{\overline{Nu}} + \overline{\lambda K\left(Nu - \overline{Nu}\right)}$. As $Kv = 0$ for all v, we have the following:

$$\overline{u} = \overline{u} + \overline{Nu}.$$

So again, $\overline{Nu} = 0$, and u is a solution of

$$u = \overline{u} + \lambda K(Nu).$$

Applying L, we have $u'' = \lambda Nu$, and the result holds.

∎

This two statements are also equivalent to the existence of a zero of the operator $F_\lambda = I - T_\lambda$.

As $F\lambda$ is a Fredholm operator, Leray-Schauder Degree can be applied.

Taking $\lambda \in [0,1]$, if we now consider the family of operators F_λ such that

$$F_\lambda u = u - \left[\overline{u} - \overline{Nu} + \lambda K\left(Nu - \overline{Nu}\right) = (I - T_\lambda)u\right]$$

we have that $F = F_1$ and $F_0 u = u - (u - \overline{Nu})$.

Note also that $Rg(T_0) = \mathbb{R} \subset C([0,T],\mathbb{R})$, the constant functions, with $dim(Rg(T_0)) = 1$.

Given $U = B_R(0) \subset \mathbb{C}([0,T],\mathbb{R})$ we have, by the definition of the degree:

$$\deg_{LS}(F_0, U, 0) = \deg(F_0|_{U\cap\mathbb{R}}, U \cap \mathbb{R}, 0),$$

this last being the Brouwer degree.

Note also that in $\mathbb{R}$, $\overline{u} = u$ so, $F_0|_{U\cap\mathbb{R}} u = \overline{Nu}$.

Then we can consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(u) = F_0|_{\mathbb{R}} = \overline{Nu} = \frac{1}{T}\int_0^T p(t) - g(u)dt = \overline{p} - g(u) = -g(u).$$

Then, if the first condition of Theorem 1.2.26 holds, the only thing to prove to assure existence of a solution of (1.2) is that:

$$deg(g, (-R, R), 0) \neq 0$$

where $(-R, R) = U \cap \mathbb{R}$, as $U = B_R(0)$ is a bounded open domain.

A classical condition, that will be described in the next chapter is due to Landesman and Lazer in [23]:

Assume that the limits $\lim_{s\to\pm\infty} g(s) = g_\pm$ exist and are finite and that the inequality $g_{-\infty} < 0 < g_{+\infty}$ hold.

For example, if $g(u) = \arctan(u)$ the result will be valid. Indeed, the fact that $\deg(g, (-R, R), 0) \neq 0$ is trivial because in this case, as seen in (1.2.15), the degree of a real function is the sum of the signs of the slopes of the tangent at the points where $g(u) = 0$. In this case, $g(u) = 0$ only at $u = 0$ and g increases, so $\deg(g, (-R, R), 0) = 1$ for every $R > 0$. Another way to show this is to see that $g \sim id$.

Now, we need an R for which the other condition holds: Let R be large enough, and consider $\lambda \in (0, 1]$. Let $u \in \partial U$, with

$$U = \{u \in C([0, T], \mathbb{R}), \|u\|_\infty \leq R\},$$

a $T-$periodic solution of

$$u'' = \lambda(p(t) - g(u)).$$

Suppose that this R does not exists, hence, there exists u_n and λ_n such that

$$u_n'' = \lambda_n N u_n \qquad \|u_n\|_\infty \to \infty.$$

Taking the average, as $\int_0^T u_n'' dt = 0$ and $\lambda_n \neq 0$, for all n

$$0 = \frac{1}{T}\int_0^T Nu_n(t)dt = \overline{p} - \frac{1}{T}\int_0^T g(u_n(t))dt = \frac{1}{T}\int_0^T g(u_n(t))dt.$$

This implies that $\int_0^T g(u_n)dt = 0$, but Landesman-Lazer conditions imply that $g_{-\infty} < 0 < g_{+\infty}$. This is a contradiction because one can prove that $\|u_n - \overline{u_n}\|$ is bounded, so that $\|u_n\|_{L^\infty} \to \infty$ implies that $|u_n| \to \pm\infty$.

Chapter 2

A brief survey of the problems

2.1 Resonant Problems

2.1.1 The Landesman-Lazer Conditions

The pioneer work on resonant problems in the direction of our studies is from Landesman and Lazer [23]. They studied the following scalar problem: Let $\Omega \subset \mathbb{R}^d$ a bounded domain, we find a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} Lu + \alpha u + g(u) = h(x) & \text{in } \Omega \\ u = 0 \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a^{ij} \left(\frac{\partial}{\partial x_j} \right)$ is a second order, self adjoint, uniformly elliptic operator.

By a weak solution of (2.1) the authors mean an $H_0^1(\Omega)$ solution of

$$u = \alpha T u + T[g(u) - h], \tag{2.2}$$

where $T : L^2(\Omega) \to L^2(\Omega)$ and Tf is the unique solution of the linear problem:

$$\begin{cases} Lu = -f & \text{in } \Omega \\ u = 0 \quad \partial\Omega. \end{cases} \tag{2.3}$$

The following result is proven:

Theorem 2.1.1. *Let $w \in H_0^1(\Omega)$, a non trivial solution ($w \neq 0$) of $u = \alpha T u$, that is, a weak solution of*

$$\begin{cases} Lu + \alpha u = 0 & \text{in } \Omega \\ u = 0 \quad \partial\Omega \end{cases} \tag{2.4}$$

Assume that the space of solutions of $u = \alpha T u$ has dimension 1, i.e. every solution is of the form cw; that the limits

$$\lim_{s\to+\infty} g(s) = g_+, \qquad \lim_{s\to-\infty} g(s) = g_-$$

exist and are finite and that

$$g_- \le g(s) \le g_+ \qquad \forall s. \tag{2.5}$$

Define $\Omega^+ = \{x \in \Omega : w(x) > 0\}$, $\Omega^- = \{x \in \Omega : w(x) < 0\}$. The inequalities

$$g_- \int_{\Omega^+} |w| dx - g_+ \int_{\Omega^-} |w| dx \le \langle h, w\rangle \le g_+ \int_{\Omega^+} |w| dx - g_- \int_{\Omega^-} |w| dx, \tag{2.6}$$

are necessary and the strict inequalities are sufficient for the existence of a weak solution of the boundary value problem (2.1).

Moreover, if (2.5) is replaced by the slightly stronger condition:

$$g_- < g(s) < g_+ \qquad \forall s, \tag{2.7}$$

then the strict inequalities are both necessary and sufficient for the existence of at least one solution of the boundary value problem (2.1).

Go to the last section of Chapter 1 for a proof of this result in an example.

Remark 2.1.2. The assumption that there exists a nontrivial solution of (2.4) is not that strict. It has been proved by the authors that if L is such that for $\alpha_1 \le \alpha \le \alpha_2$, the boundary value problem (2.4) has no nontrivial solution. Let $p(x, u), h(x, u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$. If h is uniformly bounded and $\alpha_1 \le p(x, u) \le \alpha_2$ in $\Omega \times \mathbb{R}$, then the boundary problem

$$\begin{cases} Lu + p(x, u)u = h(x, u) & \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

has at least one weak solution. In particular, if (2.4) has no nontrivial weak solutions and g is merely assumed to be continuous and bounded, then the problem (2.1) has a weak solution.

Note also that the case $g \equiv 0$ is included, and (2.6) reduces to the well known orthogonality condition $\langle h, w\rangle = 0$ for the linear boundary problem.

The authors finally give the following example: If K is a constant function and $\Omega = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$:

$$\begin{cases} \Delta u + 2u + arctg(u) = K & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

here, $L = \Delta$, $g = arctg$, $\alpha = 2$ and $h \equiv K$.

It is not difficult to show that the linear boundary problem:

$$\begin{cases} \Delta u + 2u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has the strictly positive solution $w(x,y) = \frac{2}{\pi}\sin(x)\sin(y)$, that any other solution is of the form cw and that $g(u) = arctg(u)$ clearly satisfies condition (2.7). Noting that $\Omega^+ = \Omega$, and Ω^- is empty, the nonlinear problem has a weak solution if and only if $-\frac{\pi}{2} < K < \frac{\pi}{2}$.

Another important remark is that if only (2.5) is assumed, then the strict inequalities need not hold. They give the following example:

$$g(s) = \chi_{\{s\geq 0\}} + \chi_{\{s<0\}}e^{-s^2},$$

where $\chi_A(x) = 1$ if $x \in A$ and otherwise, $\chi_A(x) = 0$. Let Ω, w be as before and suppose that all solutions are of the form cw. If we define $h(x,y) = g(w(x,y))$, then w is also a strict solution of

$$\begin{cases} \Delta u + 2u + g(u) = h(x,y) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As before, $\Omega^+ = \Omega$, Ω^- is empty and $h \equiv 1$ on Ω, hence we have:

$$\langle h, w\rangle = g_+ \int_{\Omega^+} |w|dx - g_- \int_{\Omega^-} |w|dx,$$

so the strict inequalities do not hold, although the problem has a weak solution.

2.1.2 Nirenberg's Extension to Systems

In [29], Nirenberg showed the use of some topological techniques for solving nonlinear problems. In the introduction, a simple problem is stated: Let $B \subset \mathbb{R}^N$ be the closed unit ball and $T : B \to \mathbb{R}^M$ a continuous mapping, the problem is to obtain an x such that $Tx = 0$.

There are some conditions on the boundary values of $T_0 = T|_{\partial B}$ which ensure that for every extension T of T_0 inside B, the equation $Tx = 0$ is always solvable. Assume that $T_0 \neq 0$ in ∂B, then one has the following result, in terms of the normalized map $\Psi : \partial B \subset S^{N-1} \to S^{M-1}$, defined as:

$$\Psi(x) = \frac{T_0(x)}{|T_0(x)|}.$$

.

Proposition 2.1.3. *For every extension T of T_0, the equation $Tx = 0$ is always solvable if and only if the homotopy class (see 1.2.6) of Ψ is nontrivial.*

This theorem gives useful results only in the case $M \leq N$. If $N = M$, the fact that the homotopy class of Ψ is nontrivial, means that the degree of the map Ψ, i.e. the number of times the image sphere is covered (counted algebraically), $deg(\Psi)$ (see Definition 1.2.20 in Chapter 1) is different form zero. This number is also equal to the degree of a map T at the origin in the image space, i.e, the number of times the origin is covered (counted algebraically) $\deg(T, B, 0)$, as it was stated in the Proposition 1.2.22.

In an infinite dimensional Banach space X, the previous result can be generalized. Let $B \subset X$ be the closed unit ball (B could be the closure of any open set in X), and $T : B \to X$, with $K = (I - T)$, a compact operator. The Leray-Schauder theory states that if $T_0 \neq 0$, then the mapping T has an integral valued degree at the origin and if it is different than zero, then $Tx = 0$ is solvable in B. The degree depends only on T_0 (the value at the boundary), in fact only in the homotopy class of T_0 within the class of operators such that $(I - T_0)$ is compact and $T_0 \neq 0$ in ∂B.

It is useful to remark that if the $Rg(T) \subset Y \subset X$ where $Y \neq X$ is a linear subspace, then the degree of T at the origin is zero, since it is the same for all points in a neighborhood of the origin and, at a point outside Y, i.e. outside the range of T, it vanishes.

Here the author describes a generalization of the Leray-Schauder Theorem (Theorem 1.2.5) to such a situation and an application to a nonlinear elliptic boundary value problem.

Definition 2.1.4. Let $T : B \to Y \subset X$ with $I - T$ a compact operator, $Tx \neq 0$ in ∂B, and Y a closed subspace having finite co-dimension i. If $T_0 = T_{\partial B}$ is such that the equation $Tx = 0$ is solvable in B for any extension T of T_0 inside B of the form $I - K$ with K compact, and $Rg(T) \subset Y$, we call T_0 *essential*. Whether T_0 is essential or not depends only on its homotopy class, always of the form $I - K$, of maps into $Y^* = Y \backslash \{0\}$.

It can be shown that T_0 has this very special form (with $V \oplus Z = W$, $Y = W_1 \oplus V$ and $X = Y \oplus Z$):

$$T_0 x = T_0(w_1 + w) = w_1 + \Phi(w),$$

with Φ a continuous map of the closed unit ball in W into the linear subspace V of W. We shall express the condition for T_0 to be essential in terms of the map Φ which does not vanish for $\|w\| = 1$. Suppose $dim(W) = N$, $dim(V) = M$, $i = N - M$, set

$$\Psi(w) = \frac{\Phi(w)}{\|\Phi(w)\|}, \qquad \text{for } \|w\| = 1.$$

We may consider $\Psi : S^{N-1} \to S^{M-1}$.

Theorem 2.1.5. *T_0 is essential if and only if the map Ψ has nontrivial stable homotopy.*

Let us explain the main application of the above Theorem: We recall the problem given in [23], and Theorem 2.1.1 with $\alpha = 0$ and the strict inequalities.

Here, the author gives a generalization of the result, based on Theorem 2.1.5 concerning elliptic systems of N equations for $u = (u^1, \cdots, u^N)$, $u^j : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, with Ω an open domain. Let L be a linear elliptic operator of order m, and consider vector functions u satisfying the homogeneous conditions $Bu = 0$.

An important fact is that $ker(L) =< w_1, w_2, \cdots, w_d >$, furthermore, $Rg(L) =< w'_1, w'_w, \cdots, w'_{d^*} >^{\perp}$. Then, the elliptic operator L has index $i = ind(L) = d - d^*$.

We shall assume that $i \geq 0$. We shall also make the following hypothesis concerning the Kernel: $w \equiv 0$ is the only $w \in ker(L)$ that vanishes on a set of positive measure in Ω.

We note that this is the analogue of asking in [23] the existence of a nontrivial solution of the linearized problem. The nonlinear system to be solved is of the form:

$$\begin{cases} Lu = g(x, D^\alpha u) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $D^\alpha g \in C(\overline{\Omega}, \mathbb{R}^N)$ for $0 \leq |\alpha| \leq m - 1$ and $D^\alpha g \in C(\Omega, \mathbb{R}^N)$ for $|\alpha| = m$. For all arguments $\eta = \{\eta^\alpha\} \neq 0$ (and D^α is symmetric). We suppose that

$$h(x, \eta) = \lim_{s \to \infty} g(x, s\eta), \tag{2.9}$$

and that the convergence is uniform in $\overline{\Omega} \times \{|\eta| = 1\}$. Nirenberg gives sufficient conditions on h to ensure solvability of (2.8).

For $a \in S^{d-1}$ define the map $\phi : S^{d-1} \to \mathbb{R}^{d^*-1}$ by

$$\phi_k(a) = \langle h(x, D^\alpha \sum_{j=1}^{d} a_j w_j(x)), w'_k \rangle \qquad k = 1, \cdots, d^*.$$

As a consequence of the hypothesis that the only $w \in ker(L)$ that vanishes in a set of positive measure is the trivial solution, one may prove that the mapping ϕ is continuous. Now assume that $\phi(a) \neq 0$ for $a \in S^{d-1}$ and set:

$$\psi : S^{d-1} \to S^{d^*-1}, \qquad \psi(a) = \frac{\phi(a)}{|\phi(a)|}.$$

Theorem 2.1.6. *If ψ has nontrivial stable homotopy then (2.8) is solvable.*

By a solution, we mean a function in C^{m-1} with derivatives of order m in $L^p(\Omega)$ for large p. If g is smooth then using regularity theory, it follows that these solutions are smooth.

Remark 2.1.7.

- If $d = d^*$, then ψ has a nontrivial stable homotopy and it means that ψ is homotopically nontrivial ($deg(\psi) \neq 0$). In this case, the result is proven using the Leray-Schauder degree.

- When $N = 1$, $d = d^* = 1$, and $g = g(x, u)$, then $h(x, \eta)$ corresponds to

  $$h_\pm(x) = h(x, \pm 1) = \lim_{u \to \pm\infty} g(x, u),$$

  and in this case, being homotopically nontrivial means that

  $$A_1 = \int_{\Omega^+} h_+ w' dx + \int_{\Omega^-} h_- w' dx,\; A_2 = \int_{\Omega^-} h_+ w' dx + \int_{\Omega^+} h_- w' dx$$

  have opposite signs, so the theorem contains the result of Landesman and Lazer described above (Theorem 2.1.1) as a special case.

- Since it is not known how to determine whether a map ψ has nontrivial stable homotopy, the theorem is not readily applicable.

2.1.3 Generalizations of the Nirenberg Result

Another interesting work is due to Krasnoselskii and Mawhin [22]. It has an introduction that gives a perfect insight of the interesting problems of the area. In this work they consider the 2π-periodic problem for the equation

$$-x'' - n^2 x + g(x) = p(t), \tag{2.10}$$

where n is a positive integer, $p(t)$ is continuous and 2π-periodic, and $g(x)$ is bounded and continuous. They give a new formulation for the Lazer-Leach conditions for the existence of 2π-periodic solutions, and new sufficient conditions for the existence of unbounded sequences of such solutions.

The corresponding pioneering work is due to Lazer and Leach [25], who proved the existence of at least one 2π-periodic solution under one of the conditions

$$|\overline{p}| < 2\big(\liminf_{x\to+\infty} g(x) - \limsup_{x\to-\infty} g(x)\big)$$

or

$$|\overline{p}| < 2\big(\liminf_{x\to-\infty} g(x) - \limsup_{x\to+\infty} g(x)\big),$$

where $\overline{p} = \int_0^{2\pi} e^{it} p(t)dt$.

In the same paper [25], Lazer and Leach have also proved that if g is not constant and if

$$|\overline{p}| \geq 2\big(\sup_{\mathbb{R}} g - \inf_{\mathbb{R}} g\big),$$

then equation (2.10) has no $2\pi-$periodic solution. Alonso and Ortega in [1] have shown that when local uniqueness of the Cauchy problem holds, this last condition implies that every solution of (2.10) satisfies

$$\lim_{|t|\to\infty} [x^2(t) + x'^2(t)] = +\infty,$$

and that the unboundness of sufficiently large solutions follows from a weaker condition involving the asymptotic properties of g.

Here, we considered important to comment two important works for this thesis, extensions of the seminal Nirenberg results [29]. One is due to Amster and De Nápoli [6] in the context of a $p-$Laplacian type operator (1.1.17) in an ordinary differential system and the other is a work from Ortega and Ward Jr [32] in the context of an ellpitic problem with Neumann boundary conditions. These important results will be explained in detail in the next Chapter, as they were the motivation of some of the results discussed in this work.

