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Resumen

Este trabajo se centra en el estudio de la conjetura de Whitehead y de la asfericidad de los complejos LOT, aplicando nuevos métodos y herramientas basados principalmente en la teoría de espacios topológicos finitos.

Sea L un complejo asférico de dimensión 2. Los subcomplejos de L son también asféricos? Esta pregunta fue formulada por J. H. C. Whitehead en 1941 [Whi41], y todavía no tiene respuesta. Entre los avances más importantes en este tema está el teorema de J. Howie a partir del cual el problema se separa en dos casos donde se consideran, respectivamente, complejos compactos y no compactos [How83]. A partir de este resultado, traducimos el caso compacto al contexto de los espacios finitos, y podemos atacar el problema con un enfoque nuevo, distinto de las estrategias aplicadas hasta ahora.

Los complejos LOT (*labeled oriented tree*) aparecen en el estudio de ciertas variedades que surgen como complementos de los llamados ribbon discs. Howie probó que si un complejo se 3-deforma a un punto, entonces el subcomplejo que surge de quitarle una 2-celda se 3-deforma a un complejo LOT [How83]. Es por esto que los complejos LOT forman un nexo entre la conjetura de Whitehead y la conjetura de Andrews-Curtis. Por otro lado, todo complejo LOT se puede ver como subcomplejo de un complejo contráctil. Es por esto que los complejos LOT son considerados casos testigos de la conjetura de Whitehead.

La teoría de espacios finitos comenzó en los años 30 con un trabajo de P. S. Alexandroff donde se los relaciona con los conjuntos parcialmente ordenados (posets) finitos [Ale37]. Esta teoría tuvo un avance importante en el año 1966 con el trabajo de M. C. McCord a partir del cual se constituyen como modelos combinatorios de poliedros [McC66]. En los últimos años, J. A. Barmak y E. G. Minian hicieron importantes avances en esta teoría. Entre otros aportes, desarrollaron la teoría de homotopía simple para espacios finitos, y aplicaron los espacios finitos al estudio de las conjeturas de Quillen y de Andrews-Curtis [BM07, BM08b, BM08a, Bar11].

En este trabajo desarrollamos nuevos métodos combinatorios de espacios finitos, diseñados espacialmente para el estudio de la asfericidad. Probamos la validez de la conjetura para dos amplias familias de poliedros compactos: los cuasi construibles contráctiles, y los fuertemente asféricos. Utilizando resultados recientes de Barmak y Minian sobre G-coloreos de posets, obtenemos una descripción eficiente del segundo grupo de homotopía de un complejo LOT, como un submódulo de un módulo libre, con generadores indexados por las aristas, y ecuaciones indexadas por los vértices. A partir de esta descripción, hallamos un método para el análisis de la asfericidad de estos complejos. Con este nuevo método se obtenemos resultados sobre la asfericidad de importantes familias de LOTs.

Palabras clave: CW-complejos, asfericidad, complejos LOT, espacios topológicos finitos, métodos de reducción.

A new approach to the Whitehead conjecture and the asphericity of LOT complexes

Abstract

This work is focused on the study of the Whitehead conjecture and the asphericity of LOT complexes, by applying new methods based on the theory of finite topological spaces.

Is every subcomplex of an aspherical, two-dimensional complex itself aspherical? This question was stated by J. H. C. Whitehead in 1941 [Whi41], and it sill hasn't been answered. Using a result of J. Howie [How83] one can treat separately the compact and the non-compact case (both are open and interesting). We concentrate on the compact case using, among other tools, methods of the theory of finite topological spaces.

LOT complexes appear in the study of certain manifolds that arise as complements of the so called ribbon discs. Howie proved that if a CW-complex can be 3-deformed to a point, then the subcomplex obtained by eliminating a 2-cell can be 3-deformed to a LOT complex [How83]. This is why these complexes constitute a link between the Whitehead conjecture and the Andrews-Curtis conjecture. On the other hand, every LOT complex can be seen as a subcomplex of a contractible complex. This is why LOT complexes are considered test cases for the Whitehead conjecture.

The theory of finite spaces started in 1937 with a work of P. S. Alexandroff, who related them with finite partially ordered sets (posets) [Ale37]. This theory had an important breakthrough in 1966 with the work of M. C. McCord, which established them as combinatorial models for compact polyhedra [McC66]. In the last few years, J. A. Barmak and E. G. Minian made important advances in this theory. Among other contributions, they developed the simple homotopy theory of finite spaces, and applied methods of finite spaces to the study of the conjecture of Quillen and to the Andrews-Curtis conjecture [BM07, BM08a, BM08b, Bar11].

In the present work we develop new combinatorial methods of finite spaces, specially designed for the study of asphericity. We prove the validity of the conjecture for two large families of compact polyhedra: the contractible quasi-constructible complexes, and the strong aspherical complexes. Making use of recent results of Barmak and Minian about G-colorings of posets, we obtain an efficient description of the second homotopy group of a LOT complex, as a submodule of a free $\mathbb{Z}[\pi_1]$-module, with generators indexed by the edges, and equations indexed by the vertices. Using this description, we find a method for the analysis of the asphericity of these complexes. With these new methods we obtain results about the asphericity of important families of LOTs.

Kew words: CW-complexes, asphericity, LOT complexes, finite topological spaces, reduction methods.

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Contents

Introduction

In the early 1940s J. H. C. Whitehead studied the relation between the homotopy groups $\pi_n(X)$ and $\pi_n(Y)$ for spaces X, Y such that Y is obtained from X by the adjunction of n-cells. Based on this study, he stated the following question [Whi41]

Is every subcomplex of an aspherical, two-dimensional complex itself aspherical?

This question is usually treated as a conjecture, and it still has not been answered.

A path-connected topological space X is said to be aspherical if its homotopy groups $\pi_n(X)$ are trivial for $n > 1$. In the case of a CW-complex, this happens if and only if its universal cover $\tilde{X}$ is contractible. If in addition X is a 2-complex (a 2-dimensional CW-complex), this condition is equivalent to requiring $\pi_2(X) = 0$ (or $\pi_2(\tilde{X}) = H_2(\tilde{X}) = 0$).

Important progress has been achieved in the problem of Whitehead, by means of different methods. One of the tools that have been applied is the theory of crossed modules, introduced by Whitehead himself [Whi41, Whi46]. The crossed module structure is an algebraic systematization of the properties of the action of $\pi_1(X)$ in $\pi_2(Y, X)$ for a pair of spaces $X \subseteq Y$. Whitehead [Whi49] proved that if Y is obtained from X by attaching 2-cells, then $\pi_2(Y, X)$ is a free $\pi_1(X)$-crossed module, with a basis in correspondence with the attached 2-cells. This result was applied in many later works on the conjecture. For example, Cockcroft used it in conjunction with a theorem of Lyndon to prove the conjecture in the case where the subcomplex K has only one 2-cell [Coc54]. Brown and Higgins [BH78], and later Brown [Bro80, Bro84] worked in this theory, obtaining an efficient description of the crossed module associated to a union of 2-complexes with a common 1-skeleton, introducing the notion of the coproduct of crossed modules. Gutierrez and Ratcliffe also contributed to the conjecture by applying methods of crossed modules [GR81].

The *Simple Identity Theorem* of R. C. Lyndon [Lyn50], stated in the context of group presentations, was a fundamental contribution to the study of asphericity of 2-complexes. This is because of the application made by W. H. Cockcroft [Coc54], who used it to prove the conjecture in the case where the subcomplex has only one 2-cell.

Theorem 2.4.2. *If K is a finite connected non-aspherical 2-complex with only one 2-cell, then the complex L obtained from K by attaching 2-cells to the 1-skeleton of K is non-aspherical.*

Lyndon's theorem involves presentations with a single relator, and the presentations corresponding to complexes with a single 2-cell are of this kind. Howie generalized these results [How82] by means of the notion of a reducible presentation, and of a reducible complex. These presentations have similar properties to those with a single relator.

Several partial answers to the conjecture involve hypotheses on the fundamental group of the subcomplex K of L. Cockcroft obtained another important advance in his article [Coc54] when he proved the conjecture in the case where $L-K$ has only 2-cells and $\pi_1(K)$ is free, abelian, or finite. To do this, he applied the classical formulas of Hopf about homology and homotopy of groups. He also proved that if G is a free group, then for every 2-complex K having $\pi_1(K) = G$, the condition $H_2(K) = 0$ implies $\pi_2(K) = 0$. This result was later generalized to a large class of groups: the conservative groups of Adams [Ada55], which turned out to be the locally indicable groups of Howie [How82] after the results of Howie and Schneebeli [HS83].

The methods applied in the proofs of these results involve constructions of covering projections and *towers*, that is, functions which can be factorized into covering projections and inclusions of complexes. The tower methods where introduced and applied frequently by Howie [How79, How81b, How81a]. The following theorem of Howie [How83] separates the Whitehead conjecture into two particular cases, in which compact and non-compact complexes are respectively considered.

Theorem 2.5.4. *If the answer to Whitehead's question is* negative*, then there exists a counterexample $K \subset L$ of one of the following two types:*

(a) L is finite and contractible, $K = L - e$ for some 2-cell e of L, and K is non-aspherical.

(b) L is the union of an infinite chain of finite non-aspherical subcomplexes $K = K_0 \subset K_1 \subset \cdots$ such that each inclusion map $K_i \to K_{i+1}$ is nullhomotopic.

The articles of Bogley [Bog93] in 1993 and of Rosebrock [Ros07] in 2007 are valuable reviews of the advances in the Whitehead conjecture.

In the present work we approach the conjecture combining the known techniques which have been applied until now to the problem (coming from topology, geometry and algebra) with new tools arising from the homotopy theory of finite spaces.

The theory of finite spaces starts in 1937 with the work of P. S. Alexandroff [Ale37], who described them from a combinatorial point of view, comparing them with the finite partially ordered sets (posets). For more information about this theory we refer the reader to [Bar11, BM08b, May03c, McC66, Sto66]. A *finite topological space* is a space where the underlying set is finite. These spaces have a natural relation with finite posets.

Given a finite space X, we define a relation as follows

$$x \leq y \Leftrightarrow x \in U \quad \text{for every open set } U \text{ that contains } y.$$

It is easy to see that this relation is a preorder (i.e. a transitive and reflexive relation) for every topology. However, its antisymmetry is equivalent to the topological T_0 property: for every pair of elements of X there is an open set that contains only one of them.

Similarly, if $\leq$ is an order relation on X, then the sets

$$U_x := \{y \in X : \; y \leq x\} \subseteq X$$

form a basis for a topology on X that satisfies the T_0 axiom. In this work we always assume that finite spaces fulfil this axiom.

The relevance of finite spaces is based on the fact that they serve as models for all compact polyhedra, after the results of M. C. McCord in 1966. Given a finite space X, the *order complex* $\mathcal{K}(X)$ associated to X is a simplicial complex with vertex set X and whose simplices are the chains of elements of X (i.e. totally ordered subsets). Given a simplicial complex K, the *face poset* $\mathcal{X}(K)$ associated to K is the set of simplices of K ordered by inclusion.

Theorem 3.1.9. *Given a finite space X, there is a weak homotopy equivalence between $\mathcal{K}(X)$ and X. Analogously, given a simplicial complex K, there is a weak homotopy equivalence between K and $\mathcal{X}(K)$.*

As a consequence, the weak homotopy types of finite spaces cover all weak homotopy types of compact polyhedra.

R. E. Stong [Sto66], via the notion of *beat point*, found an effective method to decide whether two finite spaces are homotopy equivalent. Beat points constitute the first of the reduction methods for finite spaces, which were later developed further. Concretely,

Theorem. *Two finite spaces are homotopy equivalent if and only if one can be obtained from the other by adding and removing beat points.*

This result is a corollary of proposition 3.1.12, and of the classification of *minimal* finite spaces (see [Bar11] for more details).

In the year 2003, J. P. May compiled everything known on this subject until that moment in a series of notes [May03a, May03b, May03c], where he also stated conjectures that relate the homotopy theory of finite spaces with classical homotopy theory. In the last few years, J. A. Barmak and E. G. Minian resumed this theory, solving some of the conjectures of May. One of their fundamental contributions was built upon the notion of *weak point*, which generalizes that of beat points. Using this notion they developed the simple homotopy theory for finite spaces. An element x of a finite space X is said to be a weak point if the subspace $\hat{C}_x = \{y \in X : y < x \text{ or } y > x\}$ of X is contractible. Given two finite spaces X and Y, Y is said to be *simply equivalent* to X if it can be obtained from X by adding or removing weak points.

Theorem. *[BM08b] Given two finite spaces X and Y, X is simply equivalent to Y if and only if $\mathcal{K}(X)$ is simply equivalent to $\mathcal{K}(Y)$. Given two simplicial complexes K and L, K is simply equivalent to L if and only if $\mathcal{X}(K)$ is simply equivalent to $\mathcal{X}(L)$.*

The results obtained for finite spaces were applied to the study of the Quillen conjecture and the Andrews-Curtis conjecture [Bar11]. The Andrews-Curtis conjecture was proved for quasi-constructible complexes, a class of complexes that includes constructible complexes. In order to do this, the equivalent class of finite spaces was defined, the *qc-reducible* spaces. In the present work we show that qc-reducible spaces also satisfy the Whitehead conjecture.

Basing ourselves in the result of Howie 2.5.4, we translated the compact case of the conjecture in terms of finite spaces. In this way we can apply methods of finite spaces to the study of this problem.

Conjecture 3.3.1. *Let X be a homotopically trivial finite space of height 2 and let $a \in X$ be a maximal point such that $X - a$ is connected. Then $X - a$ is aspherical*

If the conjecture 3.3.1 is true, then there are no counterexamples of type (a) of Howie's theorem. On the other hand, if the Whitehead conjecture holds, then so does conjecture 3.3.1.

Note that the original question of Whitehead is equivalent, in the case of compact polyhedra, to asking whether every subspace of an aspherical finite space of height 2 is also aspherical.

One of the main results of chapter 3 is the validity of this conjecture for the class of qc-reducible spaces. To show this we combine combinatorial methods of finite spaces with the result of Cockcroft which proves the conjecture in the case that $\pi_1(K)$ is free.

Theorem 3.4.4. *[CM14] Let X be a qc-reducible finite space of height 2 and let $a \in X$ be a maximal point such that $X - a$ is connected. Then $X - a$ is aspherical.*

This implies that the Whitehead conjecture is true for all quasi-constructible complexes, a large class of complexes that includes constructible complexes.

We also introduce a new reduction method, based on the notion of a-points, which serves us to prove the conjecture for a new class of complexes. Recall that given an element $x \in X$, we denote by $\hat{C}_x$ the subspace $\hat{C}_x = \{y \in X : y < x \text{ or } y > x\}$ of X. An element is an *a-point* if $\hat{C}_x$ is a disjoint union of contractible spaces. In this case, the transformation $X \mapsto X - x$ is called an *a-reduction*. A finite space X is said to be *strong aspherical* if there exists a sequence of a-reductions that transforms X to a point.

Proposition 3.5.2. *[CM14] Let X be a finite space and let $a \in X$ be an a-point. Then X is aspherical if and only if $X - a$ is aspherical. In particular, strong aspherical spaces are aspherical.*

This result provides a new tool for the study of asphericity of finite spaces. We can manipulate a given space adding or removing a-points, and analyze the asphericity of the resulting space. Another important result of chapter 3 is the following.

Theorem 3.5.5. *[CM14] Let X be a strong aspherical space of height 2 and let $a \in X$. Then $X - a$ is strong aspherical.*

Theorem 3.5.6. *The answer to Whitehead's original question is positive for strong aspherical finite spaces of height 2.*

As a consequence, we have that the answer to Whitehead's question is positive for every 2-dimensional simplicial complex L whose face poset is strong aspherical. A simplicial complex L satisfies this condition if and only if L collapses to a 1-dimensional subcomplex.

In this work we introduce further new combinatorial methods developed for the study of asphericity, in addition to the method of a-points. These methods arise from generalizations and variations of the methods of beat points, weak points, and qc-reductions. Some previous generalizations can be found in [Cer10, Fer11]. The methods developed in the present work involve transformations in which the weak homotopy type of the space is changed, but its asphericity is preserved. That is, if the space Y is obtained by applying one of these moves to a space X, then Y is aspherical if and only if X is. However, the homotopy groups of X and Y are not in general the same. These moves are more flexible than the previous ones, and provide a larger range of spaces on which the asphericity of a given space can be analyzed. We also show some programs that were developed for a computational implementation of these methods by means of the software SAGE [S⁺15]. The first programs constructed with this purpose can be found in [Fer11].

The second problem treated in this thesis is the conjecture about the asphericity of complexes associated to LOTs (labeled oriented trees), which establish a connection between the Whitehead conjecture and other problems in algebraic topology.

These complexes appear in the study of certain manifolds that arise as complements of the so called ribbon discs, in a generalization of classical knot theory. A *ribbon disc* is a proper immersion of an n-dimenional disc into an $(n+2)$-dimensional disc $D^n \hookrightarrow D^{n+2}$ such that the map $r : D^{n+2} \to \mathbb{R},\ x \mapsto \|x\|$ restricts to a Morse function on the disc D^n. A *ribbon disc complement* is the complement of the image of D^n in D^{n+2} for such an immersion. The asphericity of knot complements was an open conjecture for a long time, and it was finally proved by Papakyriakopoulos [Pap57] using methods of 3-manifolds. This problem was probably one of Whitehead's motivations when he stated his conjecture, a proof of which would imply an alternative proof for the asphericity of knot complements.

It is currently conjectured that ribbon disc complements are also aspherical. The homotopy of these spaces was studied by numerous mathematicians [Yan69, AMY81, Has83, How85, HR03]. We note that there are some incomplete proofs of this conjecture in the literature (see [How83] for further details).

Howie proved in [How83] a correspondence between ribbon disc complements and LOT complexes. A *labeled oriented tree*, or LOT, is an oriented graph without cycles, additionally equipped with a map λ from the set of edges to the set of vertices, called *label*. This means that every edge of a LOT has a source, a target, and a label.

To every LOT Γ we associate a group presentation $P(\Gamma)$ with a generator for every vertex and a relator for every edge. If an edge e has source s_e, target t_e and label λ_e, it induces the relator $\lambda_e t_e \lambda_e^{-1} s_e^{-1}$, to that in the presented group we have $s_e\lambda_e = \lambda_e t_e$. In the usual way, to such a presentation we associate a 2-complex K_Γ, which we call a *LOT complex*.

Howie associates a LOT Γ to every ribbon disc in such a way that K_Γ has the (simple) homotopy type of the ribbon disc complement [How85]. Therefore to study the asphericity of these spaces, we study that of the LOT complexes. In fact, every LOT can be associated to a ribbon disc complement, so the asphericity of all ribbon disc complements is equivalent to the asphericity of all LOT complexes.

Howie proved that if a 2-complex L can be 3-deformed to a point, and e is a 2-cell of L, then the subcomplex $K = L - e$ can be 3-deformed to a LOT complex. A consequence of this fact is that if the Andrews-Curtis conjecture were true, then the asphericity of LOT complexes would imply the compact case of the Whitehead conjecture. Recall that the Andrews-Curtis conjecture asserts that every contractible 2-complex can 3-deformed to a point (see [AC65]). Conversely, it is easy to see that every LOT complex can be transformed to a contractible complex by the addition of a 2-cell. This is why LOT complexes are considered test cases of the Whitehead conjecture.

Howie [How85] obtained relevant results on this problem. Among other contributions, he showed that to prove the asphericity of all LOTs it is sufficent to consider the case in which the underlying graph is an interval, that is, it has only 2 extremal points. He also proved the asphericity of LOTs of diameter ≤ 3 [How85].

In the last few years Huck, Rosebrock and Harlander made progress in this problem. They used combinatorial tools, such as the *spherical diagrams* as generators for the second homotopy group, and combinatorial notions of asphericity, such as that of a *diagrammatically reducible* space. Some results on the asphericity of LOTs can be found in [How85, HR01, Ros10, HR12b] and in the preprint [HR12a]. For more details on this subject we refer the reader to [Ros07].

In the present work we study the asphericity of LOTs applying as a main tool the recent results of Barmak and Minian about the fundamental group and the second homotopy group of 2-complexes in terms of colorings of the edges of the Hasse diagram of the associated poset [BM12a, BM14].

A *coloring* of a poset X is a map $c : \mathcal{E}(X) \to G$ from the set of edges of the Hasse diagram of X to a group G. The fundamental result of [BM12a] that we make use of in this work is that one can obtain a presentation of the fundamental group of any finite space X in the following way.

Let D be a subdiagram of the Hasse diagram of X that contains all the elements of X and such that D corresponds to a simply connected finite space (for example, a spanning tree of the Hasse diagram satisfies these conditions). Then a presentation of $\pi_1(X)$ is obtained by setting a generator for each edge of the Hasse diagram that is not in D, and a relator for every *simple digon*, that is, for every pair of monotonic edge paths that only meet in the extremal points.

Applying this result, and using a characterization of covering projections of a poset in terms of colorings obtained in the same article, they obtain in [BM14] a description of the universal covering space of a given space X. If, in addition, the space X is the poset associated to a regular CW-complex (for example, the face poset of a simplicial complex), they provide a description of the second homology group of the universal covering space of X, which is isomorphic to $\pi_2(X)$, using the results of [Min12].

Concretely, for the case of a regular 2-complex X with $\pi_1(X) = G$, they obtain the following description of $\pi_2(X)$ as a submodule of a free $\mathbb{Z}[G]$-module generated by the elements of degree 2, and with equations associated to the elements of degree 1. For a pair of elements $x \succ y$, $c(x,y)$ denotes the color of the edge (x,y) and $\epsilon(x,y)$ denotes the *incidence number* of x in y, which takes the values ± 1 depending on certain choices of orientation, similar to the usual geometric ones.

$$\pi_2(X) = \{ \sum_{\substack{\deg x = 2 \\ g \in G}} n_g^x g x : \ n_g^x \in \mathbb{Z} \text{ and } \sum_{x > y} \epsilon(x,y) n^x_{gc(x,y)} = 0 \quad \forall y \in X, \deg y = 1 \}.$$

We associate to every LOT Γ a LOT poset $\mathcal{X}(\Gamma)$ which is a finite model of K_Γ. In this way, we can study the asphericity of the LOT through the LOT poset, to which we can apply methods of finite spaces. We give a description of this poset, and we exhibit the code of a function which computes the LOT poset of a given LOT.

Starting from the LOT poset associated to a LOT Γ, we apply the results of Barmak and Minian to describe the second homotopy group of LOT complexes. In order to do that, we choose a specific subdiagram D of the Hasse diagram of a LOT poset, whose associated poset is a simply connected space. The presentation of the fundamental group we obtain is equivalent to the original LOT presentation. The description obtained for $\pi_2(\mathcal{X}(\Gamma))$ is the following.

Theorem 5.3.1. *Let Γ be a LOT with vertex set V and edge set E. Then the second homotopy group of Γ can be described as follows.*

$$\pi_2(\Gamma) = \{ \sum_{x \in E, g \in G} n_g^x g \overline{x} : \ n_g^x \in \mathbb{Z} : \quad \forall v \in V, \forall g \in G$$
$$- \sum_{s_x = v} n^x_{gv^{-1}} + \sum_{t_x = v} n^x_{gv^{-1}\lambda_x^{-1}} + \sum_{\lambda_x = v} (n^x_{gv^{-1}} - n^x_{gt_x^{-1}v^{-1}}) = 0 \}.$$

Using this description we achieve the main results obtained in this subject, which asserts the asphericity of a large class of LOTs.

Remark 5.4.2. Let Γ be a LOT and let $\sum_{x \in E, g \in G} n_g^x g \overline{x}$ be a fixed element of $\pi_2(\Gamma)$. Suppose that for an edge a of Γ we have an equation on the coefficients n_g^a of the form

$$n^a_{gu_1} + n^a_{gu_2} + \cdots + n^a_{gu_i} + \cdots + n^a_{gu_k} = 0 \quad \forall g \in G$$

and suppose that u_i has strictly greater weight than u_j for all $j \neq i$. Since there are finitely many non-zero coefficients, and assuming that not all of them are 0, we may choose ξ such that $n^a_\xi \neq 0$ and such that $w(\xi)$ is minimum. The equation above can be rewritten as

$$n^a_{hu_i^{-1}u_1} + n^a_{hu_i^{-1}u_2} + \cdots + n^a_h + \cdots + n^a_{hu_i^{-1}u_k} = 0 \quad \forall h \in G.$$

The equation must hold when $h = \xi$, but $w(\xi u_i^{-1} u_j) < w(\xi)$, so $n^a_{hu_i^{-1}u_j} = 0$ for all $j \neq i$. Thus the equation becomes $n^a_\xi = 0$, which is a contradiction.

Similarly the coefficients must be all 0 if there exists an u_i with strictly less weight than the others. Therefore it will be useful to analyze the weights of the subscripts in the coefficients of the elements of $\pi_2(\Gamma)$.

Using this observation, we develop a method for the study of asphericity of LOTs. With the equation of each vertex v we can express the coefficient associated to one of its incident edges in terms of the coefficients of the other incident edges, and those that are labeled by v.

In order to state the results we have to introduce some notation. If a given vertex is the label of one or more edges, we call it *label vertex*. A LOT is said to be *injective* if λ is an injective map, that is, if no vertex is a label of more than one edge. Note that in an injective LOT there is precisely one vertex that is not a label vertex.

If we fix a vertex a of Γ, called the center, then for every other vertex $v \in \Gamma$ there is one incident edge that belongs to the only path connecting a and v. We call this the *inner* edge of v. The other incident edges are called *outer* edges of v. A LOT is said to be *reduced* if one cannot perform any of the elementary moves defined by Howie in [How85] (see Definition 4.4.1). These operations preserve the simple homotopy type of the associated complex. This is why it suffices to consider reduced LOTs. in order to investigate the conjecture about their asphericity.

Theorem 5.4.3. *Let Γ be a reduced injective LOT and let the center a be the only vertex that is not an edge label. Suppose there is an order $v_1, \dots, v_n$ on the label vertices satisfying the following conditions.*

1. *For every $1 \le i \le n-1$, the inner edge of v_i (called x_i) is labeled by v_{i+1}.*
2. *The inner edge of v_n, x_n is labeled by v_1.*
3. *For every $1 \le i \le n$, the outer edges of v_i are labeled by previous vertices (that is, by vertices v_k with $k < i$).*

Then the second homotopy group $\pi_2(\Gamma)$ is generated as a $\mathbb{Z}[\pi_1(\Gamma)]$-module by a single element.

Applying this result, together with the previous observations, we can prove the asphericity of numerous LOTs. In order to generalize this result to non-injective LOTs, the ordering of the edges must be replaced by the following hypothesis.

There exists an enumeration of the edges of Γ into consecutive rows

$$\begin{array}{c} a_{11}, a_{12}, \dots, a_{1k_1} \\ a_{21}, a_{22}, \dots, a_{2k_2} \\ \dots \\ a_{r1}, a_{r2}, \dots, a_{rk_r}, \end{array}$$

satisfying the following rule. A new edge a_{ij} can be enumerated if it has an endpoint v_{ij} such that all the other incident edges and the edges labeled by v_{ij} were enumerated previously by a_{kl} with $k < i$ or $k = i, l < j$, or if the edge starts a new row (i.e. if $j = 1$).

With this hypothesis we obtain a similar result for non-injective LOTs, and also a large number of LOTs to which the method applies. However, in order to ensure that the method works without having to analyze the weights of the subscripts, we restrict the hypotheses and add a condition on the fundamental group of the LOT.

A group G satisfies the *unique product property* (UPP) if for every pair of non-empty subsets A, B of G there exists an element $g \in G$ that can be expressed in precisely one way as a product $g = ab$ with $a \in A$ and $b \in B$.

This property is weaker than the condition of being locally indicable, related to the problem of asphericity. If a LOT has locally indicable fundamental group, then it is aspherical. This is because locally indicable groups are conservative [HS83], and LOTs have the homology of the 1-sphere S^1 [How85].

A group G satisfies UPP if and only if for every pair of non-empty subsets X, Y of G there exists an element $g \in G$ such that $gX \cap Y$ has precisely one element. Combining this property with other hypotheses, we could prove the asphericity of certain classes of LOTs. It is not known whether the fundamental groups of LOTs are locally indicable, or if they satisfy UPP.

Definition. Let Γ be a reduced LOT. A *good enumeration* of Γ is an enumeration of the edges of Γ in consecutive rows

$$\begin{array}{c} a_{11}, a_{12}, \dots, a_{1k_1} \\ a_{21}, a_{22}, \dots, a_{2k_2} \\ \dots \\ a_{r1}, a_{r2}, \dots, a_{rk_r}, \end{array}$$

such that the following conditions are satisfied.

1. A new edge can be enumerated a_{ij} if it has one endpoint v_{ij} such that all the remaining incident edges, and the edges labeled by v_{ij} were enumerated a_{kl} with $k < i$ or $k = i, l < j$, or if it starts a new row (i.e. if $j = 1$).

2. For every $1 \le s \le r$ there exists a vertex w_s different from all the v_{ij} such that all the edges incident to w_s and all the edges labeled by w_s are enumerated a_{kl} with $k \le s$.

3. For every $1 \le s \le r$, the path in Γ between w_s and the edge a_{s1} is a sequence of edges a_{sj}. The vertices of the path have all their incident and labeled edges enumerated with $kl, k \le s$.

Theorem 5.4.14. *Let Γ be a reduced LOT which admits a good enumeration. If, in addition, $G(\Gamma)$ is UPP, then Γ is aspherical.*

This result implies the asphericity of a large new class of LOTs, and examples including non-injective LOTs of any diameter and complexity.

Introducción

A principios de los años 40, J. H. C. Whitehead estudió la relación entre los grupos de homotopía $\pi_n(X)$ y $\pi_n(Y)$ para espacios X e Y, donde Y se obtiene de X adjuntando n-celdas. A partir de ese estudio, formuló la siguiente pregunta

Sea L un 2-complejo asférico, y sea $K \subseteq L$. ¿Es K también asférico? [Whi41]

Esta pregunta es tratada usualmente como una conjetura, y todavía no fue respondida.

Un espacio topológico arcoconexo X se dice asférico si sus grupos de homotopía $\pi_n(X)$ son triviales para $n > 1$. Cuando se trata de un CW-complejo, esto es equivalente a que su revestimiento universal $\tilde{X}$ sea contráctil. Si además X es un 2-complejo (un CW-complejo de dimensión 2), es equivalente a que se cumpla $\pi_2(X) = 0$ (ó $\pi_2(\tilde{X}) = H_2(\tilde{X}) = 0$).

Se han obtenido importantes avances en el problema de Whitehead, utilizando diversas estrategias. Una de las herramientas que se aplicaron es la teoría de módulos cruzados, introducida por el mismo Whitehead [Whi41, Whi46]. La estructura de módulo cruzado es una codificación en términos algebraicos de las propiedades que satisface la acción de $\pi_1(X)$ en $\pi_2(Y, X)$ para un par de espacios $X \subseteq Y$. Whitehead [Whi49] probó que si Y se obtiene de X adjuntando celdas de dimensión 2, entonces $\pi_2(Y, X)$ es un $\pi_1(X)$-módulo cruzado libre, con una base en correspondencia con las 2-celdas adjuntadas. Este resultado fue usado en muchos trabajos. Entre ellos, Cockcroft lo aplicó, junto con el teorema de Lyndon que veremos más abajo, para probar la conjetura en el caso en que el subcomplejo K tiene una sola 2-celda [Coc54]. Brown y Higgins [BH78], y luego Brown [Bro80, Bro84] trabajaron en esta teoría, obteniendo una descripción eficiente del módulo cruzado de una unión de 2-complejos que se intersecan en el 1-esqueleto, introduciendo la noción de coproducto de módulos cruzados. Gutierrez y Ratcliffe también contribuyeron a la conjetura aplicando métodos de módulos cruzados [GR81].

El teorema de R. C. Lyndon, *Simple Identity Theorem* [Lyn50], formulado en el contexto de presentaciones de grupos, fue un aporte fundamental en el estudio de asfericidad de 2-complejos. Esto se debió principalmente a la aplicación que hizo W. H. Cockcroft [Coc54], al usarlo para demostrar la conjetura en el caso en que el subcomplejo tiene una sola 2-celda.

Teorema 2.4.2. *Si K es un 2-complejo finito y no asférico con una sola 2-celda, y L se obtiene de K adjuntando 2-celdas al 1-esqueleto de K, entonces L no es asférico.*

El teorema de Lyndon involucra presentaciones con una sola relación, y las presentaciones asociadas a complejos con una sola 2-celda son de este tipo. Más adelante, J. Howie generalizó estos resultados [How82] a partir de la noción de presentación reducible, y de complejo reducible. Las presentaciones de este tipo presentan propiedades similares a las de una relación.

Muchos de las respuestas parciales a la conjetura involucran hipótesis sobre el grupo fundamental del subcomplejo K de L. Cockcroft obtuvo otro avance importante en su artículo [Coc54] al probar la validez de la conjetura para los pares $K \subseteq L$ tales que $L - K$ sólo son celdas de dimensión 2, y $\pi_1(K)$ es libre, abeliano, o finito. Para esto, aplicó las fórmulas clásicas de Hopf sobre homología y homotopía de grupos. También probó que si G es un grupo libre, entonces para todo 2-complejo K con $\pi_1(K) = G$, $H_2(K) = 0$ implica $\pi_2(K) = 0$. Más adelante, este resultado se generalizó a una amplia clase de grupos: los grupos conservativos de Adams [Ada55], que resultaron ser los localmente indicables de Howie [How82] a partir de los resultados de Howie y Schneebeli [HS83].

Las técnicas usadas para probar estos resultados involucran construcciones con revestimientos, y *torres*. Las torres fueron introducidas y utilizadas recurrentemente por Howie [How79, How81b, How81a], y son funciones que se descomponen como composiciones de revestimientos e inclusiones. Destacamos el siguiente teorema de Howie [How83], que separa la conjetura de Whitehead en dos casos particulares, uno compacto y uno no compacto.

Teorema 2.5.4. *Si la conjetura de Whitehead no es cierta, entonces existe un contraejemplo $K \subset L$ de uno de los siguientes dos tipos:*

(a) L es finito y contráctil, $K = L - e$ para una 2-celda e de L, y K no es asférico.

(b) L es la unión de una cadena infinita de subcomplejos finitos no asféricos $K = K_0 \subset K_1 \subset \cdots$, tales que cada inclusión $K_i \to K_{i+1}$ es homotópica a una constante.

Los artículos de Bogley [Bog93] en 1993 y de Rosebrock [Ros07] en 2007 son valiosas recopilaciones sobre los avances en la conjetura de Whitehead.

En el presente trabajo abordamos la conjetura combinando las técnicas ya conocidas y utilizadas hasta el momento (provenientes de la topología, geometría y el álgebra) con nuevas herramientas provenientes de la teoría de homotopía de espacios finitos.

La teoría de los espacios finitos comienza en los años 30 con el trabajo de P. S. Alexandroff [Ale37], quien los describió desde un punto de vista combinatorio comparándolos con los conjuntos parcialmente ordenados (posets) finitos. Para más detalles sobre la teoría de espacios finitos referimos al lector a [Bar11, BM08b, May03c, McC66, Sto66]. Un *espacio topológico finito* es un espacio para el cual el conjunto subyacente es finito. Estos espacios guardan una relación natural con los posets finitos.