A much more recent work, by Amster and Clapp [5] studies in depth the geometric nature of the conditions for the nonlinearity g. They start form a work of Lazer, [24], who considered the scalar differential equation

$$x'' + cx' + g(x) = p(t), \tag{2.11}$$

where c is a constant and $p(t)$ is a continuous and $T-$periodic function with zero average ($\overline{p} = 0$). Lazer, in [24] proved the existence of a $T-$periodic solution of (2.11) assuming that $g \in C(\mathbb{R}, \mathbb{R})$ satisfying

$$xg(x) \geq 0 \qquad \text{for } |x| \text{ sufficiently large,} \tag{2.12}$$

and

$$\frac{g(x)}{x} \to 0 \qquad \text{as } |x| \to \infty. \tag{2.13}$$

When one interprets the equation as an oscillator, condition (2.12) means that the force $-g(x)$ points toward the origin outside a compact set. Condition (2.13) is required in order to avoid the linear resonance occurring at $c = 0$ and $g(x) = \lambda_n x$, $n = 1, 2 \cdots$, where $\lambda_n = \left(\frac{2\pi n}{T}\right)^2$ is the n-th eigenvalue of the T-periodic problem for the linear operator $Lx = -x''$.

Very soon after the publication of [24], a work by Mawhin [27] appeared, extending the result to systems. If one considers (2.11) as a system in $\mathbb{R}^N$, $p(t) = (p_1(t)), \cdots, p_N(t))$ with $\overline{p}_i = 0$ for all $1 \leq i \leq N$ and $g = (g_1, \cdots, g_N) \in C(\mathbb{R}^N, \mathbb{R}^N)$, Mawhin's result replaced (2.12) and (2.13) by

$$u_k g_k(u_1, \cdots, u_N) \geq 0, \text{ or } u_k g_k(u_1, \cdots, u_N) \leq 0 \text{ for } |u_k| \text{ sufficiently large.} \tag{2.14}$$

There are of course many other possible extensions of (2.12) with strict condition, and we refer to the literature around the seventies. From a topological point of view, a natural extension to $\mathbb{R}^N$ of the condition $ug(u) > 0$ for $|u_j|$ large could be:

$$g(u) \neq 0 \qquad \text{for } |u| \geq R \tag{2.15}$$

and

$$\deg(g, B_R(0), 0) \neq 0, \tag{2.16}$$

where deg is the Brouwer degree (see 1.2.15).

Let us finally mention some generalizations of the Landesman and Lazer conditions for systems. In [31] Ortega and Sanchez, study the analogous problem as before (2.11):

$$u'' + cu' + g(u) = p(t), \tag{2.17}$$

where $u \in \mathbb{R}^{\mathbb{N}}$, $c \geq 0, g \in C(\mathbb{R}^N, \mathbb{R}^N)$ bounded, and $p \in C(\mathbb{R}, \mathbb{R}^N)$ is $T-$periodic.

Their starting point is a well-known result which is valid for scalar equations:

Theorem 2.1.8. *Assume that $N = 1$ and that g has limits at infinity,*

$$g(\pm\infty) := \lim_{s \to \pm\infty} g(s),$$

then (2.17) has a $T-$periodic solution if

$$g_- < \overline{p} < g_+.$$

Moreover, if $g_- < g(s) < g_+$ for all $s \in \mathbb{R}$, then the previous condition is also necessary.

This condition is of course of the Landesman-Lazer type, which we have studied in the Dirichlet problem for an elliptic equation in [23], in Theorem 2.1.1. In [29], Theorem 2.1.5 extended the result to system of elliptic equations, this theorem was adapted to the $T-$periodic setting in the work of Ortega and Sanchez:

Theorem 2.1.9. *Assume that $N > 1$, and that the radial limits*

$$g_v := \lim_{s \to \infty} g(sv)$$

exist uniformly with respect to $v \in S^{N-1}$, then (2.17) has a $T-$periodic solution if the following conditions hold:

(N_1) *$g_v \neq \overline{p}$, $\forall v \in S^{N-1}$.*

(N_2) *$deg(\Phi) \neq 0$, where $\Phi : S^{N-1} \to S^{N-1}$*

$$\Phi(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}.$$

They show an example where this result applies:

$$z'' + cz' + \frac{z^n}{1 + |z|^n} = p(t),$$

where $n = 1, 2, \cdots$ and $z \in \mathbb{C}$ (identified with $\mathbb{R}^2$). In this case $z \in S^1$ implies $z = e^{i\theta}$, $\theta \in [0, 2\pi)$ so:

$$g_v = g_{e^{i\theta}} = e^{in\theta}$$

and Theorem 2.1.9 can be used whenever $|\overline{p}| < 1$.

Notice also that this condition is sharp, because if $z(t)$ is a $T-$periodic solution, then by the periodic boundary conditions,

$$\left| \int_0^T p(t)dt \right| = \left| \int_0^T \frac{z(t)^n}{1 + |z(t)|^n} dt \right| < T.$$

So, the conditions given are also necessary for the existence of the solutions (the strict inequalities hold).

Remark 2.1.10. The preceding estimate can also be obtained in a more indirect way by applying the Mean-Value Theorem for Vector-Valued Integrals (1.1.19). With this result, the arguments from the previous example can be extended to the general equation (2.17). In this way, one can deduce that if (2.17) has a $T-$periodic solution, then $\overline{p}$ must lie in the closed convex hull of $g(\mathbb{R}^N)$. For $N = 1$, this convex hull coincides with $g(\mathbb{R})$ because connected sets of $\mathbb{R}$ (intervals) are always convex. Obviously, this is not true for $N \geq 2$, and this geometrical fact must be taken into account when studying (2.17) for $N \geq 2$.

In this paper, Ortega and Sanchez intended to generalize Theorem 2.1.8 and Theorem 2.1.9:

First, they considered a class of functions g having a convex range and such that $\overline{p} \in g(\mathbb{R}^N)$ becomes a necessary and sufficient condition for the existence of $T-$periodic solutions. This can be seen as an extension of Theorem 2.1.8 to $N \geq 2$. They also showed that if $g(\mathbb{R}^N)$ is not convex, then one can not decide the solvability of the periodic problem only in terms of $\overline{p}$.

Finally, they discuss some tentative extensions of Theorem 2.1.9 which are motivated by classical results for the scalar case.

Remark 2.1.11. The origin of Theorem 2.1.8 can be traced back to the theory of forced oscillations developed in the sixties. In fact, it can be obtained as a corollary of the main result in [24]. Here it is shown that, in the scalar case, the existence of a $T-$periodic solution is guaranteed by the condition:

$$g(-u) < \overline{p} < g(u), \qquad u \geq R \qquad \text{for some } R > 0. \tag{2.18}$$

This is an improvement of Theorem 2.1.8, because (2.18) is less restrictive than the condition by Landesman and Lazer, and the existence of the limits $g(\pm\infty)$ is not required.

Going back to the case $N \geq 2$, they state two conditions which seem a natural extension of (2.18) to systems. Namely, for some $R > 0$,

$(N_1)_w$ $g(u) \neq \overline{p}$, if $|u| \geq R$.

$(N_2)_w$ $deg(\Phi) \neq 0$, where $\Phi : S^{N-1} \to S^{N-1}$

$$\Phi(v) = \frac{g(Rv) - \overline{p}}{|g(Rv) - \overline{p}|}.$$

Despite the analogy, it is shown that there are systems of the type (2.17) in $\mathbb{R}^2$ which satisfy $(N_1)_w$ and $(N_2)_w$ but have no $T-$periodic solutions. These examples show, in some sense, the necessity of the existence of radial limits of g in Theorem 2.1.9. They also indicate that some results in the theory of scalar periodic problems can not be translated literally to systems. They prove the following result:

Theorem 2.1.12 (Ortega-Sanchez)**.** *Assume that g is bounded continuous with $g(0) = 0$ and satisfying*

(OS_1) *For each $v \in S^{N-1}$, the limit $g_v := \lim_{s\to+\infty} g(sv)$ exists and is uniform with respect to v in S^{N-1}.*

(OS_2) *$g(S^{N-1}_\infty) \cap g(\mathbb{R}^N) =$, where $g(S^{N-1}_\infty) = \{g_v : v \in S^{N-1}\}$.*

(OS_3) *$deg(\Phi) \neq 0$, for $\Phi : S^{N-1} \to S^{N-1}$ given by*

$$\Phi(v) = \frac{g_v}{|g_v|}.$$

then, (2.17) has at least one $T-$periodic solution if $\overline{p} \in g(\mathbb{R}^N)$. Moreover, if $g(\mathbb{R}^N)$ is convex, this condition is also necessary.

A similar result of that of Lazer [24], can be obtained for a force pointing to infinity, that is when (2.12) is replaced by

$$xg(x) \leq 0, \qquad \text{if } |x| \geq R. \tag{2.19}$$

When the inequality is strict in (2.12) or (2.19), one is led to the condition:

$$g(x) \neq 0, \qquad \text{if } |x| \geq R \text{ and } g(R)g(-R) < 0. \tag{2.20}$$

For $N = 1$, Theorem (2.1.12) is a corollary of Lazer's result. This is easily seen because, if g and p satisfy the conditions, then $g^*(x) = g(x) - \overline{p}$ satisfies (2.20), while $p^*(t) = p(t) - \overline{p}$ has zero mean value.

In view of this, it seems a good idea to look for an extension of Lazer's result to systems. Such an extension should contain Theorem 2.1.12 as a corollary, and the assumptions should be natural extensions of (2.12), (2.19) or (2.20). The authors have been unable to find a result like this, and the following example shows why:

Example 2.1.13. Let $N = 2$ and consider in $\mathbb{C} \equiv \mathbb{R}^2$ the following equation:

$$z'' + g(z) = p(t) \qquad t \in \mathbb{R},$$

with $p \in C(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C})$, $\overline{p} = 0$ and $g \in C(\mathbb{C}, \mathbb{C})$ bounded and such that

$$g(z) \neq 0 \qquad \text{if } z \in \mathbb{C}\backslash D \text{ and } \deg(g, D, 0) \neq 0,$$

where D is certain open disk in the complex plane centered at the origin. This condition is in a way comparable to condition (2.20).

Let $g(z) = g_0(z) - \gamma$, where γ is a fixed complex number, $0 < |\gamma| < 1$ and

$$g_0(z) = e^{i\mathrm{Re}\, z} \frac{z}{\sqrt{1 + |z|^2}}.$$

It is not hard to verify that $\deg(g_0, D, 0) = 1$ in any disk containing the origin and that $g \sim g_0$ in large disks, so one has:

$$\deg(g, D, 0) = \deg(g_0, D, \gamma) = \deg(g_0, D, 0) = 1.$$

The authors prove that if the p is chosen to be $p(t) = \lambda \sin(t)$, then the problem has no $2\pi-$periodic solutions for λ large.

The problem of extending and generalizing the Landesman-Lazer conditions for systems was the first big problem we studied for this thesis and Chapter 3 is dedicated to it.

2.2 Singular Problems

There exists a vast bibliography on this kind of dynamical systems. Here, we try to show which are the main problems when dealing with singularities for this kind of systems. Since we introduced a set of boundary conditions (the Nonlocal Boundary Conditions) not that common in the field, we could not find results of systems of elliptic equations with that kind of boundary conditions in the case of singular nonlinearities. We give a series of results for the periodic case, the first being from the Italian school from the late 80'. Consider the following problem:

$$\begin{cases} u'' + g(u) = p(t) & t \in \mathbb{R} \\ u \quad T-\text{periodic}, \end{cases}$$

with, $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$. We also note that in the references here mentioned, only the case $S = \{0\}$ is studied, the case of an isolated singularity.

One of the pioneer works in this line of research is due to Lazer and Solimini [26]. They considered the scalar case $N = 1$, with $g(u) \to -\infty$ as $u \to 0$, and $\int_0^1 g(t)dt = -\infty$. Using a result by Lazer [24], it is shown that a necessary and sufficient condition for the existence of a weak solution when $g < 0$ and $p \in L^1([0, T], \mathbb{R})$, is that $\overline{p} < 0$.

In [34], Solimini studied the case $g = \nabla G$, where the potential G has a singularity of repulsive type at zero: for example, the electrostatic potential between two charges of the same sign. More precisely, they work with two sets of conditions.

In one of them it is assumed that there exist constants c_1, c_2 such that

$$\forall x \in \mathbb{R}^d \backslash \{0\} : \langle g(x), x \rangle \leq c_1 + c_2 |x|. \tag{2.21}$$

In the other one it is assumed that $G \in C^1(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$ satisfies $\lim_{|u| \to 0} G(u) = +\infty$, and $g = \nabla G$ is *strictly* repulsive at the origin, namely:

$$\limsup_{u \to 0} \left\langle g(u), \frac{u}{|u|} \right\rangle < 0.$$

and that

$$\exists\, \delta > 0 \,\text{such that, if}\, \left| \frac{u}{|u|} - \frac{v}{|v|} \right| < \delta, \text{ then } \langle g(u), v \rangle < 0. \tag{2.22}$$

Condition (2.21) is, in a sense, weaker than condition (2.22), it says that the outward radial component of $g(x)$ can grow at most as $|x|^{-1}$ as $x \to 0$.

In this work, the existence is shown of a constant $\eta > 0$ such that if $\|p\|_{L^\infty} < \eta$ and $\overline{p} = 0$, then the problem has no classical solution if g satisfies the second condition (2.22). This includes the case of the repulsive central motion, where $G(u) = \frac{1}{|u|}$.

In the same work, the existence of a solution for $\overline{p} \neq 0$ under the weaker assumption (2.21) is proved.

Also, it is remarked that if $\|p\|_{L^\infty}$ is large enough, then condition $\overline{p} = 0$ does not imply that the problem is unsolvable. This is different from what happens in the case $N = 1$, in which u cannot turn around zero; thus, if the repulsive condition $g(u)u < 0$ is assumed for all $u \neq 0$, then the condition $\overline{p} \neq 0$ is necessary. Saddle Point variational techniques are used throughout this work.

In a recent paper, Fonda and Toader [18] made an exhaustive analysis on radially symmetric Keplerian-like systems $u'' + g(t, |u|)u = 0$, where $g : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is T-periodic in t. Using a topological degree approach, the existence of classical T-periodic solutions is studied. This work provides also an excellent survey of the known results on the subject. It is focused in the attractive case, in which the main difficulty consists in avoiding collisions. It is also remarked that, for the repulsive case, the difficulty relies in the case $\overline{p} = 0$, which is consistent with our studies.

In general the first works in this area worked with this Strong Force condition:

Definition 2.2.1. The system $u'' + \nabla G(u) = p(t)$ is said to satisfy the *strong force condition* if and only if there exist a neighborhood $\mathcal{C}$ of $\mathcal{S}$ and a C^2 function H on $\mathcal{C} \backslash \mathcal{S}$ such that:

i) $U(x) \to -\infty$ as $x \to \mathcal{S}$.

ii) $-G(x) \geq |\nabla H(x)|^2$ for all $x \in \mathcal{C} \backslash \mathcal{S}$.

Roughly speaking, this condition means that the potential G behaves as $\frac{1}{|u|^\gamma}$ near the origin, with $\gamma \geq 2$; thus, it is not satisfied by the Keplerian potential.

In [40], Zhang employed topological techniques in order to study the T-periodic problem for the system

$$u'' + (\nabla F(u))' + \nabla G(u) = p(t). \tag{2.23}$$

When $F \equiv 0$, the problem has variational structure and, as mentioned, the repulsive case was studied in [34]. The attractive case with $p \equiv 0$ and $N = 2$ was solved by Gordon [19], using critical point theory and imposing a strong force condition on G (see 2.2.1) in order to get compactness properties for the involved functionals.

We here state the main result on this work:

Theorem 2.2.2. *If the following conditions hold (we call $g = \nabla G$):*

(G_1) $\lim_{x \to 0} \langle u, g(u) \rangle = -\infty$.

(G_2) *Habets-Sanchez's Strong Force Condition at 0: there exists a function $\varphi \in C^1(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$ such that:*

i) $\lim_{u \to 0} \varphi(u) = +\infty$.

ii) $|\nabla\varphi|^2 \le \langle u, g(u)\rangle + c_1$*, near 0, with* $c_1 > 0$*.*

(G_3) *There exist constants* c_2, c_3 *such that*

$$\langle u, g(u)\rangle \le c_2|u|^2 + c_3 \qquad \forall\, u \neq 0.$$

(G_4) *There exists a constant* R_1 *such that for any solution* u *of the* $T-$*periodic problem*

$$u'' + \lambda\left((\nabla F(u))' + \nabla G(u)\right) = \lambda p(t) \qquad \lambda \in (0,1],$$

one has $|u(\tau)| < R_1$ *for some* $\tau \in [0,T]$*.*

(G_5) $\deg(g, D_{r,R}, 0) \neq 0$*, for all* $0 < r < 1$ *and some sufficiently large* R*, where* $D_{r,D}$ *is the annulus* $\{r < |x| < R\}$*.*

Then, the $T-$*periodic problem (2.23) has at least one solution.*

Note that the condition (G_2) is of the same nature as the one given by Definition 2.2.1.

Condition (G_1) says that the singularity is of repulsive type. Condition (G_3) is concerned with the growth of ∇G at infinity. Conditions $(G_4), (G_5)$ are the type of conditions when using Mawhin's Continuation Theory (see Chapter 1). They are difficult to verify though, especially (G_4), that says that there are no solutions in the inner boundary of the domain. We found this result really interesting because it combined singularities and Continuation Theory. The idea of applying the theory in more general sets will be the same idea we use in this thesis when dealing with a general set of singularities $\mathcal{S}$, in Chapter 5.

Zhang remarks the following: The result says that if $G(u)$ satisfies some strong force condition at the singularity 0, the existence of periodic solutions can be obtained provided that the potential $G(u)$ is smaller than the first eigenvalue of the corresponding Dirichlet problem at infinity. Meanwhile, no restriction on the damping term $F(u)$ is imposed.

The same kind of assumptions (Strong Force) are made in a work from Coti Zelati [15] for the repulsive case.

Chapter 3

Nonsingular Problems

3.1 Introduction

Throughout this chapter we will leave for a moment the idea of a singular nonlinearity. We will focus instead on problem stated in the introduction: $Lu = Nu$, when $Nu = f - g(u)$ with $g \in C(\mathbb{R}^N m\mathbb{R}^N)$ and appropriate Boundary Conditions, depending on the context.

The periodic problem,

$$\left\{ \begin{array}{rcll} u'' + g(u) & = & p(t) & t \in \mathbb{R} \\ u(t+T) & = & u(t) & t \in \mathbb{R}, \end{array} \right. \tag{3.1}$$

was in fact the first problem that we studied in the early stages of this work. As mentioned in the Chapter 2, there were many results extending a well-known result by Nirenberg [29]: Theorem 2.1.5, or Theorem 2.1.9 , which in this context can be stated as follows:

Theorem 3.1.1. *Let $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ be bounded. Then problem (3.1) has a solution, provided that:*

(N_1) *The radial limits $g_v := \lim_{r \to +\infty} g(rv)$ exist uniformly for $v \in S^{N-1}$ and*

$$g_v \neq 0 \qquad \forall v \in S^{N-1}.$$

(N_2) *There exists a constant $R_0 > 0$ such that $deg(\Phi_r) \neq 0$ for $r \geq R_0$, where $\Phi_r : S^{N-1} \to S^{N-1}$ is given by $\Phi_r(v) := \frac{g(rv)}{|g(rv)|}$.*

Here, $deg(\Phi_r)$ is the degree defined as a function of the sphere, as explained in (1.2.20).

Our main result in this Chapter is based on two previous extensions of Theorem 3.1.1. On the one hand, a result by Ortega and Ward Jr [32], originally in the context of partial differential equations, where (N_1) is replaced by the following condition, that allows g to vanish at infinity:

(H_1) The radial limits $\lim_{r\to+\infty}\Phi_r(v)$ exist uniformly for $v\in S^{N-1}$.

On the other hand, a result by Amster and De Nápoli [6], for a p-Laplacian type operator (1.1.17), in which the asymptotic condition (N_1) is weakened to:

(F_1) There exists a family $\{(U_j,w_j)\}_{j=1}^{K}$, with U_j open subsets of S^{N-1} and $w_j\in S^{N-1}$ such that $\{U_j\}_j$ covers S^{N-1}, the upper limit

$$\limsup_{r\to+\infty}\langle g(ru),w_j\rangle := S_j(u)$$

is uniform for $u\in U_j$, and $S_j(u)<0$.

Remark 3.1.2. (N_2) is similar to the original condition $\deg(\Phi)\neq 0$ in [29], where $\Phi : S^{N-1}\to S^{N-1}$ is given by $\Phi(v) := \frac{g_v}{|g_v|}$ in the first case, and by $\Phi(v) := \lim_{r\to+\infty}\Phi_r(v)$ in the second case. However, (N_2) makes sense also when the weaker assumption (F_1) is assumed, for which radial limits for g or $\frac{g}{|g|}$ do not necessarily exist.

It is worth mentioning that, using the equivalence (1.2.22), (N_2) can be also expressed in terms of the Brouwer degree of g, namely:

(N_2') There exists a constant $R_0>0$ such that $\deg(g,B_r(0),0)\neq 0$ for $r\geq R_0$.

Indeed, the equivalence between (N_2) and (N_2') is clear from the identity (1.2.22) introduced in Chapter 1 valid for any mapping $g\in C(\overline{B_1(0)},\mathbb{R}^N)$ such that g does not vanish on S^{N-1}:

$$deg(g,B_1(0),0)=deg(\phi),$$

where $\phi : S^{N-1}\to S^{N-1}$ is given by $\phi(v) := \frac{g(v)}{|g(v)|}$.

We will adapt a condition used by Amster and De Nápoli in [6]. They introduced condition (F_1), that weakened condition (N_2). They reached to an interesting geometrical condition, weaker than the classical condition in Nirenberg [29]. It involves covering S^{N-1} with a finite number of open sets U_j and taking directions $w_j\in S^{N-1}$ such that the uniform limit exists for each $u\in U_j$:

(P_1) There exists a family $\mathcal{F}=\{(U_j,w_j)\}_{j=1}^{K}$ where $\{U_j\}_{j=1}^{K}$ is an open cover of S^{N-1} and $w_j\in S^{N-1}$, such that for some $R_j>0$ and $j=1,\dots,K$:

$$\langle g(ru),w_j\rangle<0 \qquad \forall r>R_j \qquad \forall u\in U_j.$$

3.2 A generalization of a Nirenberg result

We will work with continuous nonlinearities g bounded at infinity, i.e. $g \in L^\infty(\mathbb{R}^N \backslash B_1(0), \mathbb{R}^N)$. For convenience, the boundedness condition on g shall be expressed as:

(B) $\limsup_{|u|\to\infty} |g(u)| < \infty$.

Note that if g is nonsingular, condition (B) is equivalent to g being bounded at infinity. Moreover, it shall be seen that (B) may be replaced by

(B') $\limsup_{|u|\to\infty} \langle g(u), u\rangle < \infty$.

In particular, if $\liminf_{|u|\to\infty} |g(u)| > 0$, then condition (B') says that

$$\liminf_{|u|\to\infty} A(u) \geq \frac{\pi}{2}, \tag{3.2}$$

where $A(u)$ denotes the angle between $g(u)$ and u, as $\cos(A(u)) = \frac{\langle g(u),u\rangle}{|g(u)||u|}$. This is because

$$\limsup_{|u|\to\infty} \cos(A(u)) \leq 0.$$

Our result for the nonsingular case, reads:

Theorem 3.2.1 (Amster, M. - I)**.** *Let $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfy either (B) or (B'). Then problem (3.1) has a solution, provided that (N_2) and (P_1) hold.*

Remark 3.2.2. It is easily seen that (P_1) generalizes (F_1), since the upper limits may vanish, or may not be uniform as $r \to +\infty$.

On the other hand, following the ideas in [33] it is seen that (P_1) can be replaced by the following condition, of geometric nature:

$(\tilde{P}_1)$ There exists an open cover $\{U_j\}_{j=1,\dots,K}$ of S^{N-1} such that for some $R_j > 0$ and $j = 1, \dots, K$:

$$0 \notin co\left(g\left(C_j\right)\right), \qquad C_j := \bigcup_{r>R_j} rU_j,$$

where $co(A)$ denotes the convex hull of $A \subset \mathbb{R}^N$ (see Definition 1.1.20).

Indeed, from the geometric version of the Hahn-Banach theorem, for any compact subset $C \subset C_j$ we deduce the existence of a vector w_j such that $\langle g(u), w_j \rangle < 0$ for every $u \in C$ and, as we shall see, this suffices for obtaining a priori bounds for the average of the solution, $|\overline{u}|$. This, plus the a priori bound for $\|u - \overline{u}\|_\infty$ will give us the a priori bound needed for the solutions.

We are now in condition to show a proof of Theorem 3.2.1:
Proof:
It suffices to verify that the hypotheses of Mawhin's Continuation Theorem [27], studied in Chapter 1, are satisfied over the domain Ω, with $\Omega = \{u \in C([0,T], \mathbb{R}^N) : \|u\|_{L^\infty} < R\}$. As (N_2) holds, we know that $\deg(g, B_R(0), 0) \neq 0$ for large values of R. Thus, we only need to prove that for $\lambda \in (0,1]$, the problem

$$u'' = \lambda(p(t) - g(u)) \tag{3.3}$$

does not have a T-periodic solution on $\partial B_R(0) \subset C([0,T], \mathbb{R}^N)$, for some R large enough.

Assume firstly that (B) holds, and let us suppose that problem (3.3) has an unbounded sequence of solutions; namely, there exist $\lambda_n \in (0,1]$ and T-periodic functions u_n such that $\|u_n\|_\infty \to \infty$ and

$$u_n''(t) = \lambda_n(p(t) - g(u_n(t)).$$

Taking average on both sides, we have

$$\frac{1}{T}\int_0^T u_n''(t)dt = \frac{1}{T}\int_0^T \lambda_n(p(t) - g(u_n(t))dt = \lambda_n \overline{p} - \lambda_n \frac{1}{T}\int_0^T g(u_n(t))dt.$$

As u_n is periodic, $\int_0^T u_n''(t)dt = 0$ for all t. It follows that

$$\int_0^T g(u_n(t))dt = 0. \tag{3.4}$$

On the other hand, from the boundedness of g we obtain:

$$\|u_n'\|_{L^\infty} \leq T\|u_n''\|_{L^\infty} \leq T(\|p\|_{L^\infty} + \|g\|_{L^\infty}) = M.$$

Hence, $u_n - \overline{u}_n$ is bounded; in particular, as $\|u_n\|_{L^\infty} \to \infty$, writing

$$u_n(t) = (u_n(t) - \overline{u_n}) + \overline{u_n},$$

we conclude that $|\overline{u_n}| \to \infty$ and $r_n(t) := |u_n(t)| \geq |\overline{u_n}| - \|u_n - \overline{u}_n\|_{L^\infty}$. It is clear that this expression goes to infinity uniformly.

Next, define

$$z_n(t) = \frac{u_n(t)}{|u_n(t)|} \in S^{N-1}.$$

Passing to a subsequence, we may assume that $\frac{\overline{u}_n}{|\overline{u}_n|}$ converges to some $u \in S^{N-1}$, and hence $z_n \to u \in S^{N-1}$ uniformly. From (P_1), $u \in U_j$ for some $j = 1, \ldots, K$.

Also, fixing n_0 such that $r_n(t) > R_0$, with R_0 coming from hypothesis (N_2), implies that $z_n(t) \in U_j$ for all $n \geq n_0$ and all $t \in [0, T]$. For $n \geq n_0$, we deduce that

$$\langle g(r_n(t) z_n(t)), w_j \rangle < 0 \tag{3.5}$$

for all $t \in [0, T]$. Hence, as (3.4) holds, it also holds that

$$0 = \left\langle \int_0^T g(u_n(t))dt, w_j \right\rangle = \int_0^T \langle g(u_n(t)), w_j \rangle dt.$$

The last equality holds because w_j does not depend on t. Next as

$$u_n(t) = r_n(t) z_n(t),$$

and using (3.5), we arrive to the desired contradiction:

$$0 = \int_0^T \langle g(r_n(t) z_n(t)), w_j \rangle dt < 0 \qquad \text{for } n \geq n_0.$$

Finally, if condition (B') holds instead of (B), we multiply the equality

$$u_n'' = \lambda_n (p - g(u_n)),$$

by $u_n - \overline{u}_n$ and integrate:

$$\int_0^T \langle u_n'', u_n - \overline{u}_n \rangle dt = \lambda_n \int_0^T \langle p - g(u_n), u_n - \overline{u}_n \rangle dt.$$

Just like we did before, using that the fact that $\overline{g(u_n)} = 0$ (see 3.4) we deduce:

$$\|u_n'\|_{L^2}^2 \leq \|p\|_{L^2} \|u_n - \overline{u}_n\|_{L^2} + \lambda_n \int_0^T \langle g(u_n), u_n \rangle dt.$$

Using now the inequality (3.2), we have:

$$\|p\|_{L^2} \|u_n - \overline{u}_n\|_{L^2} + \lambda_n \int_0^T \langle g(u_n), u_n \rangle dt \leq \frac{T}{2\pi} \|p\|_{L^2} \|u_n'\|_{L^2} + kT.$$

Hence, $\|u_n'\|_{L^2}$ is bounded which, in turn implies that $\|u_n - \overline{u}_n\|_{L^\infty}$ is bounded, and the rest of the proof follows as before.

■

Remark 3.2.3. Under an appropriate Nagumo [28] type condition, a more general result could be obtained for $g = g(t, u, u')$.

Perhaps it is hard to see the improvement in the previous technical hypothesis (P_1). The crucial point is that we can guarantee existence of solutions in the absence of radial limits for g or even for $\frac{g}{|g|}$. To visualize this fact, let us consider the following Landesman-Lazer type condition (see [23]), introduced and studied in Chapter 2 motivated by an analogous result in the work from Amster and De Nápoli [6]:

(P_1') Let $\{e_i\}_{i=1}^N, \{w_j\}_{j=1}^N \subset S^{N-1}$ be two bases of $\mathbb{R}^N$, and assume there exists $s_0 > 0$ such that

$$\langle g(x - se_i), w_i\rangle > 0 > \langle g(x + se_i), w_i\rangle, \qquad \forall\, s \geq s_0,$$

for all $x \in \text{span}\{e_j : j \neq i\}$ and $1 \leq i \leq N$.

It is easy to prove that one condition implies the other:

Proposition 3.2.4. *Condition (P_1') implies condition (P_1).*

Proof:
Indeed, let $u \in S^{N-1}$, $u = x + \alpha e_i$, with $x \in \text{span}\{e_j : j \neq i\}$, $\alpha \neq 0$. Now, fix $\delta < |\alpha|$ and consider $\tilde{u} = \tilde{x} + \tilde{\alpha} e_i \in U := B_\delta(u) \cap S^{N-1}$. If $\alpha > 0$, then as $s\tilde{x} \in \text{span}\{e_j : j \neq i\}$ we obtain:

$$\langle g(s\tilde{u}), w_i\rangle = \langle g(s\tilde{x} + s\tilde{\alpha} e_i), w_i\rangle < 0 \qquad \text{for } s\tilde{\alpha} \geq s_0.$$

In the same way, for $\alpha < 0$:

$$\langle g(s\tilde{u}), -w_i\rangle = -\langle g(s\tilde{x} - s|\tilde{\alpha}|e_i), w_i\rangle < 0 \qquad \text{for } s|\tilde{\alpha}| \geq s_0.$$

As $|\tilde{\alpha}| > \alpha - \delta$, both inequalities hold for $\tilde{u} \in U$ when $s \geq \frac{s_0}{\alpha - \delta}$. The result follows now from the compactness of S^{N-1}.

■

With this implication, we state another existence result.

Theorem 3.2.5 (Amster M. - II)**.** *Let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfy either (B) or (B'), and $p \in C(\mathbb{R}, \mathbb{R}^N)$ be T-periodic with $\overline{p} = 0$. If condition (P_1') is satisfied, then problem (3.1) has at least one solution.*

Proof:
From the previous proposition, we only need to prove (N_2). Without loss of generality we may assume that $\{w_i\}_{i=1}^N = \{e_i\}_{i=1}^N$ is the canonical basis. Hypothesis (P_1') says that there exists s_0 such that if $s \geq s_0$, then

$$g_i(x - se_i) > 0 > g_i(x + se_i) \qquad \forall x \in \operatorname{span}\{e_j : j \neq i\},\ i = 1, \dots, N.$$

Let $R \geq s_0$, and consider the cube $\mathcal{Q}_R := [-R, R]^N$ and following homotopy:

$$h(\lambda, u) := \lambda g(u) - (1 - \lambda)u.$$

Suppose there exists $u \in \partial \mathcal{Q}_R$ such that $h(\lambda, u) = 0$ for some $\lambda \in [0, 1]$; for example $u = x + Re_i$ with $x \in \operatorname{span}\{e_j : j \neq i\}$. Then, looking at the $i-$th coordinate:

$$\lambda g_i(x + Re_i) = (1 - \lambda)R,$$

and we have that, from (P_1'), the left hand-side term is negative, unless $\lambda = 0$, a contradiction. An analogous argument can be used in the case $u = x - Re_i$. We then conclude that for any $R \geq s_0$, as we found an homotopy between g and $-Id$ and because of the homotopy invariance property of the degree (see Chapter 1):

$$\deg(g, \mathcal{Q}_R, 0) = \deg(-Id, \mathcal{Q}_R, 0) \neq 0.$$

This is obviously equivalent to (N_2), and so all the assumptions of Theorem 3.2.1 hold. ■

We show in the next Example a case in which our result gives us a solution, but it does not fulfill the conditions of either [6] nor [32].

Example 3.2.6. Let $N = 2$ and g given by

$$g(x, y) = \left(\frac{1 + x + r(y)}{1 + x^2}, \frac{1 + y}{1 + y^2} \left(1 + \frac{\sin x}{1 + |y|} \right) \right),$$

where $r \in C(\mathbb{R}, \mathbb{R})$ is a bounded function.

Taking $e_1 = (1, 0) = -w_1; e_2 = (0, 1) = -w_2$, we have that for all y:

$$\langle g(s, y), w_1 \rangle = -\frac{1 + s + r(y)}{1 + s^2} < 0 \quad \forall s > \|r\|_{L^\infty} - 1,$$

$$\langle g(-s, y), w_1 \rangle = \frac{s - 1 - r(y)}{1 + s^2} > 0 \quad \forall s > \|r\|_{L^\infty} + 1,$$

and for all x:

$$\langle g(x,s), w_2\rangle = -\frac{1+s}{1+s^2}\left(1+\frac{\sin x}{1+s}\right) < 0 \quad \forall s > 0,$$

$$\langle g(x,-s), w_2\rangle = \frac{s-1}{1+s^2}\left(1+\frac{\sin x}{1+s}\right) > 0 \quad \forall s > 1.$$

Thus, g verifies (P_1'), although it does not verify the assumptions of Ortega and Ward Jr [32]. Indeed, the radial limits for $\frac{g}{|g|}$ do not necessarily exist. For example, let us consider the direction $(1,0) \in S^1$: then, $(sx, sy) = (s, 0)$ and

$$g(s,0) = \left(\frac{1+s+r(0)}{1+s^2}, 1+\sin s\right);$$

$$|g(s,0)| = \sqrt{\left(\frac{1+s+r(0)}{1+s^2}\right)^2 + (1+\sin s)^2}.$$

Let $s = \frac{4k-1}{2}\pi$, $k \in \mathbb{N}$, $\gamma_{\frac{4k-1}{2}} = \frac{g(\frac{4k-1}{2}\pi,0)}{|g(\frac{4k-1}{2}\pi,0)|}$. Here, $\sin(\frac{4k-1}{2}\pi) = -1$, then

$$\gamma_{\frac{4k-1}{2}} = (1,0) \qquad \text{for } k \text{ large enough.}$$

Now, let $s = k\pi$, $k \in \mathbb{N}$, $\gamma_k = \frac{g(k\pi,0)}{|g(k\pi,0)|}$. As $\sin(k\pi) = 0$,

$$\gamma_k \to (0,1) \qquad \text{as } k \to \infty.$$

This shows that the limit of $\frac{g(s,0)}{|g(s,0)|}$ as $s \to +\infty$ does not exist. Note also that g does not satisfy the assumptions in [6], because g vanishes as $|x|$ and $|y|$ tend to infinity.

3.3 A result involving a geometrical condition

Consider now the Elliptic problem with the nonlocal boundary condition, previously discussed in the Introduction:

$$\begin{cases} \Delta u + g(u) &= f(x) \quad \text{in } \Omega \\ u &= C \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS &= 0. \end{cases} \tag{3.6}$$

As the nature of this problem is on the resonance of the operator, as was in the periodic problem, an analogous for Theorem 3.2.1 can be easily proven. In Chapter 5 we shall prove a more general version of this result.

When studying the more general elliptic singular problem (3.6) with g having a set $\mathcal{S}$ of singularities, we changed a little the conditions on g and studied some other generalization of the classical condition given by Nirenberg in [29] which implied that g cannot rotate around the origin when $|u|$ is large. These conditions were first introduced by Ruiz and Ward Jr in [33] and extended by Amster and Clapp in [5]. They have a geometric nature and involve the convex hull of the image of g over a certain ball. In this section we state a result for the nonsingular case using this type of conditions.