Dado un espacio finito X, se define una relación de la siguiente manera

$$x \leq y \Leftrightarrow x \in U \quad \text{para todo abierto } U \text{ que contiene a } y.$$

Es fácil ver que esta relación es un preorden (reflexiva y transitiva) para cualquier topología. Sin embargo, en general no resulta antisimétrica. Esta condición es equivalente a que la topología de la cual proviene cumpla el axioma T_0: para cualquier par de puntos, hay un abierto que contiene sólo a uno de ellos. Análogamente, si $\leq$ es una relación de orden en el conjunto X, los conjuntos

$$U_x := \{y \in X : \ y \leq x\} \subseteq X$$

forman una base para una topología T_0 en X. En este trabajo siempre suponemos que los espacios finitos cumplen el axioma T_0.

La importancia de los espacios finitos radica en que, gracias a los resultados de M. C. McCord de 1966, sirven de modelos para todos los poliedros compactos. Esto se da a partir de las siguiente construcciones.

Dado un espacio finito X, el *complejo de cadenas* $\mathcal{K}(X)$ asociado a X es un complejo simplicial cuyos vértices son los elementos de X y cuyos símplices son las cadenas de elementos de X (es decir, subconjuntos totalmente ordenados). Dado un complejo simplicial K, el *poset de símplices* $\mathcal{X}(K)$ asociado a K es el conjunto de los símplices de K ordenados por la inclusión.

Teorema 3.1.9. *Dado un espacio finito X, hay una equivalencia homotópica débil entre $\mathcal{K}(X)$ y X. Análogamente, dado un complejo simplicial K, hay una equivalencia homotópica débil entre K y $\mathcal{X}(K)$.*

Como consecuencia, se tiene que los tipos homotópicos débiles de los espacios finitos cubren los tipos homotópicos débiles de todos los poliedros compactos.

R. E. Stong [Sto66] explotó la naturaleza combinatoria de los espacios finitos. A partir de la noción de *beat point*, encontró un método efectivo para decidir si dos espacios finitos son homotópicamente equivalentes. El método de los beat points fue el primero de los métodos de reducción para espacios finitos, que más adelante se siguieron desarrollando. Concretamente,

Teorema. *Dos espacios finitos son homotópicamente equivalentes si y sólo si se puede llegar de uno a otro agregando y quitando beat points.*

Este resultado se deduce de la proposición 3.1.12, y de la clasificación de espacios finitos *minimales* (ver [Bar11] para más detalles).

En el año 2003, J. P. May recopiló lo sabido en este tema hasta el momento en una serie de notas [May03a, May03b, May03c], donde además formuló conjeturas que relacionan la teoría de homotopía de espacios finitos con la teoría de homotopía clásica. En los últimos años, J. A. Barmak y E. G. Minian retomaron esta teoría, resolviendo algunas de las conjeturas de May.

Uno de sus aportes fundamentales se dio a partir de la noción de *weak point*, que generaliza a la de beat points, con la cual desarrollaron la teoría de homotopía simple para espacios finitos. Un elemento x de un espacio finito X se dice weak point si el subespacio $\hat{C}_x = \{y \in X : y < x \text{ o } y > x\}$ de X es contráctil. Dados dos espacios finitos X, Y, se dice que Y es *simplemente equivalente* a X si se obtiene de X agregando o quitando weak points.

Teorema. *[BM08b] Dados dos espacios finitos X, Y, X es simplemente equivalente a Y si y sólo si $\mathcal{K}(X)$ es simplemente equivalente a $\mathcal{K}(Y)$. Dados dos complejos simpliciales K, L, K es simplemente equivalente a L si y sólo si $\mathcal{X}(K)$ es simplemente equivalente a $\mathcal{X}(L)$.*

Los resultados obtenidos para espacios finitos fueron aplicados al estudio de la conjetura de Quillen y la conjetura de Andrews-Curtis [Bar11]. Dentro de los avances en este último problema, se probó que la conjetura de Andrews-Curtis se verifica en los complejos cuasi-construibles, una clase de complejos que incluye a los construibles. Para obtener este resultado, se definieron los espacios finitos *qc-reducibles.* En el presente trabajo se prueba que los espacios qc-reducibles satisfacen la conjetura de Whitehead.

A partir del resultado de Howie 2.5.4, traducimos el caso compacto (contraejemplo de tipo (a)) de la conjetura en términos de espacios finitos. De esta manera podremos aplicar métodos de espacios finitos al estudio de este problema.

Conjetura 3.3.1. *Sea X un espacio finito de altura 2 débilmente equivalente a un punto, y sea $a \in X$ un elemento maximal, tal que $X - a$ es conexo. Entonces $X - a$ es asférico.*

Si la conjetura 3.3.1 es cierta, entonces no hay contraejemplos del tipo (a) mencionado por Howie para la conjetura de Whitehead. Por otro lado, si la conjetura de Whitehead es cierta, entonces también es cierta la conjetura 3.3.1.

Notemos que la pregunta original de Whitehead es equivalente, en el caso de poliedros compactos, a preguntarse si todo subespacio de un espacio finito asférico de altura 2 es también asférico.

Uno de los resultados principales del capítulo 3 es la validez de esta conjetura para los espacios qc-reducibles. Para esto, combinamos métodos combinatorios de espacios finitos junto con el resultado de Cockcroft en el que prueba la conjetura en el caso en que $\pi_1(K)$ es libre.

Teorema 3.4.4. *[CM14] Sea X un espacio finito de altura 2 qc-reducible y sea $a \in X$ un elemento maximal tal que $X - a$ es conexo. Entonces $X - a$ es asférico.*

Esto implica que la conjetura de Whitehead es cierta para los complejos cuasi-construibles, una amplia clase de complejos compactos que incluye a todos los complejos construibles.

También introducimos un nuevo método de reducción, el de los a-puntos, que nos sirve para probar la conjetura en una nueva clase de complejos. Recordemos que dado un elemento $x \in X$, denotamos por $\hat{C}_x$ al subespacio $\hat{C}_x = \{y \in X : \ y < x \text{ o } y > x\}$ de X. Un elemento $x \in X$ es un *a-punto* si $\hat{C}_x$ es una unión disjunta de espacios contráctiles. En este caso, la transformación $X \mapsto X - x$ se llama *a-reducción.* Un espacio finito X se dice *fuertemente asférico* si existe una sucesión de a-reducciones que transforma X en un punto.

Prosposición 3.5.2. *[CM14] Si x es un a-punto, entonces X es asférico si y sólo si $X - x$ es asférico. En particular, los espacios fuertemente asféricos son asféricos.*

Este resultado brinda una nueva herramienta para el estudio de la asfericidad de espacios finitos. Se puede manipular el espacio original agregando o quitando a-puntos, y estudiar la asfericidad del espacio obtenido. Otro de los resultados principales del capítulo 3 es el siguiente.

Teorema 3.5.5. *[CM14] Sea X un espacio de altura 2 fuertemente asférico, y sea $a \in X$. Entonces $X - a$ es fuertemente asférico.*

Teorema 3.5.6. *La respuesta a la pregunta original de Whitehead es positiva para espacios finitos fuertemente asféricos de altura 2.*

Como consecuencia, se tiene que la respuesta a la pregunta de Whitehead es positiva si se considera un 2-complejo simplicial L cuyo poset de símplices es fuertemente asférico. Un complejo simplicial L satisface esta condición si y sólo si L colapsa a un subcomplejo de dimensión 1.

En este trabajo introducimos nuevos métodos combinatorios desarrollados para el estudio de la asfericidad, además del mencionado método de los a-puntos. Estos métodos provienen de generalizaciones y variaciones de los métodos de beat points, weak points, y qc-reduciones [Cer10, Fer11]. Los desarrollados en este trabajo son métodos en los cuales el tipo homotópico débil del espacio cambia, preservándose su asfericidad. Es decir, si el espacio Y es obtenido a partir de aplicar uno de estos movimientos a un espacio X, entonces Y es asférico si y sólo si X es asférico. Sin embargo, en general, los grupos de homotopía de Y no coinciden con los de X. Estos movimientos, al ser menos rígidos, dan lugar a un espectro más amplio de espacios donde se puede analizar la asfericidad de un espacio dado. También mostramos algunos de los programas desarrollados para implementar computacionalmente estos métodos a través del software SAGE [S$^+$15]. Los primeros programas construidos con este objetivo se pueden encontrar en [Fer11].

El segundo problema tratado en esta tesis es la conjetura de asfericidad de los complejos LOT (labeled oriented tree), que forman un nexo entre la conjetura de Whitehead y otros problemas de la topología algebraica.

Estos complejos aparecen en el estudio de ciertas variedades que surgen como complementos de los llamados *ribbon discs*, en una generalización de la teoría de nudos clásica. Un ribbon disc es una inmersión propia de un disco de dimensión n en uno de dimensión $n+2$, $D^n \hookrightarrow D^{n+2}$ que satisface que la función $r : D^{n+2} \to \mathbb{R},\ x \mapsto \|x\|$ se restringe a una función de Morse en el disco D^n. Un *complemento de ribbon disc* es el complemento de la imagen de D^n en D^{n+2} por una inmersión de este tipo. Durante mucho tiempo estuvo abierta la conjetura sobre la asfericidad de los complementos de nudos, que fue finalmente probada por Papakyriakopoulos [Pap57] usando métodos de 3-variedades. Este problema fue probablemente una de las motivaciones de Whitehead al formular su conjetura, cuya validez implicaría una demostración alternativa de la asfericidad de los complementos de nudos.

Actualmente se conjetura que, al igual que en el caso de nudos, los complementos de ribbon discs también son asféricos. La homotopía de estos espacios fue estudiada por numerosos matemáticos [Yan69, AMY81, Has83, How85, HR03]. Observamos que en la literatura se encuentran algunas pruebas icompletas de esta conjetura (ver [How83] para más detalles).

Howie probó en [How83] una correspondencia entre los complementos de ribbon discs y los complejos LOT. Un *labeled oriented tree* (árbol orientado y etiquetado), o LOT, es un grafo orientado sin ciclos, provisto adicionalmente de una función λ del conjunto de aristas al conjunto de vértices, llamada *etiqueta*. Es decir que las aristas de un LOT tienen un vértice origen, un vértice destino y un vértice etiqueta.

A cada LOT Γ se le asocia una presentación $P(\Gamma)$ de un grupo, con un generador por cáda vértice, y una relación por cada arista. Si la arista e tiene origen s_e, destino t_e y etiqueta λ_e, la relación asociada es $\lambda_e t_e \lambda_e^{-1} s_e^{-1}$, de modo que en el grupo presentado se tiene $s_e \lambda_e = \lambda_e t_e$. Y de manera usual, a partir de esta presentación se construye un 2-complejo K_Γ, llamado *complejo LOT*.

Howie le asocia, a cada ribbon disc, un LOT Γ de tal manera que K_Γ tiene el tipo homotópico (simple) del complemento del ribbon disc [How85]. De esta manera, para estudiar la asfericidad de estos espacios, basta estudiar la de los complejos LOT. De hecho, todo LOT es el asociado a un ribbon disc adecuado, de modo que la asfericidad de los complejos LOT es equivalente a la de los complementos de ribbon discs.

Howie probó que si un 2-complejo L se 3-deforma a un punto y e es una 2-celda de L, entonces el subcomplejo $K = L - e$ se 3-deforma a un complejo LOT. Una consecuencia de esto es que, si la conjetura de Andrews-Curtis es cierta, entonces la asfericidad de los complejos LOT implica la conjetura de Whitehead para el caso compacto. Recordemos que la conjetura de Andrews-Curtis afirma que todo 2-complejo contráctil se 3-deforma a un punto (ver [AC65]). A su vez, es fácil ver que a todo complejo LOT se le puede agregar una 2-celda y obtener un complejo contráctil. Es por esto que los complejos LOT son considerados casos testigos de la conjetura de Whitehead.

Howie [How85] obtuvo resultados relevantes sobre este problema. Entre otros aportes, mostró que para probar la asfericidad de los LOTs, basta considerar el caso en que el grafo subyacente tiene solamente 2 puntos extremales, es decir, es un intervalo. También probó la asfericidad de LOTs de diámetro ≤ 3 [How85].

En los últimos años Huck, Rosebrock y Harlander obtuvieron avances en este problema. Utilizaron para esto métodos más combinatorios, como por ejemplo los *spherical diagrams* como generadores del segundo grupo de homotopía, y nociones combinatorias de asfericidad relacionadas, como la de *diagrammatically reducible*. En [How85, HR01, Ros10, HR12b, HR12a], se encuentran algunos resultados sobre asfericidad de complejos LOT. Para más detalles en este tema, referimos al lector a [Ros07].

En este trabajo estudiamos la asfericidad de los complejos LOT aplicando, como herramienta principal, los resultados obtenidos por Barmak y Minian respecto del grupo fundamental y del segundo grupo de homotopía de complejos de dimensión 2 en términos de coloreos del diagrama de Hasse de su poset de celdas [BM12a, BM14].

Un *coloreo* de un poset X es una función $c : \mathcal{E}(X) \to G$ del conjunto de aristas del diagrama de Hasse de X a un grupo G. El resultado fundamental de [BM12a] que utilizamos en este trabajo es que uno puede obtener una presentación del grupo fundamental de cualquier espacio finito X de la siguiente manera. Sea D un subdiagrama del diagrama de Hasse de X que contiene a todos los elementos de X y tal que corresponde a un espacio finito simplemente conexo (por ejemplo, un árbol maximal del diagrama satisface estas condiciones). Entonces se tiene una presentación de $\pi_1(X)$ con un generador por cada arista del diagrama de Hasse que no está en D, y una relación por cada *simple digon*, es decir, por cada para de caminos monótonos de aristas que se tocan sólo en los extremos.

Utilizando este resultado, y a partir de una caracterización que obtienen de los revestimientos de un poset en términos de coloreos, obtienen, en [BM14], una descripción del revestimiento universal de un espacio finito X. Además, en el caso en que X sea el espacio asociado a un CW-complejo regular (por ejemplo a un complejo simplicial), obtienen una descripción del segundo grupo de homología del revestimiento universal de X, o sea, de $\pi_2(X)$, utilizando los resultados de [Min12].

Concretamente, para el caso del poset asociado a un 2-complejo regular con $\pi_1(X) = G$, se obtiene la siguiente descripción de $\pi_2(X)$ como submódulo del $\mathbb{Z}[G]$-módulo libre generado por los elementos de altura 2, y con ecuaciones asociadas a los elementos de altura 1. Para un par $x > y$, $\epsilon(x,y)$ denota la *incidencia* de x en y, que toma los valores ± 1 según ciertas elecciones de orientación similares a las usuales a nivel geométrico.

$$\pi_2(X) = \{ \sum_{\substack{\deg x = 2 \\ g \in G}} n_g^x g x : \ n_g^x \in \mathbb{Z} \text{ y } \sum_{x > y} \epsilon(x,y) n_{gc(x,y)}^x = 0 \quad \forall y \in X, \deg y = 1 \}.$$

Nosotros asociamos a cada LOT Γ un LOT poset $\mathcal{X}(\Gamma)$ que es un modelo finito del complejo K_Γ. De esta manera, podemos estudiar la asfericidad del LOT a través del LOT poset, al cual le podemos aplicar métodos de espacios finitos. Damos una descripción de este poset, y exponemos el código de una función que, a partir de la descripción del LOT, calcula el LOT poset asociado.

A partir del LOT poset asociado a un LOT Γ, aplicamos los resultados de Barmak y Minian [BM12a, BM14] para describir el segundo grupo de homotopía de los complejos LOT. Para eso, elegimos un subgiagrama D del diagrama de Hasse de un LOT poset dado, cuyo poset asociado siempre resulta simplemente conexo. La presentación del grupo fundamental obtenida resulta equivalente a la presentación original del LOT. Y la descripción de $\pi_2(\mathcal{X}(\Gamma))$ obtenida es la siguiente.

Teorema 5.3.1. *Sea Γ un LOT con vértices V y aristas E. Entonces*

$$\pi_2(\Gamma) = \{ \sum_{x \in E, g \in G} n_g^x g \overline{x} : \ n_g^x \in \mathbb{Z} : \quad \forall v \in V, \forall g \in G$$
$$- \sum_{s_x = v} n_{gv^{-1}}^x + \sum_{t_x = v} n_{gv^{-1}\lambda_x^{-1}}^x + \sum_{\lambda_x = v} (n_{gv^{-1}}^x - n_{gt_x^{-1}v^{-1}}^x) = 0 \}.$$

A partir de esta descripción, alcanzamos los resultados principales obtenidos sobre este tema, acerca de la asfericidad de una amplia clase de LOTs.

Observación 5.4.2. Sea Γ un LOT y sea $\sum_{x \in E, g \in G} n_g^x g \overline{x}$ un elemento dado de $\pi_2(\Gamma)$. Supongamos que para una arista a de Γ se tiene una ecuación en términos de los coeficientes n_g^a de la forma

$$n_{gu_1}^a + n_{gu_2}^a + \cdots + n_{gu_i}^a + \cdots + n_{gu_k}^a = 0 \quad \forall g \in G$$

y supngamos además que u_i tiene peso estrictamente mayor que $u_j, j \neq i$. Como hay finitos coeficientes no nulos (suponiendo que no son todos nulos), podemos elegir ξ tal que $n^a_\xi \neq 0$ y tal que $w(\xi)$ es mínimo. La ecuación puede reescribirse como

$$n^a_{hu_i^{-1}u_1} + n^a_{hu_i^{-1}u_2} + \cdots + n^a_h + \cdots + n^a_{u_i^{-1}u_k} = 0 \quad \forall h \in G.$$

Esta ecuación debe valer cuando $h = \xi$, pero $w(\xi u_i^{-1} u_j) < w(\xi)$, so $n^a_{hu_i^{-1}u_j} = 0$ para todo $j \neq i$. Por lo tanto, la ecuación resulta $n^a_\xi = 0$, lo cual es una contradicción. Similarmente, los coeficientes serán todos 0 si se tiene un u_i con peso menor estricto que el resto.

Usando esta observación, elaboramos un método para el estudio de la asfericidad de LOTs. Con la ecuación de cada vértice v se puede despejar el coeficiente asociado a una de sus aristas incidentes respecto de las demás, y de las que estén etiquetadas por v. Se intenta, por lo tanto, encontrar un orden adecuado de despeje.

Para poder enunciar los resultados debemos introducir un poco de notación. Si un vértice es etiqueta de una o más aristas, lo llamaremos *vértice etiqueta*. Un LOT se dice *inyectivo* si ningún vértice es etiqueta de más de una arista. Observemos que en un LOT inyectivo hay exactamente un vértice que no es etiqueta.

Si se fija un vértice a de Γ, llamado centro, entonces para cualquier otro vértice $v \in \Gamma$ hay una arista incidente que lo acerca hacia el centro (la que forma parte del único camino en Γ que los une). El resto de sus aristas incidentes lo alejan del centro. Las llamaremos, respectivamente, arista *interna* de v y aristas *externas* de v. Un LOT se dice *reducido* si no se le pueden aplicar las operaciones elementales definidas por Howie en [How85] (ver definición 4.4.1). Estas operaciones preservan el tipo homotoópico simple del complejo asociado. Es por eso que basta considerar LOTs reducidos a la hora de investigar la conjetura de asfericidad de LOTs.

Teorema 5.4.3. *Sea Γ un LOT inyectivo y reducido y sea a, el único vértice que no es etiqueta, fijado como centro de Γ. Supongamos que hay un orden $v_1, \ldots, v_n$ de los vértices etiqueta que satisface lo siguiente.*

1. *Para cada $1 \leq i \leq n-1$, la arista interna de v_i (que llamaremos x_i) está etiquetada con v_{i+1}.*

2. *La arista interna de v_n, x_n está etiquetada con v_1.*

3. *Para cada $1 \leq i \leq n$, las aristas externas de v_i están etiquetadas por vértices previos (es decir, vértices v_k con $k < i$).*

Entonces $\pi_2(\Gamma)$ está generado, como $\mathbb{Z}[\pi_1(\Gamma)]$-módulo, por un único elemento.

Utilizando este resultado, y lo observado previamente, podemos probar la asfericidad de muchos LOTs. Para poder generalizar este resultado a LOTs no inyectivos, el orden de las aristas debe ser reemplazado por la siguiente hipótesis.

Existe una enumeración de las aristas de Γ en filas

$$a_{11}, a_{12}, \ldots, a_{1k_1}$$
$$a_{21}, a_{22}, \ldots, a_{2k_2}$$
$$\cdots$$
$$a_{r1}, a_{r2}, \ldots, a_{rk_r},$$

satisfaciendo la siguiente regla. Se puede enumerar una nueva arista a_{ij} *si tiene un extremo* v_{ij} *tal que las demás aristas incidentes en* v_{ij} *y las etiquetadas con* v_{ij} *fueron enumeradas como* a_{kl} *con* $k < i$ *o* $k = i, l < j$*, o si la arista comienza una nueva fila (i.e. si* $j = 1$*).*

Con esta hipótesis, obtenemos un resultado similar al anterior, para LOTs no necesariamente inyectivos, y obtenemos también una gran cantidad de ejemplos a los que se aplica el método. Sin embargo, para poder garantizar que el método funciona, sin necesidad de analizar los pesos de los subíndices, como lo observado previamente, restringimos las hipótesis, y agregamos una hipótesis sobre el grupo fundamental del LOT.

Un grupo G satisface la *propiedad del único producto* (UPP) si para cada par de subconjuntos finitos no vacíos A, B de G existe un elemento $g \in G$ que se puede expresar de manera única como un producto $g = ab$ con $a \in A$ y $b \in B$.

Esta propiedad es más débil que la condición de ser localmente indicable, asociada al problema de asfericidad. Si un LOT tiene grupo fundamental localmente indicable, entonces es asférico. Esto se debe a que los grupos localmente indicables son conservativos [HS83], y a que los LOT tienen la homología de S^1 [How85].

Un grupo G satisface UPP si y sólo si para todo par de subconjuntos no vacíos X, Y de G existe un elemento $g \in G$ tal que $gX \cap Y$ tiene exactamente un elemento. Combinando esta propiedad con otras hipótesis, pudimos probar la asfericidad de ciertas clases de LOTs. No se sabe si los grupos asociados a LOTs son localmente indicables, o si satisfacen UPP.

Definición. Sea Γ un LOT reducido. Una *buena enumeración* de Γ es una enumeración de las aristas de Γ en filas

$$\begin{gathered} a_{11}, a_{12}, \ldots, a_{1k_1} \\ a_{21}, a_{22}, \ldots, a_{2k_2} \\ \ldots \\ a_{r1}, a_{r2}, \ldots, a_{rk_r}, \end{gathered}$$

satisfaciendo las siguientes reglas

1. Se puede enumerar una nueva arista a_{ij} si tiene un extremo v_{ij} tal que las demás aristas incidentes en v_{ij} y las etiquetadas con v_{ij} fueron enumeradas como a_{kl} con $k < i$ o $k = i, l < j$, o si la arista comienza una nueva fila (i.e. si $j = 1$).

2. Para cada $1 \leq s \leq r$ existe un vértice w_s distinto de todos los v_{ij} tal que todas las aristas incidentes en w_s y etiquetadas con w_s fueron enumeradas como a_{kl} con $k \leq s$.

3. Para cada $1 \leq s \leq r$, el camino en Γ entre w_s y la arista a_{s1} es una sicesión de aristas a_{sj}, tal que los vértices del camino tienen todas sus aristas incidentes y etiquetadas enumeradas con $kl, k \leq s$.

Teorema 5.4.14. *Sea* Γ *un LOT reducido que admite una buena enumeración, y tal que* $G(\Gamma)$ *es UPP. Entonces* Γ *es asférico.*

A partir de este resultado se prueba la asfericidad de LOTs que hasta ahora no se había demostrado, incluyendo LOTs no inyectivos de cualquier diámetro y complejidad.

Chapter 1

Homotopy of 2-complexes and group presentations

1.1 Homotopy theory

In this section we briefly recall some basic constructions and results of homotopy theory. For a detailed exposition, we refer the reader to [Hat02, Spa94, Swi02, Mun84, Whi78, FP90, Bro06].

CW-complexes form a class of topological spaces which can be constructed as a union of cells. These spaces, introduced by J. H. C. Whitehead, have a more relaxed structure than simplicial complexes, conserving some combinatorial nature that makes them easier to handle than general spaces. They play an essential role in homotopy theory, since they serve as weak homotopy models for all spaces. We will later concentrate on 2-complexes, that is, 2-dimensional CW-complexes, and their relationship with group presentations.

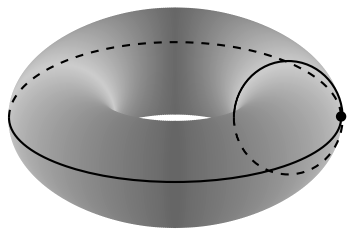
A *CW-complex* is a topological space X with a cellular decomposition. Start with a discrete space X^0, and inductively construct X^n, the n-th *skeleton* of X, by attaching n-cells $\{c^n_\alpha\}_\alpha$ to X^{n-1} via attaching maps $\varphi^n_\alpha : S^{n-1} \to X^{n-1}$. The n-th skeleton of X is the adjunction space

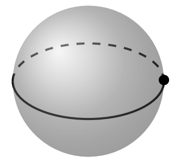
$$X^{n-1} \bigcup_{\sqcup \varphi_\alpha} \sqcup D^n_\alpha.$$

X is *finite dimensional* if $X = X^d$ for some d. Otherwise, it is *infinite dimensional*, and it is the direct limit of its skeleta, which means $X = \bigcup_d X^d$ with the final topology with respect to the inclusions $X^d \hookrightarrow X$.

For example, a 0-dimensional complex is a discrete space, and a 1-dimensional complex is a graph. A space X can be given different CW-structures. For example, the n-th sphere can be given a CW-structure with one 0-cell and one n-cell, or it can be constructed with two cells of dimension k for each $k \leq n$. In the second case, the filtration by skeleta is $S^0 \subseteq S^1 \subseteq \cdots \subseteq S^n$, where $S^k \subseteq S^{k+1}$ is embedded as the equator.

The torus T can be constructed as a CW-complex as follows. Start with one 0-cell, and two 1-cells, say a and b, attached to the 0-cell by their endpoints (these can be seen as the vertical and horizontal circles). Then attach one 2-cell with a path which follows $aba^{-1}b^{-1}$.





A *characteristic map* for an n-cell c_α^n is an extension of the attaching map φ_α, $\Phi_\alpha : D^n \to X^n$, that is a homeomorphism from the interior of D^n onto the interior of the cell.

The class of CW-complexes is closed under most topological operations (taking the convenient topology and CW-structure) such as quotients over subcomplexes, products, suspension, wedge sum. Any covering space X' of a CW-complex X is a CW-complex of the same dimension. The preimage of each open k-cell of X is a disjoint union of open k-cells in X'.

It is not difficult to see that a CW-complex is compact if and only if it has finitely many cells. A *subcomplex* X of a CW-complex Y is a CW-complex whose cells are some of the cells of Y. A CW-complex is said to be *regular* if the (closed) cells are subcomplexes, and the attaching maps are homeomorphisms with their images. For example, the second structure we gave for the n-th sphere is a regular complex.

A map $f : X \to Y$ between CW-complexes is said to be *cellular* if $f(X^n) \subseteq Y^n$ for all n. The cellular approximation theorem asserts that every map between CW-complexes is homotopic to a cellular map.

As a generalization of the fundamental group, the *higher homotopy groups* $\pi_n(X, x_0)$ with $n \geq 2$ of a given space X with basepoint $x_0 \in X$ are defined as the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$. Here I^n stands for the n-dimensional cube and ∂I^n for its boundary. The following operation, which generalizes the path concatenation, makes this set a group.

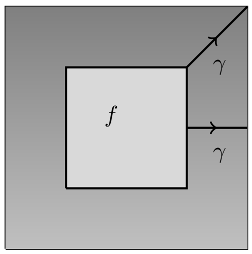
$$fg(x_1, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n) & \text{if } x_1 \leq 1/2 \\ g(2x_1 - 1, x_2, \dots, x_n) & \text{if } x_1 \geq 1/2 \end{cases}$$

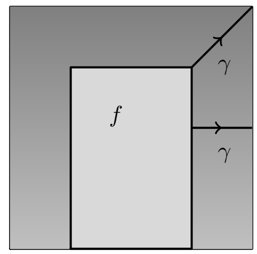
As in the case $n = 1$, the basepoint can be changed when X is a path-connected space, obtaining an isomorphic group. A change of base point $\pi_n(X, x_0) \to \pi_n(X, x_1)$ can be constructed with a path γ from x_0 to x_1. Given $f : (I^n, \partial I^n) \to (X, x_0)$, γf takes the value of f in a smaller cube $I^{n'} \subseteq I^n$ and the values of γ on each radial segment from $\partial I^{n'}$ to ∂I^n. If the given path γ is a loop, this induces a map of groups $\varphi_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_0)$. This construction yields an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Equivalently, $\pi_n(X, x_0)$ can be defined as the set of homotopy classes of maps $f : (S^n, s_0) \to (X, x_0)$, where S^n stands for the n-sphere and s_0 is a basepoint in S^n. In this context, the operation uses the fact that S^n is an h-cogroup. Given two maps $f, g : (S^n, s_0) \to (X, x_0)$, their product is the composition $S^n \to S^n \vee S^n \to X$, where the first map collapses the equator, and the second map takes the values f and g on each copy of S^n (we must take s_0 in the equator).

Given a pair of spaces $X \subseteq Y$, with basepoint $x_0 \in X$, the *relative homotopy groups* are defined as the set of homotopy classes of maps $f : (I^n, \partial I^n, J^{n-1}) \to (Y, X, x_0)$. Here we consider $J^{n-1} = \overline{I^n \smallsetminus I^{n-1}}$, where $I^{n-1} = \{x \in I^n : x_n = 0\}$. The operation and the action of π_1, are defined in a similar way on the relative homotopy groups.

We illustrate below the action of π_1 on higher homotopy and relative homotopy groups. Note that it generalizes the action of π_1 on itself given by the conjugation.





It is easy to see that maps between spaces naturally induce maps in their homotopy groups, making π_n a functor. The groups $\pi_n(X, x_0)$ are abelian for $n \geq 2$ and $\pi_n(Y, X, x_0)$ are abelian for $n \geq 3$, and they are related by a long exact sequence.

$$\dots \to \pi_n(X, x_0) \to \pi_n(Y, x_0) \to \pi_n(Y, X, x_0) \to \pi_{n-1}(X, x_0) \to \pi_{n-1}(Y, x_0) \to \dots$$

An important fact on higher homotopy groups is that, if $p : \tilde{X} \to X$ is a covering map, then $p_* : \pi_n(\tilde{X}, \tilde{x}) \to \pi_n(X, p(\tilde{x}))$ is an isomorphism for all $n \geq 2$ and $\tilde{x} \in \tilde{X}$. This is a consequence of the lifting property, the homotopy lifting property, and the fact that S^n is simply connected if $n \geq 2$.

The homotopy groups play a special role when working with CW-complexes, since in this case they determine in a certain way the homotopy type of the space. More precisely, the classical *Whitehead theorem* states that if a map $f : X \to Y$ between CW-complexes induces isomorphisms in all homotopy groups $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$ and all possible base points $x_0 \in X$, then f is a homotopy equivalence.

A map inducing isomorphisms in all homotopy groups is called a *weak homotopy equivalence*. When dealing with general spaces, weak homotopy equivalences need not be homotopy equivalences. In fact, there are spaces with trivial homotopy groups (which we call homotopically trivial spaces) which are not contractible.

Another fundamental result of homotopy theory is the *CW-approximation theorem*, which states that CW-complexes are weak models for all spaces, that is, for every topological space X there exists a CW-complex X' and a weak homotopy equivalence $f : X' \to X$.

Recall that a space X is said to be *n-connected* if its homotopy groups $\pi_k(X)$ are trivial for $k \leq n$. The *theorem of Hurewicz* relating the homotopy and homology groups is one of the basis tools in algebraic topology. It asserts that if a space X is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_k(X) = 0$ for all $k < n$, and $\pi_n(X) \simeq H_n(X)$. Here we denote by $H_k(X)$ the homology groups with integer coefficients and by $\tilde{H}_k(X)$ the reduced homology groups.

Note that if map $f : X \to Y$ between simply connected CW-complexes induces isomorphisms in all homology groups, then f is a homotopy equivalence.

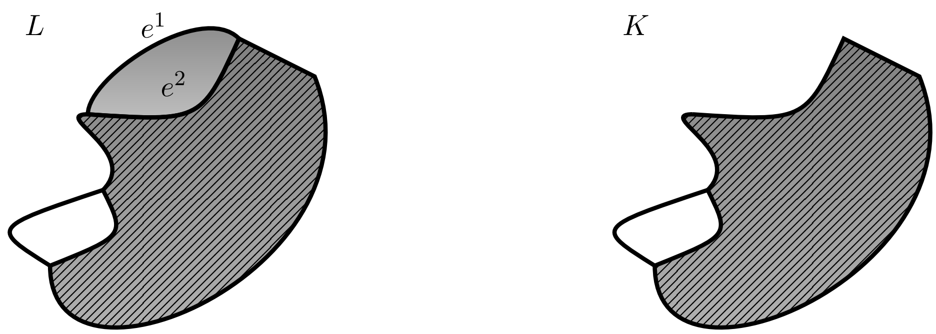
We shall now recall the fundamental notions of simple homotopy theory for CW-complexes. This theory was introduced by Whitehead as a geometric counterpart of Tietze's moves for group presentations.

Consider the $(n-1)$-dimensional disc D^{n-1} as a subspace of ∂D^n, $D^{n-1} \subseteq \partial D^n$. We denote by J^{n-1} the closure of $D^n - D^{n-1}$.

Definition 1.1.1. Given a complex K, an $(n-1)$-cell $e^{n-1} \in K$ is said to be a *free face* of the n-cell $e^n \in K$ provided that the characteristic map $\Phi : D^n \to K$ of e^n restricts to a characteristic map for e^{n-1}, $\Phi\big|_{D^{n-1}} : D^{n-1} \to K$, and $\Phi(J^{n-1}) \subseteq K^{n-1} - e^{n-1}$.

In this case there is an *elementary collapse* $K \overset{e}{\searrow} K - \{e^{n-1}, e^n\}$, or an *elementary expansion* $K - \{e^{n-1}, e^n\} \overset{e}{\nearrow} K$.

Example 1.1.2. In the following 2-complex L, the cell e^1 is a free face of e^2 and they can be therefore collapsed to obtain K.



A sequence of elementary collapses (resp. expansions) is called a *collapse* and denoted $K \searrow L$ (resp. *expansion* and denoted $K \nearrow L$), and a sequence of collapses and expansions is called a *deformation* and denoted $K \nearrow\!\!\!\!\searrow L$. Sometimes it is important to indicate that the deformation $K \nearrow\!\!\!\!\searrow L$ involves cells of up to certain dimension n. In this case we say that K n-deforms to L and we denote $K \overset{n}{\nearrow\!\!\!\!\searrow} L$. Note that in this case the dimensions of K and L cannot be greater than n.