We use from now on the geodesic distance on Ω, namely:

$$d(x,y) := \inf\{lenght(\gamma) : \gamma \in C^1([0,1],\Omega) : \gamma(0) = x, \gamma(1) = y\}. \quad (3.7)$$

Indeed, we shall fix a number r:

$$r := k \operatorname{diam}_d(\Omega)(\|f\|_{L^\infty} + \|g\|_{L^\infty}), \quad (3.8)$$

where k is a constant such that

$$\|\nabla u\|_{L^\infty} \leq k\|\Delta u\|_{L^\infty}$$

for all $u \in C^2(\overline{\Omega}, \mathbb{R}^N)$ satisfying the nonlocal boundary conditions of (3.6). The existence of this k will be shown in Lemma 5.2.1 in Chapter 5.

Then we shall assume, for a certain $D \subset \mathbb{R}^N$:

(D_1) For all $v \in \partial D$, $0 \notin co(g(B_r(v)))$.

(D_2) $\deg(g, D, 0) \neq 0$.

Condition (D_1) is weaker than Nirenberg's in the sense that it allows g to rotate, although not too fast since r cannot be arbitrarily small. Condition (D_2) is an analogous of condition (N_2). It is worth mentioning that (D_1) is even weaker than (P_1) because of the following argument: Suppose g satisfy (P_1). Taking $R_0 = \max_{\{1\leq j\leq K\}} R_j$ and $D = B_{R_0}(0)$, If $v \in \partial D$, $v = R_0 u$ with $u \in S^{N-1}$ and there is a j_0 such that $u \in U_{j_0}$, we then have that $\langle g(v), w_{j_0}\rangle < 0$. As this is uniform in U_{j_0} one can always take a slightly bigger R_0 such that $0 \notin co(g(B_r(v)))$.

The main result of this section is the following:

Theorem 3.3.1 (Amster M. - III)**.** *Let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfying* (B) *and $f \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\overline{f} = 0$. Let r be as in* (3.8)*. If there exists a bounded domain $D \subset \mathbb{R}^N$ such that (D_1) and (D_2) hold, then (3.6) has at least one solution u with $\overline{u} \in \overline{D}$ and $\|u - \overline{u}\|_{L^\infty} \leq r$.*

Proof:
Let $U = \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \|u - \overline{u}\|_{L^\infty} < r, \overline{u} \in D\}$ and consider, for $\lambda \in (0,1]$, the problem

$$\begin{cases} \Delta u + \lambda g(u) & = \lambda f(x) \quad \text{in } \Omega \\ u & = C \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS & = 0, \end{cases} \tag{3.9}$$

It is clear that if $u \in \overline{U}$ solves (3.9) for $\lambda = 1$ then u is a solution of (3.6).

Indeed, if $u \in \overline{U}$ then $\overline{u} \in \overline{D}$ and $\|u - \overline{u}\|_\infty \leq r$. These both things imply that $u(x) \in \overline{D}$.

For the reader's convenience, let us briefly describe how the standard continuation methods [27], Theorem (1.2.26), explained in Chapter 1 can be adapted to our problem.

Let $\tilde{C} := \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \overline{u} = 0\}$ and $K : \tilde{C} \to \tilde{C}$ be defined as a right inverse of Δ; specifically, for $\varphi \in \tilde{C}$ we define $u := K\varphi$ as the unique solution of the linear problem

$$\begin{cases} \Delta u & = \varphi \quad \text{in } \Omega \\ u & = C \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS & = 0, \\ \overline{u} & = 0. \end{cases} \tag{3.10}$$

A classical way to show the existence of a unique solution of the above problem is by considering the Linear Dirichlet problem:

$$\begin{cases} \Delta u_0 & = \varphi \quad \text{in } \Omega \\ u_0 & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.11}$$

This problem has a unique solution u_0. Defining $u = u_0 - \overline{u_0}$, it is easy to see that it that satisfies (3.10). Note here that $C = -\overline{u_0}$.

The compactness of K follows from the standard Sobolev embeddings as seen also in Chapter 1.

Next, let $Nu = f - g(u)$ and define the homotopy $h(u, \lambda)$ as

$$h(u, \lambda) = u - \left[\overline{u} + \overline{Nu} + \lambda K(Nu - \overline{Nu})\right].$$

For $\lambda > 0$, it is easy to check that $u \in C(\overline{\Omega}, \mathbb{R}^N)$ is a solution of (3.9) if and only if $h(u, \lambda) = 0$. Thus, it suffices to prove that (3.9) has no solutions on ∂U for $0 < \lambda < 1$. Indeed, in this case problem (5.2) has a solution on ∂U or either

$$\deg(h(\cdot, 1), U, 0) = \deg(h(\cdot, 1), U, 0) = \deg(g, D, 0) \neq 0.$$

Let $u \in \partial U$ be a solution of (3.9), then $\overline{u} \in \overline{D}$ and $\|u - \overline{u}\|_{L^\infty} \leq r$.

This implies that

$$\|\nabla u\|_{L^\infty} \leq k\|\Delta u\|_{L^\infty} < k(\|f\|_{L^\infty} + \|g\|_{L^\infty}),$$

and thus

$$\|u - \overline{u}\|_{L^\infty} \leq \mathrm{diam}_d(\Omega)\|\nabla u\|_{L^\infty} < r.$$

Hence, $\overline{u} \in \partial D$. Moreover, it follows from the Mean-Value Theorem for Vector Integrals (1.1.19) that

$$\frac{1}{|\Omega|}\int_\Omega g(u(x))\,dx \in co(g(u(\overline{\Omega}))) \subset co(g(B_r(\overline{u}))).$$

On the other hand, simple integration shows that

$$\int_\Omega g(u(x))\,dx = 0,$$

so $0 \in co(g(B_r(\overline{u})))$, a contradiction.

■

Chapter 4

Singular Periodic Problems

4.1 Motivation, The Central Motion Problem

When studying the problem presented in the previous Chapter, in a visit to Buenos Aires, Professor Ortega pointed out that a really important nonlinearity that had the property of vanishing at infinity was the Keplerian Central Motion Problem in Classical Mechanics, or the Coulomb Central Motion Problem in Electrostatic.

A fact that both of these problems have in common, and that was never considered in our previous studies is the fact that the nonlinearities involved have singularities. Refer to Chapter 2 for a brief survey on singular problems.

Let us firstly recall the T-periodic Perturbed Central Motion Problem in $\mathbb{R}^3$:

$$\begin{cases} u'' \mp \frac{u}{|u|^3} &= p(t) \qquad t \in \mathbb{R} \\ u(t+T) &= u(t) \qquad t \in \mathbb{R} \end{cases} \tag{4.1}$$

where $u : \mathbb{R} \to \mathbb{R}^3$. We shall assume that the perturbation p has null average, that is $\overline{p} := \frac{1}{T}\int_0^T p(t)dt = 0$, and that p is $T-$periodic, namely $p(t+T) = p(t)$. The $\mp$ sign leads to two essentially different physical problems; we shall focus on the '$-$' sign, which corresponds to the repulsive case. This is the case of the electrostatic Coulomb Central Motion Problem with a charge being repelled by the source.

With this problem in mind, we study the more general problem for a function $u : \mathbb{R} \to \mathbb{R}^N$:

$$\begin{cases} u'' + g(u) &= p(t) \qquad t \in \mathbb{R} \\ u(t+T) &= u(t) \qquad t \in \mathbb{R} \end{cases} \tag{4.2}$$

where $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic, $\overline{p} = 0$, and $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ has a repulsive type singularity at $u = 0$. By this, we mean that $\langle g(u), u \rangle < 0$ when u is near the origin. We will define this formally in the next section (see Definition 4.2.1).

In order to present our results, let us start making some simple comments on the central motion repulsive problem stated above (4.1).

We started working with this problem when studying the 2-body periodic problem:

$$\begin{cases} x'' - \frac{y-x}{|x-y|^3} & = & p_1(t) & t \in \mathbb{R} \\ y'' - \frac{x-y}{|x-y|^3} & = & p_2(t) & t \in \mathbb{R} \\ x(t+T) & = & x(t) & t \in \mathbb{R} \\ y(t+T) & = & y(t) & t \in \mathbb{R} \end{cases} \tag{4.3}$$

with $p_1, p_2 \in C(\mathbb{R}, \mathbb{R}^N)$, and $\overline{p}_1 = \overline{p}_2 = 0$.

Here, $u(t) = (x(t), y(t)) \in C(\mathbb{R}, \mathbb{R}^{2N})$ and the nonlinearity reads

$$g(x,y) = -\frac{1}{|x-y|^3}\left(x-y, y-x\right), \quad p(t) = (p_1(t), p_2(t)). \tag{4.4}$$

This is easily transformed into a central motion problem by the change of variables

$$\begin{cases} w = & x - y \\ v = & x + y \\ P = & p_1 + p_2 \\ Q = & p_1 - p_2. \end{cases}$$

Then, we have:

$$\begin{cases} v'' = P(t) \\ w'' - 2\frac{w}{|w|^3} = Q(t). \end{cases}$$

The first equation is easily integrable, and the second one is non other than the Central Motion Problem. Degree Theory would not be possible to apply directly without some restrictions, since there are no a-priori bounds for the first equation, namely $v'' = P(t)$ with periodic conditions. In fact, if v is a solution, $v + const$ is also a solution for every constant. Also, an interesting remark is that, besides the singularity of g at 0, it's asymptotic behavior makes it different from the Nirenberg case [29], as the nonlinearity goes to zero at infinity.

The first problem that arises is that when $|x - y|$ goes to zero, g goes to infinity. So we consider continuous perturbations of the nonlinearity. Letting $\varepsilon > 0$, we take a continuous g_ε. Next, we try to avoid the fact that g_ε is zero in the diagonal subspace $\{x = y\}$ of dimension N. We do so by restraining ourselves to the subspace:

$$V = \{u \in C_{per}(\mathbb{R}, \mathbb{R}^{2N}) : \overline{x} + \overline{y} = 0\},$$

whith $C_{per}(\mathbb{R}, \mathbb{R}^{2N}) := \{v : \mathbb{R} \to \mathbb{R}^{2N} : v(t) = v(t+T), \forall\, t \in \mathbb{R}\}$ are the $T-$periodic continuous functions.

Working only in this subspace we attack two problems at once: On one hand we avoid possible collisions. On the other hand, viewing the problem as two different problems after changing variables, we would be able to find a-priori bounds for v, in V. That is somehow the idea behind the degree approach we will use. The perturbation g_ε is carefully defined later on in (4.10).

The second equation, $w'' - 2\frac{w}{|w|^3} = Q(t)$, lead us to the Central Motion Problem taking $u = \frac{w}{2^{3/2}}$. The first difficulty arises on the fact that g is singular at 0; a reasonable way to overcome it consists in considering, for $\varepsilon > 0$, the function $g_\varepsilon(u) = -\frac{u}{\varepsilon + |u|^3}$ and then studying the convergence of the solutions u_ε of the perturbed systems

$$\begin{cases} u'' - \frac{u}{\varepsilon+|u|^3} &= p(t) \qquad t \in \mathbb{R} \\ u(t+T) &= u(t) \qquad t \in \mathbb{R}. \end{cases} \tag{4.5}$$

The second difficulty relies on the fact that g_ε vanishes at infinity; however, in this case the existence of at least one solution u_ε of (4.5) for each $\varepsilon > 0$ follows as an immediate consequence of the results we obtained for the nonsingular case, Theorem 3.2.1. Indeed, as

$$\langle g_\varepsilon(u), u\rangle = \left\langle -\frac{u}{\varepsilon + |u|^3}, u \right\rangle = -\frac{|u|^2}{\varepsilon + |u|^3} < 0,$$

for $u \neq 0$, it follows that conditions (B') and (N_2) are trivially satisfied. Moreover, for every $w \in S^{N-1}$ define $U_w = \{u \in S^{N-1} : \langle u, w\rangle > 0\}$. Then $\{U_w\}_w$ covers S^{N-1}, and clearly $\langle g(ru), w\rangle < 0$ for $u \in U_w$ and $r > 0$.

It is important here to recall the different conditions we studied in Chapter 3 as we are going to refer to them often in this Chapter and the next one.

From the compactness of S^{N-1}, condition (P_1) is satisfied. Thus, we may pass to the next step. The following computations provide some information concerning the behavior of the family $\{u_\varepsilon\}_\varepsilon$ as $\varepsilon \to 0$:

Multiplying in L^2 the equation in (4.5) by u_ε, we have:

$$\langle u_\varepsilon'', u_\varepsilon\rangle - \left\langle \frac{u_\varepsilon}{\varepsilon + |u_\varepsilon|^3}, u_\varepsilon \right\rangle = \langle p(t), u_\varepsilon\rangle\,.$$

Integrating by parts the first term on the left and rearranging the terms we get:

$$\langle u_\varepsilon', u_\varepsilon' \rangle = -\left\langle \frac{u_\varepsilon}{\varepsilon + |u_\varepsilon|^3}, u_\varepsilon \right\rangle - \langle p(t), u_\varepsilon \rangle .$$

Noting that $\left\langle -\frac{u_\varepsilon}{\varepsilon+|u_\varepsilon|^3}, u_\varepsilon \right\rangle \leq 0$, we reach to:

$$\|u_\varepsilon'\|_{L^2}^2 \leq - \langle p(t), u_\varepsilon \rangle .$$

Here, note that $\langle p, u_\varepsilon \rangle = 0$, as $\overline{p} = 0$, so last equation can be written:

$$\|u_\varepsilon'\|_{L^2}^2 \leq - \langle p(t), u_\varepsilon - \overline{u_\varepsilon} \rangle .$$

Finally, taking absolute value we get the bound:

$$\|u_\varepsilon'\|_{L^2}^2 \leq \|p\|_{L^2} \|u_\varepsilon\|_{L^2}.$$

Wirtinger inequality (Theorem 1.1.15) tells us that the following bound also holds:

$$\|u_\varepsilon - \overline{u_\varepsilon}\|_{L^\infty} \leq C \|u_\varepsilon'\|_{L^2}.$$

So we have the following important uniform bounds:

$$\|u_\varepsilon'\|_{L^2} \leq C, \qquad \|u_\varepsilon - \overline{u_\varepsilon}\|_{L^\infty} \leq C \tag{4.6}$$

where the constant C does not depend on ε. On the other hand, it is easy to prove that the family $\{\overline{u_\varepsilon}\}_\varepsilon \subset \mathbb{R}^N$ is also bounded. Indeed, integrating the main equation in (4.5) we obtain

$$\int_0^T \frac{u_\varepsilon}{\varepsilon + |u_\varepsilon|^3} dt = 0,$$

and we deduce that

$$-\int_0^T \frac{\overline{u_\varepsilon}}{\varepsilon + |u_\varepsilon|^3} dt = \int_0^T \frac{u_\varepsilon - \overline{u_\varepsilon}}{\varepsilon + |u_\varepsilon|^3} dt.$$

Now, taking norm in $\mathbb{R}^N$:

$$|\overline{u_\varepsilon}| \int_0^T \frac{1}{\varepsilon + |u_\varepsilon|^3} dt \leq \|u_\varepsilon - \overline{u_\varepsilon}\|_{L^\infty} \int_0^T \frac{1}{\varepsilon + |u_\varepsilon|^3} dt.$$

Thus, $|\overline{u_\varepsilon}| \leq C$ for all $\varepsilon > 0$. Hence, for every sequence $\varepsilon_n \to 0$ we may choose a solution $u_n := u_{\varepsilon_n}$ and from the previous bounds there exists a subsequence (still denoted $\{u_n\}_n$) and a function u such that $u_n \to u$ uniformly and weakly in H^1. Moreover, the following property is easy to see in this case and will be generalized later in this Chapter (see Lemma 4.3.1).

Proposition 4.1.1. *If u is obtained as before and $u \neq 0$ over an open interval I, then $u'' - \frac{u}{|u|^3} = p$ in I, in the classical sense.*

Our last problem concerns the study of the set of zeros of the limit function u. As we shall prove for a more general case (Lemma 4.3.3), the boundary of the zero set $Z = \{t \in [0, T] : u(t) = 0\}$ is finite. However, in the central motion problem it can be seen, further, that if $u \not\equiv 0$ then the zero set is empty, i.e. u is a classical solution.

A detailed proof of these last remarks will be done in the next section, for the previously stated general singular case (4.2).

4.2 Main Results

From now on, we shall always consider nonlinearities with singularities of repulsive type at the origin, namely:

Definition 4.2.1. The function $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ is said to be *repulsive at the origin* if, for some $\kappa > 0$

$$\langle g(u), u \rangle < 0 \qquad \text{for } 0 < |u| < \kappa. \tag{4.7}$$

If, furthermore

$$\limsup_{u \to 0} \left\langle g(u), \frac{u}{|u|} \right\rangle := -c, \tag{4.8}$$

with c a positive constant, then g shall be called *strictly repulsive* at the origin.

In order to study the general problem (4.2), we shall proceed in two steps. Firstly, given $\varepsilon > 0$ we introduce the approximated problem

$$\left\{ \begin{array}{rcll} u'' + g_\varepsilon(u) & = & p(t) & t \in \mathbb{R} \\ u(t+T) & = & u(t) & t \in \mathbb{R}, \end{array} \right. \tag{4.9}$$

where g_ε is a continuous (nonsingular) perturbation of g, and obtain sufficient conditions for the existence of a family of solutions $\{u_\varepsilon\}_\varepsilon$.

In this Chapter we work mainly with the following type of approximations:

Definition 4.2.2. We shall say that a family of nonsingular approximations $\{g_\varepsilon\}_\varepsilon \in C(\mathbb{R}^N, \mathbb{R}^N)$ of a function $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ is *admissible* if $g_\varepsilon \to g$ uniformly over compact subsets of $\mathbb{R}^N \backslash \{0\}$ as $\varepsilon \to 0$.

Secondly, we study the convergence of particular sequences $\{u_{\varepsilon_n}\}_n$ as $\varepsilon_n \to 0$, and study some properties of the limit function u. If $u \not\equiv 0$, then it shall be defined as a *generalized solution* of the problem:

Definition 4.2.3. A function $u \in H^1_{per}(\mathbb{R},\mathbb{R}^N)$ is said to be a *generalized solution* of (4.2) if $u \not\equiv 0$, and for some admissible choice of g_ε there exists a sequence $\varepsilon_n \to 0$ and $\{u_{\varepsilon_n}\}_n$ solutions of (4.9) for ε_n such that $u_{\varepsilon_n} \to u$ uniformly and weakly in H^1.