A *simple homotopy equivalence* is a map that is homotopic to a deformation. If there exists a simple homotopy equivalence between K and L, we will also note $K \nearrow\!\!\!\!\searrow L$.

It is easy to see that if $K \overset{e}{\nearrow} L$ then K is a deformation retract of L. As a consequence, $K \nearrow\!\!\!\!\searrow L$ imples $K \simeq L$ (by $\simeq$ we mean homotopy equivalent), but the converse does not hold. The obstruction to this implication can be measured with the Whitehead group $\mathrm{Wh}(K)$, an invariant that depends on $\pi_1(K)$ (see [Coh73]). Since the Whitehead group of a simply connected space is trivial, $K \simeq * \Leftrightarrow K \nearrow\!\!\!\!\searrow *$.

C. T. C. Wall proved that for $n \geq 3$, if K and L are complexes of dimension n (or less), then $K \nearrow\!\!\!\!\searrow L$ implies $K \overset{n+1}{\nearrow\!\!\!\!\searrow} L$ [Wal66].

Note that if K and L are 2-complexes such that $K \nearrow\!\!\!\!\searrow L$, we can see them as 3-complexes and deduce $K \overset{4}{\nearrow\!\!\!\!\searrow} L$. But whether $K \overset{3}{\nearrow\!\!\!\!\searrow} L$ holds in such a case is still an open question. This problem is known as the *generalized Andrews-Curtis conjecture.*

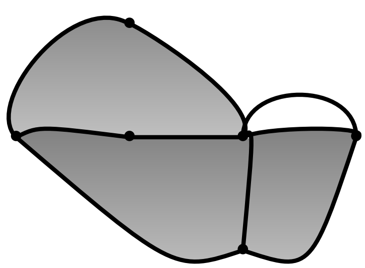
1.2 Group presentations

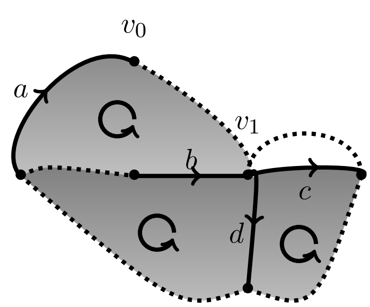
In this section we recall the connection between 2-complexes (i.e. 2-dimensional CW-complexes) and group presentations, arising from the presentation of the fundamental group of a complex in terms of its edges and 2-cells. We refer the reader to [HM93] for a detailed introduction to this matter. In this section we consider only finite connected CW-complexes.

Because of the cellular approximation theorem, the fundamental group of a given complex X coincides with that of the 2-skeleton X^2, and because of the Van Kampen theorem it can be described as follows. Choose a vertex v^0 and a spanning tree $T \subseteq X^1$, that is, a tree that includes all the vertices of X. Choose an orientation for each edge (1-cell) a in $X - T$. Then the unique path in T from v_0 to the source of a, followed by a and the path from its target to v_0 is a closed path in X. This path determines an element g_a in $\pi_1(X^1, v_0)$. In fact, this group is free with basis $\{g_a\}_{a \in X-T}$.

Next, choose for every 2-cell e in X an attaching map, a base point and orientation for ∂D^2, and a path in X^1 from v_0 to the image in X of the base point of ∂D^2. This determines an element of $\pi_1(X^1, v_0)$, which can be expressed a product r_e of the elements $\{g_a\}_{a \in X-T}$ and its inverses. This element is trivial in the quotient $\pi_1(X^2, v_0)$. In fact, the group $\pi_1(X, v_0) = \pi_1(X^2, v_0)$ is the quotient of the free group $\pi_1(X^1, v_0)$ modulo the normal subgroup generated by the relators $\{r_e\}_{\text{2-cells } e}$. For simplicity, we will use a to note the element g_a, so the presentation of the fundamental group has the edges of $X - T$ as generators.

Example 1.2.1. Consider the following complex X with six 0-cells, nine 1-cells and three 2-cells. On the right, a base point v_0 and a maximal tree consisting of the dotted edges have been chosen. The remaining edges have been labeled with the letters a, b, c, d, and oriented. All the 2-cells have v_1 as a base point and the chosen orientation is illustrated.





Thus the free group $\pi_1(X, v_0)$ has basis $\{a, b, c, d\}$. And the 2-cells e yield the relators $a^{-1}b, b^{-1}d^{-1}, dc^{-1}$. Therefore the presentation obtained is $\langle a, b, c, d \mid a^{-1}b, b^{-1}d^{-1}, dc^{-1} \rangle$.

In this way, a 2-complex gives rise to a group presentation for its fundamental group. Conversely, given a group presentation P, one can construct a 2-complex K_P with the presented group as its fundamental group. Let $P = \langle a_1, \dots, a_k \mid r_1, \dots, r_l \rangle$ be a group presentation. We construct $K(P)$ with only one vertex, one edge for each generator a_i, and one 2-cell for each relator r_j. The edges are oriented and have the only 0-cell as endpoint. For the attaching map of the 2-cell associated to r_i, choose a representative $\prod_s a_{i_s}^{\epsilon_{i_s}}$, then the attaching map goes through the edges a_{i_s} forwards or backwards depending on the value of $\epsilon_{i_s} \in \{-1, 1\}$.

For example, the presentation $P = \langle a \mid a \rangle$ of the trivial group yields a disc D^2. However the complex associated to the presentation $\langle a \mid a, a \rangle$ of the same group is the 2-sphere S^2. The presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$ of $\mathbb{Z}$ gives rise to the torus.

These assignments constitute a correspondence between 3-deformation types of 2-complexes and classes of group presentations modulo the Q^{**}-transformations, which we will define next.

The Q^{**}-transformations are a subclass of the so called Tietze transformations, a set of operations on group presentations, which preserve the presented group.

Q^{**}-transformations, also called extended Nielsen transformations, preserve the deficiency of the presentation (number of generators minus number of relators). These transformations generalize Nielsen transformations and Q-transformations.

Definition 1.2.2. The Q^{**}*-transformations* on a presentation $P = \langle a_1, \dots, a_k \mid r_1, \dots, r_l \rangle$ consist of:

- Replace a relator r_i by its inverse r_i^{-1}.
- Replace r_i by wr_iw^{-1}, where $w \in F(a_1, \dots, a_k)$.
- Replace r_i by r_ir_j or r_jr_i, $(i \neq j)$.
- Replace all occurrences of a generator a_i throughout the relators by a_i^{-1}.
- Replace all occurrences of a generator a_i throughout the relators by a_ia_j, $(i \neq j)$.
- Replace all occurrences of a generator a_i throughout the relators by a_ja_i, $(i \neq j)$.
- Add a new generator a_{k+1} and a new relator $r_{l+1} = a_{k+1}$.
- The opposite of the previous transformation, when possible.

The last transformation can be performed provided that there is a relator $r_j = a_i$ and the generator a_i does not appear in any of the other relators.

Note that, for example, the effect of the first transformation on the associated complex is the replacement of the attaching map of a 2-cell for a similar map in which the orientation is inverted. The last two operations correspond to an elementary expansion or collapse.

Remark 1.2.3. The Q^{**}-transformations can be reformulated as follows.

- Replace a relator r_i by $wr_i^{\epsilon}w^{-1}$, where $\epsilon = \pm 1, w \in F(a_1, \dots, a_k)$.
- Replace r_i by r_ir_j or r_jr_i, $(i \neq j)$.
- Add a new generator a_{k+1} and a new relator $r_{l+1} = w^{-1}a_{k+1}$, where $w \in F(a_1, \dots, a_k)$, or the opposite transformation.

Let us see how to replace all occurrences of the generator a_1 throughout the relators by a_1^{-1}, using these operations. We introduce a new generator $\tilde{a}_1$ with a relator $r_{l+1} = a_1\tilde{a}_1$. If the generator a_1 occurs in the relator $r_1 = ua_1w$ with exponent 1, we proceed as follows.

$\langle a_1 \ldots, a_k, \tilde{a}_1 \mid ua_1w, r_2, \ldots, r_k, a_1\tilde{a}_1\rangle$

$\langle a_1 \ldots, a_k, \tilde{a}_1 \mid a_1wu, r_2, \ldots, r_k, \tilde{a}_1^{-1}a_1^{-1}\rangle$ we inverted r_{l+1} and conjugated r_1

$\langle a_1 \ldots, a_k, \tilde{a}_1 \mid \tilde{a}_1^{-1}a_1^{-1}a_1wu, r_2, \ldots, r_k, \tilde{a}_1^{-1}a_1^{-1}\rangle$

$\langle a_1 \ldots, a_k, \tilde{a}_1 \mid \tilde{a}_1^{-1}wu, r_2, \ldots, r_k, \tilde{a}_1^{-1}a_1^{-1}\rangle$ we replaced r_1 by $r_{l+1}r_1$

$\langle a_1 \ldots, a_k, \tilde{a}_1 \mid u\tilde{a}_1^{-1}w, r_2, \ldots, r_k, \tilde{a}_1^{-1}a_1^{-1}\rangle$ we conjugated r_1

We proceed similarly for other relators where a_1 occurs. If the exponent is -1, we modify (invert an conjugate) the new relator to obtain $\tilde{a}_1a_1$. After all the occurrences have been replaced, the generator a_1 only appears in the relator r_{l+1}, and we can eliminate the generator and the relator. Finally, we can rename the generator $\tilde{a}_1$ as a_1 (this can also be done with a sequence of the considered transformations).

If we expand Q^{**}-transformations and allow the addition (or removal) of the trivial word 1 to the set of relators, we obtain all *Tietze transformations*:

- Add the relator $r_{l+1} = w$, where $w \in N(r_1, \ldots, r_l)$, or the opposite transformation.
- Add a new generator a_{k+1} and a new relator $r_{l+1} = w^{-1}a_{k+1}$, where $w \in F(a_1, \ldots, a_k)$, or the opposite transformation.

Note that in these transformations the deficiency is no longer preserved. The effect in the associated complex of adding a relator 1 is that a 2-sphere is adjoined at the base point.

Tietze's theorem states that two presentations P and Q yield the same group if and only if one can obtain Q from P by a sequence of Tietze transformations.

We will say that two group presentations are *equivalent* if they are related by a sequence of Q^{**}-transformations. Note that equivalent presentations yield the same group, but the same group can be presented by non-equivalent transformations. For example, $\langle a \mid a\rangle$ and $\langle a \mid a, a\rangle$ present the trivial group but they are not equivalent (they have different deficiency).

Although the presentation of the fundamental group of a 2-complex described above depends on several choices (like the spanning tree and adjunction maps of the cells), different choices give rise to equivalent presentations. Therefore, given a 2-complex K we can define $\tilde{P}(K)$, the class of group presentations of its fundamental group obtained with this process.

Example 1.2.4. Consider the presentation $\langle a, b, c, d \mid a^{-1}b, b^{-1}d^{-1}, dc^{-1}\rangle$ from Example 1.2.1. We can perform the following sequence of Q^{**}-transformations.

$\langle a, b, c, d \mid a^{-1}b, b^{-1}d^{-1}, dc^{-1}\rangle$
$\langle a, b, c, d \mid b^{-1}a, b^{-1}d^{-1}, dc^{-1}\rangle$ (inverted r_1)
$\langle a, b, c, d \mid ab^{-1}, b^{-1}d^{-1}, dc^{-1}\rangle$ (conjugated r_1 by a)
$\langle a, b, c, d \mid a, b^{-1}d^{-1}, dc^{-1}\rangle$ (replaced all occurrences of a by ab)
$\langle b, c, d \mid b^{-1}d^{-1}, dc^{-1}\rangle$ (eliminated the generator a and the relator r_1)
$\langle b, c, d \mid b^{-1}d^{-1}, cd^{-1}\rangle$ (inverted r_3)
$\langle b, c, d \mid b^{-1}d^{-1}, c\rangle$ (replaced all ocurrences of c by cd)
$\langle b, d \mid b^{-1}d^{-1}\rangle$ (eliminated the generator c and the relator r_3)
$\langle b, d \mid db\rangle$ (inverted r_2)
$\langle b, d \mid db^{-1}\rangle$ (replaced all ocurrences of b by b^{-1})
$\langle b, d \mid d\rangle$ (replaced all ocurrences of d by db)
$\langle b \mid \rangle$ (eliminated the generator d and the relator r_2)

In this way, we obtain a much simpler presentation of the fundamental group, which is infinite cyclic.

The following result establishes the connection between group presentations and 2-complexes. Recall that two complexes K, L are simply homotopy equivalent ($K \nearrow\!\searrow L$) if there is a sequence of expansions and collapses transforming K into L, and that if K and L are 2-complexes, and such a sequence can be taken to involve only cells up to dimension 3, we say that K 3-deforms to L ($K \overset{3}{\nearrow\!\searrow} L$). A proof of this theorem can be found in [HM93].

Theorem 1.2.5. *The assignment $K \mapsto \tilde{P}(K)$, together with the construction $P \mapsto K_P$ induce a bijection between 3-deformation types of finite connected 2-complexes and Q^{**}-classes of group presentations.*

It is because of this result that many problems of combinatorial group theory can be translated to a geometrical context. An example of this is the Andrews-Curtis conjecture. In group-theoretical terms, the conjecture states that every balanced (i.e. with deficiency 0) presentation of the trivial group is equivalent to the empty presentation $\langle \mid \rangle$. But the deficiency $d(P)$ of a presentation P is equal to the number of 1-cells minus the number of 2-cells of the associated complex K_P. Therefore we have

$$d(P) = 1 - \chi(K_P).$$

As a consequence, P is a balanced presentation of the trivial group iff K_P is a simply-connected acyclic (that is, contractible) 2-complex. So the geometrical translation of the conjecture states that every contractible 2-complex 3-deforms to a point.

$$K \simeq * \iff K \overset{3}{\nearrow\!\searrow} *$$

Note that Tietze's theorem means that K and L have the same fundamental group if and only if the wedge of K with a set of 2-spheres is 3-deformable to the wedge of L with a set of 2-spheres.

$$\pi_1(K) \simeq \pi_1(L) \;\Leftrightarrow\; K \bigvee_{i=1}^{k} S_i^2 \overset{3}{\nearrow\!\!\searrow} L \bigvee_{j=1}^{l} S_j^2 \quad \text{for some } k, l \in \mathbb{N}_0$$

Example 1.2.6. A simple application of the equivalence given by the previous theorem is that the transformations performed in Example 1.2.4 show that the complex of Example 1.2.1 can be 3-deformed to S^1.

We will now show a 2-dimensional version of the classical theorem of Whitehead, for which there is an elementary proof using techniques of group presentations.

Lemma 1.2.7. *Let $f : K \to L$ be a map between 2-complexes such that $K^1 = L^1$. If f restricts to the identity map on the 1-skeleta and induces an isomorphism on π_2, then f is a homotopy equivalence.*

Proof. The idea is to prove that f also induces an isomorphism on the relative homotopy groups $\pi_2(K, K^1) \to \pi_2(L, L^1)$ and then use this isomorphism to define an homotopy inverse for f on the 2-cells, using that the characteristic maps of the 2-cells represent elements of the second relative homotopy group. To prove that f induces an isomorphism on the relative homotopy groups, consider the homotopy sequences for the pairs $(K, K^1), (L, L^1)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(K) & \longrightarrow & \pi_2(K,K^1) & \longrightarrow & \pi_1(K^1) & \longrightarrow & \pi_1(K) & \longrightarrow & 0 \\
 & & \downarrow f_* & & \downarrow f_* & & \| & & \| & & \\
0 & \longrightarrow & \pi_2(L) & \longrightarrow & \pi_2(L,L^1) & \longrightarrow & \pi_1(L^1) & \longrightarrow & \pi_1(L) & \longrightarrow & 0
\end{array}$$

□

Theorem 1.2.8. *Let $f : K : L$ be a map between 2-complexes that induces isomorphisms in π_1 and π_2. Then f is a homotopy equivalence.*

Proof. We can suppose that K and L are the associated complexes of presentations $P = \langle a_1, \dots, a_k \mid r_1, \dots, r_l \rangle$ and $Q = \langle b_1, \dots, b_m \mid s_1, \dots, s_n \rangle$. In order to reduce the problem to the case where K and L have the same 1-skeleton, we will construct two new presentations of the same group, with all the generators a_i and b_i, which serve as sorts of unions of the presentations P and Q.

Set

$$\tilde{P} = \langle a_1, \dots, a_k, b_1, \dots, b_m \mid r_1, \dots, r_l, g(b_1)b_1^{-1}, \dots, g(b_m)b_m^{-1} \rangle,$$

where g is the inverse map of $f_* : \pi_1(K) \to \pi_1(L)$. Analogously, set

$$\tilde{Q} = \langle a_1, \dots, a_k, b_1, \dots, a_m \mid s_1, \dots, s_n, f_*(a_1)a_1^{-1}, \dots, f_*(a_k)a_k^{-1} \rangle.$$

Let $\tilde{K}$ and $\tilde{L}$ be the associated complexes of $\tilde{P}, \tilde{Q}$ respectively. Then it is clear that K expands to $\tilde{K}$ and L expands to $\tilde{L}$. The induced map $\tilde{f} : \pi_1(\tilde{L}) \to \pi_1(\tilde{L})$ is the identity. Thus, it can be deformed to a map $F \simeq \tilde{f}$ that restricts to the identity on the 1-skeleton of $\tilde{K}$ and $\tilde{L}$. Furthermore, the map $\tilde{f}$ induces an isomorphism on π_2, and therefore F is a homotopy equivalence. □

Resumen del capítulo 1: Homotopía de 2-complejos y presentaciones de grupos

En este primer capítulo repasamos algunas construcciones básicas y resultados fundamentales de la teoría de homotopía, concentrándonos principalmente en los 2-complejos (CW-complejos de dimensión 2). Luego, presentamos la correspondencia entre estos complejos y las presentaciones de grupos, que da lugar a una interacción entre la topología algebraica y la teoría combinatoria de grupos.

La sección 1.1 es sobre teoría de homotopía de CW-complejos. Estos complejos, introducidos por J. H. C. Whitehead, son una clase de espacios topológicos que pueden ser construidos a partir de bloques, llamados celdas. La estructura de celdas de estos complejos, que es más laxa que la de los complejos simpliciales, les brinda una naturaleza combinatoria. La importancia de estos espacios radica en que sirven como modelos para espacios topológicos en general. Problemas sobre espacios topológicos de todo tipo pueden estudiarse a partir de su traducción a la categoría de los CW-complejos, donde se pueden aplicar métodos y herramientas desarrollados en este contexto. Para más detalles en este tema, referimos al lector a [Hat02, Spa94, Swi02, Mun84, Whi78, FP90, Bro06].

Un *CW-complejo* X se construye a partir de un espacio discreto, X^0. Inductivamente, el *n-esqueleto* X^n se obtiene a partir de X^{n-1} adjuntando celdas de dimensión n, $\{c^n_\alpha\}_\alpha$ a X^{n-1} a través de fuciones de adjunción $\varphi^n_\alpha : S^{n-1} \to X^{n-1}$. El n-esqueleto de X es el espacio de adjunción

$$X^{n-1} \bigcup_{\sqcup \varphi_\alpha} \sqcup D^n_\alpha.$$

X es de *dimensión finita* si $X = X^d$ para algún d. De lo contrario, es de *dimensión infinita*, y es el límite directo de sus esqueletos. Es decir, es la unión $X = \bigcup_d X^d$, con la topología final respecto de las inclusiones $X^d \hookrightarrow X$.

Por ejemplo, los CW-complejos de dimensión 0 son los espacios discretos y los de dimensión 1 son los grafos. Un espacio X puede tener diferentes estructuras celulares. Por ejemplo, la esfera S^n puede construirsea partir de una sola 0-celda (vértice), adjuntando y una sola n-celda, con una función de adjunción que identifica todo el borde de D^n en el punto. Otra construcción posible es con dos celdas de cada dimensión $k \leq n$. La filtración por esqueletos de esta estructura celular de S^n es $S^0 \subseteq S^1 \subseteq \cdots \subseteq S^n$, donde $S^k \subseteq S^{k+1}$ es el ecuador.

Los *grupos de homotopía de orden superior* son una generalización del grupo fundamental. Dado un espacio X con punto base x_0, $\pi_n(X, x_0)$ con $n \geq 2$, se define como el conjunto de clases homotópicas de funciones $f : (I^n, \partial I^n) \to (X, x_0)$, donde I^n denota el cubo n-dimensional, y ∂I^n su borde.

De manera similar se definen los grupos de homotopía relativos, para un para un par de espacios $X \subseteq Y$ y un punto base $x_0 \in X$. Y estos grupos se relacionan a partir de una sucesión exacta larga

$$\cdots \to \pi_n(X, x_0) \to \pi_n(Y, x_0) \to \pi_n(Y, X, x_0) \to \pi_{n-1}(X, x_0) \to \pi_{n-1}(Y, x_0) \to \dots$$

Los teoremas de Whitehead y Hurewicz, junto con el de CW-aproximación, son los resultados fundamentales de la teoría de homotopía y de CW-complejos.

El *teorema de Whitehead* afirma que si una función continua $f : X \to Y$ entre CW-complejos induce isomorfismos $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ en todos los grupos de homotopía, $n \geq 0$ para todo punto base $x_0 \in X$, entonces f is una *equivalencia homotópica*, es decir, tiene una *inversa homotópica* $g : Y \to X$ tal que las composiciones $fg : Y \to Y$ y $gf : X \to X$ son homotópicas a las respectivas identidades. Este resultado dice que cuando se trata de CW-complejos, los grupos de homotopía determinan de cierta manera el tipo homotópico del espacio.

El *teorema de CW-aproximación* afirma que para todo espacio topológico X existe un CW-complejo X' y una función $f : X' \to X$ que induce isomorfismos en todos los grupos de homotopía. Es a partir de este resultado que los CW-complejos son considerados modelos para todos los espacios topológicos.

El *teorema de Hurewicz* relaciona los grupos de homotopía de un espacio con los grupos de homología. El mismo afirma que si para un espacio X se tiene $\pi_k(X) = 0$ para todo $k \leq n$, con $n \geq 2$, entonces $\tilde{H}_k(X) = 0$ para todo $k < n$, y se tiene un isomorfismo $\pi_n(X) \simeq H_n(X)$. Aquí $H_n(X)$ denota el n-ésimo grupo de homología con coeficientes en $\mathbb{Z}$ y $\tilde{H}_n(X)$, el de homología reducida.

Recordamos luego las nociones fundamentales de la teoría de homotopía simple para CW-complejos. Esta teoría fue introducida por Whitehead como un equivalente geométrico de los movimientos de Tietze para presentaciones de grupos.

Un complejo K es *simplemente equivalente* a L ($K \diagup\!\!\searrow L$) si hay una sucesión de colapsos y expansiones elementales que transforman K en L. Un colapso elemental en un complejo K consiste en omitir dos celdas e^{n-1}, e^n de K tales que e^{n-1} es una *cara libre* de e^n, es decir que ninguna otra celda de K toca el interior de la celda e^{n-1} y que la función de adjunción de e^n manda una parte del borde de D^n homeomórficamente en e^{n-1} y el resto no toca e^{n-1}. El dibujo ilustra la situación, con $n = 2$.

Es fácil ver que si dos complejos son simplemente equivalentes, entonces son homotópicamente equivalentes. Pero la implicación inversa no es cierta. Es decir, hay complejos homotópicamente equivalentes que no son simplemente equivalentes. La obstrucción a esta implicación se mide con un invariante que de pende del grupo fundamental, llamado el *grupo de Whitehead*.

Un problema importante de esta teoría es acotar las dimensiones de las celdas involucradas en una equivalencia simple. C. T. C. Wall probó que si $n \geq 3$ y K, L son complejos de dimensión n tales que $K \nearrow\!\!\!\searrow L$, entonces $K \overset{n+1}{\nearrow\!\!\!\searrow} L$, es decir, hay una equivalencia simple que no involucra celdas de dimensión mayor que $n+1$ [Wal66].

Si K, L son 2-complexes tales que $K \nearrow\!\!\!\searrow L$, se los puede considerar como 3-complejos, y deducir $K \overset{4}{\nearrow\!\!\!\searrow} L$. Pero aún hoy no se sabe si en este caso se tiene $K \overset{3}{\nearrow\!\!\!\searrow} L$. Este problema se conoce como la *conjetura de Andrews-Curtis generalizada*.

En la sección 1.2 repasamos la conexión entre los 2-complejos y las presentaciones de grupos, que proviene de la presentación del grupo fundamental de un complejo en términos de sus celdas. Referimos al lector a [HM93] para una introducción detallada en este tema.

Se puede calcular una presentación del grupo fundamental de un 2-complejo compacto X de la siguiente manera. Se debe elegir un vértice como punto base y un árbol maximal de su 1-esqueleto. La presentación de $\pi_1(X)$ tendrá un generador por cada 1-celda que no forme parte de este árbol. Luego, para cada 2-celda, se elige una función de adjunción que se descomponga como lazos en el punto base. Las 1-celdas recorridas fuera del árbol maximal, en el sentido correspondiente, forman una palabra en los generadores. De esta manera se definen las relaciones de las presentación.

Análogamente, dada una presentación finita de un grupo, se puede construir un 2-complejo asociado, que tiene por grupo fundamental al grupo presentado. Éste tiene una sola 0-celda, una 1-celda por cada generador, y una 2-celda por cada relación, que se adjunta recorriendo las 1-celdas según lo indica la palabra.

Esto consituye una correspondencia entre clases de 3-deformaciones de 2-complejos, y clases de presentaciones identificadas por las Q^{**}-transformaciones. Éstas transformaciones son una clase de movimientos sobre presesentaciones de grupos que generalizan a las llamadas transformaciones de Nielsen, y son más rígidas que las transformaciones de Tietze. Se pueden resumir en las siguientes operaciones, aplicables a una presentación $P = \langle a_1, \dots, a_k \mid r_1, \dots, r_l \rangle$:

- Reemplazar una relación r_i por $w r_i^{\epsilon} w^{-1}$, donde $\epsilon = \pm 1, w \in F(a_1, \dots, a_k)$.

- Reemplazar r_i por $r_i r_j$ o $r_j r_i$, $(i \neq j)$.
- Agregar un generador a_{k+1} y una relación $r_{l+1} = w^{-1} a_{k+1}$, donde $w \in F(a_1, \ldots, a_k)$, o la transformación inversa.

Si se permite, además de estas operaciones, agregar una relación 1 a la presentación, se obtienen las transformaciones de Tietze. El teorema de Tietze afirma que dos presentaciones se transforman vía transformaciones de Tietze si y sólo si definen el mismo grupo. En cambio, las transformaciones que consideramos nosotros caracterizan el tipo de 3-deformación de los complejos, no sólo la clase de isomorfismo del grupo fundamental. Esta correspondencia da lugar a una enriquecedora interacción entre la topología y la teoría de grupos. Problemas de toería combinatoria de grupos se pueden estudiar métodos geométricos, y vice versa, como se puede ver por ejemplo en la conjetura de Andrews-Curtis, o en los trabajos sobre los *word problem* o *conjugacy problem* relacionados con resultados sobre asfericidad.

Chapter 2

Whitehead's asphericity question

2.1 Introduction

Whitehead studied, in [Whi41] and later works, the relation between $\pi_n(X)$ and $\pi_n(Y)$ for spaces X and Y, where Y is obtained from X by attaching n-cells. He worked with what we now call the relative homotopy groups $\pi_q(Y,X)$ and the action of $\pi_1(X)$ on $\pi_q(Y,X)$, which later gave rise to the theory of crossed modules. We present this theory later in this chapter. He also worked with portions of the exact homotopy sequence of the pair (Y,X),

$$\dots \to \pi_n(X) \to \pi_n(Y) \to \pi_n(Y,X) \to \pi_{n-1}(X) \to \dots$$

For $n \geq 3$, $\pi_n(Y,X)$ is a free $\mathbb{Z}[\pi_1(X)]$-module on the set of n-cells missing in X, and Whitehead even procured generators for $\ker(i_* : \pi_n(X) \to \pi_n(Y))$ as a $\mathbb{Z}[\pi_1(X)]$-module.

For $n = 2$, $\pi_2(Y,X)$ is not in general an abelian group, and this fact makes it the most interesting case. Whitehead gave a description for $\pi_2(Y,X)$ in terms of generators and relations, which makes his question, stated below, a problem of geometric and algebraic nature.

Is every subcomplex of an aspherical 2-complex itself aspherical?

A connected space X is said to be *aspherical* if its higher homotopy groups $\pi_n(X), n \geq 1$ are trivial. This question, usually treated as a conjecture, is the main motivation for our investigation, and the cause of numerous and important works since 1941. In the following sections, we present some of the techniques that were applied to the problem, and of the results which were thereby achieved.

One of the main motivations for this question was the problem of asphericity of knot complements. The complement of a knot in the 3-sphere has the homotopy type of a subcomplex of an aspherical 2-complex. Therefore an affirmative answer to Whitehead's question would provide a proof of the asphericity of knot complements. But these spaces have already been proved to be aspherical in 1957 by Papakyriakopoulos (see [Pap57]). However, a proof of the Whitehead conjecture would give an alternative proof of this fact, and it would also prove the asphericity of ribbon disc complements, which are spaces of similar properties to the classical knot complements. This is an important open problem, and we will treat this subject in chapter 4.

Given a connected CW-complex X, a universal cover of X will be denoted by $\tilde{X}$. Since all of the universal covers of a connected CW-complex are homeomorphic (as coverings over X), we will refer to *the* universal cover of X.

Remark 2.1.1. Given a 2-complex X, the condition $\pi_n(X) = 0$ for all $n \geq 2$ is equivalent to any of the following conditions:

- $\pi_n(\tilde{X}) = 0$ for all $n \geq 1$,
- $\pi_2(\tilde{X}) = 0$,
- $\pi_2(X) = 0$,
- $H_2(\tilde{X}) = 0$,
- $H_n(\tilde{X}) = 0$ for all $n \geq 1$,
- $\tilde{X}$ is contractible.

Here we use the fact that $p_* : \pi_q(\tilde{X}) \to \pi_q(X)$ is an isomorphism for $q > 1$, and the theorems of Hurewicz and Whitehead previously mentioned. This means that Whitehead's question can be restated as follows. Given L a 2-complex with $\pi_2(L) = 0$, is $\pi_2(K) = 0$ for all possible subcomplexes $K \subseteq L$? Or, can π_2 of a 2-complex be killed by attaching cells of dimensions 0,1,2?

Note that these questions have a positive answer if one changes π_2 by H_2. Just consider the homology exact sequence and the fact that 2-complexes have trivial homology in dimensions $n \geq 3$. Note also that changing 2 by 3 gives a very simple question (with negative answer), since $\pi_3(S^2) \neq 0$, and therefore D^3 has subcomplexes with non-trivial third homotopy group.

The following result will be useful in the following chapters.

Lemma 2.1.2. *Let K be a 2-complex and let L be a complex obtained from K by attaching cells of dimensions 0 and 1 to the 1-skeleton of K. Then L is aspherical if and only if K is.*

Proof. Suppose first that K is aspherical. We may assume that L is connected. Let $p : \tilde{L} \to L$ be the universal cover for L and let $K' = p^{-1}(K)$ be the portion covering K. Usually K' is not connected, but p restricted to each of its connected components is a regular cover associated to the subgroup $\ker(i_* : \pi_1(K) \to \pi_1(L)) \triangleleft \pi_1(K)$. But the construction of L implies that this subgroup is trivial, so each component of K' is contractible and therefore acyclic. Thus $H_q(K') = 0$ for $q > 0$. Besides, $\tilde{L} \smallsetminus K'$ has only cells in dimensions 0 and 1, so $H_2(\tilde{L}, K') = 0$ and by the homology exact sequence of the pair $(\tilde{L}, K')$ we have $H_2(\tilde{L}) = 0$.

For the converse implication it is sufficient to prove that if $K \vee S^1$ is aspherical, then so is K. Moreover, it is sufficient to prove that if the subspace $A \subseteq X$ of a space X is a retract, and X is aspherical, then so is A. Let $i : A \to X$ be the inclusion and $r : X \to A$ such that $r \circ i = 1_A$. Then the identity $\pi_2(1_A) : \pi_2(A) \to \pi_2(A)$ factors through $\pi_2(L)$ and is therefore trivial. □

Remark 2.1.3. It is easy to see that the conjecture holds when the inclusion $K \to L$ induces an injective morphism on π_1, since then the universal cover of K can be seen as a subcomplex of the universal cover of L, as in the previous observation, and therefore $H_2(\tilde{L}) = 0$ implies $H_2(\tilde{K}) = 0$, since $H_3(\tilde{L}, \tilde{K}) = 0$.

2.2 Alternative notions of asphericity

As we have observed in chapter 1, the study of asphericity of 2-complexes is strongly connected with group theoretical problems, because of the correspondence established with the assignment of the standard complex of a presentation. A group presentation $P = \langle X|R\rangle$ is defined to be aspherical if its standard complex K_P is aspherical. In the present thesis we will only work with this notion of asphericity of a presentation. However, we shall present here some alternative notions of asphericity for group presentations which appear in the literature, and the relations between them. We refer the reader to [CCH81] for further details.

Definition 2.2.1. A presentation P is said to be *aspherical* (A) if K_P is aspherical.

Recall that a 2-complex is aspherical if and only if its universal cover has a trivial second homology group (see section 2.1), so P is aspherical if and only if $H_2(\tilde{K}_P) = 0$.

The *Cayley complex* C_P associated to the presentation $P = \langle X \mid R\rangle$ is a quotient of the universal covering space $\tilde{K}_P$ of the standard complex K_P. For each $r \in R$ with root s, identify every cell $\tilde{e}_r^2$ over e_r^2 with $\gamma_{\overline{s}}(\tilde{e}_r^2)$. Here $\gamma_{\overline{s}}$ denotes the deck transformation associated to the class $\overline{s}$ of s in $G = F(X)/N(R)$. Note that, if no relator is a proper power, then $C_P = \tilde{K}_P$.

A presentation P is said to be *concise* if every relator of P is a reduced non-empty word, and no relator is a conjugate of another relator or its inverse. A presentation P' is said to be a *concise refinement* of P if it can be obtained from P by reducing its relators and eliminating conjugates and inverses. Elimination of empty relators is not allowed, thus a presentation P which has a relator $r = 1$ has no concise refinement. Note also that P and P' are in general not Q^{**}-equivalent.

Definition 2.2.2. A presentation P is defined to be *combinatorially aspherical* (CA) if there exists a concise refinement P' of P such that the Cayley complex $C_{P'}$ of P' is aspherical.

Actually, it can be seen that the definition does not depend on the choice of the concise refinement.

Since the Cayley complex of a presentation P, as well as $\tilde{K}_P$, is always simply connected, P is CA if and only if $H_2(C_{P'}) = 0$ for some concise refinement P'.

This notion of asphericity is weaker than the topological one. The following result comparing these two notions can be found in [CCH81].

Proposition 2.2.3. *Consider a presentation $P = \langle X|R\rangle$ where every $r \in R$ is reduced. Then P is aspherical if and only if P is CA, and concise and no relator is a proper power in $F(X)$.*

Note that the condition about proper powers considers only proper powers in the free group. As we shall see later in this chapter, some results need a similar hypothesis involving proper powers in other associated groups.

We introduce now the Peiffer operations, in order to give a combinatorial characterization of combinatorially aspherical presentations due to Chiswell et al [CCH81].

Given a tuple $(g_1, g_2, \ldots, g_n)$ of elements in a group G, the *Peiffer operations* are defined as follows.

- *Exchange* consists of the replacement of g_i, g_{i+1} by $g_i g_{i+1} g_i^{-1}, g_i$ or by $g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}$.
- *Deletion* consists of the omission of g_i, g_{i+1} if $g_i g_{i+1} = 1$.
- *Insertion* is the inverse of the deletion operation.

Note that the two possible exchange operations are mutually inverse operations. An *identity sequence* over P is a sequence $(g_1, \ldots, g_n)$ of elements in $S = \{g \in F(X) : g = wr^{\epsilon}, w \in F(X), r \in R, \epsilon \in \{-1, 1\}\}$ such that $g_1 \cdots g_n = 1$ in $F(X)$.

Proposition 2.2.4. *A presentation P is CA if and only if every identity sequence over P can be transformed into the empty sequence by Peiffer operations.*

Definition 2.2.5. A presentation P is said to be *diagrammatically aspherical* (DA), if every identity sequence over P can be transformed into the empty sequence by exchanges and deletions.

It is clear that DA implies CA. The following example shows that the converse does not hold.

Example 2.2.6. (Chiswell) [CCH81] The presentation $P = \langle a, b \mid b^{-2}aba, a \rangle$ is CA but not DA.

Since P is balanced and presents the trivial group, the complex K_P is simply connected, that is, $K_P = \tilde{K}_P$. In addition $\chi(K) = 1$ (χ denotes the Euler characteristic). Therefore $\pi_2(K_P) = H_2(K_P) = 0$ and P is CA. To see that P is not DA, Chiswell exhibits the identity sequence

$$\sigma = (b^{-2}aba, (ba)^{-1}a^{-1}(ba), bab^{-1}, b(b^{-2}aba)^{-1}b^{-1}),$$

and proves that it cannot be transformed into the empty sequence by exchanges and deletions.

Given a presentation $P = \langle X \mid R \rangle$, we denote by $N(R)$ the normal closure of R in $F(X)$, and $C(g)$ the centraliser of an element $g \in F(X)$. A *Cohen-Lyndon* basis for the group $N(R)$ is a basis of the form $\bigcup_{r \in R} \{uru^{-1} : u \in U(r)\}$, where $U(r)$ is a full left transversal for $N(R)C(r)$ in $F(X)$.

Definition 2.2.7. A presentation P is called *Cohen-Lyndon aspherical* (CLA) if for some refinement $P' = \langle X \mid R' \rangle$ of P, $N(R')$ has a Cohen-Lyndon basis.

Lyndon and Schupp [LS01] proved that CLA implies DA. It is not known whether the converse of this implication holds.

It is proved in [CCH81] that every subpresentation of a CLA presentation is also CLA. It is clear from the definition of DA that the same holds for DA presentations. If this were also true for CA presentations, then it would also hold for aspherical presentation, that is, the Whitehead conjecture would be proved for finite 2-complexes.

2.3 Crossed modules

We present the theory of crossed modules, introduced by Whitehead in [Whi46], although the ideas were already present in [Whi41]. Let $X \subseteq Y$ be a pair of connected spaces with a common base point. Consider the end of the homotopy exact sequence of the pair (Y, X).

$$\pi_2(X) \longrightarrow \pi_2(Y) \longrightarrow \pi_2(Y,X) \xrightarrow{\partial} \pi_1(X) \longrightarrow \pi_1(Y) \longrightarrow \pi_1(Y,X)$$

The action of $\pi_1(X)$ on the second relative homotopy group $\pi_2(Y,X)$ satisfies the following properties.

- $\partial(\gamma \cdot f) = \gamma(\partial f)\gamma^{-1}$ for all $f \in \pi_2(Y,X), \gamma \in \pi_1(X)$.
- $(\partial f) \cdot g = gfg^{-1}$ for all f, g in $\pi_2(Y,X)$.

If we consider the pair consisting of a 2-complex K and its 1-skeleton, we obtain the *fundamental sequence of* K:

$$0 \longrightarrow \pi_2(K) \longrightarrow \pi_2(K,K^1) \xrightarrow{\partial} \pi_1(K^1) \longrightarrow \pi_1(K) \longrightarrow 1$$

Definition 2.3.1. A *G-crossed module* (C, ∂, G) consists of two groups C, G, a map of groups $\partial : C \to G$ and a group action of G on C satisfying

(cm1) $\partial(g \cdot c) = g(\partial c)g^{-1}$ for all $c \in C, g \in G$.

(cm2) $(\partial c) \cdot d = cdc^{-1}$ for all c, d in C.

Therefore $(\pi_2(K,K^1), \pi_1(K^1), \partial)$ is a $\pi_1(K^1)$-crossed module, the crossed module associated to the 2-complex K.

The relation between the theory of crossed modules and homotopy of 2-complexes derives from the fact, proved by Whitehead [Whi49], that if a space Y is obtained from X by attaching 2-cells, then $\pi_2(Y,X)$ is a free crossed $\pi_1(X)$-module on the set of characteristic maps of the attached cells. A crossed module is said to be free on a given basis if it satisfies a universal property for morphisms of crossed modules. For these definitions and further information about crossed modules, see for example [Dye93, BHS11]. Whitehead also proved that every free crossed module can be realised as the associated crossed module of a pair of such spaces. Cockcroft used this result for the proof of theorem 2 [Coc54], the validity of the conjecture in the case where the subcomplex has only one 2-cell.

Brown and Higgins [BH78] proved a 2-dimensional analogue of the van Kampen theorem which had several consequences, for example an alternative proof that the crossed module associated to a 2-complex is free. The results of that article were later generalized by Brown [Bro80, Bro84]. One of the fundamental contributions is the following. If a 2-complex K is the union of two subcomplexes K_1, K_2 with a common 1-skeleton, then the crossed module associated to K is the coproduct of those associated to K_1, K_2. In addition, Brown gave a simple description of the coproduct of two crossed modules.

Gutierrez and Ratcliffe [GR81] also proved a similar theorem about the second homotopy group of a union of 2-complexes with common 1-skeleton. In the same article they use these methods together with Lyndon's Identity Theorem to prove that a 2-complex associated to a group presentation $\langle X \mid R\rangle$ is aspherical provided that the relators are independent and not proper powers. A relator r is *independent* if $N(r) \cap N(R \smallsetminus \{r\}) = [N(r), N(R \smallsetminus \{r\}]$, where $N(-)$ means the smallest normal generated subgroup.

2.4 Simple Identity Theorem

In 1950, Lyndon gave a fundamental contribution to group theory when he computed the cohomology of groups which are presented with a single relator [Lyn50]. To do this, he constructed a free resolution of a given finitely presented group G with presentation P. This resolution was later recognized by Reidemeister as the chain complex of a contractible CW-complex, namely the universal covering space of the Eilenberg-Mac Lane space $K(G,1)$, which is obtained from K_P by attaching cells of dimensions $k \geq 3$ to kill the higher homotopy groups. In the case of one-relator presentations, this resolution is of a particular form. Concretely, the theorem states the following.

Theorem 2.4.1. *[Lyn50, Simple Identity Theorem] If $r = s^q$, for q maximal, is a word in the free group F, and $N(r)$ is the smallest normal subgroup of F containing r, then*

$$\prod_i u_i r^{\epsilon_i} u_i^{-1} = 1$$

implies that the indices can be grouped into pairs (i,j) such that $\epsilon_i = -\epsilon_j$ and, for certain integers c_i, $u_i \equiv u_j s^{c_i}$ modulo $N(r)$.

An important topological application of Lyndon's theorem is the following result of Cockcroft, which solves the conjecture for the case when the subcomplex K is finite and has only one 2-cell.

Theorem 2.4.2. *[Coc54, Theorem 2] If K is a finite connected non-aspherical 2-complex with only one 2-cell, then the complex L obtained from K by attaching 2-cells to the 1-skeleton of K is non-aspherical.*

For the proof, he studies the attaching map for the 2-cell of K as an element in the fundamental group of $\pi_1(K)$, which is the unique relator of the associated presentation. He considers whether the relator can be written as a power of a word in the free group $\pi_1(K^1)$.

This result was later generalized by Howie [How82], who introduced the notions of reducible complexes and reducible presentations.

An *elementary reduction* consists of a pair (L, K) of 2-complexes with $K \subseteq L$ such that L is obtained from K either by adding a unique 1-cell, or by adding a pair of cells e^1, e^2 satisfying that the attaching map of the new 2-cell e^2 *strictly involves* e^1, that is, it is not homotopic in $K \cup L^1$ to a map into K. Elementary reductions can be seen as a generalization of elementary collapses.

Definition 2.4.3. A 2-complex L is said to be *reducible* if for every finite subcomplex L' of L either $L' \subseteq L^0$ or there exists an elementary reduction (L', K'). A presentation P is said to be *reducible* if the associated complex K_P is reducible.

Of course any 2-complex with a unique 2-cell is reducible.

A presentation P is reducible if and only if every finite subpresentation $P' = \langle X \mid R\rangle$ with $R \neq \varnothing$ satisfies the following. There is a generator $x_0 \in X$ and a relator $r_0 \in R$ such that, if $G = \langle X - \{x_0\} \mid R - \{r_0\}\rangle$, then r_0 is not a conjugate in $G * \langle x_0\rangle$ to any element of G.

An element g of a group G is called a *proper power* if there exist $h \in G$ and $p > 1$ such that $g = h^p$. A relator r of a presentation P is said to be a *proper power* if it represents a proper power in the group presented by $P' = \langle X \mid R - \{r\}\rangle$. A presentation is said to have *no proper powers* if none of its defining relators is a proper power. Note that this condition is different to the one defined previously, involving proper powers in the free group $F(X)$.

Theorem 2.4.4. *[How82, Corollary 4.5] If a group G has a reduced presentation with no proper powers, then G is locally indicable.*

A 2-cell e^2 of a complex X is said to be *attached by a proper power* if the attaching map of e^2 is a proper power in $\pi_1(X - e^2)$. So the theorem implies that a reducible 2-complex in which no 2-cell is attached by a proper power is aspherical.

It can be seen (see [Bog93]) that if a complex K is a subcomplex of an aspherical 2-complex L, then the cells K cannot be attached with proper powers. This means that the theorem of Cockcroft follows from Howie's result.

The theorems of Cockcroft and Lyndon are actually equivalent. Dyer and Vasquez [DV73] proved Cockcroft's theorem using topological methods, and applied it to give alternative proofs for serveral results, including Lyndon's Identity Theorem.

See [ARS84] and [Bog93] for further details on this methods.

2.5 Covering maps and conditions on $\pi_1(K)$

One of the most important advances in the problem of Whitehead was achieved by Cockcroft in 1954, who proved the asphericity of the subcomplex K when its fundamental group satisfies certain conditions. His main tools for the proof of this theorem were two well known formulas of Hopf that compare the homology and homotopy groups of a 2-complex with the homology groups of its fundamental group. Namely,

$$H_2(X)/\Sigma_2(X) \simeq H_2(\pi_1(X)), \qquad \Gamma_2(X)/\pi_2^0(X) \simeq H_3(\pi_1(X)),$$

where $\Sigma_2(X)$ and $\Gamma_2(X)$ respectively denote the image and kernel of the Hurewicz morphism $\pi_2(X) \to H_2(X)$, and $\pi_2^0(X)$ is the subgroup of $\pi_2(X)$ generated by $\{gf - f : g \in \pi_1(X), f \in \pi_2(X)\}$. Note that $\pi_2^0(X)$ is in fact included in $\Gamma_2(X)$. The quotient $\Gamma_2(X)/\pi_2^0(X)$ trivializes of the action of $\pi_1(X)$. The result we are refering to is the following.

Theorem 2.5.1. *[Coc54, Theorem 1] If K is a finite connected non-aspherical 2-complex and $\pi_1(K)$ is either finite, abelian or free, then the complex L obtained from K by attaching 2-cells is also non-aspherical.*

He also proved that given a finite 2-complex K such that $H_2(K) = 0$, if $\pi_1(K)$ is free, then K is aspherical. We will see that this result was later generalized to a very large class of groups.

The following diagram yields a simple but important observation made by Cockcroft in the same article. The horizontal maps are the Hurewicz maps for K and L and the vertical ones are induced by the inclusion $K \to L$.

$$\begin{array}{ccc} \pi_2(K) & \xrightarrow{h_K} & H_2(K) \\ \downarrow & & \downarrow \\ \pi_2(L) & \xrightarrow{h_L} & H_2(L) \end{array}$$

If we assume $\pi_2(L) = 0$, then we have that $h_K = 0$. A 2-complex for which this map is trivial is now said to be a *Cockcroft complex*. There are generalizations to this notion and several applications (see for example [Dye93, Bog93]).

Definition 2.5.2. A group G is *conservative* if for every covering $E \to B$ with deck transformation group isomorphic to G, $H_2(B) = 0$ implies $H_2(E) = 0$.

Clearly a 2-complex K with $H_2(K) = 0$ and conservative fundamental group is aspherical, since the universal covering has deck transformation group isomorphic to $\pi_1(K)$.

The notion of conservative group was introduced by Adams [Ada55], who used these groups to generalize the above theorem of Cockcroft. Given any non-aspherical subcomplex K of an aspherical simply connected 2-complex, he found a regular acyclic covering, namely the covering corresponding to a certain perfect subgroup of $\pi_1(K)$, for which the quotient group is conservative. A group P is said to be *perfect* if it coincides with its commutator subgroup, that is, $P_{ab} = 1$. He also proved that torsion-free abelian groups are conservative and that extensions and limits of conservative groups are conservative. Cohen [Coh78] and Howie [How79] gave new versions and proofs for the theorems of Cockcroft and Adams.

Locally indicable groups first arose in the work of Higman in 1940, who was investigating the units and zero-divisors of group rings. Later, in 1970, Karras and Solitar proved some properties of this class of groups. In the 80's Howie, Brodskii and Short independently resumed the work on these groups, proving the existence of solutions for independent systems of equations over a locally indicable group.

Definition 2.5.3. A group G is said to be *indicable* if there exists a surjective morphism $G \to \mathbb{Z}$ to the group of integers. This is equivalent to having infinite first homology group $H_1(G) = G_{ab} = G/[G,G]$ (or equivalently $H^1(G) \neq 0$).

A group G is said to be *locally indicable* if every non-trivial finitely generated subgroup of G is indicable.

Howie and Schneebeli [HS83] studied homological properties of locally indicable groups and proved that this notion coincides with that of conservative groups.

Howie and Short [HS85] have proved that knot groups are locally indicable, using that knot complements had already been proved to be aspherical by Papakyriakopoulos. The condition of local indicability will appear later in this work.

The notion of towers and the technique of tower constructions were introduced by Howie in [How81a], although he had been using tower methods in his previous work [How79, How81b]. A *tower* is a map between complexes that can be written as a composition of inclusions and coverings. When the coverings of a tower $E \to B$ are requested to be $\mathbb{Z}$-coverings, then $H_n(B) = 0$ implies $H_n(E) = 0$ [How81a]. This result is similar to the theorem of Adams [Ada55] stating that $\mathbb{Z}$ is a conservative group. The work in [How81a] is about a group theoretic conjecture, namely whether independent systems over groups have solutions. This conjecture is proved for locally indicable groups, using topological methods regarding tower constructions.

One of the most important contributions to the study of the Whitehead conjecture is the following theorem, where Howie reduced the problem to the following two particular cases.

Theorem 2.5.4. *[How83, Theorem 3.4] If the answer to Whitehead's question is* negative*, then there exists a counterexample $K \subset L$ of one of the following two types:*

(a) L is finite and contractible, $K = L - e$ for some 2-cell e of L, and K is non-aspherical.

(b) L is the union of an infinite chain of finite non-aspherical subcomplexes $K = K_0 \subset K_1 \subset \cdots$ such that each inclusion map $K_i \to K_{i+1}$ is nullhomotopic.

Later, Luft [Luf96] proved that the existence of a counterexample of type (a) actually implies the existence of a counterexample of type (b).

Considering the first possibility, Howie states the following question: Assuming L is a finite contractible 2-complex and e^2 is a 2-cell of L, is $L - e^2$ aspherical? In this direction, he obtains the result below [How83, Theorem 4.2], which links this problem with the conjecture about the asphericity of ribbon disc complements. Recall that a spine of a complex X is a subcomplex $Y \subseteq X$ such that X collapses to Y.

Theorem 2.5.5. *If a finite 2-complex L can be 3-deformed to a point, then for every 2-cell e^2 of L the subcomplex $K = L - e^2$ can be 3-deformed to a spine of a ribbon disc complement.*

To prove this, he assumes L is the 2-complex associated to a group presentation. Using the correspondence of 3-deformations and Q^{**}-transformations, he translates the operations on L to a sequence of operations on K that transform it into a weak LOT presentation. But LOTs and weak LOTs are spines of ribbon disc complements. These spaces and the conjecture about their asphericity will be ellaborated with more details in chapter 4.

Howie defined a 2-complex K to be *almost-acyclic* if $H_2(K) = 0$ and $H_1(K)$ is torsion-free, and observed that this class of complexes is closed under subcomplexes and $\mathbb{Z}$-coverings. Also using tower methods, he proved a new version of Adams' theorem, namely that if the subcomplex K of an aspherical 2-complex L is non-aspherical, then the kernel of $i_* : \pi_1(K) \to \pi_1(L)$ has a non-trivial finitely generated perfect subgroup. Later, in [How81b], he proved that the fundamental group of an almost-acyclic 2-complex is locally indicable if and only if it has no finitely generated perfect subgroups. Then in [How82] he proved that for a 2-complex K with locally indicable fundamental group, $H_2(K) = 0$ implies $\pi_2(K) = 0$. This result became trivial given the equivalence between conservability and locally indicability. It was in the same article, and also using tower constructions, that he proved the result mentioned above on the asphericity of reducible complexes for which the attaching maps are not proper powers. He proved this showing that they have locally indicable fundamental groups.

Also using local indicability, Howie proved in [How85] that a class of ribbon disc complements are aspherical.

2.6 Small cancellation theory

There are various results about asphericity of group presentations involving *small cancellation*. We present here the small cancellation hypotheses and some of the applications to asphericity. More information about this subject can be found in [Lyn66, Sch71, Sch73, Ger87, Hue79, LS01, Gro03].

The idea of a group presentation $P = \langle X \mid R \rangle$ satisfying small cancellation is that, given a pair of elements $r, s \in R$, one can reduce the word rs by canceling a small part. The precise hypotheses are stated below. The geometric applications of these conditions involve spherical diagrams and combinatorial notions of asphericity, such as diagrammatic reducibility.

Definition 2.6.1. Given a finite group presentation $P = \langle X \mid R \rangle$, let R^* be the symmetrization of R, that is, the set of all distinct cyclic permutations of the relators $r \in R$ and their inverses. A word w is a *piece* relative to R if it is a common prefix of two distinct words in R^*. In the following hypotheses, p and q are integers, $p \geq 1$, $q \geq 3$, and λ is a positive real number.

The *small cancellation hypotheses* are defined as follows.

- $C(p)$: no element of R^* is a product of less than p pieces.
- $C'(\lambda)$: $|u| < \lambda |r|$ for every prefix u of r that is a piece.
- $T(q)$: for every $k \in \{3, \ldots, q-1\}$ and every sequence $(r_1, \ldots, r_k)$ of elements in R^*, such that $r_i \neq r_{i+1}^{-1}$ for all $i \leq k-1$, $r_k \neq r_1^{-1}$, at least one of the products $r_i r_{i+1}$ or $r_k r_1$ is freely reduced.

$C'(\lambda)$ are called *metric* conditions and $C(p)$ are called *non-metric*.

Note that $T(3)$ is satisfied by every presentation, and that $C'(1/n)$ implies $C(n+1)$.

Proposition 2.6.2. *[Ger87, Remark 4.18]. Let $\langle X|R\rangle$ be a presentation where each relator is a cyclically reduced word. Suppose that no relator is a proper power or a conjugate of another relator or of its inverse (in the free group $F(X)$). Additionally, assume that the symmetrized presentation $\langle X|R^*\rangle$ satisfies conditions $C(p)$ and $T(q)$ for a pair $(p,q) \in \{(6,3),(4,4),(3,6)\}$. Then the associated complex is diagrammatically reducible (and therefore aspherical).*

In order to state the following result, called *weight test*, we need to recall some definitions and notations.

Definition 2.6.3. Let K be a combinatorial 2-complex. Let e be a 2-cell of K with characteristic map $\Phi : D^2 \to K$. Then there is a subdivision of ∂D^2 such that the attaching map $\varphi = \Phi\,|_{\partial D^2}$ of e sends each edge of this subdivision homeomorphically onto a 1-cell of K. The neighborhood in D^2 of a vertex of that subdivision is called a *corner* of e and of K.

A *weight* on K is the assignment of a real number to every corner of K.

For a vertex v in K, $d(x)$ denotes the number of corners incident at v, and for a 2-cell e of K, $d(e)$ denotes the number of corners of e.

Theorem 2.6.4. *[Ger87, Theorem 4.7] (Weight test) Suppose a combinatorial 2-complex K has a weight w satisfying the following conditions.*

- *For each non-trivial reduced circuit z in the link complex of K, $\sum_{\gamma < z} w(\gamma) \geq 2$.*
- *For each 2-cell e of K, $\sum_{\gamma < e} w(\gamma) \leq d(e) - 2$.*

Then K is diagrammatically reducible.

Huck and Rosebrock [HR95] presented a new version of this test, using the *Whitehead graph* of the complex K_P, which has edges in correspondence to the corners of the complex. The result obtained involves the notion of *vertex asphericity* (or VA), which is weaker than DR but also implies asphericity.

They also introduce a *hyperbolic weight test*, using a variation of the Whitehead graph, called the *Star graph*. They showed that if a LOT presentation satisfies one of these tests, then the presentation obtained by reorienting one edge also satisfies it.

G. Ellis recently developed software with applications to asphericity involving these results. In the GAP [GAP15] package HAP (*homological algebra programming*) [HAP13], there is a function called `IsAspherical(F,R)` that examines whether the complex K_P associated to the presentation $P = \langle F \mid R\rangle$ is piecewice Euclidean non-positively curved. This is a sufficent (but not necessary) condition for K_P to be aspherical (see [BH99]), and it can be tested with a set of inequalities.

Resumen del capítulo 2: La conjetura de asfericidad de Whitehead

En este capítulo presentamos la conjetura de asfericidad de Whitehead, y los avances alcanzados hasta ahora en el tema, junto con las estrategias que se usaron para ello.

En la sección 2.1 introducimos la noción de asfericidad y el contexto en el que fue formulada la conjetura. Un espacio topológico arcoconexo X se dice asférico si sus grupos de homotopía $\pi_n(X)$ son triviales para $n > 1$. Cuando se trata de un 2-complejo (un CW-complejo de dimensión 2), esto es equivalente a que su segundo grupo de homotopía sea trivial.

J. H. C. Whitehead se encontraba trabajando en la relación entre $\pi_n(X)$ y $\pi_n(Y)$ para espacios X e Y, donde Y se obtiene de X adjuntando n-celdas cuando se preguntó si todo subcomplejo K de un 2-complejo asférico L sería a su vez asférico [Whi41]. Esta pregunta suele ser tratada como una conjetura, y aún hoy sigue abierta.

Hay diferentes caracterizaciones para 2-complejos asféricos. Se puede ver que las siguientes condiciones son equivalentes para un 2-complejo X ($\tilde{X}$ denota el revestimiento universal de X).

- X es asférico ($\pi_n(X) = 0$ para todo $n \geq 2$),
- $\pi_n(\tilde{X}) = 0$ para todo $n \geq 1$,
- $\pi_2(\tilde{X}) = 0$,
- $\pi_2(X) = 0$,
- $H_2(\tilde{X}) = 0$,
- $H_n(\tilde{X}) = 0$ para todo $n \geq 1$,
- $\tilde{X}$ es contráctil.

Presentamos luego dos resultados elementales conocidos que serán útiles más adelante. Si el 2-complejo L se obtiene de K agregando celdas de dimensión 0 y 1, entonces L es asférico si y sólo si K lo es. Por otro lado, vemos que la conjetura es cierta para un par de complejos $K \subseteq L$ si se requiere que la inclusión $K \hookrightarrow L$ induzca un monomorfismo en los grupos fundamentales.

A partir de la correspondencia entre 2-complejos y presentaciones de grupos vista en el primer capítulo, decimos una presentación P es *asférica* (A) si su complejo asociado K_P es un espacio asférico, o equivalentemente, si $H_2(\tilde{K}_P) = 0$. Dada la interacción que se da en entre distintas áreas de la matemática al rededor de esta correspondencia entre complejos y presentaciones, el tema ha sido abordado desde puntos de vista muy diversos, y es por eso que en la literatura se encuentran distintas nociones de asfericidad. En la sección 2.2 repasamos algunas de estas nociones. Para más detalles en este tema, referimos al lector a [CCH81].

La noción de presentación *combinatoriamente asférica* (CA) se define a partir del complejo de Cayley C_P de la presentación, que es un cociente de $\tilde{K}_P$, y que coincide con $\tilde{K}_P$ si las relaciones de la presentación no son potencias propias. Esta noción es más débil que la noción topológica de asfericidad (A). De hecho, (A) es equivalente a (CA) si se requieren dos hipótesis adicionales: que las relaciones no sean potencias propias, y que no haya una de ellas que es conjugada o inversa de otra [CCH81].

Luego definimos dos nociones más, la de *diagramáticamente asférica* (DA) y la de *Cohen-Lyndon asférica* (CLA), y mencionamos las implicaciones que valen entre ellas.

Las propiedades (CLA) y (DA) son hereditarias en el siguiente sentido. Si una presentación P es (CLA) (resp. (DA)), entonces toda subpresentación de P lo es [CCH81]. Si esto valiera también para (CA), valdría la conjetura de Whitehead en el caso compacto.

En las siguientes secciones repasamos las principales herramientas utilizadas hasta ahora en el estudio de asfericidad y de la conjetura de Whitehead, y los resultados más importantes que se obtuvieron.

Comenzamos con la teoría de módulos cruzados, que fue introducida por el mismo Whitehead [Whi41, Whi46]. La estructura de módulo cruzado es una codificación en términos algebraicos de las propiedades que satisface la acción de $\pi_1(X)$ en $\pi_2(Y, X)$ para un par de espacios $X \subseteq Y$. En [Dye93, BHS11] se puede encontrar información sobre módulos cruzados. Whitehead [Whi49] probó que si Y se obtiene de X adjuntando celdas de dimensión 2, entonces $\pi_2(Y, X)$ es un $\pi_1(X)$-módulo cruzado libre, con una base en correspondencia con las 2-celdas adjuntadas. Este resultado fue usado en muchos trabajos. Entre ellos, Cockcroft lo aplicó, junto con el teorema de Lyndon que veremos más abajo, para probar la conjetura en el caso en que el subcomplejo K tiene una sola 2-celda [Coc54]. Brown y Higgins [BH78], y luego Brown [Bro80, Bro84] trabajaron en esta teoría, obteniendo una descripción eficiente del módulo cruzado de una unión de 2-complejos que se intersecan en el 1-esqueleto, introduciendo la noción de coproducto de módulos cruzados. Gutierrez y Ratcliffe también contribuyeron a la conjetura aplicando métodos de módulos cruzados [GR81].

El teorema de Lyndon, *simple identity theorem* [Lyn50], formulado en el contexto de presentaciones de grupos, fue un aporte fundamental en el estudio de asfericidad de 2-complejos. Esto se debió principalmente a la aplicación que hizo Cockcroft [Coc54], al usarlo para demostrar la conjetura en el caso en que el subcomplejo tiene una sola 2-celda. El teorema de Lyndon involucra presentaciones con una sola relación, y las presentaciones asociadas a complejos con una sola 2-celda son de este tipo. Más adelante, Howie generalizó estos resultados [How82] a partir de la noción de presentación reducible, y de complejo reducible. Las presentaciones de este tipo presentan propiedades similares a las de una relación. Dyer y Vasquez [DV73] probaron el teorema de Cockcroft con métodos topológicos, y luego dedujeron el teorema de Lyndon.

Muchos de las respuestas parciales a la conjetura involucran hipótesis sobre el grupo fundamental del subcomplejo K de L. Cockcroft obtuvo otro avance importante en su artículo [Coc54] al probar la validez de la conjetura para los pares $K \subseteq L$ tales que $L - K$ sólo son celdas de dimensión 2, y $\pi_1(K)$ es libre, abeliano, o finito. Para esto, aplicó las fórmulas clásicas de Hopf sobre homología y homotopía de grupos. También probó que si $\pi_1(K)$ es libre, para un 2-complejo K, entonces $H_2(K) = 0$ implica $\pi_2(K) = 0$. Más adelante, este resultado se generalizó a una amplia clase de grupos. Adams trabajó con los grupos *conservativos* [Ada55], y Howie con los *localmente indicables* [How82], pero luego dos clases de grupos resultaron ser una sola [HS83]. Las técnicas usadas para probar estos resultados involucran construcciones con revestimientos, y *torres*. Las torres fueron introducidas y utilizadas recurrentemente por Howie [How79, How81b, How81a], y son funciones que se descomponen como composiciones de revestimientos e inclusiones. Destacamos el siguiente teorema de Howie, que separa la conjetura de Whitehead en dos casos particulares, uno compacto y uno no compacto.

Teorema 2.5.4. *[How83, Teorema 3.4] Si la conjetura de Whitehead no es cierta, entonces existe un contraejemplo $K \subset L$ de uno de los siguientes dos tipos:*

(a) L es finito y contráctil, $K = L - e$ para una 2-celda e de L, y K no es asférico.

(b) L es la unión de una cadena infinita de subcomplejos finitos no asféricos $K = K_0 \subset K_1 \subset \cdots$, tales que cada inclusión $K_i \to K_{i+1}$ es homotópica a una constante.

Por último, mencionamos algunos resultados obtenidos a través de la teoría llamada *Small Cancellation Theory*. Estos resultados se obtienen a través de métodos combinatorios que involucran nociones combinatorias de asfericidad.

Chapter 3

Finite spaces and applications to the Whitehead conjecture

Finite spaces were first studied by Alexandroff, who found their correspondence with finite partially ordered sets (posets). This fact suggested a possible combiantorial approach to finite spaces. Later, in the year 1966, two important works were published. McCord [McC66] proved that finite spaces serve as models for compact polyhedra. For this, he assigned to each simplicial complex K a finite space $\mathcal{X}(K)$ weakly equivalent to K. Stong [Sto66] studied the homotopy theory of finite spaces with combinatorial methods, introducing the notion of beat point. He found a necessary and sufficent condition for two finite spaces to be homotopy equivalent, which is easy to verify.

In the year 2003, May wrote a series of notes [May03c, May03b, May03a] summarizing everything known at that time about finite spaces and formulating some conjectures that connect this theory with classical homotopy theory.

In the year 2005 Barmak and Minian started working at this field, answering some of the questions risen by May, such as the characterization of the smallest finite spaces with the homotopy groups of the spheres [BM07]. Later they investigated problems related to simple homotopy theory [BM08b, BM08a, BM12b] and applied the results to analyze the conjectures of Quillen and Andrews-Curtis [Bar11].

In this chapter we start by reviewing in section 3.1 the basic notions of the theory of finite topological spaces, and some of the results that show how to use them to study problems of polyhedra. In section 3.2 we recall the definitions of qc-reducible finite spaces and quasi-constructible complexes, needed in the succeeding sections. For further details on the theory of finite spaces and its applications we refer the reader to [Bar11, BM08b, May03c, McC66, Sto66]. In section 3.3 we translate the compact case of the Whitehead conjecture in terms of finite spaces. Then some of the main results of this work are presented. In section 3.4 we show the validity of the conjecture for a large class of compact complexes, including all constructible complexes. Then in section 3.5 we introduce a new method of reduction and apply it to prove the conjecture for a new class of complexes. In section 3.6 we introduce new combinatorial methods for the study of asphericity, and algorithmic implementations for these methods.

3.1 Definitions and basic results

There is a natural connection between finite topological spaces and finite posets. Given a finite set X and a topology on X, one can define a relation on the set X by

$$x \leq y \Leftrightarrow x \in U \text{ for every open set } U \text{ containing } y.$$

It is easy to see that this relation is a preorder, i.e. it is reflexive and transitive. Note that it is antisymmetric if and only if the topology satisfies the T_0 axiom: for any given pair of points of X, there is an open set which contains one and only one of them.

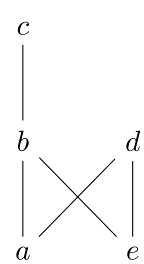
Conversely, given a preorder $\leq$ and an element $x \in X$, we set

$$U_x := \{y \in X : \ y \leq x\} \subseteq X.$$

The sets U_x (for $x \in X$) form a basis for a topology on X. These applications are mutually inverse and give rise to a one-to-one correspondence between finite T_0-spaces and finite posets. From now on we will only consider finite spaces which satisfy the T_0 axiom. In order to represent a finite T_0-space, we will use the Hasse diagram of the corresponding poset.

The Hasse diagram of a finite poset X is the digraph whose vertices are the points of X and whose edges are the pairs (x, y) such that $x \prec y$. Here $x \prec y$ means that x is covered by y, i.e. $x < y$ and there is no $z \in X$ such that $x < z < y$. In the graphical representation of the Hasse diagram, instead of drawing the edge (x, y) with an arrow, we simply put y over x.

Example 3.1.1. The poset $X = \{a, b, c, d, e\}$ with order given by $a < b, a < d, e < d, e < b, b < c$ has the following Hasse diagram.



Proposition 3.1.2. *A map between finite spaces is continuous if and only if it is order preserving.*

Proof. Let $f : X \to Y$ continuous and let $a \leq b$ in X. Since f is continuous and $U_{f(b)}$ is an open subspace of Y, $f^{-1}(U_{f(b)})$ is an open subspace of X. Now $b \in f^{-1}(U_{f(b)})$ and $a \leq b$, so $a \in f^{-1}(U_{f(b)})$.

Conversely, let f be an order preserving map and V an open subspace of Y. Given $b \in f^{-1}(V)$, we shall see that $U_b \subseteq f^{-1}(V)$. Let $a \leq b$, so $f(a) \leq f(b)$, and $f(a) \in U_{f(b)}$. Since V is an open subspace and $f(b) \in V$, then $U_{f(b)} \subseteq V$. This implies that $f(a) \in V$, that is, $a \in f^{-1}(V)$. $\square$

As a consequence, it is easy to see that, for any $Y \subseteq X$, the subspace topology of Y is the same as the topology obtained with the restricted order.

Proposition 3.1.3. *Let X be a finite space. Two elements $x, y \in X$ are in the same connected component if and only if there exists a sequence $x = x_0, x_1, \dots, x_n = y$ such that x_{i-1} and x_i are comparable elements.*

Proof. Let $x \in X$ and let $A = \{y \in X : \exists x = x_0, x_1, \dots, x_n = y, x_{i-1}$ and x_i are comparable$\}$. For every, $y \in A$, $U_y \subseteq A, F_y \subseteq A$, so A is open and closed.