In some cases, we shall consider specific choices of g_ε, for instance

$$g_\varepsilon(u) = \begin{cases} g(u) & |u| \geq \varepsilon \\ \rho_\varepsilon(|u|)g\Big(\varepsilon\frac{u}{|u|}\Big) & 0 < |u| < \varepsilon \\ 0 & u = 0, \end{cases} \tag{4.10}$$

where $\rho_\varepsilon \in C([0,\varepsilon],[0,+\infty))$ is continuous and satisfies $\rho_\varepsilon(0) = 0, \rho_\varepsilon(\varepsilon) = 1$ (more details shall be given below).

For the first step, that is proving existence results for nonsingular problems, we will use the results studied in Chapter 3.

With Theorem 3.2.1 in mind, we proceed to the second step. Our main existence results can be stated as follows. The first one:

Theorem 4.2.4 (Amster, M. - IV)**.** *Let $p \in C(\mathbb{R},\mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and let $g \in C(\mathbb{R}^N\backslash\{0\},\mathbb{R}^N)$ be repulsive at the origin. Further, assume that g satisfies (B) or (B'), and that conditions (P_1) and (N_2) hold. Then either (4.2) has a classical solution, or else for any choice of g_ε as in (4.10) there exists a sequence $\{u_n\}_n$ of solutions of problem (4.9) with $\varepsilon_n \to 0$ that converges uniformly and weakly in H^1.*

The second one, with stronger hypotheses but that gives a stronger result:

Theorem 4.2.5 (Amster, M. - V)**.** *Let $p \in C(\mathbb{R},\mathbb{R}^N)$ be T-periodic such that $\overline{p} = 0$, and assume that $g \in C(\mathbb{R}^N\backslash\{0\},\mathbb{R}^N)$ is repulsive at the origin and satisfies (B) or (B'). Further, assume that condition (P_1) holds, that*

$$\|p\|_{L^\infty} + \sup_{|u|=\tilde{r}} \left\langle g(u), \frac{u}{|u|} \right\rangle < 0 \tag{4.11}$$

for some $\tilde{r} > 0$ and that the following condition holds:

(P_2) *There exists a constant $R_0 > 0$ such that* $\deg(g, B_R(0), 0) \neq (-1)^N$ *for $R \geq R_0$,*

then either (4.2) has a classical solution, or a generalized solution u such that $\|u\|_{L^\infty} \geq \tilde{r}$.

Moreover, if g is strictly repulsive at the origin (see Definition 4.2.1), then the boundary of the set of zeros of u in $[0,T]$ is finite.

Finally, if $g = \nabla G$ with $\lim_{u\to 0} G(u) = +\infty$, then (4.2) has a classical solution.

Throughout the rest of this Chapter we shall always assume that $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic, and $\overline{p} = 0$.

In order to define the perturbed problem (4.9) in an appropriate way, let us firstly observe that the *natural* extension of the previous situation would consist in considering

$$g_\varepsilon(u) = \begin{cases} \frac{|u|}{\varepsilon|g(u)|+|u|} g(u) & u \neq 0 \\ 0 & u = 0. \end{cases} \tag{4.12}$$

Nevertheless, there are other possible choices of g_ε such as the ones defined by (4.10). In particular, for the central motion problem, taking $\rho_\varepsilon(s) = \frac{s}{\varepsilon}$ the expression simply reduces to $g_\varepsilon(u) = -\frac{u}{(\max\{|u|,\varepsilon\})^3}$.

Remark 4.2.6. For convenience, in the previous situation we shall adopt the following notation. We shall denote $u_n := u_{\varepsilon_n}$, and $g_n := g_{\varepsilon_n}$.

Remark 4.2.7. When $g = \nabla G$, a different concept of solution (called *collision solution*) was introduced in [10] (see also [2]). As we shall prove (see Proposition 4.3.4 below), under the assumption that $G(u) \to +\infty$ as $u \to 0$, both generalized and collision solutions are in fact classical. Conversely, taking g_ε as in (4.10), it is obvious that classical solutions are also generalized solutions.

4.3 The Approximation Scheme

Before giving a proof to the main results of this Chapter, we shall prove some lemmas concerning the properties of those functions defined as the limit of a sequence of perturbed problems.

Lemma 4.3.1. *Let $\{u_n\}_n$ and u be defined as before, and assume that $u \neq 0$ over an open interval I. Then u satisfies*

$$u'' + g(u) = p(t), \qquad \forall\, t \in I$$

in the classical sense.

Proof:
Let $\phi \in C_0^\infty(I, \mathbb{R}^N)$, multiplying the last equation in L^2, then

$$\int_I \langle u_n'' + g_n(u_n), \phi \rangle dt = \int_I \langle p, \phi \rangle dt.$$

Integrating by parts the first term on the left we have

$$-\int_I \langle u_n', \phi' \rangle dt + \int_I \langle g_n(u_n), \phi \rangle dt = \int_I \langle p, \phi \rangle dt.$$

Now, because of weak convergence of u_n to u in H^1, we deduce that the following is valid:

$$\int_I \langle u_n', \phi' \rangle dt \to \int_I \langle u', \phi' \rangle dt \qquad (\text{as } n \to \infty).$$

Thus, to prove that u is in fact a weak solution of the problem, it suffices to check that the following limit holds:

$$\int_I \langle g_n(u_n), \phi \rangle dt \to \int_I \langle g(u), \phi \rangle dt \qquad (\text{as } n \to \infty).$$

As $u_n \to u$ uniformly on I, we may assume that $M \geq |u_n| \geq c > 0$ on the support of ϕ. Moreover, as $g_n \to g$ uniformly on the annulus $\{c \leq |u| \leq M\} \subset \mathbb{R}^N \backslash \{0\}$, taking norm, it follows that

$$\Big| \int_I \langle g_n(u_n) - g(u), \phi \rangle dt \Big| \leq \int_I |\langle g_n(u_n) - g(u_n), \phi \rangle| dt + \int_I |\langle g(u_n) - g(u), \phi \rangle| dt.$$

And this expression goes to zero, as n goes to infinity. This proves that u is a weak solution, and the result follows from standard regularity argument. ■

Remark 4.3.2. Condition (4.8) is the same as in Solimini [34] for the case $g = \nabla G$. It is observed that it does not imply the strong force condition (2.2.1). In particular, for any value of $\gamma > -1$ the nonlinearity $g(u) = \frac{-u}{|u|^{\gamma+2}}$ is strictly repulsive, with $c = +\infty$.

In such a situation, it can be proved that the boundary of the set of zeros of the limit function u is discrete; more generally:

Lemma 4.3.3. *Let $\{u_n\}_n$ and u be defined as before, and assume that g is strictly repulsive at the origin. Then the boundary of the set defined by $Z = \{t \in [0,T] : u(t) = 0\}$ is finite, provided that $\|p\|_{L^\infty} < c$, with $c \in (0, +\infty]$ as in (4.8).*

Proof:

Suppose $u(t_0) = 0$, and fix $\mu > 0$ such that $\|p\|_{L^\infty} + \left\langle g(u), \frac{u}{|u|} \right\rangle < 0$ for $0 < |u| < \mu$.

Next, fix $\delta > 0$ such that $|u(t)| < \mu$ for $t \in (t_0 - \delta, t_0 + \delta)$, and suppose for example that u does not vanish in (a, b) for some non-trivial interval $[a, b] \subset [t_0, t_0 + \delta)$. By Lemma 4.3.1 u is a classical solution of the equation $u'' = p - g(u)$ in (a, b). Moreover, if $\phi(t) = |u(t)|^2$ then on (a, b) we have:

$$\phi'' = 2\langle u'', u \rangle + 2|u'|^2 \geq 2\langle p - g(u), u \rangle =$$

$$2[\langle p,u\rangle - \langle g(u),u\rangle] \geq -2|u|\left[\|p\|_{L^\infty} + \left\langle g(u), \frac{u}{|u|}\right\rangle\right] > 0.$$

Thus, ϕ cannot vanish both on a and b, and it follows that either u does not vanish on $(t_0, t_0+\delta)$ or $u \equiv 0$ on $[t_0,t_1]$ for some $t_1 > t_0$. The same conclusion holds for $(t_0-\delta, t_0]$, and the result follows from the compactness of $[0,T]$.

■

The following result improves Lemma 4.3.3 for the variational case studied in [34]. However, we do not make use of the variational structure of the problem: more generally, it may be assumed that $g = \nabla G$ only near the origin.

Proposition 4.3.4. *Assume there exists a neighborhood U of the origin and a function $G \in C^1(U\backslash\{0\},\mathbb{R})$ such that $g = \nabla G$ on $U\backslash\{0\}$. Further, assume that*

$$\lim_{|u|\to 0} G(u) = +\infty.$$

Then every generalized solution of (4.2) is classical.

Proof:
Let u be a generalized solution, and suppose that u vanishes at some point. Fix $\tilde{t}$ such that $u(\tilde{t}) \neq 0$, and define $t_1 = \inf\{t > \tilde{t} : u(t) = 0\}$. Next, fix a value $t_0 \in (\tilde{t}, t_1)$ such that $u(t) \in U\backslash\{0\}$ and $G(u(t)) > 0$ for $t \in [t_0,t_1)$. As u is a classical solution of the equation on $[t_0,t_1)$, multiplying by u' we deduce, for $t \in [t_0,t_1)$ that

$$\frac{|u'(t)|^2}{2} + G(u(t)) = \frac{|u'(t_0)|^2}{2} + G(u(t_0)) + \int_{t_0}^{t} \langle p(s), u'(s)\rangle\, ds. \quad (4.13)$$

As $G(u(t)) > 0$, for any $\tilde{t}_1 \in (t_0,t_1)$ and $t \in [t_0,\tilde{t}_1]$ we obtain:

$$\frac{|u'(t)|^2}{2} \leq A + B\|u'|_{[t_0,\tilde{t}_1]}\|_{L^\infty}.$$

where the constants $A := \frac{|u'(t_0)|^2}{2} + G(u(t_0))$ and $B := (t_1-t_0)\|p\|_{L^\infty}$ do not depend on the choice of $\tilde{t}_1$, so we can choose $\tilde{t}_1$ arbitrarily close from below to t_1 and have the following:

$$\|u'|_{[t_0,t_1]}\|_\infty^2 \leq |u'(t_0)|^2 + 2G(u(t_0)) + 2(t_1-t_0)\|p\|_\infty\|u'|_{[t_0,t_1^-)}\|_{L^\infty}.$$

So, $u'(t)$ is bounded on $[t_0,t_1)$. Now, taking limit as $t \to t_1^-$ in (4.13) we have that the right term is bounded and $G(u(t)) \to +\infty$ when $t \to t_1$⁻, so a contradiction yields.

■

Remark 4.3.5. It is worth noting that in this context the repulsive condition (4.7) implies that $G(u)$ increases when u moves on rays that point towards the origin. However, this specific condition was not necessary in the preceding result, which only uses the fact that $G(0) = +\infty$, since it is not required for the proof of Lemma 4.3.1.

Taking into account the previous comments on the central motion problem, we are able to establish an existence result for the particular radial case $g(u) = h(|u|)u$:

Theorem 4.3.6. *Let $g(u) = h(|u|)u$, with $h \in C((0,+\infty),(-\infty,0))$, and let*

$$g_\varepsilon(u) = \frac{h(|u|)u}{1-\varepsilon h(|u|)}.$$

Then there exists a sequence $\{u_n\}_n$ of solutions of (4.9) with $\varepsilon_n \to 0$ that converges uniformly and weakly in H^1 to some limit function u.

Furthermore, if

$$\limsup_{r\to 0^+} rh(r) + \|p\|_{L^\infty} < 0,$$

then $\partial\{t \in [0,T] : u(t) = 0\}$ is finite, and if $\int_0^1 sh(s)ds = -\infty$, then either $u \equiv 0$ or u is a classical solution.

Proof:
As in the particular case of the central motion problem, existence of solutions of (4.9) follows from Theorem 3.2.1 with condition (B'). Moreover, a bound for $\|u_\varepsilon'\|_{L^2}$ is also obtained as before and, again, the fact that $\int_0^T g_\varepsilon(u_\varepsilon)dt = 0$ implies that

$$-\int_0^T \frac{h(|u_\varepsilon|)\overline{u_\varepsilon}}{1-\varepsilon h(|u_\varepsilon|)}dt = \int_0^T \frac{h(|u_\varepsilon|)(u_\varepsilon - \overline{u_\varepsilon})}{1-\varepsilon h(|u_\varepsilon|)}dt.$$

Thus, a bound for $\overline{u}_\varepsilon$ is also obtained. If we consider a the sequence $\varepsilon_n \to 0$ we have weak convergence in H^1 and subsequence that converges strongly in L^2'. This conclusions follow from the Banach-Alaoglu and the Arzelá-Ascoli Theorems.

Moreover, if $\left\langle g(u), \frac{u}{|u|}\right\rangle = h(r)r < -\|p\|_{L^\infty}$ for $|u| = r$ small, then Lemma 4.3.3 applies. Finally, as $g = \nabla G$, with $G(u) = f(|u|)$ for the function $f(\sigma) := \int_1^\sigma sh(s)ds$, let us see that Proposition 4.3.4 applies. Indeed, here we have that $G(u) = \int_1^{|u|} sh(s)ds$, so we are in the case $g(u) = \nabla G(u) = h(|u|)u$:

$$\nabla G(u) = f'(|u|)\frac{u}{|u|} = h(|u|)u, \qquad |u| > 0.$$

Note also that changing the limits of integration $G(u) = -\int_{|u|}^{1} sh(s)ds$, so, as $h < 0$, G is aways positive, and $\limsup_{r\to 0} rh(r) < -\|p\|_{L^\infty}$ also imply that $h(r) \to -\infty$ when $r \to 0$, so for this choice of G, we have that $\lim_{|u|\to 0} G(u) = +\infty$ and the Proposition 4.3.4 applies.

■

Remark 4.3.7. It is worth mentioning that assumption $p \in C(\mathbb{R}, \mathbb{R}^N)$ could be weakened and the previous result would also hold if p is a piecewise continuous periodic function.

Example 4.3.8. The following elementary example shows that the assumption

$$\lim_{|u|\to 0} G(u) = +\infty,$$

in Proposition 4.3.4 is sharp. Let us consider the equation

$$u'' = \frac{u}{|u|^{\gamma+2}} + p(t), \tag{4.14}$$

which corresponds to the potential

$$G(u) = \begin{cases} \frac{1}{\gamma|u|^\gamma} & \text{if } \gamma \neq 0 \\ -\log|u| & \text{if } \gamma = 0. \end{cases}$$

If $\gamma > -1$, the equation is singular, although for $\gamma \in (-1, 0)$ the potential is continuous up to 0. For simplicity, let us consider the case $N = 1$, and $p = \chi_{[\frac{T}{2},T]} - \chi_{[0,\frac{T}{2})}$.

As remarked before, Proposition 4.3.4 still applies for this choice of p and G. As $\overline{p} = 0$, there are no classical solutions, in the sense of having $u \in H^2(\Omega, \mathbb{R}^N)$ satisfying the variational problem. Moreover, if we set g_ε as

$$g_\varepsilon(u) = -\frac{|u|^{\gamma-2}u}{(\varepsilon + |u|^\gamma)^2},$$

then from the energy conservation law, $E = K + V$ with $K = \frac{|u_\varepsilon'|^2}{2}$ the kinetic energy, and $V = u_\varepsilon + \frac{1}{\gamma(\varepsilon+|u_\varepsilon|^\gamma)}$ the potential energy:

$$\frac{u_\varepsilon'^2}{2} = E_\varepsilon - u_\varepsilon - \frac{1}{\gamma(\varepsilon + |u_\varepsilon|^\gamma)}, \qquad 0 < t < \frac{T}{2}.$$

A standard computation proves that if T is sufficiently large, then there exist $M_\varepsilon > 0$ and v_ε a positive solution of the equation over $(0, \frac{T}{2})$ such that $v_\varepsilon(0) = v_\varepsilon(\frac{T}{2}) = 0$, with energy $E_\varepsilon = M_\varepsilon + \frac{1}{\gamma(\varepsilon+M_\varepsilon^\gamma)}$ and with norm $\|v_\varepsilon\|_{L^\infty} = v_\varepsilon(\frac{T}{4}) = M_\varepsilon$.

We obtain a periodic solution of the perturbed problem by reflection, namely:

$$u_\varepsilon(t) = \begin{cases} v_\varepsilon(t) & \text{if} \quad 0 \le t \le \frac{T}{2} \\ -v_\varepsilon(t - \frac{T}{2}) & \text{if} \quad \frac{T}{2} < t \le T. \end{cases}$$

In particular, for $\varepsilon = 0$ we obtain a solution u of the problem with a collision at $t = \frac{T}{2}$. Furthermore, it is easily checked that $u_\varepsilon \to u$; thus, u is a generalized but non-classical solution.

Remark 4.3.9. Proposition 4.3.4 can be regarded as an alternative, in the following way: for g satisfying the assumption, if a sequence $\{u_n\}_n$ of solutions of (4.9) for $\varepsilon = \varepsilon_n \to 0$ converges uniformly and weakly in H^1 to some function u, then either $u \equiv 0$, or u is a classical solution of the problem.

It is worth seeing that both situations may occur: for instance, we may consider again equation (4.14), now with $\gamma \ge 0$. If $p \equiv 0$, then there are no generalized solutions, since they should be classical, because of Lemma 4.3.1. In some sense, this is expectable since if g_{ε_n} is given as in (4.12) or (4.10), then $u_\varepsilon \equiv 0$ is the unique solution of the perturbed problem. On the other hand, for $N = 2$ we may consider $p(t) = -\lambda(\cos(\omega t), \sin(\omega t))$ with $\omega = \frac{2\pi}{T}$, and the circular solution given by $u(t) = r(\cos(\omega t), \sin(\omega t))$, where $\lambda = r\omega^2 + \frac{1}{r^{\gamma+1}}$. After a simple computation, we conclude that the problem has classical solutions for $\lambda \ge (\gamma + 2)\left(\frac{\omega^2}{(\gamma+1)}\right)^{\frac{\gamma+1}{\gamma+2}}$.

Following the ideas in [34], for the preceding case (4.14) with $\gamma \ge 0$ a non-existence result holds when $\|p\|_{L^\infty}$ is small. It is interesting to observe that this result can be extended for the L^1-norm: if $\|p\|_{L^1} \le \eta$ for some η sufficiently small, then the problem admits no classical solutions.

For simplicity, we shall consider only the case $\gamma = 1$ and prove that $\eta \ge \left(\frac{16}{T}\right)^{1/3}$. On the other hand, as we always have circular solutions for any $\lambda \ge 3\left(\frac{2\pi^2}{T^2}\right)^{2/3}$ (and any $N \ge 2$), we also know that $\eta \le 3\left(\frac{4\pi^4}{T}\right)^{1/3}$.

In order to obtain the previously mentioned explicit lower bound for η, let us assume that u is a classical solution, and fix t_0 the maximal time; i.e. such that $|u(t_0)| = \|u\|_{L^\infty}$. Multiplying the equation by u and integrating, it follows that

$$\|u'\|_{L^2}^2 = -\int_0^T \left(\frac{1}{|u|} + \langle p, u\rangle\right) dt \le -\frac{T}{\|u\|_{L^\infty}} + \|p\|_{L^1}\|u\|_{L^\infty},$$

and in particular, as u is non-constant,

$$\|p\|_{L^1} > \frac{T}{\|u\|_\infty^2}.$$

Also, for the j-th coordinate of u we have:

$$u_j(t) - u_j(t_0) = \int_{t_0}^{t} u_j'(s)ds \le \int_0^T (u_j')^+(s)ds = \frac{1}{2}\|u_j'\|_{L^1} \le \frac{T^{1/2}}{2}\|u_j'\|_{L^2},$$

and an analogous inequality follows using $(u_j')^-$. Then

$$\|u - u(t_0)\|_{L^\infty}^2 \le \frac{T}{4}\|u'\|_{L^2}^2 \le \frac{T}{4}\left(\|p\|_{L^1}\|u\|_{L^\infty} - \frac{T}{\|u\|_{L^\infty}}\right)$$

and in particular

$$|u(t)| \ge |u(t_0)| - \left[\frac{T}{4}\left(\|p\|_{L^1}\|u\|_{L^\infty} - \frac{T}{\|u\|_{L^\infty}}\right)\right]^{1/2}.$$

Thus,

$$\langle u(t), u(t_0)\rangle = \frac{1}{2}\left(|u(t)|^2 + |u(t_0)|^2 - |u(t) - u(t_0)|^2\right) \ge$$

$$\ge \|u\|_{L^\infty}\left(\|u\|_{L^\infty} - \left[\frac{T}{4}\left(\|p\|_{L^1}\|u\|_{L^\infty} - \frac{T}{\|u\|_{L^\infty}}\right)\right]^{1/2}\right).$$

If $\|p\|_{L^1}^3 \le \frac{16}{T}$, we deduce that $\|p\|_{L^1}^2 \le \frac{16}{T^2}\frac{T}{\|p\|_{L^1}} < \left(\frac{4}{T}\|u\|_{L^\infty}\right)^2$. Hence we have the inequality $\frac{T}{4}\|p\|_{L^1}\|u\|_{L^\infty} < \|u\|_{L^\infty}^2$, and we conclude that $\langle u(t), u(t_0)\rangle > 0$ for every t.

Finally, integrating the equation we obtain

$$0 = \left\langle u(t_0), \int_0^T u''(t)\, dt\right\rangle = \int_0^T \frac{1}{|u(t)|^3}\langle u(t_0), u(t)\rangle\, dt > 0,$$

a contradiction.

Remark 4.3.10. It might be worth observing that the geometric idea behind the last proof is that for any $w \in \mathbb{R}^N\backslash\{0\}$ the range of a classical solution of the problem cannot be contained in the half-space

$$H_w := \{u : \langle u, w\rangle > 0\}.$$

Together with the preceding results, the previous computations imply that, for the central motion case, if $\|p\|_{L^1} \le \left(\frac{16}{T}\right)^{1/3}$ and $g_\varepsilon(u) = -\frac{u}{\varepsilon+|u|^3}$ the solutions of the perturbed problems (4.9) $\{u_\varepsilon\}_\varepsilon$ go to 0 uniformly as $\varepsilon \to 0$. Indeed, if there existed a sequence $\varepsilon_n \to 0$ such that $\{u_n\}_n$ did not go to zero, we could assume that $\|u_n\|_\infty \ge c > 0$, but with this g_ε we already showed that there were subsequences for which $u_n \to 0$, a contradiction.

However, it is worth to observe that this can always be done if we do not impose restrictions on the choice of g_ε. Indeed, we may recall that for any $\lambda > 0$, the unique T-periodic solution of the linear problem $u'' - \lambda^2 u = p$ is given by

$$u(t) = \int_0^T G(t,s)p(s)ds,$$

where G is the Green Function (see Chapter 1) defined by

$$G(t,s) = \frac{-\cosh\left(\lambda\left(\frac{T}{2} - |t-s|\right)\right)}{2\lambda \sinh\left(\lambda\frac{T}{2}\right)}.$$

A simple computation shows, moreover, that $\|G(t,\cdot)\|_{L^1} = \frac{1}{\lambda^2}$. Thus, if $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is any continuous function satisfying $\varepsilon\mu(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, then we may define, using Tietze's Theorem, $g_\varepsilon \in C(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$g_\varepsilon(u) = \begin{cases} g(u) & \text{if } |u| \geq 2\varepsilon \\ -\mu(\varepsilon)u & \text{if } |u| \leq \varepsilon. \end{cases}$$

Then, for every $\varepsilon > 0$ with $\varepsilon\mu(\varepsilon) > \|p\|_{L^\infty}$, the unique solution of the linear problem $u'' - \mu(\varepsilon)u = p$ satisfies:

$$|u(t)| \leq \frac{\|p\|_{L^\infty}}{\mu(\varepsilon)} < \varepsilon,$$

and hence it solves (4.9).

4.4 Proof of Main Results

The rest of this Chapter is devoted to the particular case in which g_ε is defined by (4.10) for some ρ_ε. This is a family of admissible approximations (see 4.2.2). The reason of this specific choice is that, unlike the case of Theorem 4.3.6, the existence of a priori bounds for u_ε cannot be established for a general nonlinearity g. Note also that, if $g(u) = h(|u|)u$, then the 'linear' cutoff function defined by $\rho_\varepsilon(s) = \frac{s}{\varepsilon}$ in (4.10) would lead to the previous situation, with $\mu = -h$, and the conclusions in our existence results would become trivial. However, we do not need to impose any restriction on the function $\rho(\varepsilon) := \rho_\varepsilon$.

Theorem 4.4.1. *Let $g \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R}^N)$ and assume that (4.7) holds. Further, assume that g satisfies (B) or (B'). Then either problem (4.2) has a classical solution, or else for every sequence $\{u_n\}_n$ of solutions of (4.9) with $\varepsilon_n \to 0$ and g_n as in (4.10), there exists a subsequence that converges uniformly and weakly in H^1.*

Proof:
If the problem has a classical solution, then there is nothing to prove. Next, assume that (4.9) admits no classical solutions, and let u_n be a T-periodic solution of

$$u_n'' + g_n(u_n) = p(t).$$

Multiplying by $u_n - \overline{u}_n$ and integrating:

$$\int_0^T \langle u_n'', u_n - \overline{u}_n \rangle dt + \int_0^T \langle g_n(u_n), u_n - \overline{u}_n \rangle dt = \int_0^T \langle p(t), u_n - \overline{u}_n \rangle dt.$$

Hence

$$-\int_0^T |u_n'|^2 dt + \int_0^T \langle g_n(u_n), u_n - \overline{u_n} \rangle dt = \int_0^T \langle p(t), u_n - \overline{u}_n \rangle dt.$$

Then, we have the following inequality:

$$\|u_n'\|_{L^2}^2 \leq \|p\|_{L^2} \|u_n - \overline{u}_n\|_{L^2} + \int_0^T \langle g_n(u_n), u_n - \overline{u_n} \rangle dt. \tag{4.15}$$

If (B) holds, then we may split the last term in two terms as:

$$\int_{\{|u_n| > \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt + \int_{\{|u_n| \leq \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt,$$

with κ given by the repulsiveness in (4.7).

For the first term, we use the definition of g_n:

$$g_n(u) = g(u) \qquad \text{if } |u| > \varepsilon_n.$$

We may assume that $\varepsilon_n < \kappa$:

$$\left| \int_{\{|u_n| > \kappa\}} \langle g_n(u_n), u_n - \overline{u_n} \rangle dt \right| \leq \int_{\{|u_n| > \kappa\}} |g(u_n)| |u_n - \overline{u_n}| dt \leq C \|u_n - \overline{u_n}\|_{L^2}.$$

The remaining term can be written as:

$$\int_{\{|u_n| \leq \kappa\}} \langle g_n(u_n), u_n \rangle dt - \left\langle \int_{\{|u_n| \leq \kappa\}} g_n(u_n) dt, \overline{u_n} \right\rangle.$$

Condition (4.7) implies that the first term is non-positive; moreover, as $\overline{g_n(u_n)} = 0$ we deduce that

$$\int_{\{|u_n| \leq \kappa\}} g_n(u_n) dt = -\int_{\{|u_n| > \kappa\}} g_n(u_n) dt$$

.

Hence,

$$\int_{\{|u_n|\le\kappa\}} \langle g_n(u_n), u_n - \overline{u_n}\rangle dt \le |\overline{u_n}| \int_{\{|u_n|>\kappa\}} |g_n(u_n)| dt.$$

Again, the integral in the right-hand side term is bounded, because g_n may be replaced by g. Gathering all together:

$$\|u_n'\|_{L^2}^2 \le C_1\|u_n - \overline{u_n}\|_{L^2} + C_2|\overline{u_n}|.$$

Finally, using Wirtinger's inequality we obtain:

$$\|u_n'\|_{L^2} \le C|\overline{u_n}|^{\frac{1}{2}}, \qquad \|u_n - \overline{u_n}\|_{L^\infty} \le C|\overline{u_n}|^{\frac{1}{2}}.$$

At this point we can state that $\{\overline{u}_n\}_n$ is bounded.

If this was not the case, we would have, for some value of n, that $|\overline{u_n}|^{\frac{1}{2}} > C+1 \ge \varepsilon_n$. Then

$$|u_n(t)| \ge |\overline{u_n}| - \|u_n - \overline{u_n}\|_{L^\infty} \ge |\overline{u_n}| - C|\overline{u_n}|^{\frac{1}{2}} > C+1.$$

Thus, u_n would be a classical solution of the original problem, a contradiction.

If, instead, we assume that (B') holds, from the fact that $\overline{g_n(u_n)} = 0$ we deduce that the last term of (4.15) is bounded, and uniform bounds for $\|u_n'\|_{L^2}$ and for $\|u_n - \overline{u}_n\|_\infty$ yield. As before, this implies that $\{\overline{u}_n\}_n$ is also bounded. Hence, there is a subsequence (still denoted $\{u_n\}_n$) and a function $u \in H^1$ such that $u_n \to u$ uniformly and weakly in H^1. ∎

In the previous proof, note that the bounds for $\|u_n\|_{H^1}$ do not depend on the choice of ρ_ε. This is the reason why Theorem 4.2.4, with ρ arbitrarily chosen, follows as an immediate consequence of the preceding results:

Proof of Theorem 4.2.4:

Given $0 < \varepsilon_n \to 0$ then either $g_n \in C(\mathbb{R}^N, \mathbb{R}^N)$ is bounded or satisfies (B') for each n. Theorem 3.2.1 guarantees the existence of a sequence $\{u_n\}_n$ of classical solutions of problem (4.9). Finally, Theorem 4.4.1 is applied.

∎

The last part of this section is devoted to Theorem 4.2.5, which assumes a different asymptotic condition on g. In order to understand its meaning, let us firstly observe that if

$$\|p\|_{L^\infty} + \sup_{|u|=\varepsilon} \left\langle g_\varepsilon(u), \frac{u}{|u|} \right\rangle \le 0 \tag{4.16}$$

then a Hartman type condition (see [21]) holds, and the existence of a solution u_ε of (4.9) with $\|u_\varepsilon\|_{L^\infty} \leq \varepsilon$ is deduced. In particular, if g satisfies (4.8) with $c > \|p\|_{L^\infty}$, then condition (4.16) holds strictly when ε is small and, again, there exists a sequence of solutions of (4.9) that converges to 0. However, in this case we may take advantage of the fact that $deg(\Phi_\varepsilon) = (-1)^N$, and replace condition (N_2) by (P_2), namely that $deg(\Phi_R) \neq (-1)^N$ for R sufficiently large. Indeed, if we consider now the Brouwer degree of g_ε, from the excision property it follows that

$$\deg(g_\varepsilon, B_R(0)\backslash B_\varepsilon(0), 0) = deg(\Phi_R) - deg(\Phi_\varepsilon) \neq 0.$$

Thus, Mawhin's Continuation Theorem (Theorem 1.2.26) [27] implies the existence of a second solution u_ε of (4.9) such that $\|u_\varepsilon\|_{L^\infty} > \varepsilon$, provided that the homotopy does not vanish when $\|u\|_{L^\infty} = \varepsilon$ or $\|u\|_{L^\infty} = R$. More generally, if we assume only that (4.16) holds strictly for some fixed $\tilde{r}$, then we are able to prove Theorem 4.2.5.

Proof of Theorem 4.2.5:

From Theorem 4.4.1, it suffices to show that for each $\varepsilon \leq \tilde{r}$ problem (4.9) has a solution u_ε such that $\|u_\varepsilon\|_{L^\infty} > \tilde{r}$. To this end, we may follow the general outline of the proof of Theorem 3.2.1, from Chapter 3, but now taking the domain $U = \{u \in C([0,T], \mathbb{R}^N) : \tilde{r} < \|u\|_{L^\infty} < R\}$. The proof of the fact that $u'' \neq \lambda(p - g_\varepsilon(u))$ for any T-periodic function u with $\|u\|_{L^\infty} = R > 0$ big enough and $\lambda \in (0,1]$ follows as in the proof of Theorem 3.2.1. On the other hand, if u is T-periodic and satisfies

$$u'' = \lambda(p - g_\varepsilon(u)),$$

with $\|u\|_{L^\infty} = \tilde{r}$, then consider $\phi(t) := |u(t)|^2$ and t_0 a maximum of ϕ. Hence $|u(t_0)| = \tilde{r}$, and

$$0 \geq \phi''(t_0) \geq -2\lambda r \left[\|p\|_{L^\infty} + \left\langle g(u(t_0)), \frac{u(t_0)}{|u(t_0)|} \right\rangle \right] > 0,$$

a contradiction. Finally, from the remarks previous to this proof we deduce that $\deg(g, \{\tilde{t} < |u| < R\}, 0) \neq 0$, and the conclusion of the Theorem follows.

■

Example 4.4.2. If there exist $v \in S^{N-1}$ and $r_0 > 0$ such that $g(u) \in H_v$ for $|u| \geq r_0$, where H_v is the half-space defined as before, then condition (P_1) is satisfied taking $w = -v$ and $\mathcal{F} = \{(S^{N-1}, w)\}$. Moreover, it is also clear that $deg(\Phi_R) = 0$ for $R \geq r_0$. Hence, if g satisfies (B) or (B') and (4.8), the existence of a generalized solution follows for any p continuous and T-periodic such that $\overline{p} = 0$ and $\|p\|_{L^\infty} < c$.

More generally, if g satisfies (B) or (B'), (P_1) and condition (4.8) with $\|p\|_{L^\infty} < c$, then it suffices to assume that $g(u) \neq \lambda v$ for $|u| \geq r_0$ and $\lambda \geq 0$.

Remark 4.4.3. Under the assumptions of Theorem 4.4.1, if (P_1) and (P_2) are satisfied, and g is *Sequentially Strongly Repulsive* at the origin, namely

$$\sup_{|u|=r_n} \left\langle g(u), \frac{u}{|u|} \right\rangle \to -\infty \text{ for some } r_n \to 0, \tag{4.17}$$

then existence of a generalized solution holds for any p continuous and T-periodic such that $\overline{p} = 0$.

Remark 4.4.4. It is interesting to observe that condition (2.22), introduced by Solimini in [34] and discussed in Chapter 2:

$$\exists\, \delta > 0 \,\text{such that, if } \left| \frac{u}{|u|} - \frac{v}{|v|} \right| < \delta, \text{ then } \langle g(u), v \rangle < 0$$

implies that $deg(\Phi_r) = (-1)^N$ for all values of r; thus, Theorem 4.2.5 does not apply here. This is consistent with the non-existence result obtained in [34]. On the other hand, condition (P_1) is still satisfied if (2.22) is reversed, namely:

$$\exists\, \delta, r_0 > 0 : \text{ if } |u|, |v| \geq r_0 \text{ and } \left| \frac{u}{|u|} - \frac{v}{|v|} \right| < \delta, \text{ then } \langle g(u), v \rangle > 0. \tag{4.18}$$

In some sense, (4.18) says that g is repulsive at infinity, and that it cannot rotate too fast. We have already used the fact that repulsiveness at the origin implies that the Brouwer degree of g_ε over small balls is $(-1)^N$; on the other hand, repulsiveness at infinity implies that its degree over large balls is 1. Hence, if the assumptions of Theorem 4.4.1 are satisfied and g is (sequentially) strongly repulsive at the origin (4.17) and (4.18) holds, then there exist generalized solutions for any p continuous and T-periodic such that $\overline{p} = 0$, provided that N is odd. Here having the following two degrees:

$$\deg(g, B_\varepsilon(0), 0) = -1, \qquad \deg(g, B_R(0), 0) = 1.$$

The excision property would assure existence of solutions.

In particular, for the radial case we have:

Corollary 4.4.5. *let N be odd, p as before, and let g be given by*

$$g(u) = \varphi(|u|)\psi\left(\frac{u}{|u|}\right)$$

with $\psi \in C(S^{N-1}, S^{N-1})$*,* $\varphi \in C(\mathbb{R}^+, \mathbb{R})$ *and bounded from below, and*

$$\langle \psi(v), v\rangle < 0 \qquad \forall v \in S^{N-1},$$
$$\lim_{r\to 0^+} \varphi(r) = +\infty, \qquad \varphi(r) < 0 \quad \text{if } r > r_0$$

for some $r_0 > 0$*.*

Then, for any p *as before, either (4.2) has a classical solution, or a generalized solution* u*. Moreover, the boundary of the set of zeros of* u *is finite. For the case* $\psi(v) = -v$*, if furthermore* $\int_0^1 \varphi(s)ds = +\infty$*, then (4.2) has a classical solution.*

Proof:
Condition (B) is clear. Moreover, as ψ is continuous, for each $u \in S^{N-1}$ there exists an open neighborhood $U \subset S^{N-1}$ of u such that :

$$\langle \psi(w), u\rangle < 0 \qquad \forall w \in U.$$

Then taking $w_u = -u$, for $r > r_0$ and $w \in U$ we obtain:

$$\langle g(rw), w_u\rangle = |\varphi(r)|\langle \psi(w), u\rangle < 0.$$

From the compactness of S^{N-1}, condition (P_1) is satisfied.

Finally, define the homotopy $h : \mathbb{R}^N\backslash\{0\} \times [0,1] \to \mathbb{R}^N$ given by

$$h(u,\lambda) = \lambda g(u) + (1-\lambda)u$$

.