The connected components of a space are closed subspaces, therefore in the case of a finite space they turn out to be open as well. Given $x \in X$, its component must contain U_x and F_x, so any pair of comparable elements must be in the same component. □

Proposition 3.1.4. *A finite space is connected iff it is path connected.*

Proof. It is sufficent to see that for every $x, y \in X$, $x < y$, there is a path between them. Define $\gamma : I \to X$, as $\gamma(t) = x$ for $0 \le t < 1$ and $\gamma(1) = y$. It is easy to see that $\gamma^{-1}(U_z)$ is open for every $z \in X$. Therefore γ is continuous. □

Definition 3.1.5. Given two continuous maps $f, g : X \to Y$, we will say that $f \le g$ if $f(x) \le g(x)$ for every $x \in X$. And we will say that f and g are comparable if $f \le g$ or $g \le f$.

The following characterization of homotopic maps between finite spaces can be found in [Bar11].

Proposition 3.1.6. *Two continuous maps $f, g : X \to Y$ are homotopic iff there exists a sequence $f = f_0, f_1, \dots, f_n = g$ such that f_{i-1} and f_i are comparable for every i.*

Corollary 3.1.7. *Let X be a finite space with maximum (or minimum). Then X is contractible.*

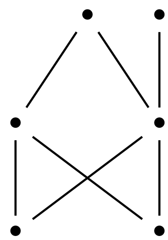
Proof. The identity map is comparable, and therefore homotopic, to a constant map. □

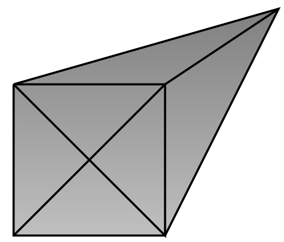
Definition 3.1.8. The *length* of a chain $x_0 < x_1 < \dots < x_n$ in a finite space X is n. The *height* $\mathrm{h}(X)$ is the maximum length of the chains of X.

Given a finite space X, the *order complex* $\mathcal{K}(X)$ is a simplicial complex with vertex set X and n-simplices given by the chains in X of $n+1$ elements.

Given a simplicial complex K, the *face poset* $\mathcal{X}(K)$ of K is the poset of simplices of K ordered by inclusion.

It is clear that $\mathrm{h}(X) = \dim(\mathcal{K}(X))$ for every finite space X and $\mathrm{h}(\mathcal{X}(K)) = \dim(K)$ for every simplicial complex K. Below we show a finite space X of 6 points, represented by its Hasse diagram, and the associated 2-complex $\mathcal{K}(X)$.





A continuous function $f : X \to Y$ gives rise to a simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ in the obvious way, and a simplicial map between complexes $\varphi : K \to L$ induces a continuous function $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$.

The functors $\mathcal{K}$ and $\mathcal{X}$ are not mutually inverse. In fact, if K is a simplicial complex, then $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision K' of K. The following results compare the topology of X with that of $\mathcal{K}(X)$, and the topology of K with that of $\mathcal{X}(K)$, can be found in [McC66], [May03b] and [Bar11], and are due to McCord.

Proposition 3.1.9.

(*i*) *Given a finite space X, there is a weak homotopy equivalence between $\mathcal{K}(X)$ and X.*

(*ii*) *Given a continuous map $f : X \to Y$, between finite spaces, f is a weak homotopy equivalence if and only if $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is.*

(*iii*) *Given a simplicial complex K, there is a weak homotopy equivalence between K and $\mathcal{X}(K)$.*

(*iv*) *Given a simplicial map $f : K \to L$, f is a weak homotopy equivalence if and only if $\mathcal{X}(f) : \mathcal{X}(K) \to \mathcal{X}(L)$ is.*

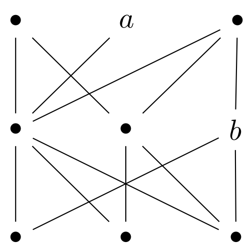
Corollary 3.1.10. *Given a finite space X, there is a weak homotopy equivalence between X and $\mathcal{X}\big(\mathcal{K}(X)\big)$.*

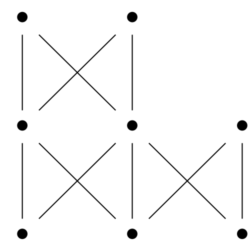
This results imply that weak homotopy types of finite spaces cover all the homotopy types of compact polyhedra. At this point, it is important to notice that the Whitehead theorem does not hold for finite spaces. Counterexamples can be found in [Bar11, BM08b]. Concretely, there are finite spaces which are homotopically trivial (i.e. weak homotopy equivalent to the singleton) but not contractible.

We recall now some of the main results of the homotopy theory of finite spaces, starting with the notion of beat point, which was introduced by R. E. Stong to give a characterization of the homotopy types of finite spaces [Sto66].

Definition 3.1.11. A point $x \in X$ is called a *down beat point* if $\hat{U}_x = \{y \in X : y < x\}$ has a maximum and an *up beat point* if $\hat{F}_x = \{y \in X : y > x\}$ has a minimum.

Note that a beat point can be recognized in the Hasse diagram because it has exactly one incident edge above, or exactly one incident edge below. We illustrate this in the following diagrams. In the first space, a is a down beat point and b is an up beat point. The second space has no beat points.





Proposition 3.1.12. *If x is a beat point of X, then $X \smallsetminus \{x\} \hookrightarrow X$ is a strong deformation retract.*

Proof. If $\widehat{U}_x$ has a maximum $\widetilde{x}$, then the map $r : X \to X \smallsetminus \{x\}$ defined by $r(x) = \widetilde{x}$ and $r|_{X \smallsetminus \{x\}} = id$ satisfies $ir \leq id$, and the result follows. If $\widehat{F}_x$ has a minimum, the proof is similar. □

Furthermore, two finite spaces X and Y have the same homotopy type if and only if Y can be obtained from X by removing and adding beat points.

The following result can be found in [Bar11].

Proposition 3.1.13. *The join of two topological spaces is contractible if and only if one of them is contractible.*

Definition 3.1.14. A point $x \in X$ is called a *weak point* if $\hat{U}_x$ or $\hat{F}_x$ is a contractible finite space, or equivalently, if $\hat{C}_x = \{y \in X : y < x \text{ or } y > x\}$ is contractible.

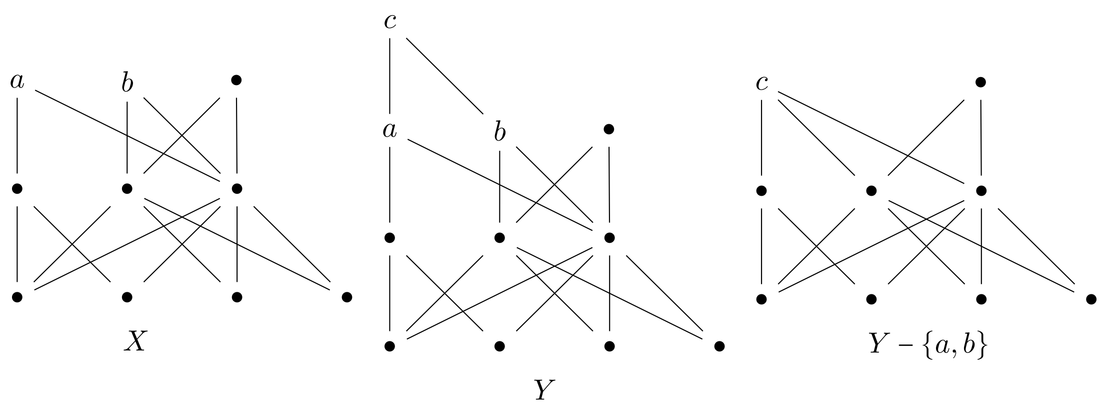
In this case the inclusion $X - x \hookrightarrow X$ is a weak homotopy equivalence. This notion was introduced in [BM08b] to study simple homotopy theory of finite spaces and its applications to problems of simple homotopy theory of polyhedra (see [Bar11, BM08b]).

Since finite spaces with maximum or minimum are contractible, the notion of weak point generalizes that of beat point. The process of removing a weak point is called an *elementary collapse*, and a space X is called *collapsible* if there is a sequence of elementary collapses that transforms X into a single point. Contractible finite spaces are collapsible and collapsible spaces are homotopically trivial. None of the converse implications hold (see [Bar11, BM08b] for more details).

3.2 QC-reducible spaces and quasi-constructible complexes

The notions of qc-reduction and qc-reducible space were introduced by Barmak to investigate the Andrews-Curtis conjecture using finite spaces. We recall here these notions and we refer the reader to [Bar11] for a detailed exposition on qc-reducible spaces and the Andrews-Curtis conjecture.

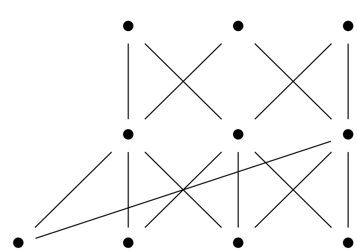
Definition 3.2.1. Let X be a finite space of height 2 and let $a, b \in X$ be two maximal points. If $U_a \cup U_b$ is contractible, we will say that there is a *qc-reduction* from X to $Y - \{a, b\}$, where $Y = X \cup \{c\}$ with $a, b < c$. The point c, which replaces a and b in $Y - \{a, b\}$, is called a *relative* of a and b. Here we illustrate a qc-reduction.



Definition 3.2.2. A finite space X of height 2 is called *qc-reducible* if one can obtain a space with maximum by performing a sequence of qc-reductions.

Remark 3.2.3. In the first step of a qc-reduction, the point c added to X is a weak point of Y. The points a, b removed in the second step are beat points of Y. It follows that the qc-reductions preserve the weak homotopy type and that qc-reducible spaces are homotopically trivial. Of course, not every homotopically trivial finite space is qc-reducible. However one can show that any contractible finite space of height 2 is qc-reducible.

Example 3.2.4. The finite space X respresented below, which was introduced in [Bar11], is collapsible and therefore weak equivalent to a point. However no qc-reduction can be performed on X.



The proof of the following result, which will be used in section 3.4, can be found in [Bar11].

Proposition 3.2.5. *Let X be a finite space of height at most 2 and such that $H_2(X) = 0$. Let a, b be two maximal elements of X. Then the following are equivalent.*

1. *$U_a \cup U_b$ is contractible.*

2. *$U_a \cap U_b$ is non-empty and connected.*

3. *$U_a \cap U_b$ is contractible.*

Here $H_2(X)$ denotes the second homology group of X with integer coefficients. Note that in fact the equivalence between assertions 1 and 3 of the last proposition does not need any assumption on height or homology of the finite space, since $U_a \cup U_b$ is homotopy equivalent to $\mathbb{S}(U_a \cap U_b)$, the *non-Hausdorff suspension* of the intersection (see [Bar11]).

By [Bar11, Theorem 11.2.10], the associated finite space $\mathcal{X}(K)$ of a 2-complex K is qc-reducible if and only if K is contractible and *quasi-constructible*. The class of quasi-constructible complexes contains all constructible complexes. We refer the reader to [Bjo95, Koz08] for more details on constructible complexes.

3.3 The Whitehead conjecture in terms of finite spaces

We will use the homotopy theory of finite spaces to investigate Whitehead's asphericity question. We will focus our attention on the finite case (i.e. counterexamples of type (a)) of Howie's Theorem 2.5.4 and restate the question in terms of finite spaces.

Suppose that there exists a counterexample of type (a). Since every finite CW-complex is homotopy equivalent to a finite simplicial complex of the same dimension (see [Coh93, (7.2)]), it follows that there is a contractible simplicial complex L of dimension 2 and a subcomplex $K = L - \sigma$ with σ a 2-simplex of L, such that K is connected and non-aspherical. Let $\mathcal{X}(L)$ and $\mathcal{X}(K)$ be their face posets. Then $\mathcal{X}(L)$ is a homotopically trivial space of height 2 and $\mathcal{X}(K)$ is a connected, non-aspherical subspace of $\mathcal{X}(L)$, obtained by removing the maximal point $\sigma \in \mathcal{X}(L)$. This motivates the following conjecture.

Conjecture 3.3.1. *Let X be a homotopically trivial finite space of height 2 and let $a \in X$ be a maximal point such that $X - a$ is connected. Then $X - a$ is aspherical.*

It is clear that if Conjecture 3.3.1 is true, then there are no counterexamples of type (a). On the other hand, suppose that the answer to Whitehead's question is *positive*. Given a homotopically trivial finite space X of height 2 and a maximal point $a \in X$ such that $X - a$ is connected, the associated simplicial complex $\mathcal{K}(X)$ is aspherical (in fact, it is contractible) and the subcomplex $\mathcal{K}(X - a)$ of $\mathcal{K}(X)$ is connected. Hence $\mathcal{K}(X - a)$, and therefore also $X - a$, are aspherical. This means that, in this case, Conjecture 3.3.1 is also true.

Note that Whitehead's original question is equivalent, in the case of compact polyhedra, to asking whether every subspace of an aspherical finite space of height two is itself aspherical.

Sometimes it is convenient to work with spaces which are not necessarily path connected. In general we will say that a space X is aspherical if every path connected component of X is aspherical.

3.4 The conjecture holds for QC-reducible spaces

In this section we prove that QC-reducible spaces satisfy the conjecture.

Remark 3.4.1. Let $x, y \in X$ be two maximal points. If X' is obtained from X by performing a qc-reduction that does not involve x nor y, then $x, y \in X'$ and $U_x^X \cup U_y^X = U_x^{X'} \cup U_y^{X'}$. Here we write U_x^X, $U_x^{X'}$ in order to distinguish whether the open subsets are considered in X or X'.

Lemma 3.4.2. *Let X be a qc-reducible finite space and let $a \in X$ be a maximal point. Then the qc-reductions can be reordered and split into two phases such that*

i) in the first phase, the point a and its relatives are not involved in any reduction,

ii) in the second phase every reduction involves a or one of its relatives.

Proof. Let $X = X_0, X_1, X_2, \dots, X_n$ be a sequence of finite spaces starting with X where each X_i is obtained from X_{i-1} by a qc-reduction q_i and such that X_n has maximum. Let $q_{i_1}, q_{i_2}, \dots, q_{i_k}$ be the reductions involving a and its relatives. Concretely, q_{i_1} involves $a = a_0$ and $a_0' \in X_{i_1-1}$, which are replaced by a_1 in X_{i_1}, q_{i_2} involves a_1 and $a_1' \in X_{i_2-1}$ which are replaced by a_2 in X_{i_2} and so on. We will now perform all the reductions which do not involve the relatives of a. In order to do that, set $\widetilde{X}_0 = X_0, \widetilde{X}_1 = X_1, \dots, \widetilde{X}_{i_1-1} = X_{i_1-1}$, then skip the reduction q_{i_1} and set $\widetilde{X}_{i_1} = \widetilde{X}_{i_1-1}$. By the previous remark, the next reduction q_{i_1+1} can be performed on $\widetilde{X}_{i_1}$, because it does not involve a_1 (unless $i_1 + 1 = i_2$, in which case we keep skipping reductions). In this way we obtain $\widetilde{X}_{i_1+1}$. Analogously, we can perform the reductions $q_{i_1+2}, \dots, q_{i_2-1}$ and skip q_{i_2}. We proceed similarly with every q_{i_j} and obtain $\widetilde{X}_n$ when the first phase of reductions is over.

Again by the previous remark, it is easy to see that the reductions $q_{i_1}, q_{i_2}, \dots, q_{i_k}$ can now be performed on $\widetilde{X}_n$ to obtain a space with maximum. This is the second phase. □

Lemma 3.4.3. *Let X be a qc-reducible finite space of height 2 and let $a \in X$ be a maximal point such that $X - a$ is connected. If the sequence of qc-reductions can be chosen so that every reduction involves a or one of its relatives, then $\hat{U}_a$ is connected and the inclusion $\hat{U}_a \hookrightarrow X - a$ induces an isomorphism $\pi_1(\hat{U}_a) \to \pi_1(X - a)$.*

Proof. Similarly as in the previous lemma, let $X = X_0, X_1, X_2, \dots, X_n$ be a sequence of finite spaces where each X_i is obtained from X_{i-1} by a qc-reduction q_i which replaces a_{i-1} and a'_{i-1} with a_i, where $a = a_0$, and such that X_n has maximum. The elements a_i' are exactly the maximal points of X other than a. Thus $X = U_a \cup \bigcup_{i=0}^{n-1} U_{a_i'}$ and $X - a = \hat{U}_a \cup \bigcup_{i=0}^{n-1} U_{a_i'}$.

Since $\hat{U}_a \cap U_{a_0'} = U_a \cap U_{a_0'}$ is non-empty and connected, by the van Kampen theorem it follows that the inclusion $\hat{U}_a \hookrightarrow \hat{U}_a \cup U_{a_0'}$ induces an isomorphism of the fundamental groups. Since a second reduction, involving a_1 and a_1', can be performed, we know that $(\hat{U}_a \cup U_{a_0'}) \cap U_{a_1'} = (\hat{U}_a \cup \hat{U}_{a_0'}) \cap U_{a_1'} = \hat{U}_{a_1} \cap U_{a_1'} = U_{a_1} \cap U_{a_1'}$ is also non-empty and connected. Again by van Kampen, it follows that $\hat{U}_a \hookrightarrow \hat{U}_a \cup U_{a_0'} \cup U_{a_1'}$ induces an isomorphism of the fundamental groups. The result now follows by iterating this reasoning.

Note that, since the intersections $U_{a_i} \cap U_{a_i'}$ are connected, $\hat{U}_a$ is also connected. Otherwise $\hat{U}_a \cup U_{a_0'}$ could not be connected, neither could $\hat{U}_a \cup \bigcup_{i=0}^{k} U_{a_i'}$ for any $1 \leq k \leq n-1$. □

Theorem 3.4.4. *Let X be a qc-reducible finite space of height 2 and let $a \in X$ be a maximal point such that $X - a$ is connected. Then $X - a$ is aspherical.*

Proof. Applying Lemma 3.4.2, let Y be the last space obtained in the first phase of the reductions. Note that $X - a$ is weak equivalent to $Y - a$ by performing the same qc-reductions. Note also that Y is qc-reducible by reductions that involve a and its relatives. By Lemma 3.4.3, $\pi_1(Y - a) = \pi_1(\hat{U}_a)$ and this group is free, since $\mathrm{h}(\hat{U}_a) \leq 1$.

Let us now consider the associated complexes of these spaces. $\mathcal{K}(X)$ is contractible and $\mathcal{K}(X - a)$ is a connected subcomplex of $\mathcal{K}(X)$ with free fundamental group. More precisely, $\mathcal{K}(X) = \mathcal{K}(X-a) \cup \mathrm{st}\, a$ with $\mathcal{K}(X-a) \cap \mathrm{st}\, a = \mathrm{lk}\, a$. Here $\mathrm{st}\, a$ denotes the (closed) star of the vertex $a \in \mathcal{K}(X)$ and $\mathrm{lk}\, a$ denotes its link. Therefore the CW-complex $\mathcal{K}(X)$ is obtained from $\mathcal{K}(X-a)$ by attaching one 0-cell, and some 1-cells and 2-cells. Let $(\mathrm{st}\, a)^{(1)}$ be the 1-skeleton of $\mathrm{st}\, a$ and let $L = \mathcal{K}(X-a) \cup (\mathrm{st}\, a)^{(1)}$ be the complex between $\mathcal{K}(X-a)$ and $\mathcal{K}(X)$ missing only the 2-cells of $\mathrm{st}\, a$. Then L is also connected and $\pi_1(L)$ is also free. By [Coc54, Theorem 1], L is aspherical. Since L is obtained from $\mathcal{K}(X - a)$ by attaching only 0-cells and 1-cells, it follows that $\mathcal{K}(X - a)$ is also aspherical. □

As an immediate consequence of this result we deduce the following.

Corollary 3.4.5. *There are no counterexamples of type* (a) *in the class of quasi-constructible and contractible 2-complexes.*

3.5 Aspherical reductions and strong aspherical spaces

We investigate now a method of reduction which preserves the asphericity of finite spaces, and prove that a new class of finite spaces, the *strong aspherical* spaces, satisfy the conjecture.

Recall that, given a point $x \in X$, we denote by $\hat{C}_x$ the subspace $\hat{C}_x = \{y \in X : y < x \text{ or } y > x\}$.

Definition 3.5.1. Let X be a finite space. A point $x \in X$ is called an *a-point* if $\hat{C}_x$ is a disjoint union of contractible spaces. In that case, the move $X \to X - x$ is called an *a-reduction.* A finite space X is called *strong aspherical* if there is a sequence of a-reductions which transforms X into the singleton.

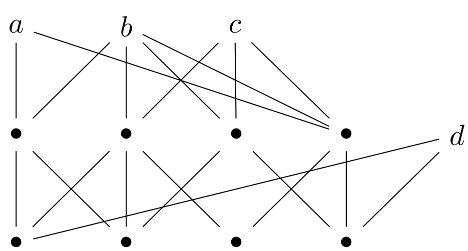
Note that the notion of a-point generalizes that of weak point. In particular, collapsible finite spaces are strong aspherical. Note also that strong aspherical spaces are not necessarily homotopically trivial. In fact, every finite space of height one is strong aspherical.

Proposition 3.5.2. *Let X be a finite space and let $a \in X$ be an a-point. Then X is aspherical if and only if $X - a$ is aspherical. In particular, strong aspherical spaces are aspherical.*

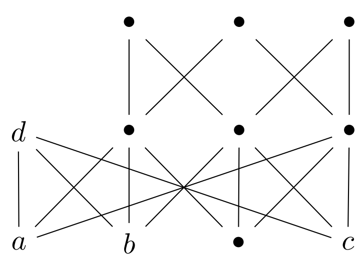
Proof. Since $\mathcal{K}(X) = \mathcal{K}(X - a) \cup a\mathcal{K}(\hat{C}_a)$ and $\mathcal{K}(\hat{C}_a)$ is a disjoint union of contractible spaces, the space $\mathcal{K}(X)$ is homotopy equivalent to a space obtained from $\mathcal{K}(X - a)$ by attaching a cone of a discrete subspace. Since attaching cells of dimensions 0 and 1 does not change the asphericity, it follows that $\mathcal{K}(X)$ is aspherical if and only if $\mathcal{K}(X-a)$ is. □

The following two examples illustrate how to use this result as a method of reduction to investigate asphericity of finite spaces. Note that, in both cases, the original spaces have no weak points.

Example 3.5.3. In the space X represented below, the elements a and d are a-points. The space Y which one obtains removing these points has only two maximal elements, b and c. Therefore Y is homotopy equivalent to the non-Hausdorff suspension of $U_b \cap U_c$. Since $U_b \cap U_c$ is not acyclic, it follows that Y is non-aspherical (in fact its associated complex is homotopy equivalent to S^2). This shows that the space X (and consequently, the associated complex $\mathcal{K}(X)$) is non-aspherical.



Example 3.5.4. In the space X represented below, the elements a, b, c and d are a-points. If we remove these points, we obtain a space with minimum, and therefore contractible. It follows that X (and consequently, $\mathcal{K}(X)$) is aspherical.



We concentrate now on strong aspherical spaces of height 2.

Theorem 3.5.5. *Let X be a strong aspherical space of height 2 and let $a \in X$. Then $X - a$ is strong aspherical.*

Proof. We proceed by induction on the cardinality of X. Choose a sequence of a-reductions from X to the singleton. If a is the first a-point to be removed, then $X - a$ is strong aspherical by definition. Otherwise, let $b \in X$ be the first reduction, with $b \neq a$. Then $X - b$ is strong aspherical and, by induction, $X - \{a, b\}$ is strong aspherical.

On the other hand, since $\hat{C}_b$ is a disjoint union of contractible subspaces, and it is of height at most 1, then $\hat{C}_b - a$ is also a disjoint union of contractible subspaces (possibly empty). This implies that b is also an a-point of $X - a$. Since $(X - a) - b$ is strong aspherical, it follows that $X - a$ is strong aspherical. □

Corollary 3.5.6. *The answer to Whitehead's original question is positive for strong aspherical finite spaces of height 2.*

Definition 3.5.7. A simplicial complex K is called *strong aspherical* if its face poset $\mathcal{X}(K)$ is strong aspherical.

By Corollary 3.5.6, the answer to Whitehead's question is positive for strong aspherical 2-complexes. We can also use Theorem 3.5.5 to give the following characterization of strong aspherical simplicial complexes of dimension 2.

Proposition 3.5.8. *A 2-dimensional simplicial complex L is strong aspherical if and only if L collapses to a 1-dimensional subcomplex.*

Proof. It is clear that if L collapses to a 1-dimensional subcomplex K, then $\mathcal{X}(L)$ a-reduces to $\mathcal{X}(K)$, which in turn a-reduces to a point.

Conversely, suppose that $\mathcal{X}(L)$ is strong aspherical and that L collapses to a 2-dimensional subcomplex K on which no more collapses can be performed. Since K has no collapses, the only possible a-points of $\mathcal{X}(K)$ are minimal points. Given such an a-point $v \in \mathcal{X}(K)$, $\hat{F}_v$ must be discrete, since it is a space of height at most 1 in which every component is contractible and it does not have beat points. This implies that no element $\sigma \in \mathcal{X}(K)$ of height 2 can become an a-point and be removed in a process of a-reductions. This contradicts the fact that, by Theorem 3.5.5, the subspace $\mathcal{X}(K)$ of $\mathcal{X}(L)$ is strong aspherical. □

3.6 Further combinatorial methods and algorithmic implementation

The combinatorial methods presented above, such as those involving beat points, weak points and qc-reductions, have many possible generalizations. Our research group has worked on several applications of such methods, and algorithmic implementations that make them very useful for analyzing examples [Cer10, Fer11].

One of the basic tools used in this subject is the following result by McCord [McC66] which gives a local characterization for weak homotopy equivalences. Recall that a basis-like open cover $\mathcal{U}$ of a space X is an open cover such that $\mathcal{U}$ is a basis for a topology on X. For example, for any finite space X the cover $\{U_x\}_{x\in X}$ is a basis-like open cover for X.

Theorem 3.6.1. *Let $f : X \to Y$ be a continuous map between finite spaces. Suppose there exists a basis-like open cover $\mathcal{U}$ of Y such that each restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a weak homotopy equivalence for every $U \in \mathcal{U}$. Then f is a weak homotopy equivalence.*

In the present work, we are interested in new combinatorial methods that are particularly useful in the study of asphericity. We will introduce moves on finite spaces that generally change the weak homotopy type of the space, preserving its asphericity. That is, the space obtained by applying these moves may have different homotopy groups, but it is aspherical if and only if the original space is aspherical.

The computational approach of these results is inspired in the work of Fernández [Fer11], who started using the software SAGE [S$^+$15] to work with finite spaces and apply them to topological problems. Also, the functions presented here are based on the first basic functions designed by her for finite spaces, such as `F(x,X)`,`U(x,X)`, `is_beat_point(x,X)`, `is_weak_point(x,X)`.

The following reduction is a variation of the qc-reduction, with the difference that the weak homotopy type of the space is not preserved.

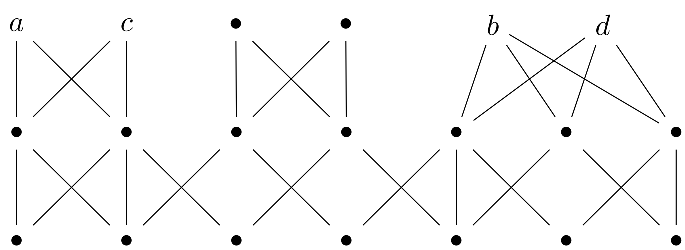
Definition 3.6.2. Given $a, b \in X$ two maximal points, there is an *a-qc-reduction* between a and b if $U_a \cap U_b = \emptyset$.

Given such two points, the a-qc-reduction consists of identifying a and b. Note that it is fairly easy to look for these reductions.

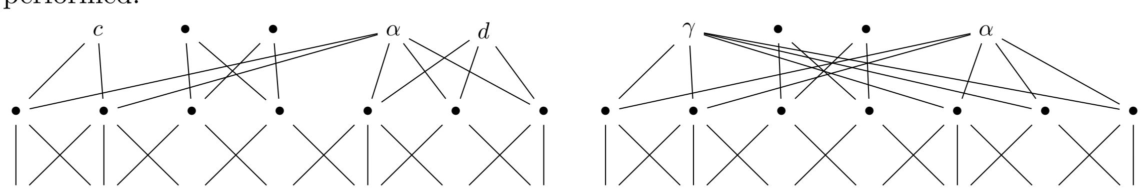
Proposition 3.6.3. *Let Y be the space obtained from X by performing an a-qc-reduction. Then Y is aspherical if and only if X is aspherical.*

Proof. The space Y can be obtained by first adding an element α covering a and b, and then removing a and b. The first step preserves asphericity because $\hat{C}_\alpha = U_a \cup U_b$ is a disjoint union of contractible subspaces, and thus α is an a-point. The second step preserves the homotopy type because a and b are beat points. □

Example 3.6.4. Let X be the space represented below.

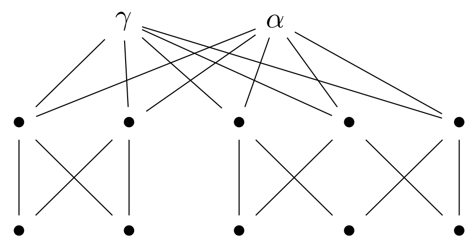


There is an a-qc-reduction between a and b, and then one between c and d can be performed.



The original space has free fundamental group of rank 3, and the space Y obtained after the reductions has free fundamental group of rank 5. However, the ashericity is preserved.

Moreover, if we perform a sequence of a-reductions on Y, we obtain the following space W.



It is easy to see that the space W is weak equivalent to the suspension of a disjoint union of two 1-spheres, and is thereofore non-aspherical.

SAGE code 3.6.5. The following functions can be applied to search for a-qc-reductions in a given finite space X, and determine an a-qc-core for X, that is, perform a-qc-reductions on X until there are no more such possible reductions. Note that, unlike in the case of removing beat points, the a-qc-core of a space is not well defined. This function looks for reductions in arbitrary order.

```
def quotient(A,X):
    l=X.list()
    for a in A:
        l.remove(a)
    Y=X.subposet(l)
    R=Y.cover_relations()
    l.append(A[0])
    for y in Y.list():
        for a in A:
            if X.covers(a,y): R.append([A[0],y])
            if X.covers(y,a): R.append([y,A[0]])
    YY=Poset([l,R])
    return YY

def is_a_qc_reduction(a,b,X):
    return interu(a,b,X).cardinality()==0

def a_qc_core(X):
    if X.has_top(): return X
    for a in X.maximal_elements():
        for b in X.maximal_elements():
            if a!=b and is_a_qc_reduction(a,b,X):
                print a,b
                Y=quotient([a,b],X)
                return a_qc_core(Y)
        return X
```

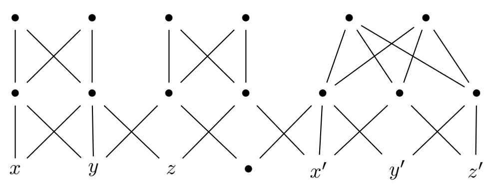
There is also a process of a-qc-op-reduction, which is the reversed analogue of a-qc-reduction. We omit the proof of proposition 3.6.7 for being similar to the previous one.

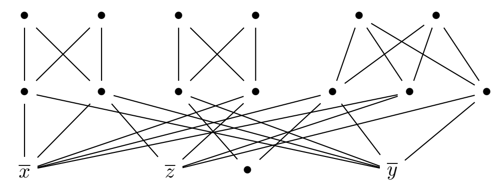
Definition 3.6.6. Given $a, b \in X$ two minimal points, there is an *a-qc-op-reduction* between a and b if $F_a \cap F_b = \varnothing$.

Given such two points, the a-qc-op-reduction consists of identifying a and b.

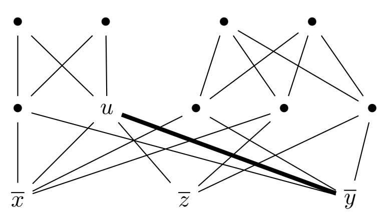
Proposition 3.6.7. *Let Y be the space obtained from X by performing an a-qc-op-reduction. Then Y is aspherical if and only if X is aspherical.*

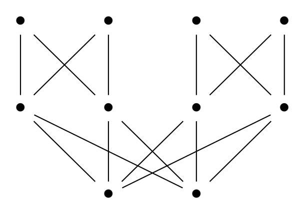
Example 3.6.8. Consider again the previous example. There are three consecutive a-qc-op-reductions in X. Between x and x', y and y', z and z'. Both X and the space Y obtained after these reductions are depicted below.





The space Y can be reduced futher. First, we can remove a-points, and obtain a space Z of 12 elements. In this new space there is a possible *a-edge-reduction*. We will define this reduction later in this section. Thus we can omit the edge $(\overline{y}, u)$, and obtain a space whose asphericity is equivalent to that of Z. After performing this reduction, a qc-op-reduction is possible (this reduction is similar to qc-reductions, but involves minimal elements), and then a weak point can be removed. After this sequence of reductions, we obtain the space W. Below we illustrate the spaces Z (left) and W (right).





In this case it is again clear that the space W is weak equivalent to the suspension of a disjoint union of two 1-spheres, and is thereofore non-aspherical.

We will now introduce a notion of mapping cylinder for maps between posets, which will be useful for our redution methods. This analogue of the classical topological mapping cylinder was first studied in [BM08b] (see also [Bar11]).

Definition 3.6.9. Let $f : X \to Y$ be a map of finite spaces. The *non-Hausdorff mapping cylinder* $B(f)$ of f is the set $X \sqcup Y$ with the order given by the inner orders of each space, adding the cover relations $x \prec f(x)$ for every $x \in X$.

As in the classical setting, this mapping cylinder has the subspace Y as a strong deformation retract. The proof of this result can be found in [Bar11], together with applications of the non-Hausdorff mapping cyilinder, which serve to generalize the theorem of McCord 3.6.1. The main technique employed to this purpose, which we shall also make use of, is to prove that $B(f)$ collapses to X by removing points satisfying certain property (for example beat points or weak points).

The method of a-qc-gen-reduction is a generalization of the methods of a-qc- and a-qc-op-reduction, applicable to pairs of elements which do not need to be both maximal or minimal.

Definition 3.6.10. Given $a, b \in X$, there is an *a-qc-gen-reduction* between a and b if the following conditions are satisfied.

- $\hat{C}_a \cap \hat{C}_b = \varnothing$,
- For every $x \in F_a \smallsetminus F_b$, $U_x \cap U_b = \varnothing$,
- For every $x \in F_b \smallsetminus F_a$, $U_x \cap U_a = \varnothing$.

Given such two points, the a-qc-reduction consists of identifying a and b.

SAGE code 3.6.11. The following functions can be applied to search for a-qc-gen-reductions.

```
def interu(a,b,X):
    l=[]
    for x in X.list():
        if X.is_gequal(a,x) and X.is_gequal(b,x):
            l.append(x)
    return X.subposet(l)

def interf(a,b,X):
    l=[]
    for x in X.list():
        if X.is_lequal(a,x) and X.is_lequal(b,x):
            l.append(x)
    return X.subposet(l)

def interchat(a,b,X):
    l=[]
    for x in X.list():
        if x in C_hat(a,X) and x in C_hat(b,X):
            l.append(x)
    return X.subposet(l)

def is_a_qc_gen_reduction(a,b,X):
    if interchat(a,b,X).cardinality()!=0: return False
    for x in F(a,X):
        if not x in F(b,X):
            if interu(x,b,X).cardinality()!=0: return False
```

```
for x in F(b,X):
    if not x in F(a,X):
        if interu(x,a,X).cardinality()!=0: return False
return True
```

Proposition 3.6.12. *Let Y be the space obtained from X by performing an a-qc-gen-reduction. Then Y is aspherical if and only if X is aspherical.*

Proof. Note that since $\hat{C}_a \cap \hat{C}_b = \varnothing$, if a and b are comparable, then $\{a, b\}$ must be a maximal chain. That is, $a \prec b$ with a a minimal point, and b a maximal point, or $b \prec a$ with b minimal and a maximal. Assuming $a \prec b$, and since by hypothesis for every $x \in F_a \smallsetminus F_b$, $U_x \cap U_b = \varnothing$, we must have $F_a = \{b\}$, thus a is a beat point. In this case the effect of the reduction is the same as removing a, so the space obtained has the same homotopy type as the original space.

Let us now consider the case where a and b are not comparable elements. Let $q : X \to Y$ be the quotient map. We will prove that X can be obtained from $B(q)$ by removing weak points and a-points, and therefore the asphericity of X is equivalent to that of $B(q)$ which is homotopy equivalent to Y. Let α be the class of a and b in the quotient (all the other classes have a unique representative). We start by removing the minimal elements of Y, then the elements of height one in Y, and finally the maximal elements of Y. In the successive steps, we will denote by $\tilde{B}(q)$ (respectively $\tilde{Y}$) the space $B(q)$ (respectively its subspace $Y \subseteq B(q)$) without the previously removed elements. In each step we consider an element $y \in \tilde{Y}$ with no lower covers left in $\tilde{Y} \subseteq \tilde{B}(q)$.

- If $y \notin F_\alpha^Y$, then $\hat{U}_y^{\tilde{B}(q)} = U_y^X$ is contractible, so $\hat{C}_y^{B(q)}$ is contractible. That is, y is a weak point.
- If $y = \alpha$, then $\hat{U}_y^{\tilde{B}(q)} = \hat{U}_\alpha^{\tilde{B}(q)} = U_a^X \cup U_b^X$ and $\hat{F}_y^{\tilde{B}(q)} = \hat{F}_\alpha^{\tilde{B}(q)} = \hat{F}_a^X \cup \hat{F}_b^X$. So $\hat{C}_\alpha^{\tilde{B}(q)} = C_a^X \cup C_b^X$ is a disjoint union of two contractible spaces. That is, y is an a-point.
- If $y \in \hat{F}_\alpha^Y$, suppose $y \in F_a^X \smallsetminus F_b^X$. Then $\hat{U}_y^{\tilde{B}(q)} = U_y^X \cup U_b^X$ and $\hat{F}_y^{\tilde{B}(q)} = \hat{F}_y^X$. So $\hat{C}_y^{\tilde{B}(q)} = C_y^X \cup U_b^X$ is a disjoint union of two contractible spaces. That is, y is an a-point. If $y \in F_b^X \smallsetminus F_a^X$, the procedure is similar.

□

The method of a-edge-reduction is a generalization of the method of edge-reduction (see [Cer10, Fer11]). This new method changes the homotopy type of the space, preserving its asphericity.

Definition 3.6.13. Let X be a finite space of height 2 and let $a, b \in X$ be an edge in X, that is, a cover relation $a \prec b$ with b a maximal element. The edge $a \prec b$ is said to be *down-reducible* if

$$U_c \cap U_a = \varnothing \quad \forall \ c \prec b, c \neq a.$$

Analogously, let $a, b \in X$ with a a minimal element. The edge $a \prec b$ is said to be *up-reducible* if

$$F_c \cap F_b = \varnothing \quad \forall\ c > a, c \neq b.$$

If an edge is up- or down- reducible, then an *a-edge-reduction* can be performed, which consists of omitting the edge in the Hasse diagram of X. That is, replacing the topology of the space for the topology generated by all the other cover relations.

Proposition 3.6.14. *If the space Y is obtained from X by performing an a-edge-reduction, then Y is aspherical if and only if X is aspherical.*

Proof. Let us consider $f : Y \to X$ defined as the identity map on the underlying sets. Note that, unlike its inverse, f is order preserving, that is, continuous. Also $f^{-1}(U_x^X) = U_x^Y$ for every $x \in X$ except for $x = b$, where $f^{-1}(U_b^X) = U_b^Y \cup U_a^Y$, is a disjoint union of two contractible subspaces. Therefore $B(f)$ a-collapses to X. The procedure is similar to the one used in proposition 3.6.12, this time using that b is a maximal element of Y and of $B(q)$. □

3.6.1 Division methods

Division methods come up as inverse moves of the reduction methods. The space Y obtained by performing a certain division on a space X is one where the corresponding reduction is permitted, through which X is recovered. The difficulty of division methods is that, in general, there are many ways to do them. Additionally, it is usually more complicated to check whether a division can be performed than it is with reductions. This makes division methods computationally more expensive. But they are certainly interesting because of the substantial effects they can generate.

Definition 3.6.15. A maximal element $x \in X$ is *a-qc-divisible* if $\hat{U}_x$ is not connected.

A division of a divisible element x consists of choosing U_1, U_2 open sets such that $\hat{U}_x = U_1 \dot{\cup} U_2$, and replacing x by two new elements x_1, x_2, setting $\hat{U}_{x_1} = U_1$ and $\hat{U}_{x_2} = U_2$.

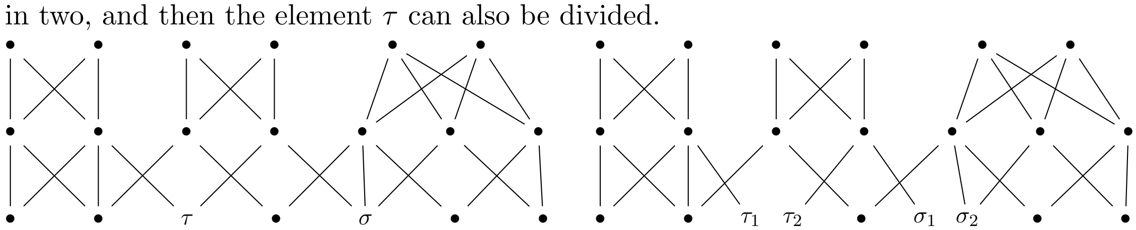
The a-qc-op-division and a-qc-gen-division are defined analogously. Note that these transformations preserve asphericity because they are inverse moves of the corresponding reductions.

SAGE code 3.6.16. We show the code for the function that performs an a-qc-op-division.

```
def components(X):
    if X.order_complex().is_connected(): return {X}
    l=[]
    for x in X.minimal_elements():
        C=X.subposet(X.order_complex().connected_component([x]).vertices())
        l.append(C)
    S=Set(l)
    return S
```

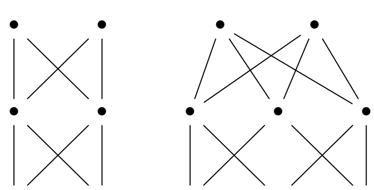
```
def is_a_qc_op_divisible(x,X):
    if not x in X.minimal_elements(): return False
    return len(components(F_hat(x,X)))>1

def a_qc_op_division(x,X):
    n=len(components(F_hat(x,X)))
    if n<=1:
        print 'error: the division is not possible'
        return
    l=[y for y in X if y!=x]
    Y=X.subposet(l)
    r=Y.cover_relations()
    if x+'_1' in Y.list():
        print 'error: there is an element named ', x+'_1', ' in the poset'
    if x+'_2' in Y.list():
        print 'error: there is an element named ', x+'_1', ' in the poset'
    l.append(x+'_1')
    l.append(x+'_2')
    for u in components(F_hat(x,X))[0]:
        r.append([x+'_1',u])
    for i in range(1,n):
        for u in components(F_hat(x,X))[i]:
            r.append([x+'_2',u])
    return Poset((l,r))
```

Example 3.6.17. We consider once more the space X, where the element σ can be divided in two, and then the element τ can also be divided.



Recall that X has free fundamental group of rank 3. The space Y obtained has infinite cyclic fundamental group, and its asphericity is equivalent to that of X.

The space Y has a-points (even beat points). Removing a sequence of a-points, we arrive at the following space Z, which is clearly weakly equivalent to $S^2 \dot\cup S^2$. We conclude that X, Y and Z are non-aspherical.



Resumen del capítulo 3: Espacios finitos y aplicaciones a la conjetura de Whitehead

En la sección 3.1 comenzamos repasando las nociones básicas de la teoría de espacios finitos. Ésta es la herramienta fundamental que usamos en el presente trabajo. Presentamos algunos de los resultados de esta teoría y de cómo se aplica al estudio de problemas sobre poliedros. También repasamos brevemente las definiciones de espacio finito qc-reducible y complejo cuasi-construible, que se necesitan más adelante. Para más detalles sobre la teoría de espacios finitos referimos al lector a [Bar11, BM08b, May03c, McC66, Sto66].

Un espacio topológico finito es un espacio para el cual el conjunto subyacente es finito. Estos espacios guardan una relación natural con los conjuntos parcialmente ordenados (posets) finitos. Las siguientes asignaciones dan lugar a una correspondencia entre estas dos categorías.

Dado un espacio finito X, se define una relación de la siguiente manera

$$x \leq y \Leftrightarrow x \in U \quad \text{para todo abierto } U \quad \text{que contiene a } y.$$

Es fácil ver que esta relación es un preorden (reflexiva y transitiva) para cualquier topología. Sin embargo, en general no resulta antisimétrica. Esta condición es equivalente a que la topología de la cual proviene cumpla el axioma T_0: para cualquier par de puntos, hay un abierto que contiene sólo a uno de ellos.

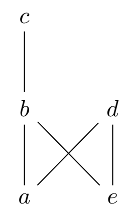
Análogamente, si $\leq$ es una relación de orden en el conjunto X, los conjuntos

$$U_x := \{y \in X : \ y \leq x\} \subseteq X$$

forman una base para una topología T_0 en X.

En este trabajo siempre suponemos que los espacios finitos cumplen el axioma T_0. Para la representación de un espacio finito X se utiliza su diarama de Hasse. Este diagrama tiene por vértices a los elementos de X, y tiene una arista entre x e y (con x por debajo de y) si y *cubre* a x (denotado $x \prec y$ o $y \succ x$), es decir, si $x < y$ y no existe u con $x < z < y$.

Ejemplo. El poset $X = \{a, b, c, d, e\}$ con el orden dado por $a < b, a < d, e < d, e < b, b < c$ tiene el siguiente diagrama de Hasse.

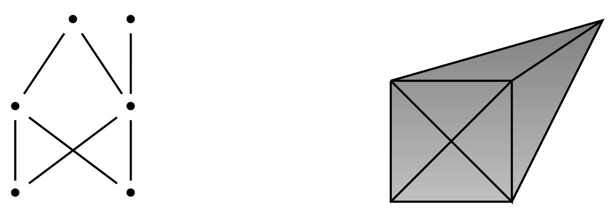


La importancia de los espacios finitos radica en que sirven de modelos para los poliedros compactos, a partir de la construcción que detallamos a continuación.

Dado un espacio finito X, el *complejo de cadenas* $\mathcal{K}(X)$ asociado a X es un complejo simplicial cuyos vértices son los elementos de X y cuyos símplices son las cadenas de elementos de X (es decir, subconjuntos totalmente ordenados).

Dado un complejo simplicial L, el *poset de símplices* $\mathcal{X}(K)$ asociado a K es el conjunto de los símplices de K ordenados por la inclusión.

En el dibujo se muestra un espacio finito X de 6 puntos, y su complejo de cadenas $\mathcal{K}(X)$.



El siguiente resultado fue probado por McCord [McC66], y se puede encontrar también en [May03b, Bar11].

Teorema 3.1.9. *Dado un espacio finito X, hay una equivalencia homotópica débil entre $\mathcal{K}(X)$ y X. Análogamente, dado un complejo simplicial K, hay una equivalencia homotópica débil entre K y $\mathcal{X}(K)$.*

Como consecuencia, se tiene que los tipos homotópicos débiles de los espacios finitos cubren los tipos homotópicos débiles de todos los poliedros compactos.

Repasamos también los resultados principales de la teoría de homotopía de espacios finitos. Comenzamos con la definición de *beat point*, introducida por Stong [Sto66]. Un punto x en un espacio finito X es un beat point si x cubre a un único elemento, o si hay un único elemento que cubre a x. A partir de esta noción, Stong caracterizó los tipos homotópicos de los espacios finitos. Dos espacios finitos son homotópicamente equivalentes si y sólo si se puede llegar de uno a otro agregando y quitando beat points.

Luego presentamos la noción de *weak point*, que generaliza a la de beat points, y que fue introducida por Barmak y Minian, quienes desarrollaron la teoría de homotopía simple para espacios finitos y sus aplicaciones [BM08b, BM08a, Bar11]. Si un espacio se obtiene de otro agregando o quitando weak points, éstos son débilmente equivalentes, pero la implicación inversa no es cierta (ver [BM08b, Bar11]).

Luego repasamos en la sección 3.2 algunas nociones relacionadas con los espacios *qc-reducible*. Estos espacios fueron introducidos por Barmak para investigar la conjetura de Andrews-Curtis con métodos de espacios finitos. En este trabajo se prueba que estos espacios satisfacen la conjetura de Whitehead.

En la sección 3.3, y a partir del resultado de Howie 2.5.4, traducimos el caso compacto (contraejemplo de tipo (a)) de la conjetura en términos de espacios finitos. De esta manera podremos aplicar métodos de espacios finitos al estudio de este problema.

Conjetura 3.3.1. *Sea X un espacio finito de altura 2 débilmente equivalente a un punto, y sea $a \in X$ un elemento maximal, tal que $X - a$ es conexo. Entonces $X - a$ es asférico.*

Si la conjetura 3.3.1 es cierta, entonces no hay contraejemplos del tipo (a) mencionado por Howie para la conjetura de Whitehead. Por otro lado, si la conjetura de Whitehead es cierta, entonces también es cierta la conjetura 3.3.1.

Notemos que la pregunta original de Whitehead es equivalente, en el caso de poliedros compactos, a preguntarse si todo subespacio de un espacio finito asférico de altura 2 es también asférico.

En las siguientes secciones presentamos algunos de los principales resultados de este trabajo. En la sección 3.4 probamos la validez de la conjetura 3.3.1 para los espacios finitos qc-reducibles. Esto implica que la conjetura de Whitehead es cierta para los complejos cuasi-construibles. Esta amplia clase de complejos compactos incluye a todos los complejos construibles.

En un primer resultado, manipulamos la sucesión de qc-reducciones necesarias para llevar a un espacio qc-reducible a un espacio con máximo. Separamos una tal sucesión de reducciones en dos etapas, de manera que el elemento a que se quiere quitar no sea involucrado en toda la primera etapa, y sí se utilice en todas las reducciones de la segunda etapa.

Recordemos Cockcroft probó la validez de la conjetura para pares $K \subseteq L$ tales que $L - K$ sólo son celdas de dimensión 2, y $\pi_1(K)$ es libre. Utilizando este resultado, y haciendo uso de la sepraración de reducciones en dos etapas, obtenemos los siguientes resultados.

Teorema 3.4.4. *Sea X un espacio finito de altura 2 qc-reducible y sea $a \in X$ un elemento maximal tal que $X - a$ es conexo. Entonces $X - a$ es asférico.*

Corolario. *No hay contraejemplos de tipo (a) en la clase de 2-complejos cuasi-constructibles y contráctiles.*

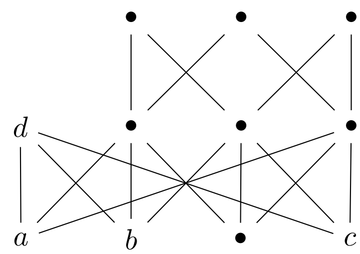
En la sección 3.5 introducimos un nuevo método de reducción, el de los a-puntos. Aplicando este método, probamos la conjetura para una nueva clase de complejos.

Recordemos que dado un elemento $x \in X$, denotamos por $\hat{C}_x$ al subespacio $\hat{C}_x = \{y \in X : \ y < x \text{ o } y > x\}$.

Definición. Sea X un espacio finito. Un elemento $x \in X$ es un *a-punto* si $\hat{C}_x$ es una unión disjunta de espacios contráctiles. En este caso, la transformación $X \mapsto X - x$ se llama *a-reducción*. Un espacio finito X se dice *fuertemente asférico* si existe una sucesión de a-reducciones que transforma X en un punto.

Probamos que si x es un a-point, entonces X es asférico si y sólo si $X - x$ es asférico. En particular, los espacios fuertemente asféricos son asféricos.

Ejemplo. En el espacio X representado a continuación los elementos a, b, c y d son a-points. Al quitarlos se obtiene un espacio con mínimo, que por lo tanto es contráctil. Por lo tanto X es asférico (y luego $\mathcal{K}(X)$ también).



El segundo resultado principal de este trabajo es el siguiente.

Teorema 3.5.5. *Sea X un espacio de altura 2 fuertemente asférico, y sea $a \in X$. Entonces $X - a$ es fuertemente asférico.*

Teorema 3.5.6. *La respuesta a la pregunta original de Whitehead es positiva para espacios finitos fuertemente asféricos de altura 2.*

Como consecuencia, se tiene que la respuesta a la pregunta de Whitehead es positiva si se considera un 2-complejo simplicial L cuyo poset de símplices es fuertemente asférico. Un complejo simplicial L satisface esta condición si y sólo si L colapsa a un subcomplejo de dimensión 1.

Finalmente, en la sección 3.6 introducimos nuevos métodos combinatorios desarrollados para el estudio de la asfericidad, y presentamos también implementaciones algorítmicas de estos métodos, para ser usadas con el software SAGE.

Estos métodos provienen de generalizaciones y variaciones de los métodos de beat points, weak points, y qc-reduciones [Cer10, Fer11]. Para el presnte trabajo, se desarrollaron métodos en los cuales el tipo homotópico débil del espacio cambie, preservándose su asfericidad. Es decir, si el espacio Y es obtenido a partir de aplicar uno de estos movimientos a un espacio X, entonces Y es asférico si y sólo si X es asférico. Sin embargo, en general, los grupos de homotopía de Y no coinciden con los de X. Estos movimientos, al ser menos rígidos, dan lugar a un espectro más amplio de espacios donde se puede analizar la asfericidad de un espacio dado. En esta sección también mostramos algunos programas desarrollados para implementar computacionalmente estos métodos a través del software SAGE. Los primeros programas construidos con este objetivo se pueden encontrar en [Fer11].

Chapter 4

Ribbon disc complements and labeled oriented trees

4.1 Introduction

A *knot* is an embedding $S^1 \hookrightarrow S^3$. The fundamental group of a knot complement is called the *knot group*. A presentation of the knot group of a given knot can be constructed using the Wirtinger Presentation. The abelianization of a knot group is known to be always isomorphic to $\mathbb{Z}$. Asphericity of knot complements was proved by Papakyriakopoulos [Pap57] using results on 3-manifolds. The asphericity of knot complements had been an open problem for a long time. In fact, it was probably one of the motivations of Whitehead's asphericity question. A proof of this conjecture would imply an alternative proof of the asphericity of knot complements.

Ribbon knots and ribbon discs arise as generalizations of classical knots. A *ribbon n-disc* is a proper embedding of an n-dimensional disc into an $(n+2)$-dimensional disc $D^n \hookrightarrow D^{n+2}$, such that the radial map $D^{n+2} \to \mathbb{R}$, $x \mapsto \|x\|$ restricts to a Morse function in D^n, which has no critical points of index greater than 1. A *ribbon disc complement* is the complement in D^{n+2} of the image of D^n by such an immersion.

Ribbon disc complements are conjectured to be aspherical as well as knot complements. The homotopy of ribbon disc complements and the question about their asphericity have been studied by numerous mathematicians [Yan69, AMY81, Has83, How85, HR03]. There are several incomplete proofs of the asphericity of ribbon disc complements (see [How83] for more details).

Every ribbon n-disc induces an immersion $S^{n-1} \hookrightarrow S^{n+1}$, thus the ribbon disc bounds a *ribbon* $(n-1)$*-knot*.

4.2 LOT complexes

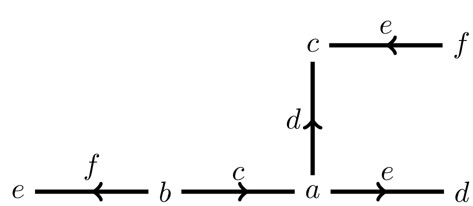
A combinatorial approach to the problem of the asphericity of ribbon disc complemnts was achieved when Howie introduced the labeled oriented tree associated to a ribbon disc [How85]. This construction is similar to the Wirtinger presentations that arise from knots, and it yields a connection between ribbon disc complements and LOT complexes, or LOT presentations.

Definition 4.2.1. A *labeled oriented graph* (*LOG*) $\Gamma = (E, V, s, t, \lambda)$ consists of two sets E, V of edges and vertices, and three maps $s, t, \lambda : E \to V$ called respectively source, target and label.

Γ is said to be a *labeled oriented tree* (*LOT*) when the underlying graph is a tree.

Associated to a LOT Γ there is a *LOT presentation* $P(\Gamma)$ with generating set in correspondence to the vertices of Γ and a relator associated to each edge in the following way. For an edge e with source s_e, target t_e and label λ_e, there is a relation $s_e\lambda_e = \lambda_e t_e$, that is, a relator $\lambda_e t_e \lambda_e^{-1} s_e^{-1}$. The group presented by $P(\Gamma)$ is called the LOT group and denoted $G(\Gamma)$.

Example 4.2.2. Consider the following LOT.



Then $P = \langle a, b, c, d, e, f \mid ece^{-1}f^{-1}, dcd^{-1}a^{-1}, ede^{-1}a^{-1}, cbc^{-1}a^{-1}, fbf^{-1}e^{-1}\rangle$ is the associated presentation.

Note that the deficiency of a LOT presentation is always 1 since the underlying graph of a LOT is a tree.

The LOT complex K_Γ associated to a given LOT Γ is the standard 2-complex of the LOT presentation. Concretely, K_Γ has a unique 0-cell, one 1-cell for each vertex of Γ, and one 2-cell for each edge of Γ. The attaching map for the 2-cell associated to an edge e spells the relator $\lambda_e t_e \lambda_e^{-1} s_e^{-1}$. The fundamental group of K_Γ is isomorphic to $G(\Gamma)$.

LOTs are a useful tool for the study of ribbon discs, since the complement of any ribbon disc has a LOT complex as a spine [How85]. Conversely, every LOT complex can be embedded as a spine of a ribbon disc complement, therefore the asphericity of LOTs is equivalent to the asphericity of ribbon disc complements.

Recall that the Q^{**}-class of the LOT presentation corresponds to the 3-deformation type of the LOT complex.

Howie proved that if a finite 2-complex L can be 3-deformed to a point, then $K = L - e^2$ can be 3-deformed to a LOT complex [How83, Theorem 4.2]. As a consequence, if one assumes the Andrews-Curtis conjecture to be true, then the asphericity of LOTs implies that there are no counterexamples of type (a) described in 2.5.4 of the Whitehead conjecture.

On the other hand, it is easy to see that a LOT complex is always a subcomplex of a contractible complex. Concretely, taking any generator x of the LOT presentation, and adding the relator x, one obtains a balanced presentation of the trivial group. The associated 2-complex is therefore simply connected and has Euler characteristic equal to 1. Thus it must be acyclic and, by the theorems of Hurewicz and Whitehead, contractible.

For these reasons LOTs are considered text cases of the Whitehead conjecture.

Remark 4.2.3. Let Γ be a LOT, with presentation $P(\Gamma) = \langle v_1, v_2, \dots, v_n | r_1, r_2, \dots, r_{n-1}\rangle$ and group $G(\Gamma)$. Then there is a *weight map* $w : G(\Gamma) \to \mathbb{Z}$ which takes the value 1 in every generator v_i. This morphism is well defined because the relators r_i have total exponent sum equal to 0.

Actually, the weight map of a LOT group G coincides with the abelianization map $G \twoheadrightarrow G/[G,G]$.

Proposition 4.2.4. *LOT complexes have the homology of S^1.*

Proof. Using cellular homology, it can be shown that for a given LOG Γ, the homology of K_Γ coincides with that of the suspension of $\Gamma \uplus *$. □

Corollary 4.2.5. *If a LOT group is known to be locally indicable, then the LOT is aspherical.*

Proof. Let Γ be a LOT with locally indicable LOT group. Then $H_2(K_\Gamma) = H_2(S^1) = 0$. Consider the universal cover $\tilde{K}_\Gamma \to K_\Gamma$. This covering has deck transformation group isomorphic to $\pi_1(K_\Gamma)$, which is by hypothesis locally indicable. Since locally indicable groups are conservative, this implies $H_2(\tilde{K}_\Gamma) = 0$. □

In the next section we present other contitions on groups that are connected to local indicability, which will be applied later in chapter 5.

4.3 Group properties

We shall now present various group properties related to the problem of asphericity of LOTs, and the implications that hold between them.

We have seen above that if a LOT has locally indicable fundamental group, then it is aspherical. A weaker notion than local indicability is that of left orderability. Left orderable groups, which arose as fundamental groups of certain 3-manifolds, have recently gained importance in different areas of mathematics. The notes of Rolfsen [Rol01] are a useful summary of the properties of left-orderable groups, and recent work on the subject, including applications to probability and dynamics, can be found in [DNR14, DKR14].

A *left order* in a group G is a left-invariant total order, that is, an order relation such that

$$g < g' \Rightarrow fg < fg' \quad \forall f, g, g' \in G.$$

Definition 4.3.1. A group G is said to be *left orderable* (or *LO*) if a left order can be defined in G.

The notions of right order and right orderable group are defined similarly, and a group is left orderable if and only if it is right orderable. Examples of LO groups are free groups and torsion-free abelian groups. These groups have useful properties. It is easy to see that they are torsion-free. If G is left-orderable, then $\mathbb{Z}[G]$ has no zero devisors (a proof can be found in [Pas11]). LO groups are closed under many operations such as subgroups, free products and extensions. In general they are not abelian. In fact a LO group is abelian if and only if every left-order is actually a bi-order, that is, an order which is both left- and right-invariant.

Burns and Hale proved that a group is LO if and only if every finitely generated subgroup has a quotient that is LO [BH72]. As a consequence this characterization, we have the following result.

Lemma 4.3.2. *Locally indicable groups are left orderable.*

The notion of the unique product property arose in the study of the problems of zero devisors and non-trivial units of group rings.

Definition 4.3.3. A group G is said to satisfy the *unique product property* (*UPP*) if for any two non-empty finite subsets A, B in G there is an element $g \in G$ that can be expressed in precisely one way as $g = ab$ with $a \in A$ and $b \in B$.

Lemma 4.3.4. *Locally indicable groups satisfy the unique product property.*

Proof. Given $A, B \subseteq G$ finite and non-empty, set g to be the maximal element of the form $g = ab, a \in A, b \in B$. Then the only way to obtain g as such a product is with b being the maximal element of B. Therefore if $g = a'b'$, then $b' = b$ and thus $a' = a$. □

Remark 4.3.5. It is also easy to see that LO groups are torsion-free. In fact, if $g^n = 1$, then the UPP fails with the sets $A = B = \{1, g, g^2, \ldots, g^{n-1}\}$.

The following characterization of UPP groups will be useful in the next chapter.

Lemma 4.3.6. *A group G satisfies UPP if and only if for any two non-empty finite subsets X, Y in G there is an element $g \in G$ such that $gX \cap Y$ has precisely one element.*

Proof. Suppose G satisfies UPP, and let X, Y be two non-empty finite subsets. Take $A = Y, B = X^{-1}$. There exists $g \in G$ such that z can be uniquely expressed as $z = yx^{-1}$ with $y \in Y, x \in X$. Thus $zX \cap Y = \{y\}$.

The converse is similar. □

Summarizing, the following implications hold for any group.

$$\text{LI} \Rightarrow \text{LO} \Rightarrow \text{UPP} \Rightarrow \text{TF}$$

Here TF stands for torsion-free. None of the converse implications hold. A LO group that is not LI was found by Bergman in 1991 [Ber91]. Old and new examples of torsion-free groups which are not UPP can be found in [RS87, Ste13, Car14]. It was not until 2014 that Dunfield found a group which is UPP and not LO [DKR14]. His example actually satisfies an intermediate condition, it is a diffuse group.

It is not known whether all LOT groups are LI, nor LO nor UPP [Howie, private communication]. As we stated above, if a LOT group is LI, then the presentation is aspherical. In chapter 5 we see that a class of LOTs are aspherical if their LOT groups are assumed to satisfy UPP.

In the next chapter we obtain results on asphericity of certain classes of LOTs, using these conditions as part of the hypotheses.

4.4 Contributions to the problem

Several families of LOT complexes, and thus of ribbon disc complements, have been proved to be aspherical (see [Ros07]). We will sometimes say that a LOT is aspherical, meaning that the associated complex is aspherical.

Howie proved the asphericity of LOTs of diameter ≤ 3, by showing that their LOT groups are locally indicable [How85]. His main tool was his previous result about reducible presentations with no proper powers having locally indicable fundamental group [How82, Corollary 4.5].

To work with LOTs, Howie introduces the following transformations, which do not change the homotopy type of the LOT complex.

Definition 4.4.1. Transformations on LOTs.

(L1) Shrink an edge e such that $\lambda(e) = s(e)$ or $\lambda(e) = t(e)$, and identify the endpoints $s(e), t(e)$.

(L2) Identify two edges e, f such that $\lambda(e) = \lambda(f)$ and either $s(e) = s(f)$ or $t(e) = t(f)$.

(L3) Omit a leaf v and its only incident edge, if v does not appear as an edge label.

A LOT is said to be *reduced* if no such moves can be performed in it.

A LOT is called *injective* if no vertex appears more than once as an edge label.

In fact, it is easily seen that these moves on LOTs do not change the Q^{**}-class of the LOT presentation (the 3-deformation type of the LOT complex).

This is Howie's definition of a reduced LOT, but in the literature this can also mean a weaker condition. For example, in [HR01], a LOT Γ is said to be reduced if there are no possible (L3) moves on Γ.

Definition 4.4.2. A *weakly labeled oriented graph* (*WLOG*) $\Gamma = (E, V, s, t, \lambda)$ is an oriented graph with the additional structure of a labeling function $\lambda : E \to F(V)$, where $F(V)$ is the free group with basis V.

If the underlying graph of a WLOG is a tree, we call it *weak LOT*, (*WLOT*).

These type of trees come from a more natural decomposition of ribbon immersions. The notion was introduced by Howie in [How85], although it appears in his previous work. In fact, in the proof of [How83, Theorem 4.2] he shows that if $L \overset{3}{\nearrow\!\!\searrow} *$, then $L - e^2 \overset{3}{\nearrow\!\!\searrow} K_\Gamma$, where Γ is a WLOT.

Weak LOTs are also assigned group presentations and 2-complexes, in a similar way as in the case of LOTs. As we shall see, 3-deformation types of LOT complexes cover those of WLOT complexes.

The three moves defined above extend to six moves defined for WLOTs as follows.

Definition 4.4.3. Transformations on WLOTs.

(W1) Shrink an edge e such that $\lambda(e) = s(e)$ or $\lambda(e) = t(e)$, and identify the endpoints $s(e), t(e)$.

(W2) Identify two edges e, f such that $\lambda(e) = \lambda(f)$ and either $s(e) = s(f)$ or $t(e) = t(f)$.

(W3) Omit a leaf (extremal vertex) and its only incident edge, if the vertex does not appear *in any* edge label.

(W4) Switch the orientation of an edge and invert its label.

(W5) Given two edges e, f such that $t(e) = s(f)$, replace the source and label of f by $s'(f) = s(e)$ and $\lambda'(f) = \lambda(e)\lambda(f)$.

(W6) Replace, in some edge label $\lambda(f)$, the letter $t(e)$ for the word $\lambda(e)^{-1}s(e)\lambda(e)$.

These moves also correspond to Q^{**}-transformation of the WLOT presentation, and 3-deformations of the associated complex.

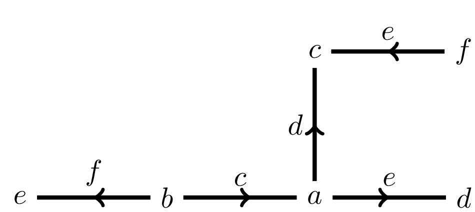
Howie shows that every WLOT can be transformed using these moves, to a LOT. In fact, it is not difficult to prove the following result.

Proposition 4.4.4. *Every equivalence class of WLOTs contains:*

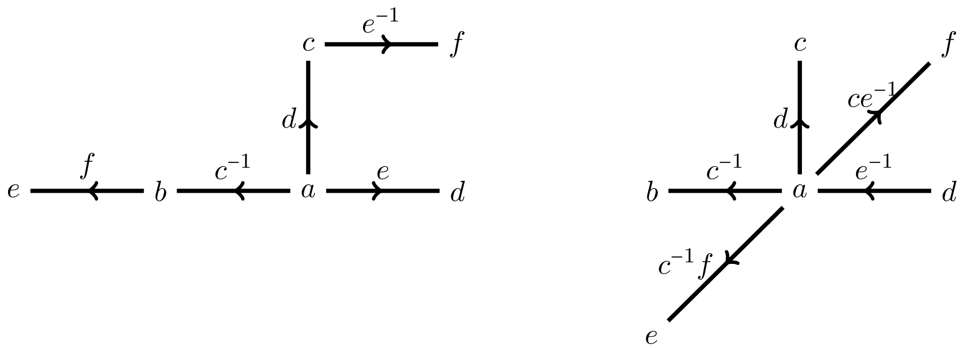
- *a WLOT of diameter at most 2,*
- *a WLOT with only two extremal vertices,*
- *a LOT,*
- *a reduced LOT,*
- *a LOT with only two extremal vertices.*

A LOT with only two extremal vertices is called a *labeled oriented interval* (*LOI*). Note that, as a consequence of this result, the asphericity of all LOIs would imply the asphericity of all LOTs, i.e. of all ribbon disc complements.

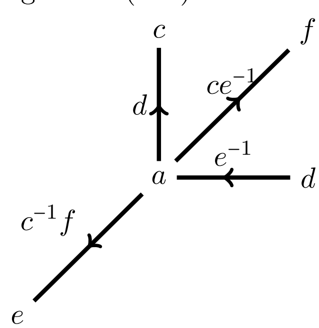
Example 4.4.5. Consider the LOT of the previous example.