Then, for $|u| = R > r_0$ we have that

$$\langle h(u,\lambda), u\rangle = \lambda\langle g(u), u\rangle + (1-\lambda)R^2 > 0.$$

By the homotopy invariance of the degree, we conclude that

$$deg(\Phi_R) = deg(Id) = 1 \neq (-1)^N,$$

as N was supposed to be odd. Hence, condition (P_2) is then also satisfied, and the conclusion follows from Theorem 4.2.5.

■

Chapter 5

Singular Elliptic Problems

5.1 Introduction and Motivation

Although the previous Chapter could have be included as a particular case of the theory developed in this Chapter, we decided to treat it as a separate one. First of all because the techniques used in both cases are different, but mainly because it is faithful to the history of how this thesis evolved. It is also true that the results obtained in the previous chapter were stronger, because some of the proofs used techniques of ordinary differential equations which are not true in Elliptic Problems

As it was told in the Introduction, after obtaining the main results for the Periodic case (see Chapter 4), we started to think in a possible generalization for Elliptic problems:

$$\Delta u + g(u) = f(x) \qquad \text{in } \Omega,$$

with $\Omega \in \mathbb{R}^d$ $f : \overline{\Omega} \to \mathbb{R}^N$ a continuous function a $g : U \subset \mathbb{R}^N \to \mathbb{R}^N$ a nonlinearity.

There are many possible ways of extending a periodic conditions for an ordinary differential equation system, to an Elliptic system of partial differential equations. As it was also stated in the Introduction, and revisited in Chapter 3, we came across to one boundary condition for the Elliptic case that seemed odd at first sight, but indeed generalized the periodic conditions in one dimension. The Nonlocal Condition, introduced in the Introduction:

$$\begin{cases} u \equiv & C \quad u \in \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = & 0. \end{cases} \tag{5.1}$$

The problem we studied was already stated in (3.6) for the nonsingular case in Chapter 3), is the following: Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and consider the following elliptic system:

$$\begin{cases} \Delta u + g(u) & = f(x) \quad \text{in } \Omega \\ \quad u & = C \quad \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS & = 0, \end{cases} \tag{5.2}$$

with $C \in \mathbb{R}^N$ an unknown constant vector, $f : \overline{\Omega} \to \mathbb{R}^N$ continuous. and $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$, with $\mathcal{S} \subset \mathbb{R}^N$ bounded. Without loss of generality we may assume that $\overline{f} := \frac{1}{|\Omega|} \int_\Omega f(x)dx = 0$

Here, we shall consider two different problems. In the next section we shall allow the (bounded) set $\mathcal{S}$ of singularities to be arbitrary and focus our attention on the behavior of the nonlinear term g over the boundary of an appropriate domain $D \subset \mathbb{R}^N \backslash \mathcal{S}$. In this case we will try to use Continuation Theory on these sets D.

Next, we will study the case in which $\mathcal{S}$ consists in a single point; without loss of generality, it may be assumed $\mathcal{S} = \{0\}$, as we did in Chapter 4. We shall focus our attention on the way g behaves near the singular point. We shall assume that g is repulsive (4.7), as defined in (4.2.1) and we will look for strong results in the direction of Theorem 4.2.4 and Theorem 4.2.5. Unfortunately, if $d > 1$ it is no longer true that $H^1(\Omega, \mathbb{R}^N)$ is compactly embedded in $C(\Omega, \mathbb{R}^N)$, so we could not be able to obtain such strong results. Nevertheless we defined a different kind of generalized solution of a distributional nature and in this context we did prove interesting results for this case.

5.2 Singular Set

In this section, $\mathcal{S} \subset \mathbb{R}^d$ will denote a bounded set of singularities and we will assume the same boundness condition we worked with in the last Chapter:

(B) $\quad \limsup_{|u| \to \infty} |g(u)| < \infty.$

The tools and ideas used in this section are similar of those used in the last section of Chapter 3. We will also use the geodesic distance introduced before (3.7):

$$d(x, y) := \inf\{lenght(\gamma) : \gamma \in C^1([0, 1], \Omega) : \gamma(0) = x, \gamma(1) = y\}. \tag{5.3}$$

Next, we shall fix a compact neighborhood $\mathcal{C}$ of $\mathcal{S}$ and a number r (recall (3.8)) defined by:

$$r := k \operatorname{diam}_d(\Omega)(\|f\|_{L^\infty} + ess \sup_{u \notin \mathcal{C}} |g(u)|,) \tag{5.4}$$

where $\operatorname{diam}_d(\Omega)$ is the diameter with respect to the distance (5.3) and k is a constant such that

$$\|\nabla u\|_{L^\infty} \leq k\|\Delta u\|_{L^\infty} \tag{5.5}$$

To verify the existence of this estimate let us prove the following:

Lemma 5.2.1. *There exist k such that (5.5) holds for all $u \in C^2(\overline{\Omega}, \mathbb{R}^N)$ satisfying the nonlocal boundary conditions of (5.2).*

<u>*Proof*</u>:

By standard regularity results (see e.g. [20]), if $u \in C(\overline{\Omega}, \mathbb{R}^N)$ is a solution of (5.2) then $u \in \mathcal{A}(\Omega)$, where

$$\mathcal{A}(\Omega) = \left\{ u \in C^1(\overline{\Omega}, \mathbb{R}^N) : \|\Delta u\|_{L^\infty} < \infty, u \equiv C \text{ on } \partial\Omega, \int_\Omega \frac{\partial u}{\partial \nu} dS = 0 \right\}.$$

with C a constant vector in $\mathbb{R}^N$. Note that $\mathcal{A}(\Omega) \subset W^{2,s}(\Omega, \mathbb{R}^N)$, for any $s < \infty$.

Next, suppose that, for a sequence $\{u_n\}_n \subset \mathcal{A}(\Omega)$, $\|\nabla u_n\|_\infty > n\|\Delta u_n\|_\infty$. Let $v_n := u_n/\|\nabla u_n\|_\infty$, then $\|\Delta v_n\|_\infty \to 0$ and hence $\|\Delta v_n\|_{L^2} \to 0$.

This implies that $\|\nabla v_n\|_{L^2} \to 0$ and, consequently, $\|v_n - \overline{v}_n\|_{H^1} \to 0$. Thus $\|v_n - \overline{v}_n\|_{H^2} \to 0$ which, in turn, implies that $\|v_n - \overline{v}_n\|_{W^{1,2^*}} \to 0$.

Again, we conclude that $\|v_n - \overline{v}_n\|_{W^{2,2^*}} \to 0$ and by a standard bootstrapping argument we deduce that $\|v_n - \overline{v}_n\|_{W^{2,s}} \to 0$ for some $s > N$.

By the Sobolev embedding (see Chapter 1): $W^{2,s}(\Omega, \mathbb{R}^N)$ is continuously embedded in $C^1(\overline{\Omega}, \mathbb{R}^N)$, this implies $\|\nabla v_n\|_\infty \to 0$, a contradiction.
■

Having done that, we shall assume, as in the last part of Chapter 3, the existence of a set $D \subset \mathbb{R}^N \backslash (\mathcal{C} + \overline{B_r}(0))$ such that the following two conditions hold:

(D_1) For all $v \in \partial D$, $0 \notin co(g(B_r(v)))$, were $co(A)$ stands for the convex hull of A (see Definition 1.1.20).

(D_2) $\deg(g, D, 0) \neq 0$.

Condition (D_1) was introduced by Ruiz and Ward in [33] and extended in [5] by Amster and Clapp. It generalizes a classical condition given by Nirenberg in [29] which, in particular, implies that g cannot rotate around the origin when $|u|$ is large. (D_1) allows g to rotate but not too fast since r cannot be arbitrarily small.

The main result of this section reads as follows:

Theorem 5.2.2 (Amster, M. - VI)**.** *Let* $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$ *satisfying* (B) *and* $f \in C(\overline{\Omega}, \mathbb{R}^N)$ *such that* $\overline{f} = 0$*. Let* $\mathcal{C}$ *be a compact neighborhood of* $\mathcal{S}$ *and let* r *be as in* (5.4)*. If there exists a bounded domain* D *such that* $D \subset \mathbb{R}^N \backslash (\mathcal{C} + \overline{B_r}(0))$ *such that* (D_1) *and* (D_2) *hold, then (5.2) has at least one solution* u *with* $\overline{u} \in \overline{D}$ *and* $\|u - \overline{u}\|_{L^\infty} \le r$*.*

Proof:

The proof has great resemblance with the proof of Theorem 3.3.1. Let

$$U = \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \|u - \overline{u}\|_{L^\infty} < r, \, \overline{u} \in D\}$$

and consider, for $\lambda \in (0, 1]$, the problem

$$\begin{cases} \Delta u + \lambda \hat{g}(u) & = \lambda f(x) \quad \text{in } \Omega \\ \qquad u & = C \qquad \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS & = 0, \end{cases} \tag{5.6}$$

where $\hat{g} \in C(\mathbb{R}^N, \mathbb{R}^N$ is a bounded extension of g with $\hat{g} = g$ over $\overline{D + B_r(0)}$. It is clear that if $u \in \overline{U}$ solves (5.6) for $\lambda = 1$ then u is a solution of (5.2).

Indeed, if $u \in \overline{U}$ then $\overline{u} \in \overline{D}$ and $\|u - \overline{u}\|_{L^\infty} \le r$. This implies that $u(x) \in \overline{D} + \overline{B_r(0)}$ so that $\hat{g}(u(x)) = g(u(x))$.

Now, using the same Continuation Method as the one used in the proof of Theorem 3.3.1, we define u as the solution of (5.6).

As $d(\overline{u}, \mathcal{C}) \ge r$, we deduce that $u(x) \in \overline{\mathbb{R}^N \backslash \mathcal{C}}$ and hence the inequality $|g(u(x))| \le ess\sup_{z \notin \mathcal{C}} |g(z)|$ holds for all x. This implies

$$\|\nabla u\|_{L^\infty} \le k \|\Delta u\|_{L^\infty} < k(\|f\|_{L^\infty} + ess\sup_{z \notin \mathcal{C}} |g(z)|),$$

and thus

$$\|u - \overline{u}\|_{L^\infty} \le \text{diam}_d(\Omega) \|\nabla u\|_{L^\infty} < r.$$

Hence, $\overline{u} \in \partial D$. Moreover, it follows from the Mean-Value Theorem for Vector Integrals (see Theorem 1.1.19 in Chapter 1) that

$$\frac{1}{|\Omega|} \int_\Omega g(u(x)) \, dx \in co(g(u(\overline{\Omega}))) \subset co(g(B_r(\overline{u}))).$$

On the other hand, simple integration shows that

$$\int_\Omega g(u(x)) \, dx = 0,$$

so $0 \in co(g(B_r(\overline{u})))$, a contradiction.

∎

Remark 5.2.3. It is worth noticing that the previous result can be extended for g sublinear:

$$\lim_{|u|\to\infty} \frac{g(u)}{|u|} = 0. \tag{5.7}$$

Indeed, for any given $\varepsilon > 0$, there exist a constant $M_{\varepsilon,\mathcal{C}}$ such that

$$|g(u)| \le \varepsilon|u| + M_{\varepsilon,\mathcal{C}} \qquad \forall u \in \mathbb{R}^N \backslash \mathcal{C}.$$

Thus, if u is a solution of problem (5.2), then

$$\|\nabla u\|_{L^\infty} \le k\|\Delta u\|_{L^\infty} \le k\left(\|f\|_{L^\infty} + \varepsilon\|u\|_{L^\infty} + M_{\varepsilon,\mathcal{C}}\right),$$

and hence

$$\|\nabla u\|_{L^\infty} \le k\left(\|f\|_{L^\infty} + M_{\varepsilon,\mathcal{C}} + \varepsilon(\mathrm{diam}_d(\Omega)\|\nabla u\|_{L^\infty} + |\overline{u}|)\right).$$

Suppose now that $|\overline{u}| = R < \alpha K \mathrm{diam}_d(\Omega)$ for some constants $\alpha > 1$, $K > 0$. If $\|\nabla u\| \ge K$, then:

$$K\left(1 - k\varepsilon \mathrm{diam}_d(\Omega)(1+\alpha)\right) \le k\left(\|f\|_{L^\infty} + M_{\varepsilon,\mathcal{C}}\right).$$

Consequently, taking these constants such that

$$\varepsilon < \frac{1}{k\mathrm{diam}_d(\Omega)(1+\alpha)},\ K > \frac{k\left(\|f\|_{L^\infty} + M_{\varepsilon,\mathcal{C}}\right)}{1 - k\varepsilon\mathrm{diam}_d(\Omega)(1+\alpha)},\ r := K\mathrm{diam}_d(\Omega), \tag{5.8}$$

it follows that any solution u such that $|\overline{u}| = R < \alpha r$ satisfies:

$$\|\nabla u\|_{L^\infty} < K, \qquad \|u - \overline{u}\|_{L^\infty} < r.$$

We then have proved the following result for the sub linear case:

Corollary 5.2.4. *Let $g \in C(\mathbb{R}^N \backslash \mathcal{S}, \mathbb{R}^N)$ be sub linear and $f \in C(\overline{\Omega}, \mathbb{R}^N)$ such that $\overline{f} = 0$. Let $\mathcal{C}$ be a compact neighborhood of $\mathcal{S}$ and assume that $\alpha > 1$, $\varepsilon > 0$, $K > 0$ and r satisfying (5.8), there exists a bounded domain $D \subset B_{\alpha r}(0)\backslash(\mathcal{C} + \overline{B_r}(0)) \subset \mathbb{R}^N$ such that (D_1) and (D_2) hold. Then (5.2) has at least one solution u with $\overline{u} \in \overline{D}$ and $\|u - \overline{u}\|_{L^\infty} \le r$.*

Let us show an example that illustrates the possibility of obtaining multiple solutions. We note $B_\rho := B_\rho(0) = \{u \in \mathbb{R}^N : |u| < \rho\}$.

Example 5.2.5. Let $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ be bounded, $a = \|A\|_{L^\infty}$ and $b > 0$. Define $g(u) = \frac{A(u)}{|u|(b-|u|)}$, so $\mathcal{S} = \{0\} \cup \partial B_b$. Let $\eta > 0$ and consider the following compact set:

$$\mathcal{C} = \overline{B_\eta} \cup \left(\overline{B_{b+\eta}} \backslash B_{b-\eta}\right).$$

We have that $\mathbb{R}^N \backslash \mathcal{C} = (B_{b-\eta} \backslash \overline{B_\eta}) \cup (\mathbb{R}^N \backslash \overline{B_{b+\eta}})$. From the previous computations, if u is a solution of the problem, the following estimate holds:

$$\|\nabla u\|_\infty \leq K := k\left(\|f\|_{L^\infty} + \frac{a}{\eta(b+\eta)}\right).$$

Thus,

$$r = \mathrm{diam}_d(\Omega) k \left(\|f\|_{L^\infty} + \frac{a}{\eta(b+\eta)}\right).$$

If also $b > 2(r+\eta)$, then we might be able to obtain two disjoint sets $D^1, D^2 \subset \mathbb{R}^N \backslash (\mathcal{C} + B_r)$ such that:

$$D^1 \subset B_{b-\eta-r} \backslash B_{\eta+r}, \qquad D^2 \subset \mathbb{R}^N \backslash B_{b+\eta+r},$$

leading to two different solutions u^1, u^2 with $\overline{u^1} \in D^1$ and $\overline{u^2} \in D^2$ respectively.

In order to apply our previous result, observe that condition (D_1) requires $\eta + 2r < b - \eta - 2r$, that is: $b > 4r + 2\eta$.

For example, let $T > 0$ be large enough and define $g : B_{b+T} \backslash \mathcal{S} \to \mathbb{R}^N$ by

$$g(u) := \frac{(|u| - x_1)(|u| - x_2)\, u}{|u|(|u| - b)},$$

for some numbers $x_1, x_2 > 0$. The numerator of this function can be extended continuously to $\mathbb{R}^N \backslash \mathcal{S}$ in such a way that $a \leq (b+T)^3$. Taking $\mathrm{diam}_d(\Omega)$ small enough, the preceding inequalities for r are satisfied, so we may fix $x_1 \in (\eta + 2r, b - \eta - 2r)$ and $x_2 \in (b + \eta + 2r, b + T - 2r)$.

Thus, all the assumptions are satisfied for D^1 and D^2; hence, by Theorem 5.2.2 we deduce the existence of classical solutions $u^1 \neq u^2$ of problem (5.2) such that $\overline{u^i} \in D^i$, for $i = 1, 2$.

Remark 5.2.6. This example shows that if the assumptions of Theorem 5.2.2 are verified, then the distance between different connected components of $\mathcal{S}$ cannot be too small, because the open sets chosen to cover them must be sufficiently large to contain them, but sufficiently small to allow sets D to be in between.

5.3 Isolated Repulsive Singularity

This section focus on the case where $\mathcal{S} = \{0\}$. Note that the same analysis can be made for a general isolated point $\mathcal{S} = \{s\}$ with $s \in \mathbb{R}^N$.

The philosophy of this section is similar to that of the first sections of Chapter 4.

We shall proceed as follows: firstly, we will appropriate the nonlinearity g with continuous ones; secondly we swill prove existence of at least one solution of the approximated problems that come from this approximations of g; and finally we will obtain accurate estimates and deduce the existence of a convergent sequence of these solutions.

In order to define the approximated problems, fix a sequence $\varepsilon_n \to 0$ and consider the problem

$$\Delta u + g_n(u) = f(x) \qquad \text{in } \Omega, \tag{5.9}$$

together with the nonlocal boundary conditions of (5.2). Although more general perturbations are admitted, in fact our results will be true for admissible families of perturbations (see 4.2.2), for convenience we shall define g_n as in (4.10), with $g_n := g_{\varepsilon_n}$.

The conditions on g shall be, as before, of geometric nature. We will again assume g is such that conditions (P_1) and (P_2) hold. These where studied in the last two Chapters, but they are worth repeating:

(P_1) There exists a family $\mathcal{F} = \{(U_j, w_j)\}_{j=1}^K$ where $\{U_j\}_{j=1}^K$ is an open cover of S^{N-1} and $w_j \in S^{N-1}$, such that for some $R_j > 0$ and $j = 1, \ldots, K$:

$$\langle g(ru), w_j \rangle < 0 \qquad \forall r > R_j \qquad \forall u \in U_j.$$

(P_2) There exists a constant $R_0 > 0$ such that $\deg(g, B_R(0), 0) \neq (-1)^N$ for $R \geq R_0$.