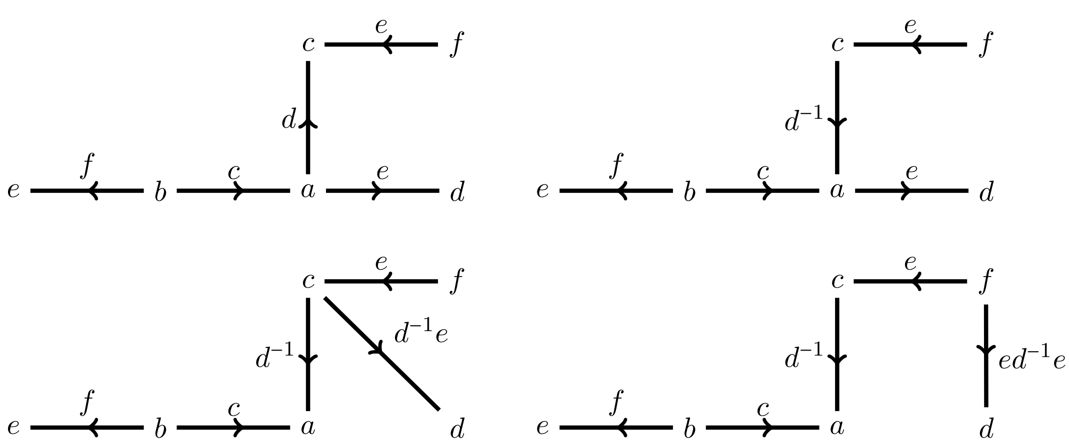
Switch the orientation and label (W4) of two edges, and then perform (W5) twice. In this way, we obtain a star.



Now, the vertex b does not appear in any edge label, and it is now a leaf, so we can omit the vertex and its incident edge with (W3).



If instead we want to transform it into a WLOI, we can do it by performing (W4) once, and then two consecutive (W5)-moves.



Another studied way to prove asphericity of LOTs, with a combinatorial point of view, is by the use of spherical diagrams. Recall that a map between CW-complexes is said to be combinatorial if it sends open cells homeomorphically onto open cells. A *spherical diagram* on a complex X is a combinatorial map from a combinatorial cell decomposition C of the 2-sphere to X. The information of such a map can be encoded by an orientation and labeling of the 1- and 2-cells of C. Since the second homotopy group of a complex is generated by spherical diagrams, it suffices to show that these represent the trivial element to show that the complex is aspherical. Concepts such as *diagrammatically reducible* or *vertex aspherical* are used in this context. They involve reducibility properties of spherical diagrams. If every diagram can be reduced, then all diagrams represent the trivial map.

For example, Huck and Rosebrock used these methods together with a classical graph theoretical lemma of Stallings to prove the asphericity of certain LOTs [HR07]. Stallings' lemma states that given a finite oriented graph G in the 2-sphere, without isolated vertices, there must be at least two consistent items [Sta87]. Here a consistent item means a source, a sink, or a region bounded by consistently oriented edges. The graph in the lemma corresponds to a combinatorial cell decomposition of the sphere.

Definition 4.4.6. Given a LOT Γ, a *sub-LOT* is a subtree Δ of Γ such that $\lambda(e) \in \Delta$ for every edge $e \in \Delta$.

It had also been proved that prime injective LOTs are DR [HR01]. Here *prime* means that Γ has no sub-LOTs where an (L3) move is possible. Later, Harlander and Rosebrock found examples of prime LOTs that are not DR [HR12b].

An interesting fact about LOTs is that for every LOT (actually for every LOG) Γ there is a reorientation of the edges of Γ such that the complex associated to the obtained LOT (resp. LOG) is DR [HR01].

Rosebrock defined the notion of complexity of a LOT, and proved that LOTs of complexity 2 are aspherical [Ros10]. The notion of a generating set of vertices had been introduced by Howie [How85].

Definition 4.4.7. Given a LOT Γ and a set $S = \{v_1, \dots, v_k\}$ of vertices of Γ, a sub-LOT Δ of Γ is *generated* by S if Δ is the smallest sub-LOT containing S such that no edge e of $\Gamma - \Delta$ has its label and one of its endpoints in Δ.

The *complexity* of a LOT Γ is the smallest number k such that Γ is generated by a set of k vertices.

In other words, Γ is *generated* S if one can visit all the vertices of Γ starting with the set S and using the following rule: if an edge has one visited endpoint and is labeled by a visited vertex, then its other endpoint can be visited. Note that the complexity of Γ is bounded by the number of vertices that appear as edge labels.

Rosebrock found an upper bound for the complexity of labeled oriented intervals, and gave examples of LOIs of maximal complexity. Later, Christmann and de Wolff generalized this bound to all LOTs [CdW14], and characterized the LOTs of maximal complexity. For this characterization, they defined a Rosebrock LOT, which is the smallest non-reducible LOT, with any orientation:

$$a \;\overset{c}{\overline{\qquad}}\; b \;\overset{a}{\overline{\qquad}}\; c$$

They show that LOTs of maximal complexity are decomposable as unions of Rosebrock LOTs.

Finally they make use of the following result of Rosebrock [Ros10, Lemma 3.4], which is an immediate consequence of a theorem of Whitehead about the homotopy groups of a union of two complexes [Whi39, Theorem 5].

Lemma 4.4.8. *Assume $P = \langle X \mid R \rangle$ and $P' = \langle Y \mid S \rangle$ are two LOT presentations, such that the corresponding 2-complexes are aspherical. Let Q be the LOT presentation obtained from the union of P and P' by identifying some x_k with some y_j Then the corresponding 2-complex K_Q is aspherical.*

Using this result, and the fact that Rosebrock LOTs are aspherical, Christmann and de Wolff prove the following.

Theorem 4.4.9. *If Γ is a LOT with maximal complexity, then its corresponding 2-complex is aspherical.*

Recently, Harlander and Rosebrock proved the asphericity of injective LOTs. This result appears in the preprint [HR12a], available in arXiv. In that article, they also use spherical diagrams, Stallings' lemma and related techniques.

Resumen del capítulo 4: Complementos de ribbon discs y LOTs

Los *ribbon discs* son una generalización de los nudos clásicos. Un ribbon disc es una inmersión propia de un disco de dimensión n en uno de dimensión $n+2$, $D^n \hookrightarrow D^{n+2}$ que satisface que la función $r : D^{n+2} \to \mathbb{R},\ x \mapsto \|x\|$ se restringe a una función de Morse en el disco D^n. Un *complemento de ribbon disc* es el complemento de la imagen de D^n en D^{n+2} por una inmersión de este tipo.

Durante mucho tiempo estuvo abierta la conjetura sobre la asfericidad de los complementos de nudos, que fue finalmente probada por Papakyriakopoulos [Pap57] usando métodos de 3-variedades. Este problema fue probablemente una de las motivaciones de Whitehead al formular su conjetura, cuya validez implicaría una demostración alternativa de la asfericidad de los complementos de nudos.

Actualmente se conjetura que, al igual que en el caso de nudos, los complementos de ribbon discs también son asféricos. En este capítulo presentamos este problema, que luego estudiamos con métodos de espacios finitos en el capítulo 5, donde probamos la asfericidad de amplias clases de complementos de ribbon discs. La homotopía de estos espacios fue estudiada por numerosos matemáticos [Yan69, AMY81, Has83, How85, HR03]. Observamos que en la literatura se encuentran algunas pruebas icompletas de esta conjetura (ver [How83] para más detalles).

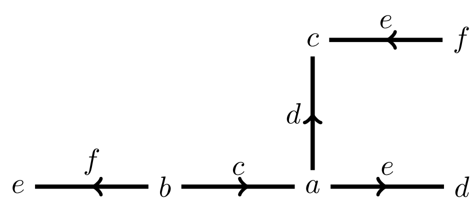
En la sección 4.2 presentamos la construcción de Howie del *labeled oriented tree* (LOT), asociado a un ribbon disc [How85]. Se trata de un grafo (árbol) orientado que adicionalmente tiene, para cada una de sus aristas, una etiqueta con el nombre de un vértice. Es a partir de esta construcción que el problema se puede estudiar desde un punto de vista combinatorio.

Definición. Un LOT $\Gamma = (E, V, s, t, \lambda)$ consiste en dos conjuntos E, V de aristas y de vértices, y tres funciones $s, t, \lambda : E \to V$ llamadas origen, destino y etiqueta, tal que el grafo subyacente es un árbol, es decir, no tiene ciclos.

A cada LOT Γ se le asocia una presentación $P(\Gamma)$ de un grupo, con un generador por cáda vértice, y una relación por cada arista. Si la arista e tiene origen s_e, destino t_e y etiqueta λ_e, la relación asociada es $\lambda_e t_e \lambda_e^{-1} s_e^{-1}$, de modo que en el grupo presentado se tiene $s_e \lambda_e = \lambda_e t_e$. Y de manera usual, a partir de esta presentación se construye un 2-complejo K_Γ, llamado *complejo LOT*.

Ejemplo. Al LOT dibujado a continuación se le asocia la presentación

$$P = \langle a, b, c, d, e, f \mid ece^{-1}f^{-1}, dcd^{-1}a^{-1}, ede^{-1}a^{-1}, cbc^{-1}a^{-1}, fbf^{-1}e^{-1} \rangle.$$



Howie le asocia, a cada ribbon disc, un LOT Γ de tal manera que K_Γ tiene el tipo homotópico (simple) del complemento del ribbon disc [How85]. De esta manera, para estudiar la asfericidad de estos espacios, basta estudiar la de los complejos LOT. De hecho, todo LOT es el asociado a un ribbon disc adecuado, de modo que la asfericidad de los complejos LOT es equivalente a la de los complementos de ribbon discs.

Howie probó que si un 2-complejo L se 3-deforma a un punto y e es una 2-celda de L, entonces el subcomplejo $K = L - e$ se 3-deforma a un complejo LOT. Una consecuencia de esto es que, si la conjetura de Andrews-Curtis es cierta, entonces la asfericidad de los complejos LOT implica la conjetura de Whitehead para el caso compacto. A su vez, es fácil ver que a todo complejo LOT se le puede agregar una 2-celda y obtener un complejo contráctil. Es por esto que los complejos LOT son considerados casos testigos de la conjetura.

En la sección 4.3 presentamos ciertas propiedades de grupos que están asociadas al problema de asfericidad de LOTs, y mencionamos las relaciones que hay entre ellas.

Definición. Un grupo G satisface la *propiedad del único producto* (*UPP*) si para cada par de subconjuntos finitos no vacíos A, B de G existe un elemento $g \in G$ que se puede expresar de manera única como un producto $g = ab$ con $a \in A$ y $b \in B$.

En la sección 5.4 utilizamos esta propiedad, que es más débil que la de ser localmente indicable, a partir de su equivalencia con la siguiente condición.

Un grupo G satisface UPP si y sólo si para todo par de subconjuntos no vacíos X, Y de G existe un elemento $g \in G$ tal que $gX \cap Y$ tiene exactamente un elemento.

Combinando esta propiedad con otras hipótesis, pudimos probar la asfericidad de ciertas clases de LOTs. No se sabe si los grupos asociados a LOTs son localmente indicables, o si satisfacen UPP.

Por último, en la sección 4.4 repasamos los avances alcanzados hasta ahora en el problema de asfericidad de LOTs. Para más detalles en este tema, referimos al lector a [Ros07]. Howie mostró que para probar la asfericidad de los LOTs, basta considerar el caso en que el grafo subyacente tiene solamente 2 puntos extremales, es decir, es un intervalo. También probó la asfericidad de LOTs de diámetro ≤ 3 [How85]. Huck y Rosebrock también obtuvieron resultados en el tema mediante métodos combinatorios, probando la asfericidad de ciertas clases de LOTs [HR01, HR07], como los primos e inyecitvos. Recientemente, Rosebrock y Harlander hicieron nuevos aportes [Ros10, HR12a], a partir de las nociones de complejidad de un LOT, y de LOTs inyectivos.

Chapter 5

Asphericity of LOTs using colorings

5.1 G-colorings

In this section we give an outline of recent works of Barmak and Minian, which we will apply in the following sections. These results, which can be found in [BM12a, BM14], provide a way of computing the first and second homotopy group of posets, using the notion of coloring.

A *coloring* of a poset X is a map $c : \mathcal{E}(X) \to G$ from the set of edges of the Hasse diagram of X to a group G. The coloring c is said to be *admissible* if for every pair of chains $x = x_1 \prec x_2 \prec \cdots \prec x_k = y$, $x = x'_1 \prec x'_2 \prec \cdots \prec x'_l = y$ that agree at the top and at the bottom, we have $c(x'_1, x'_2)c(x'_2, x'_3) \dots c(x'_{k-1}, x'_k) = c(x'_1, x'_2)c(x'_2, x'_3) \dots c(x'_{k-1}, x'_k)$.

Given a point $x_0 \in X$, if the coloring is admissible, then it gives rise to a map $\mathcal{H}(X, x_0) \to G$, where $\mathcal{H}(X, x_0)$ is the group of classes of edge-paths of the diagram, which is isomorphic to the fundamental group $\pi_1(X, x_0)$. The coloring is said to be *connected* when this map is surjective.

Using this notion they prove correspondence between admissible connected G-colorings of a poset X and regular coverings of X with deck transformation group isomorphic to G. In fact, they give a construction for the covering space associated to a given coloring.

Applying these results, they prove the following theorem, which gives an efficient method for computing a presentation of the fundamental group of a poset.

Theorem 5.1.1 (Theorem 4.4, [BM12a])**.** *Let X be a connected poset and let D be a subdiagram of the Hasse diagram of X, which corresponds to a simply-connected space. Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be the subset of $\mathcal{E}(X)$ of edges which are not in D. Let G be the group presented as follows. The generators are the set $\{e_\alpha\}_{\alpha\in\Lambda}$ and the relations are given by admissibility. Concretely, every pair of chains*

$$x = x_1 \prec x_2 \prec \cdots \prec x_k = y, \quad x = x'_1 \prec x'_2 \prec \cdots \prec x'_l = y$$

from a point x to a point y produces a relator

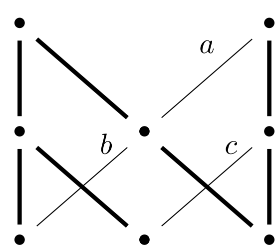
$$\prod_{(x_i, x_{i+1}) \notin D} (x_i, x_{i+1}) = \prod_{(x'_i, x'_{i+1}) \notin D} (x'_i, x'_{i+1}).$$

Suppose there is a subset $\Gamma \subseteq \Lambda$ such that the classes $\{e_\alpha\}_{\alpha\in\Gamma}$ generate G and such that for each $\alpha \in \Gamma$ there exists a closed edge-path ω_α in x_0 in which e_α appears precisely once, and the other edges of Λ do not appear. Then $\pi_1(X, x_0) \simeq G$.

Remark 5.1.2. If the diagram D in the theorem contains all the elements of X, then the existence of the edge-paths ω_α is ensured: since A is connected, simply take an edge-path in D that joins the endpoints of the edge e_α.

As a consequence of this result, one can obtain a presentation for the fundamental group of any finite space X as follows. Let D be a subdiagram of the Hasse diagram of X such that it contains all the elements of X, and corresponds to a simply connected space (for example, any spanning tree in the Hasse diagram is easily seen to be contractible since leaves in the diagram correspond to beat points). Then the presentation of the fundamental group of X has one generator for each missing edge in D and one relator for each *simple digon*, that is, each pair of monotonic edge-paths which are disjoint except at their endpoints, where they agree.

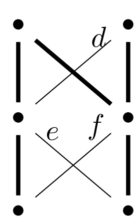
Examples 5.1.3. Consider the poset X represented below, where the diagram D (a spanning tree of the Hasse diagram) of thick edges has been chosen, and the remaining edges have been colored a, b, c.



The diagram has only 2 non-trivial digons, that give rise to the relations $b = 1$, $a = 1$, and there are no digons containing the third colored edge. This means that the fundamental group is generated by the edge colored c, with no relations. Thus

$$\pi_1(X) \simeq \langle\ a, b, c \mid a, b\ \rangle \simeq \langle\ c \mid\ \rangle \simeq \mathbb{Z}.$$

Next consider the poset Y represented below, where the diagram D is again emphasized, and the remaining edges have been named d, e, f.



The diagram has 4 non-trivial digons, and they give rise to the relations $f = 1$, $e = 1$, $d = f$, $ed = 1$, which means that the group presented is the trivial group.

We shall now recall the main results of [BM14], which provide a description for the second homotopy group of compact regular CW-complexes.

Recall that the fundamental group $\pi_1(X, x_0)$ of a given poset X with basepoint $x_0 \in X$ is naturally isomorphic to the edge-path group $\mathcal{H}(X, x_0)$, which consists of sequences of edges strating and ending at x_0, with certain identifications (see [BM07]). Theorem 5.1.1 cited above and the subsequent observations, give a procedure for obtaining a presentation of this group using colorings. Moreover, they show that the isomorphism $\pi_1(X, x_0) \simeq G$, which factors through $\mathcal{H}(X, x_0)$, is actually induced by the coloring. This means that the image of a given edge path is the product of the colors of its edges.

Using this fact and their characterization of regular coverings in terms of colorings, they provide a description of the universal cover of a given finite poset. Concretely, if c is a G-coloring of X which corresponds to the universal cover (for example, the coloring of theorem 5.1.1), then the universal covering space $\tilde{X}$ is the set $X \times G$ with the relations $(y, g) \prec (x, gc(x, y))$ whenever $y \prec x$. The covering map is the projection on the first coordinate. The second homotopy group of X is isomorphic to the second homology group of $\tilde{X}$.

If in addition X is the face poset of a regular CW-complex, then this is also true for $\tilde{X}$, and therefore its homology can be computed using its cellular chain complex,

$$C_n(\tilde{X}) = \bigoplus_{\deg(u)=n} \mathbb{Z}, \text{ with boundary map } \quad d : C_n(\tilde{X}) \to C_{n-1}(\tilde{X}), u \mapsto \sum_{v \prec u} \epsilon(u, v) v.$$

The *incidence number* $\epsilon(u, v)$ of u in v corresponds to the incidence of one cell in the other and agrees with the degree of the connecting morphism $H_{n-1}(\hat{U}_u) \to H_{n-2}(\hat{U}_v)$ of the Mayer-Vietoris sequence for the covering $\hat{U}_u = (\hat{U}_u - \{v\}) \cup U_v$. Moreover, the incidences between points of degrees 2 and 1 can be computed by choosing, for every point u of height 2, a simple oriented cycle in the subposet $\hat{U}_u$, which is a model of S^1, and for every point v of height 1, a positive and a negative lower cover ($\hat{U}_v$ consists of two minimal points). Then the incidences depend on whether the cycle chosen for u passes through v from the negative lower cover to the positive one, or in the opposite direction. See [Min12] for more details about cellular chain complexes for computing homology of posets.

The incidence numbers can also be defined selecting a number $\epsilon(x, y) \in \{1, -1\}$ for each pair $y \prec x$ such that

- $\sum_{z \prec y} \epsilon(y, z) = 0$ for every $y \in X$ of height 1,
- $\sum_{z \prec y \prec x} \epsilon(x, y)\epsilon(y, z) = 0$ for every pair $z < x$ with x of height 0 and z of height 2.

This way of choosing the incidence numbers, used in [BM14], was introduced by Massey (see [Mas91]).

Making use of this fact, they define the incidences in $\tilde{X}$ from the ones in X. Given a coherent choice of incidences for X, $\epsilon(u, v) = \epsilon(p(u), p(v))$ is a coherent choice of incidences for $\tilde{X}$. Since the fundamental group G acts freely and transitively on the fibers, $C_n(\tilde{X}) = \mathbb{Z}G \otimes C_n(X)$ identifying $(x, 1)$ with x, so the elements of $C_n(\tilde{X})$ are of the form

$\sum\limits_{g\in G}\sum\limits_{\deg(x)=n} n_g^x gx$, where the coefficients n_g^x are in $\mathbb{Z}$. Moreover, d is a $\mathbb{Z}G$-morphism, so

$$d(\sum_{g\in G}\sum_{\deg(x)=n} n_g^x gx) = \sum_{g\in G}\sum_{\deg(x)=n} n_g^x g d(x).$$

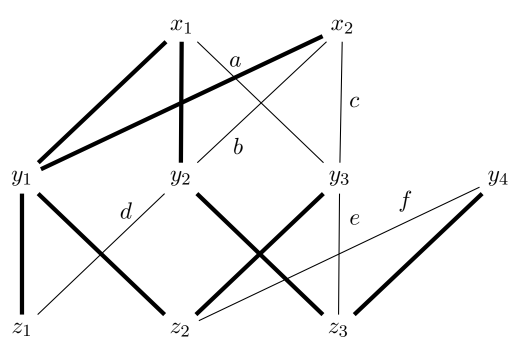
But the lower covers of $x \sim (x,1)$ in $\tilde{X}$ are the elements $(y, c(x,y)^{-1})$, with $y \prec x$ in X, therefore

$$d(\sum_{g\in G}\sum_{\deg(x)=n} n_g^x gx) = \sum_{g\in G}\sum_{\deg(x)=n}\sum_{y\prec x} \epsilon(x,y) n_g^x g c(x,y)^{-1} y.$$

Finally, if the poset is additionally the face poset of a 2-dimensional regular CW-complex, then the second homology group is just the kernel of the boundary map, since $C_3(\tilde{X}) = 0$. That is,

$$\pi_2(X) = \left\{ \sum_{\substack{\deg x=2\\ g\in G}} n_g^x gx : \; n_g^x \in \mathbb{Z} \text{ and } \sum_{x\succ y} \epsilon(x,y) n^x_{gc(x,y)} = 0 \quad \forall y \in X, \deg y = 1 \right\}.$$

Example 5.1.4. Let us consider the following space X, which is a model of $S^2 \vee S^1$. The chosen diagram D is shown with thick edges and the remaining edges are colored with a, b, c, d, e, f.

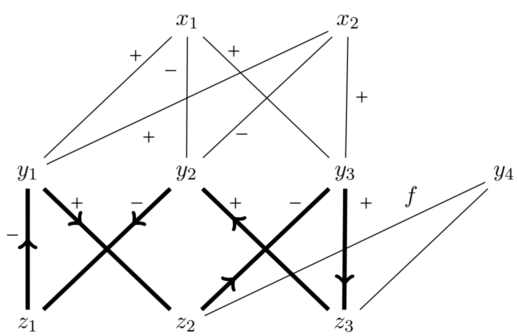


The digons in the diagram say $d = 1, a = 1, ea = 1, db = 1, c = 1, ec = b$, so the fundamental group of this poset is isomorphic to $\mathbb{Z}$, generated by f.

Now we must choose the incidence numbers. For the lower level, set

$$\epsilon(y_1,z_1) = -1, \epsilon(y_1,z_2) = 1, \epsilon(y_2,z_1) = -1, \epsilon(y_2,z_3) = 1, \epsilon(y_3,z_2) = -1, \epsilon(y_3,z_3) = 1$$

(the incidences $\epsilon(y_4, z_i)$ are irrelevant). Next we choose the cycle $z_1, y_1, z_2, y_3, z_3, y_2, z_1$ for both x_1 and x_2 (note that $\hat{U}_{x_1} = \hat{U}_{x_2}$). With these choices we have $\epsilon(x_i, y_1) = 1$, $\epsilon(x_i, y_2) = -1$ and $\epsilon(x_i, y_3) = 1$. We illustrate this cycle and the signs of the incidence numbers in the diagram below.



Thus the equations of $\pi_2(X)$ are the following.

(y_1) $\quad n_g^{x_1} + n_g^{x_2} = 0 \quad \forall\ g \in G$

(y_2) $\quad -n_g^{x_1} - n_g^{x_2} = 0 \quad \forall\ g \in G$

(y_3) $\quad n_g^{x_1} + n_g^{x_2} = 0 \quad \forall\ g \in G$

We can now write the description of the second homotopy group.

$$\pi_2(X) = \{\sum_{g\in G} n_g^{x_1} g x_1 + n_g^{x_2} g x_2 : n_g^{x_i} \in \mathbb{Z},\ \text{and}\ n_g^{x_1} + n_g^{x_2} = 0 \quad \forall\ g \in G\},$$

where $G \simeq \mathbb{Z}$. Since $n_g^{x_2} = -n_g^{x_1}$ $\forall g \in G$, we have

$$\pi_2(X) = \{\sum_{g\in\mathbb{Z}} n_g^{x_1} g(x_1 - x_2) : n_g^{x_1} \in \mathbb{Z}\}.$$

That is, $\pi_2(X)$ is a free $\mathbb{Z}[t, t^{-1}]$-module on one generator $x_1 - x_2$.

SAGE code 5.1.5. We present a program that uses the results of this section to compute the fundamental group and the second homotopy group of 2-complexes.

Input: a poset X which is the face poset of a 2-complex and a list A of cover relations of X (i.e. edges (a, b) of the Hasse diagram of X), which generate a simply connected poset. Output: a presentation of $\pi_1(X)$ and a set of equations describing $\pi_2(X)$ as a submodule of the free $\mathbb{Z}[\pi_1(X)]$-module generated by the elements of degree 2, both computed using the results above.

```
def pi1_pi2(X,A):

    from sage.groups.free_group import FreeGroup
    from sage.interfaces.gap import gap

    #first, \pi_1 using colrings:

    gen=[e for e in X.cover_relations() if tuple(e) not in A] #generators
    colorgen=[e[0] + e[1] for e in gen]
    edgegen=[tuple(e) for e in gen]
```

```
gen_dict = dict(zip(edgegen,range(len(gen))))
s=colorgen[0]
for i in range(1,len(gen)):
    s=s+',’+ colorgen[i]
FG=FreeGroup(s)

rels = [] #relators
chains=X.maximal_chains()
for i in range(len(chains)):
    for j in range(i+1,len(chains)):
        c=chains[i]
        d=chains[j]
        if c[0]==d[0] and c[2]==d[2]:
            z=dict()
            if (c[0],c[1]) in A:
                z[0]=FG.one()
            else:
                z[0]=FG.gen(gen_dict[(c[0],c[1])])
            if (c[1],c[2]) in A:
                z[1]=FG.one()
            else:
                z[1]=FG.gen(gen_dict[(c[1],c[2])])
            if (d[0],d[1]) in A:
                z[2]=FG.one()
            else:
                z[2]=FG.gen(gen_dict[(d[0],d[1])])
            if (d[1],d[2]) in A:
                z[3]=FG.one()
            else:
                z[3]=FG.gen(gen_dict[(d[1],d[2])])
            rels.append(z[0]*z[1]*z[3].inverse()*z[2].inverse())

FGq=FG.quotient(rels) #\pi_1
I=FGq.simplification_isomorphism()
print 'presentation of \pi_1: ', FG.quotient(rels).simplified()

#now, \pi_2 using golorings:

#we enumerate the set of minimal elements
ordermin=zip(X.minimal_elements(),range(len(X.minimal_elements())))
d=dict(ordermin)
h1=[x for x in X if x not in X.maximal_elements() and x not in
X.minimal_elements()]

print 'equations for \pi_2:'
```

```
    for w in h1:
        for x in X.upper_covers(w): #we define the incidence number
            mx=[d[u] for u in X.minimal_elements() if X.is_less_than(u,x)]
            mw=[d[u] for u in X.minimal_elements() if X.is_less_than(u,w)]
            mx.sort()
            if mx[1] not in mw:
                print '-',
            else:
                print '+',

            print 'n^',
            print x,
            print '_{ l',
            if (tuple([w,x]) not in A):
              print '* ',
              print (I(FG.gen(gen_dict[w,x]))),
            print "}",
        print "= 0 for all l in \pi_1"
    return FG,rels
```

To use this program, one should set F, r =`pi1_pi2(X,A)`. In this way, the presentation of $\pi_1(X)$ and the equations for $\pi_2(X)$ are printed, and all the information of the fundamental group is saved in F and r, with the possibility of later using the simplification isomorphism, which indicates how a word of the free group F is expressed in the generators that remained in the simplified presentation.

5.2 LOT posets

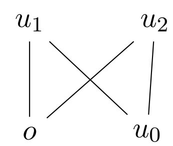
To every LOT Γ we will associate a finite space $\mathcal{X}(\Gamma)$, which is the face poset of the barycentric subdivision of the LOT complex K_Γ. This space has the weak homotopy type of the LOT complex and will be a useful tool for studying asphericity of LOTs using combinatorial methods of finite spaces. We give a description of this space (as a poset) and fix some notation.

Recall that to every LOT Γ with vertices $v_1, v_2, \dots, v_n$, one can associate a LOT presentation $P(L) = \langle v_1, v_2, \dots, v_n | r_1, r_2, \dots, r_{n-1} \rangle$, where each r_j is a relator of the form $\lambda_x t_x \lambda_x^{-1} s_x^{-1}$, arising from an edge x of Γ with source s_x, target t_x and label λ_x. And the LOT complex K_Γ has a 1-cell for every vertex of Γ, all of them attached at a unique vertex, and a 2-cell for every edge of Γ, with an attaching map that spells the relator associated to it.

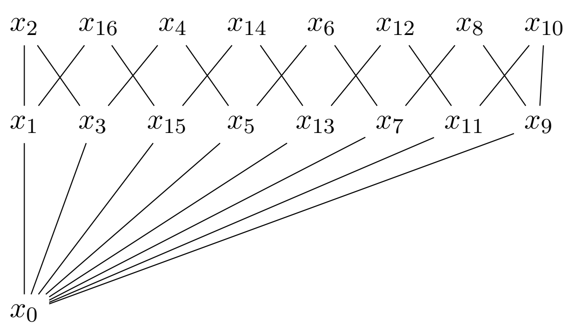
We shall now give an explicit description of the LOT poset $\mathcal{X}(\Gamma)$. Note that it is a graded homogeneous poset of height 2. Let o be the minimal point which represents the only original vertex of $\mathcal{K}_\Gamma$. We will call it the *base point* for the poset $\mathcal{X}(\Gamma)$.

For every vertex u of Γ there is a subspace of $\mathcal{X}(\Gamma)$ which is a model of S^1, consisting of 4 points, one of which is the base point. Thus there are 3 new elements associated to

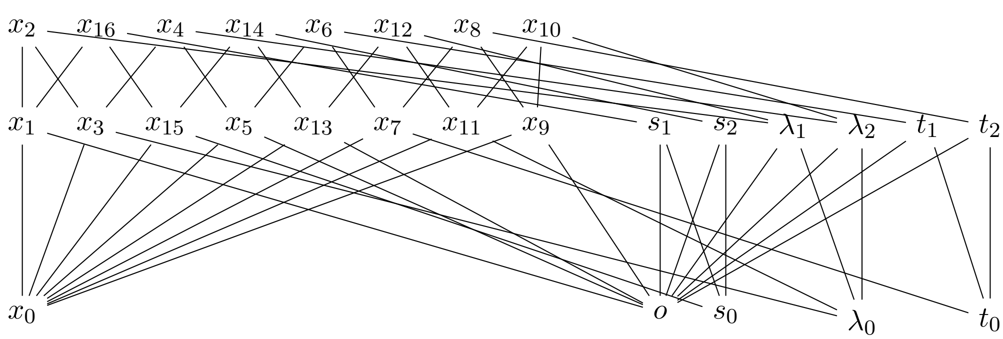
u, called *vertex elements*, u_0, u_1, u_2 in $\mathcal{X}(\Gamma)$. The point u_0 is a minimal element, and both u_1, u_2 cover o and u_0.



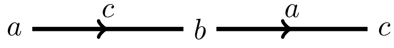
For every edge x of Γ, there ir a subspace of $\mathcal{X}(\Gamma)$ which is a model of D^2. This space is the cone with apex x_0 of a model of S^1 with 16 elements $x_1, \ldots, x_{16}$. Thus we have 17 new elements associated to x, called *edge elements*, $x_0, \ldots, x_{16}$ in $\mathcal{X}(\Gamma)$. The point x_0 is minimal and corresponds to the barycenter of the 2-cell associated to the edge. The other evenly indexed elements $x_2, x_4, \ldots, x_{16}$ are of degree 2, and the oddly indexed elements $x_1, x_3, \ldots, x_{15}$ are of degree 1. The order between these elements is represented below.



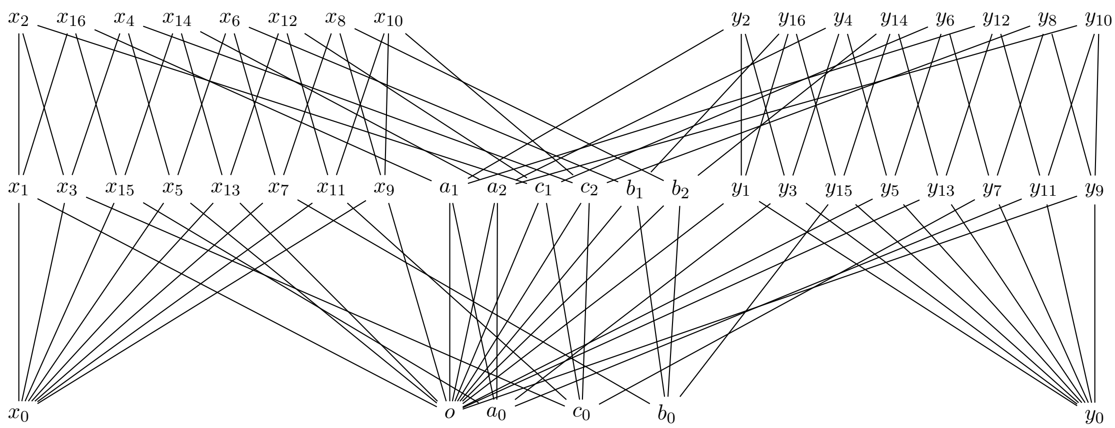
There are also cover relations between the vertex elements and the edge elements. If the edge x has source s, target t and label λ, then the elements $x_1, \ldots, x_{16}$ spell the corresponding relator $\lambda t \lambda^{-1} s^{-1}$ in the diagram, that is, the elements $x_1, x_2, \ldots, x_{16}$ respectively cover $o, \lambda_1, \lambda_0, \lambda_2, o, t_1, t_0, t_2, o, \lambda_2, \lambda_0, \lambda_1, o, s_2, s_0, s_1$.



Example 5.2.1. Consider the following LOT Γ.



The LOT poset associated to Γ is illustrated below. We call x the edge with endpoints a and b, and y the edge with endpoints b and c.



Clearly LOT posets are very large. We remark that we usually do not need to sketch these posets, but we use them for theoretical purpose. When we do want to compute or handle a LOT poset, we do it using a computer program.

SAGE code 5.2.2. The following program produces the LOT poset associated to a given LOT Γ.

The input consists of list V of vertices, for example $V = ['a','b','c']$, and a list A of edges described as [source, label, target], for example $A = [['a','c','b'],['b','a','c']]$.