We will also ask the singularity at the origin to be of a repulsive kind (see 4.2.1). However, a stronger assumption than the ones we worked on the last Chapter is needed in order to obtain uniform estimates: The Strongly Repulsive condition:

Definition 5.3.1. The nonlinearity g is said to be *strongly repulsive* at the origin if:

$$\lim_{u \to 0} \langle g(u), u \rangle = -\infty. \tag{5.10}$$

Note that this kind of repulsiveness is stronger than the one defined as sequentially strongly repulsiveness condition (4.17), which ensured that the degree over certain small balls centered at the origin is $(-1)^N$. This will allow as to work with condition (P_2) instead of condition (D_2).

We remark that, although g is not defined in 0, we may still use the expression $\deg(g, B_R(0), 0)$ as a notion to refer to the Brouwer degree

$\deg(\hat{g}, B_R(0), 0)$, where $\hat{g} : \overline{B}_R(0) \to \mathbb{R}^N$ is any continuous function such that $\hat{g} = g$ on $\partial B_R(0)$, as the degree only depends on the value in the boundary (refer to Definition 1.2.9 in Chapter 1).

The preceding conditions will allow us to construct a sequence $\{u_n\}_n$ of solutions of the approximated problems that converges weakly in H^1 to some function u. It is easy to see that if u does not vanish on Ω, then u is a classical solution of the problem.

If $u \not\equiv 0$ but possibly vanishes in Ω, then we shall call it a *generalized solution*. It will be a different concept of generalized solution as that of the last Chapter (4.2.3), and it will be clear which definition is used given the context.

Let us first make some comments on what we are going to ask the generalized solution to be.

Let $\{u_n\}_n$ be a sequence of weak solutions of (5.9) such that $u_n \to u$ weakly in H^1. From the equality

$$\int_\Omega \Delta u_n \varphi dx + \int_\Omega g_n(u_n)\varphi dx = \int_\Omega f\varphi dx \qquad \forall \varphi \in H$$

we deduce that the operator $A : H \to \mathbb{R}^N$ given by

$$A\varphi = \lim_{n\to\infty} \int_\Omega g_n(u_n)\varphi dx$$

is well defined and continuous, that is: $A \in H^{-1}$ (refer to Chapter 1 to the basics of the Dual Sobolev Spaces). In fact,

$$A\varphi = \int_\Omega f\varphi dx + \sum_{j=1}^{d} \int_\Omega \nabla u^j \nabla \varphi^j dx.$$

We may regard it as a pair $(f, \nabla u) \in H^{-1}$, namely

$$A\varphi := (f, \nabla u)[\varphi].$$

Thus, we are able to define the operator $\mathcal{G} : H \to H^{-1}$ by

$$\mathcal{G}(u) := (f, \nabla u); \qquad i.e. \qquad \mathcal{G}(u)[\varphi] = A\varphi. \tag{5.11}$$

Remark 5.3.2. As shown in Chapter 4, it is always possible to find approximations in such a way that $u \equiv 0$, this is why we need to exclude this case in the definition of generalized solution.

Indeed, for $\lambda > 0$ let G_λ be the Green's Function associated to the operator $-\Delta u + \lambda u$ for the nonlocal boundary conditions. Defining the function $c(\lambda) := \sup_{x\in\Omega} \|G_\lambda(x,\cdot)\|_{L^1}$, then $c(\lambda)$ is well defined and tends to 0 as $\lambda \to +\infty$. Next, define g_n in such a way that

$$g_n(u) = \begin{cases} g(u) & \text{if } |u| \geq \frac{2}{n} \\ -\lambda_n u & \text{if } |u| \leq \frac{1}{n} \end{cases},$$

with λ_n satisfying $c(\lambda_n)\|f\|_{L^\infty} \leq \frac{1}{n}$ for all n. Let u_n be the unique solution of the linear problem $\Delta u - \lambda_n u = f$ satisfying the nonlocal boundary conditions, then

$$|u_n(x)| = \left|\int_\Omega G_{\lambda_n}(x,y)f(y)\,dy\right| \leq c(\lambda_n)\|f\|_{L^\infty} \leq \frac{1}{n}.$$

Thus, u_n is a solution of (5.9) and $u_n \to 0$ uniformly.

A similar statement was done in (4.3.10) for the periodic problem. Here the argument it is slightly different:

Let $G_\lambda(x,y)$ the Green function associated to $-\Delta u + \lambda u$ and let us see that $\sup_{x\in\Omega} \|G_\lambda(x,\cdot)\|_{L^1} \to 0$ for $\lambda \to +\infty$.

It can be seen that $G_\lambda > 0$, and this implies that

$$\varphi = \frac{G}{|G|} = 1 \in \mathbb{R}^N.$$

Then, the norm we wanted to calculate is the unique solution of problem $-\Delta u + \lambda u = 1$ with the boundary condition. This is indeed the constant function $u \equiv 1/\lambda$. The order of convergence to 0 is then $1/\lambda$.

Also, observe that if u does not vanish in Ω then for any $\varphi \in H$, then

$$\mathcal{G}(u)[\varphi] = A\varphi = \lim_{n\to\infty} \int_\Omega g_n(u_n)\varphi dx = \int_\Omega g(u)\varphi dx.$$

Definition 5.3.3. A *generalized solution* is a nontrivial distributional solution of the equation

$$\Delta u + \mathcal{G}(u) = f.$$

We now state the main result of this section.

Theorem 5.3.4 (Amster, M. VII)**.** *Let $g \in C(\mathbb{R}^N\backslash\{0\}, \mathbb{R}^N)$ satisfying the boundary condition (B), repulsive at the origin (4.7), sequentially strongly repulsive at the origin (4.17) and let $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\overline{f} = 0$. Suppose that (P_1) and (P_2) hold and let $\{g_n\}_n$ be as in (4.10).*

Then there exist a sequence $\{u_n\}_n$ of solutions of (5.9), a positive constant $\tilde{r}$ such that $\|u_n\|_{L^\infty} \geq \tilde{r}$ and a subsequence of $\{u_n\}_n$ that converges weakly in H^1 to some function u.

If furthermore, the singularity is strongly repulsive (5.10), then u is a generalized solution of the problem.

In order to prove this theorem, firstly let us state an existence result for the approximated problems.

Proposition 5.3.5. *Let $\Omega \subset \mathbb{R}^d$ a bounded domain with $\partial\Omega \in C^2$. Let $g \in C(\mathbb{R}^N\backslash\{0\}, \mathbb{R}^N)$ satisfying (B), (4.7), (4.17) and let $f \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\overline{f} = 0$. Suppose that (P_1) and (P_2) hold and let $\{g_n\}_n$ be as in (4.10). Then there exist $\{u_n\}_n$ solutions of (5.9) and a constant $\tilde{r} > 0$ such that $\|u_n\|_{L^\infty} \geq \tilde{r}$.*

<u>*Proof*</u>:
With the help of (4.17), fix $\tilde{r} > 0$ such that

$$\left\langle g(u), \frac{u}{|u|} \right\rangle + \|f\|_{L^\infty} < 0 \text{ for } |u| = \tilde{r}, \tag{5.12}$$

and $n_0 \in \mathbb{N}$ such that $g_n \equiv g$ in $\mathbb{R}^N\backslash B_{\tilde{r}}(0)$ for $n \geq n_0$. As before, we shall apply once again Mawhin's Continuation Method, now over the set

$$U := \{u \in C(\overline{\Omega}, \mathbb{R}^N) : \tilde{r} < \|u\|_{L^\infty} < R\}$$

for some $R > \tilde{r}$ to be specified.

Suppose that for some $\lambda \in (0, 1)$ there exists $u \in \partial U$ a solution of (5.6) with $\hat{g} = g_n$.

If $\|u\|_\infty = \tilde{r}$, then we may fix x_0 such that $\|u\|_\infty = |u(x_0)| = \tilde{r}$ and define $\phi(x) := \frac{|u(x)|^2}{2}$.

As $g_n(u(x_0)) = g(u(x_0))$, if $x_0 \in \Omega$ then it can be seen that

$$\Delta\phi(x_0) = |\nabla u(x_0)|^2 + \langle u(x_0), \Delta u(x_0)\rangle \geq \langle u(x_0), \lambda(f(x_0) - g(u(x_0)))\rangle =$$

$$= \lambda \left[\langle u(x_0), f(x_0)\rangle - |u(x_0)|\langle g(u(x_0)), \frac{u(x_0)}{|u(x_0)|}\rangle \right] \geq$$

$$\geq \lambda\tilde{r} \left[-\|f\|_{L^\infty} - \left\langle g(u(x_0)), \frac{u(x_0)}{|u(x_0)|} \right\rangle \right] > 0,$$

a contradiction.

If $x_0 \in \partial\Omega$, then $\tilde{r} = |C|$. Moreover,

$$\int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} dS = \int_{\partial\Omega} \left\langle u, \frac{\partial u}{\partial\nu} \right\rangle dS = \left\langle C, \int_{\partial\Omega} \frac{\partial u}{\partial\nu} dS \right\rangle = 0. \tag{5.13}$$

From the continuity of ϕ, arguing as before we deduce that, $\Delta\phi > 0$ in $B_{2\delta}(x_0) \cap \Omega$ for some $\delta > 0$.

From standard regularity theory, it follows that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Moreover, we may consider a C^2 domain $\Omega_0 \subset \Omega$ such that $B_\delta \cap \Omega \subset \Omega_0$ and $\Omega_0 \subset B_{2\delta} \cap \Omega$; then $\phi(x_0) > \phi(x)$ for every $x \in \Omega_0$, and from Hopf's Lemma (see (1.1.18) in Chapter 1) we obtain

$$\frac{\partial \phi}{\partial \nu}(x_0) > 0.$$

As $u \equiv C$ on the boundary, then $|u(x)| \equiv \tilde{r}$ and so $\frac{\partial \phi}{\partial \nu}(x) > 0$ for each $x \in \partial\Omega$. This contradicts (5.13) and thus $\|u\|_{L^\infty} = R$.

Also, $\|u - \overline{u}\|_{L^\infty} < r$, then from condition (P_1) we deduce (D_1) with $D = B_R(0)$ when R is sufficiently large.

Indeed, assume that (P_1) holds and fix a positive constant $c < c_j$ for all j and R_0 such that

$$\langle g(Ru), w_j\rangle < -c, \qquad \text{for all } u \in U_j, R \geq R_0.$$

In particular, for $|v| = R$ with $R > R_0 + r$ large enough, there exists $j \in \{1, \ldots, J\}$ such that if $z \in \overline{B_r(v)}$ then $\frac{z}{|z|} \in U_j$, and $\langle g(z), w_j\rangle \leq -c$.

By taking the hyperplane $v + < w_j >^{\perp}$, with $v = -\alpha w_j$, $0 < \alpha << 1$, it is clear that it separates 0 and $g(\overline{B_r(v)})$. Thus, condition (D_1) holds for $D = B_R(0)$. Then, using the same arguments as in Theorem 5.2.2, it follows that $\|u\|_\infty < R$.

Finally, observe that the repulsiveness condition implies that the degree

$$\deg(g_n, B_{\tilde{r}}(0), 0) = (-1)^N$$

so, by the excision property of the degree, condition (P_2) ensures that

$$\deg(g_n, \{\tilde{r} < |u| < R\}, 0) \neq 0$$

and so, the proof is complete.

■

The following Lemma shows that the solutions of the perturbed problems are also bounded for the H^1 norm.

Lemma 5.3.6. *In the situation of Proposition 5.3.5, there exists a constant $\mathfrak{C}$ independent of n such that $\|u_n\|_{H^1} \leq \mathfrak{C}$ for all n.*

Proof:
As $\Delta u_n + g_n(u_n) = f(x)$ in Ω and $u_n \equiv C_n$ on $\partial\Omega$, we may multiply by $u_n - C_n$ and integrate to obtain:

$$\int_\Omega \langle \Delta u_n + g_n(u_n), u_n - C_n\rangle \, dx = \int_\Omega \langle p, u_n - C_n\rangle \, dx.$$

Integrating by parts, the left hand side is equal to:

$$-\int_\Omega |\nabla u_n|^2 \, dx + \int_{\partial\Omega} \left\langle \frac{\partial u_n}{\partial \nu}, u_n - C_n \right\rangle dS + \int_\Omega \langle g_n(u_n), u_n - C_n\rangle \, dx$$

As $u_n \equiv C_n$ on $\partial\Omega$, it follows that

$$\|\nabla u_n\|_{L^2}^2 = \int_\Omega \langle g_n(u_n), u_n - C_n\rangle \, dx - \int_\Omega \langle p, u_n - C_n\rangle \, dx.$$

Now, taking absolute value and using the Cauchy-Schwarz inequality, we get

$$\|\nabla u_n\|_{L^2}^2 \leq \left| \int_\Omega \langle g_n(u_n), u_n - C_n\rangle \, dx \right| + \|p\|_{L^2} \|u_n - C_n\|_{L^2}.$$

Let c be the constant in condition (4.7) and write:

$$\left| \int_\Omega \langle g_n(u_n), u_n - C_n\rangle \, dx \right| \leq \left| \int_{\{|u_n|<c\}} \langle g_n(u_n), u_n - C_n\rangle \, dx \right| + \\ + \left| \int_{\{|u_n|\geq c\}} \langle g_n(u_n), u_n - C_n\rangle \, dx \right|$$

Fix $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < c$ for every $n \geq n_0$. Then we have that $g_n(u_n(x)) = g(u_n(x))$ if $|u_n(x)| > c > \frac{1}{n}$ and hence, on the one hand

$$\left| \int_{\{|u_n|\geq c\}} \langle g_n(u_n), u_n - C_n\rangle \, dx \right| \leq |\Omega|^{1/2} \gamma_c \|u_n - C_n\|_{L^2},$$

where $\gamma_c := ess \sup_{|u|>c} |g(u)|$ and, on the other hand:

$$\int_{\{|u_n|<c\}} \langle g_n(u_n), u_n - C_n\rangle \, dx \leq - \int_{\{|u_n|<c\}} \langle g_n(u_n), C_n\rangle \, dx.$$

Moreover, as $\int_\Omega g_n(u_n) \, dx = 0$, we deduce that

$$\int_{\{|u_n|<c\}} \langle g_n(u_n), u_n - C_n\rangle \, dx \leq \left\langle C_n, \int_{\{|u_n|\geq c\}} g_n(u_n) dx \right\rangle \leq |\Omega|^{1/2} \gamma_c |C_n|.$$

Gathering all together,

$$\left| \int_\Omega \langle g_n(u_n), u_n - C_n\rangle \, dx \right| \leq |\Omega|^{1/2} \gamma_c \left(\|u_n - C_n\|_{L^2} + |C_n| \right).$$

Thus,

$$\|\nabla u_n\|_{L^2}^2 \leq \mathfrak{C}_1 \|u_n - C_n\|_{L^2} + \mathfrak{C}_2 |C_n|,$$

for some constants $\mathfrak{C}_1, \mathfrak{C}_2$. Using Poincaré inequality, we deduce the existence of a constant $\mathfrak{C}$ such that

$$\|\nabla u_n\|_{L^2}^2 \leq \mathfrak{C}|C_n|$$

and hence

$$\|u_n - C_n\|_{H^1}^2 \leq A + B|C_n| \qquad \text{for some } A, B > 0.$$

Suppose that $|C_n|$ is unbounded, then taking a subsequence (still denoted C_n) we may assume that $|C_n| \to +\infty$, $\frac{C_n}{|C_n|} \to \eta \in S^{N-1}$. From the inequality

$$\left\| \frac{u_n - C_n}{\sqrt{|C_n|}} \right\|_{H^1}^2 \leq \frac{A}{|C_n|} + B \qquad \forall n \geq n_0,$$

we may take again a subsequence and thus assume that $\frac{u_n - C_n}{\sqrt{|C_n|}}$ converges almost everywhere and weakly in H^1 to some $w \in H^1$.

Let $\varepsilon > 0$ and fix M large enough so that $|\Omega \backslash \Omega_M| < \varepsilon$, where

$$\Omega_M := \{x \in \Omega : |w(x)| \leq M\}.$$

Then $\frac{u_n - C_n}{|C_n|} \to 0$ and $\frac{u_n}{|u_n|} \to \eta$ almost everywhere in Ω_M.

Fix $U_k \subset S^{N-1}$ as in (P_1) such that $\eta \in U_k$, then writing

$$\langle g(u_n(x)), w_k \rangle = \left\langle g\left(|u_n(x)| \frac{u_n(x))}{|u_n(x)|} \right), w_k \right\rangle,$$

we deduce that

$$\limsup_{n\to\infty} \langle g(u_n(x)), w_k \rangle \leq -c_k,$$

almost everywhere in Ω_M. Thus we obtain, from Fatou's Lemma:

$$\limsup_{n\to\infty} \int_{\Omega_M} \langle g(u_n(x)), w_k \rangle \, dx \leq \int_{\Omega_M} \limsup_{n\to\infty} \langle g(u_n(x)), w_k \rangle \, dx \leq -c_k|\Omega_M|.$$

We may assume that $M \geq c$, then taking $\varepsilon < \frac{c_k|\Omega|}{\gamma_c}$ we conclude:

$$\limsup_{n\to\infty} \int_{\Omega} \langle g(u_n(x)), w_k \rangle \, dx \leq -c_k|\Omega_M| + \limsup_{n\to\infty} \int_{\Omega \backslash \Omega_M} \langle g(u_n(x)), w_k \rangle \, dx$$

$$\leq -c_k|\Omega_M| + \gamma_c |\Omega \backslash \Omega_M| < 0,$$

which contradicts the fact that $\int_\Omega g(u_n(x)) \, dx = 0$.

■

We now are in condition to give a proof of Theorem 5.3.4.

Proof of Theorem 5.3.4:
From the preceding results, there exists a sequence (still denoted $\{u_n\}_n$) of solutions of the approximated problems converging almost everywhere and weakly in H^1 to some function u, and also such that $\|u_n\|_\infty \geq \tilde{r}$. It remains to prove that if (5.10) holds then $u \not\equiv 0$.

Suppose that $u \equiv 0$, then from (5.9) we obtain

$$\int_\Omega \langle \Delta u_n(x), u_n(x)\rangle + \langle g(u_n(x)), u_n(x)\rangle \, dx = \int_\Omega \langle p(x), u_n(x)\rangle \, dx \to 0,$$

as $n \to \infty$. Moreover,

$$\int_\Omega \langle \Delta u_n(x), u_n(x)\rangle \, dx = -\int_\Omega |\nabla u_n(x)|^2 \, dx$$

is bounded, and from (5.10) and Fatou's Lemma we obtain

$$\limsup_{n\to\infty} \int_\Omega \langle g(u_n(x)), u_n(x)\rangle \, dx \leq \int_\Omega \limsup_{n\to\infty} \langle g(u_n(x)), u_n(x)\rangle \, dx = -\infty,$$

a contradiction.

We have proved that u is in fact a generalized solution (5.3.3) of the problem.

■

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