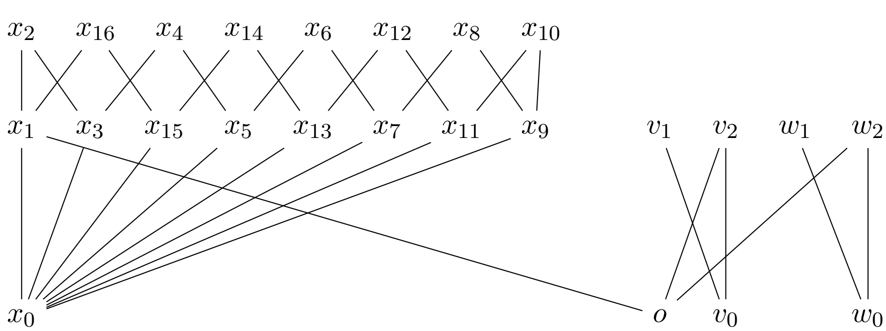
```
def lot_poset(V,A):
    elem=['o']
    rel=[]
    for v in V:
        for j in range(3):
            elem.append(v+'_'+str(j))
    for i in range(len(A)):
        for j in range(17):
            elem.append( 'edge'+A[i][0]+A[i][2]+'_'+str(j))
    for v in V:
        rel.append(['o',v+'_1'])
        rel.append(['o',v+'_2'])
        rel.append([v+'_0',v+'_1'])
        rel.append([v+'_0',v+'_2'])
    for i in range(len(A)):
        for j in range(1,9):
            rel.append(['edge'+A[i][0]+A[i][2]+'_0','edge'+A[i][0]+A[i][2]+'_'+str(j)])
        rel.append(['edge'+A[i][0]+A[i][2]+'_1','edge'+A[i][0]+A[i][2]+'_9'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_1','edge'+A[i][0]+A[i][2]+'_10'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_2','edge'+A[i][0]+A[i][2]+'_9'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_2','edge'+A[i][0]+A[i][2]+'_11'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_3','edge'+A[i][0]+A[i][2]+'_10'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_3','edge'+A[i][0]+A[i][2]+'_12'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_4','edge'+A[i][0]+A[i][2]+'_11'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_4','edge'+A[i][0]+A[i][2]+'_13'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_5','edge'+A[i][0]+A[i][2]+'_12'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_5','edge'+A[i][0]+A[i][2]+'_14'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_6','edge'+A[i][0]+A[i][2]+'_13'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_6','edge'+A[i][0]+A[i][2]+'_15'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_7','edge'+A[i][0]+A[i][2]+'_14'])
```

```
        rel.append(['edge'+A[i][0]+A[i][2]+'_7','edge'+A[i][0]+A[i][2]+'_16'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_8','edge'+A[i][0]+A[i][2]+'_15'])
        rel.append(['edge'+A[i][0]+A[i][2]+'_8','edge'+A[i][0]+A[i][2]+'_16'])
    for i in range(len(A)):
        rel.append([A[i][1]+'_1','edge'+A[i][0]+A[i][2]+'_9'])
        rel.append([A[i][1]+'_2','edge'+A[i][0]+A[i][2]+'_11'])
        rel.append([A[i][1]+'_1','edge'+A[i][0]+A[i][2]+'_14'])
        rel.append([A[i][1]+'_2','edge'+A[i][0]+A[i][2]+'_16'])
        rel.append([A[i][2]+'_1','edge'+A[i][0]+A[i][2]+'_13'])
        rel.append([A[i][2]+'_2','edge'+A[i][0]+A[i][2]+'_15'])
        rel.append([A[i][0]+'_1','edge'+A[i][0]+A[i][2]+'_10'])
        rel.append([A[i][0]+'_2','edge'+A[i][0]+A[i][2]+'_12'])
        rel.append(['o','edge'+A[i][0]+A[i][2]+'_1'])
        rel.append(['o','edge'+A[i][0]+A[i][2]+'_4'])
        rel.append(['o','edge'+A[i][0]+A[i][2]+'_5'])
        rel.append(['o','edge'+A[i][0]+A[i][2]+'_8'])
        rel.append([A[i][1]+'_0','edge'+A[i][0]+A[i][2]+'_2'])
        rel.append([A[i][0]+'_0','edge'+A[i][0]+A[i][2]+'_3'])
        rel.append([A[i][2]+'_0','edge'+A[i][0]+A[i][2]+'_6'])
        rel.append([A[i][1]+'_0','edge'+A[i][0]+A[i][2]+'_7'])
    return Poset((elem,rel))
```

5.3 Description of π_2 via colorings

Now we will choose a subdiagram D of the Hasse diagram of $\mathcal{X}(\Gamma)$ which contains all the elements and which corresponds to a simply connected space. Different choices of D can give different presentations for the fundamental group and different descriptions of the second homotopy group of Γ. Here we consider a particular choice of D and we will use it to obtain our main results. In fact our subdiagram D corresponds to a contractible subposet.

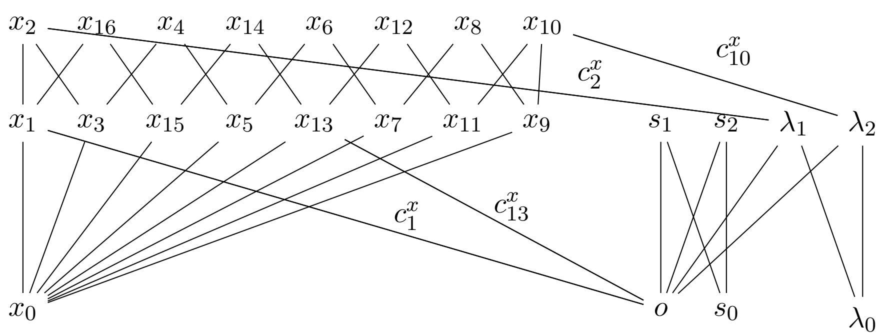
As we said, the diagram D has all the vertices of $\mathcal{X}(\Gamma)$. The edges of $\mathcal{X}(\Gamma)$ that are missing in D are one for each vertex u, the one joining o and u_1, and 15 for each edge x, the ones joining $x_i, i > 1$ with the corresponding vertex elements. A fragment of the diagram is illustrated below.



The poset represented by the diagram D is contractible. To see this, first observe that for every edge x of Γ, the elements $x_3, x_5, x_7, x_9, x_{11}, x_{13}, x_{15}$ are down beat points, since the only cover x_0 in D. After removing these points, the elements $x_4, x_6, x_8, x_{10}, x_{12}, x_{14}$ cover only x_0 and are new beat points, and x_2, x_{16} are beat points, since they cover only

x_1. After removing all of them, the element x_0 is an up beat point, and after removing it x_1 becomes beat point. Finally, for every vertex v of Γ, v_1, v_0, and v_2 can be removed (in that order) being beat points. Thus the entire poset collapses to the base point.

Now the presentation of the fundamental group of Γ associated to this diagram has one generator for each missing edge in D. The color of every edge of D is the trivial element, 1. The color of the non-trivial edges that have upper end x_i will be denoted by c_i^x. And the color of the non-trivial edges that have upper end v_1 will be denoted by v.

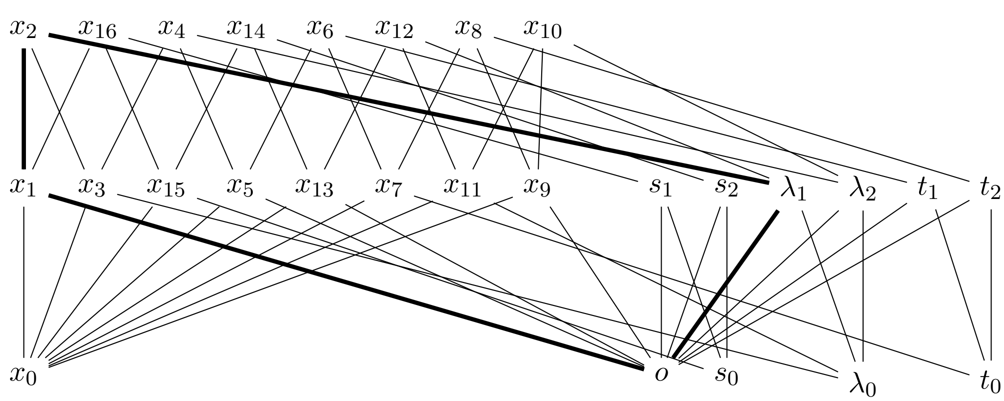


So the generators are $\{c_i^x, v\}$, with $2 \leq i \leq 16,\ x$ edge of $L,\ v$ vertex of L. That is, the presentation obtained will be

$$Q = \langle v_1, v_2, \dots, v_n, c_1^{x_1}, c_2^{x_1}, \dots, c_{16}^{x_1}, c_1^{x_2}, c_2^{x_2}, \dots, c_{16}^{x_2}, \dots, c_1^{x_{n-1}}, c_2^{x_{n-1}}, \dots, c_{16}^{x_{n-1}} \mid \text{relators} \rangle.$$

The relators of the presentation are given by the simple digons of the poset. Recall that a simple digon in a poset of height 2 is a pair of maximal chains which share maximum and minimum but not the middle point. The relation given by a digon $p \prec q \prec r,\ p \prec \tilde{q} \prec r$ is that $c(p,q)c(q,r) = c(p,\tilde{q})c(\tilde{q},r)$.

In a LOT poset, there are digons that involve $x_0,\ x_{2k-1},\ x_{2k}$ and x_{2k+1}. These give a trivial relator, since all the edges between them are in D.



The remaining digons involve $x_m,\ x_{m+1}$ and one vertex element and the base point, or two vertex elements.

The relators obtained by these digons are the following.

$$\begin{array}{llll}
1 = \lambda c_2^x & c_5^x = t c_6^x & c_9^x = c_{10}^x & c_{13}^x = c_{14}^x \\
c_2^x = c_3^x & c_6^x = c_7^x & c_{10}^x = c_{11}^x & c_{14}^x = c_{15}^x \\
c_3^x = c_4^x & c_7^x = c_8^x & c_{11}^x = c_{12}^x & c_{15}^x = c_{16}^x \\
c_4^x = c_5^x & c_8^x = c_9^x & \lambda c_{12}^x = c_{13}^x & s c_{16}^x = 1
\end{array}$$

So if we omit the trivial relations, the presentation is

$$Q = \langle v_1, v_2, \dots, v_n, c_1^{x_1}, c_2^{x_1}, \dots, c_{16}^{x_1}, c_1^{x_2}, c_2^{x_2}, \dots, c_{16}^{x_2}, \dots, c_1^{x_{n-1}}, c_2^{x_{n-1}}, \dots, c_{16}^{x_{n-1}} | \\ \{\lambda_x^{-1} c_2^x, (c_5^x)^{-1} t_x c_6^x, \dots, (c_{15}^x)^{-1} c_{16}^x, s_x c_{16}^x\}_{x \text{ edge of } L}\rangle.$$

With these equations, the colors $c_j^{x_i}$ can be expressed in terms of the colors v_i. In fact, we can eliminate these generators and relators using Q^{**} operations as follows. For each edge x, using the relator $\lambda_x^{-1} c_2^x$, we can replace the other appearence of c_2^x by λ_x, and then eliminate this generator and this relator. Repeating this process, we end up with the presentation

$$Q' = \langle v_1, v_2, \dots, v_n | \{s_x \lambda_x t_x^{-1} \lambda_x^{-1}\}_{x \text{ edge of } L}\rangle.$$

This presentation is clearly equivalent to the LOT presentation $P(L)$. Note that the colors c_i^x are expressed in terms of the colors v_i as follows:

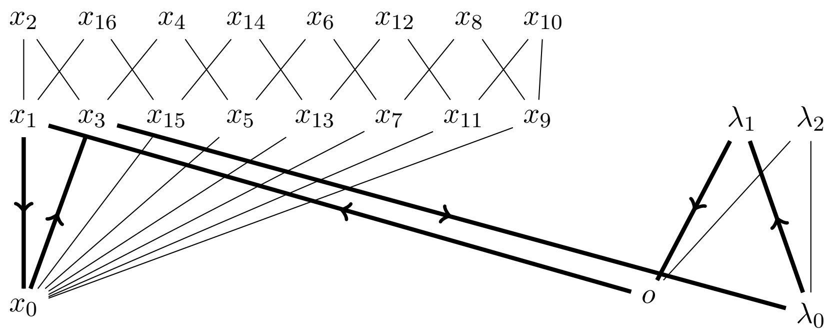
$$\begin{aligned}
c_2^x, c_3^x, c_4^x, c_5^x, &= \lambda_x^{-1} \\
c_6^x, \dots, c_{12}^x &= t_x^{-1} \lambda_x^{-1} \\
c_{13}^x, c_{14}^x, c_{15}^x, c_{16}^x &= \lambda_x t_x^{-1} \lambda_x^{-1} = s_x^{-1}
\end{aligned}$$

Recall that, given such a coloring of a poset P that is the face poset of a complex, the second homotopy group has the following description.

$$\pi_2(P) = \left\{ \sum_{\deg p = 2 g \in G} n_g^p g p : \ n_g^p \in \mathbb{Z} \text{ and } \sum_{p \succ q} \epsilon(p,q) n_{gc(p,q)}^p = 0 \quad \forall q \in P, \deg q = 1 \right\}$$

Where $\epsilon(p,q)$ means the incidence number of the edge (p,q), which can take the values 1 and -1, and $c(p,q)$ means the color of the edge (p,q), thus the second homotopy group of the LOT Γ is the submodule of the free $\mathbb{Z}[G]$-module generated by $\{x_{i,2k} : 1 \le i \le n-1, 1 \le k \le 8\}$.

To determine the values of the incidence numbers, we choose the orientations as follows. For every element x_{2k} of degree 2, the poset $\hat{U}_{x_{2k}}$ is a model of S^1. We will choose one cycle which generates its first homology group. The cycle begins at the base points and follows with x_{2k-1} or x_{2k+1} (only one of them covers o). After this the cycle is uniquely determined.



For an element p of degree 1, the orientation consists of a sign for each component of $\hat{U}_p$. In the case of an element v_i the signs will be − for the base point and + for v_0. And in the case of an element x_{2k+1}, the signs will be − for x_0 and + for the other element in $\hat{U}_{x_{2k+1}}$, which can be the base point or some v_0.

Given the orientation, the incidence number $\epsilon(p,q)$ depends on whether the cycle chosen for p passes through q positively (coming from the negative component of $\hat{U}_p$ and going to the positive component), or negatively.

With our choice of orientations the incidence numbers turn out to be as follows. For m an odd index,

$$\epsilon(x_{m+/-1}, x_m) = 1 \text{ if } m = 3 \bmod 4, \quad \epsilon(x_{m+/-1}, x_m) = -1 \text{ if } m = 1 \bmod 4.$$

And for any even index m,

$$\epsilon(x_m, v_i) = -1 \text{ for any } v \text{ and } i = 1, 2.$$

This implies that the equations for the elements of degree 1 of type x_m with m an odd index are:

$$\begin{aligned}
\text{for } x_1 : \quad & -n_g^{x_{16}} - n_g^{x_2} = 0 \quad \forall g \in G \\
\text{for } x_3 : \quad & n_g^{x_2} + n_g^{x_4} = 0 \quad \forall g \in G \\
\text{for } x_5 : \quad & -n_g^{x_4} - n_g^{x_6} = 0 \quad \forall g \in G \\
\text{for } x_7 : \quad & n_g^{x_6} + n_g^{x_8} = 0 \quad \forall g \in G \\
\text{for } x_9 : \quad & -n_g^{x_8} - n_g^{x_{10}} = 0 \quad \forall g \in G \\
\text{for } x_{11} : \quad & n_g^{x_{10}} + n_g^{x_{12}} = 0 \quad \forall g \in G \\
\text{for } x_{13} : \quad & -n_g^{x_{12}} - n_g^{x_{14}} = 0 \quad \forall g \in G \\
\text{for } x_{15} : \quad & n_g^{x_{14}} + n_g^{x_{16}} = 0 \quad \forall g \in G
\end{aligned}$$

This means that the coefficients $n^{x_{2k}}$ can all be written in terms of n^{x_2}. To simplify notation, we define $n_g^x := n_g^{x_2}$, then we have

$$\begin{aligned}
n_g^{x_2} = n_g^{x_6} = n_g^{x_{10}} = n_g^{x_{14}} = n_g^x \quad \forall g \in G \\
n_g^{x_4} = n_g^{x_8} = n_g^{x_{12}} = n_g^{x_{16}} = -n_g^x \quad \forall g \in G
\end{aligned}$$

This implies that $\pi_2(L)$ is generated by $\{\overline{x}_i : 1 \le i \le n-1\}$, where $\overline{x}_i = x_{i,2} - x_{i,4} + x_{i,6} - x_{i,8} + x_{i,10} - x_{i,12} + x_{i,14} - x_{i,16}$.

Now we consider the remaining equations. For an element v_1 of height 1 in $\mathcal{X}(\Gamma)$, the involved coefficients correspond to elements of height 2 associated to the edges with an endpoint or label v. If an edge x_i has v as its source (that is, $s_x = v$), then $x_{i,16}$ covers v_1, the incidence is $\epsilon(x_{i,16}, v_1) = -1$, and the color is $c(x_{i,16}, v_1) = v^{-1}$. If an edge x_i has v as its target (that is, $t_x = v$), then $x_{i,6}$ covers v_1, the incidence is $\epsilon(x_{i,6}, v_1) = -1$, and the color is $c(x_{i,6}, v_1) = v^{-1}\lambda_x^{-1}$, where λ_x is the label of x. And if an edge x_i has v as its label (that is, $\lambda_x = v$), then $x_{i,2}$ and $x_{i,12}$ cover v_1 with incidences $\epsilon(x_{i,2}, v_1) = -1$, $\epsilon(x_{i,12}, v_1) = -1$, and colors $c(x_{i,2}, v_1) = v^{-1}$, $c(x_{i,12}, v_1) = t_x^{-1}v^{-1}$, where t_x is the target of x. Thus the equations for v_1 are

$$-\sum_{s_x=v} n^{x_{16}}_{gv^{-1}} - \sum_{t_x=v} n^{x_6}_{gv^{-1}\lambda_x^{-1}} - \sum_{\lambda_x=v} \left[n^{x_2}_{gv^{-1}} + n^{x_{12}}_{gt_x^{-1}v^{-1}} \right] = 0 \qquad \forall g \in G.$$

The equations for v_2 are similar:

$$-\sum_{s_x=v} n^{x_{14}}_{gv^{-1}} - \sum_{t_x=v} n^{x_8}_{gv^{-1}\lambda_x^{-1}} - \sum_{\lambda_x=v} \left[n^{x_4}_{gv^{-1}} + n^{x_{10}}_{gt_x^{-1}v^{-1}} \right] = 0 \qquad \forall g \in G.$$

Replacing $n^{x_k}_g = +/-n^x_g$, both give the following:

$$-\sum_{s_x=v} n^{x}_{gv^{-1}} + \sum_{t_x=v} n^{x}_{gv^{-1}\lambda_x^{-1}} + \sum_{\lambda_x=v} \left[n^{x}_{gv^{-1}} - n^{x}_{gt_x^{-1}v^{-1}} \right] = 0 \qquad \forall g \in G$$

Summarizing, we have the following result.

Theorem 5.3.1. *Let Γ be a LOT with vertex set V and edge set E. Then the second homotopy group of Γ can be described as follows.*

$$\pi_2(\Gamma) = \{ \sum_{x \in E, g \in G} n^x_g g\overline{x} : \; n^x_g \in \mathbb{Z} : \quad \forall v \in V, \forall g \in G$$
$$-\sum_{s_x=v} n^{x}_{gv^{-1}} + \sum_{t_x=v} n^{x}_{gv^{-1}\lambda_x^{-1}} + \sum_{\lambda_x=v} (n^{x}_{gv^{-1}} - n^{x}_{gt_x^{-1}v^{-1}}) = 0\}$$

In the next section we will use this description to obtain the main results of this chapter.

Remark 5.3.2. Note the effect on this description of π_2 of the three transformations defined for LOTs. For example, if v is a leaf with incident edge e, then the move $(L3)$ eliminates the generator $\overline{e}$ of π_2 and the equation of v stating $n^x_{g\circledast} = 0$ for all $g \in G$ (here $\circledast$ denotes some element of the group G).

5.4 Applications

In this section we make use of the description of $\pi_2(\Gamma)$ to prove asphericity in many cases and families of LOTs. The method consists of using the equations of the vertices to express all the coefficients of the edges in terms of the coefficients associated to one particular edge, and then use one extra vertex to prove that the latter are all 0.

Remark 5.4.1. Let Γ be a LOT, and let $a \in \Gamma$ be a chosen vertex which we call the center of Γ. For any other vertex $v \in \Gamma$ there exists a unique simple edge-path from v to a in Γ. We shall call the first edge (which is adjacent to v) as the *inner edge* of v. The other edges that are adjacent to v in Γ will be called *outer edges* of v.

Recall that the weight map of a given LOT presentation maps each generator to 1 and each word on the total exponent sum.

Remark 5.4.2. Let Γ be a LOT and let $\sum_{x\in E, g\in G} n_g^x g\overline{x}$ be a fixed element of $\pi_2(\Gamma)$. Suppose that for an edge a of Γ we have an equation on the coefficients n_g^a of the form

$$n^a_{gu_1} + n^a_{gu_2} + \cdots + n^a_{gu_i} + \cdots + n^a_{gu_k} = 0 \quad \forall g \in G$$

and suppose that u_i has strictly greater weight than $u_j, j \neq i$. Since there are finitely many non-zero coefficients, and assuming not all of them are 0, we may choose ξ such that $n^a_\xi \neq 0$ and such that $w(\xi)$ is minimum. The equation above can be rewritten as

$$n^a_{hu_i^{-1}u_1} + n^a_{hu_i^{-1}u_2} + \cdots + n^a_h + \cdots + n^a_{hu_i^{-1}u_k} = 0 \quad \forall h \in G.$$

The equation must hold when $h = \xi$, but $w(\xi u_i^{-1}u_j) < w(\xi)$, so $n^a_{hu_i^{-1}u_j} = 0$ for all $j \neq i$. Thus the equation becomes $n^a_\xi = 0$, which is a contradiction.

Similarly the coefficients must be all 0 if there exists an u_i with strictly less weight than the others. Therefore it will be useful to analyze the weights of the subscripts in the coefficients of the elements of $\pi_2(\Gamma)$.

Proposition 5.4.3. *Let Γ be a reduced injective LOT and let the center a be the only vertex that is not an edge label. Suppose there is an order $v_1, \ldots, v_n$ on the label vertices satisfying the following conditions.*

1. *For every $1 \leq i \leq n-1$, the inner edge of v_i (called x_i) is labeled by v_{i+1}.*

2. *The inner edge of v_n, x_n is labeled by v_1.*

3. *For every $1 \leq i \leq n$, the outer edges of v_i are labeled by previous vertices (that is, by vertices v_k with $k < i$).*

Then the second homotopy group $\pi_2(\Gamma)$ is generated as a $\mathbb{Z}[\pi_1(\Gamma)]$-module by a single element.

Proof. Let $\alpha = \sum_{x\in E, g\in G} n_g^x g\overline{x}$ be an element of $\pi_2(\Gamma)$.

Note that v_1 must be a leaf, since it cannot have outer edges labeled by previous vertices.

We will follow the order of the vertices, and in each step i we will use the equations associated to the vertex v_i to express the coefficient of its inner edge, n^{x_i}, in terms of that of the first edge. At the end all the coefficients will be expressed in terms of n^{x_1}, and α will be expressed as a multiple of a certain combination of the elements $\overline{x_i}$.

In the first step we consider the equation for the vertex v_1. The edges whose coefficients appear in this equation are the edge incident to v_1, called x_1, and the edge labeled by v_1, called x_n. Depending on the orientation of the edge x_1, the equation might be

$$-n^{x_1}_{gv_1^{-1}} + (n^{x_n}_{gv_1^{-1}} - n^{x_n}_{gt_{x_n}^{-1}v_1^{-1}}) = 0 \quad \text{or} \quad n^{x_1}_{gv_1^{-1}\lambda_{x_1}^{-1}} + (n^{x_n}_{gv_1^{-1}} - n^{x_n}_{gt_{x_n}^{-1}v_1^{-1}}) = 0 \;\; \forall g \in G.$$

It is clear that in both cases we can rearrange the indexes to force the subscript of n^{x_1} to be g and solve the equation for $n^{x_1}_g$, obtaining

$$n^{x_1}_g = n^{x_n}_g - n^{x_n}_{gs_{x_n}^{-1}} \quad \text{or} \quad n^{x_1}_g = -n^{x_n}_{g\lambda_{x_1}} + n^{x_n}_{g\lambda_{x_1}s_{x_n}^{-1}} \quad \forall g \in G.$$

In the $(k+1)$-th step we assume that the coefficients n^{x_j} with $j \le k$ have all been expressed in terms of n^{x_n}. Then we consider the equation for the vertex v_{k+1}. The edges whose coefficients are involved in the equation are the inner edge x_{k+1}, labeled by v_{k+2}, the edge x_k with label v_{k+1}, and the outer edges of v_{k+1}, which are labeled by vertices v_j, with $j \le k$ (that is, they are x_j with $j < k$). The equation is of the following kind (the sum in the middle is taken over all outer edges y).

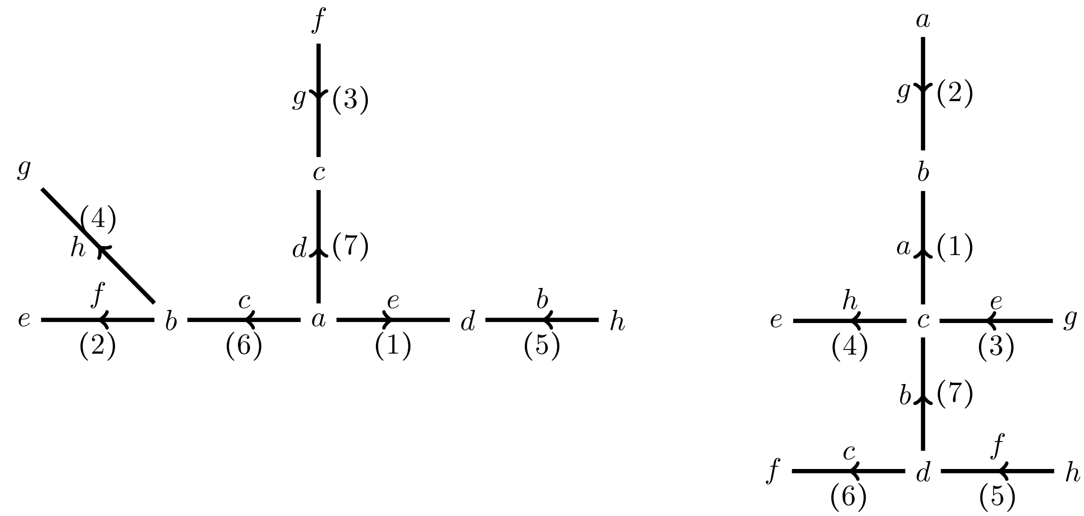
$$\pm n^{x_{k+1}}_{g\circledast} + \sum_y \pm n^y_{g\circledast} + (n^{x_k}_{g\circledast} - n^{x_k}_{g\circledast}) = 0$$

Here $\circledast$ stands for some (possibly different) elements of the group G. Since the outer edges are assumed to be x_j with $j \le k-1$, their coefficients, together with that of x_k, are assumed to be expressed in terms of n^{x_n}. Thus the coefficient $n^{x_{k+1}}$ can be expressed, using the equation above, in terms of n^{x_n}.

□

Remark 5.4.4. The result above is interesting since it suggests the following method for studying asphericity of injective LOTs. Find, if possible, such an ordering of the edges. Then use the equations as shown in the proof. Finally, consider the equation of any of the two remaining vertices. Replace, in that equation, all the coefficients associated to the edges by combinations of coefficients associated to the edge x_n, and find a condition that ensures $n^{x_n} = 0$. For example, if the maximum or minimum weight is reached only once, then the equation implies $n^{x_n} = 0$, and the LOT is aspherical.

Examples 5.4.5. With this method it can be shown that the following are aspherical LOTs. The numbers show the order of the edges.



When dealing with non-injective LOTs, the condition of the ordering of the edges must be replaced by a more complex hypothesis.

Proposition 5.4.6. *Let* Γ *be a reduced LOT such that there is an enumeration of the edges of* Γ *in consecutive rows*

$$\begin{gathered} a_{11}, a_{12}, \dots, a_{1k_1} \\ a_{21}, a_{22}, \dots, a_{2k_2} \\ \dots \\ a_{r1}, a_{r2}, \dots, a_{rk_r}, \end{gathered}$$

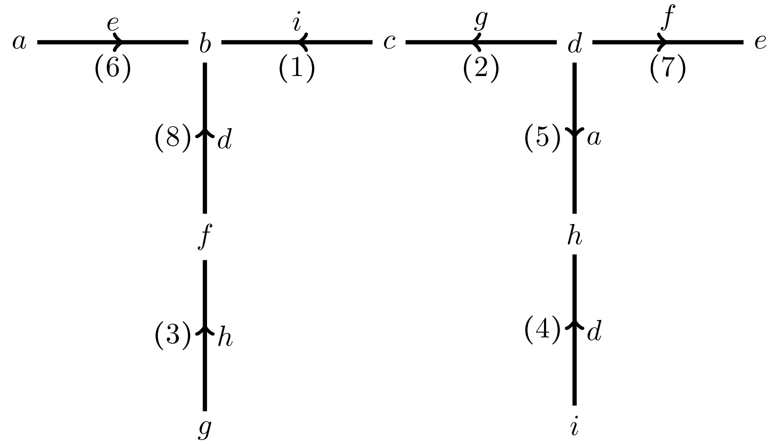
satisfying the following rule. A new edge can be enumerated a_{ij} *if it has one endpoint* v_{ij} *such that all the remaining incident edges, and the edges labeled by* v_{ij} *were enumerated* a_{kl} *with* $k < i$ *or* $k = i, l < j$*, or if it starts a new row (i.e. if* $j = 1$*).*

Then the second homotopy group $\pi_2(\Gamma)$ *is generated as a* $\mathbb{Z}[\pi_1(\Gamma)]$*-module by* r *elements.*

Proof. The procedure is similar to the one used in 5.4.3. At the end of the j-th row, the coefficients of the edges of that row are expressed in terms of the first edges of all i-th rows with $i \leq j$. □

Remark 5.4.7. The result above is interesting because it suggests a method for studying asphericity, in a similar way as in the case of injective LOTs. At the end of each row, one has to find a condition that ensures that the coefficients corresponding to the first edge of the row are all 0, and this implies that the coefficients of all the edges of the row are 0.

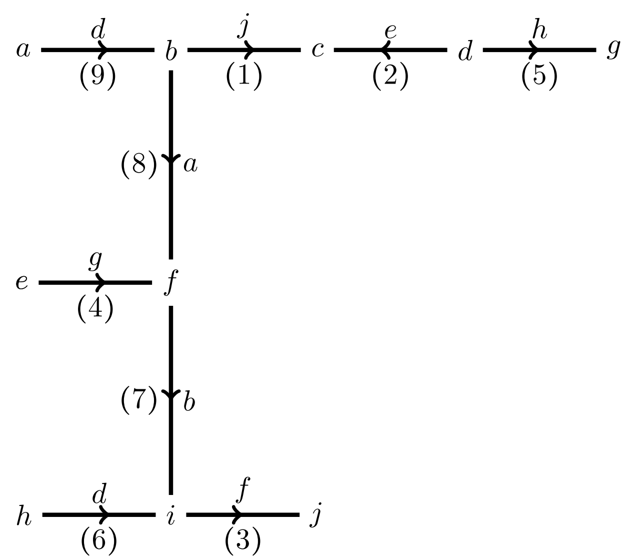
Example 5.4.8. With this method, and using only one row, it is shown that the following LOT is aspherical. The enumeration of the edges is written in brackets.



The last step consists of replacing all coefficients in the equation of the vertex b, obtaining an equation involving only coefficients of the first edge. To prove that all of these are 0, we observe that the maximum weight of the subscripts is reached exactly once.

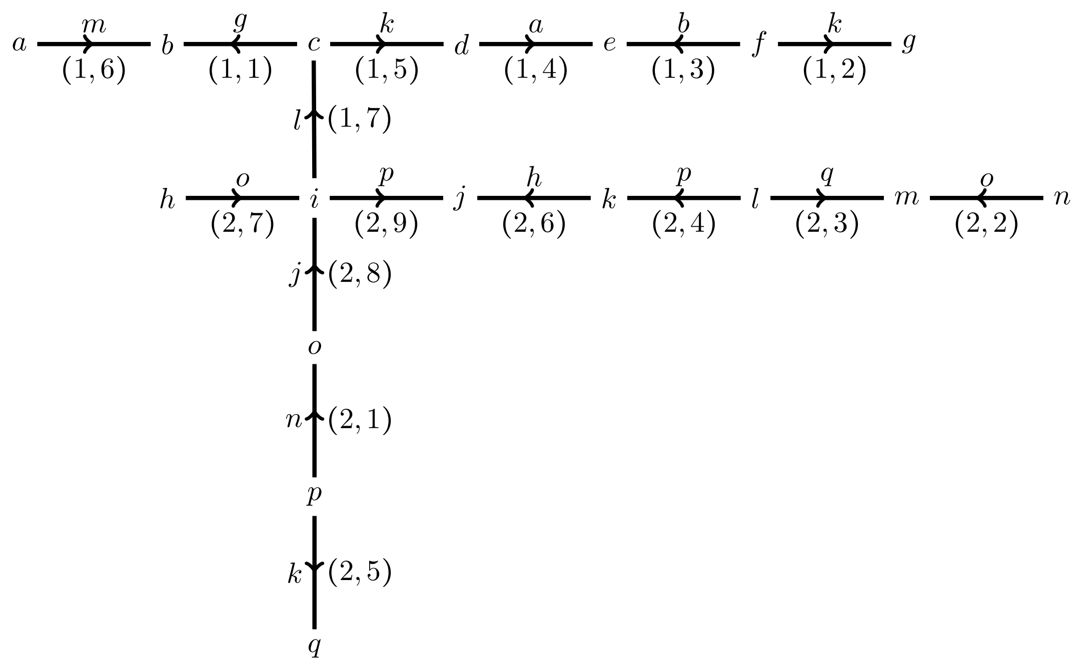
Example 5.4.9. Also using only one row, it is shown that the following LOT is aspherical.

In this example, the last step consists of replacing all coefficients in the equation of the vertex a, obtaining an equation involving only coefficients of the first edge. To prove that all of these are 0, we observe that the maximum weight of the subscripts is reached exactly once.

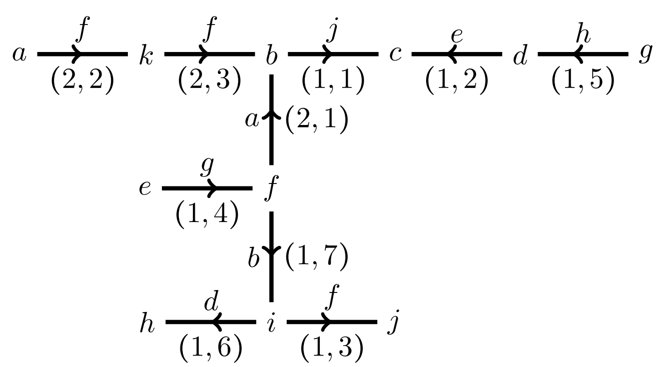


Now we present some examples where more than one row is needed.

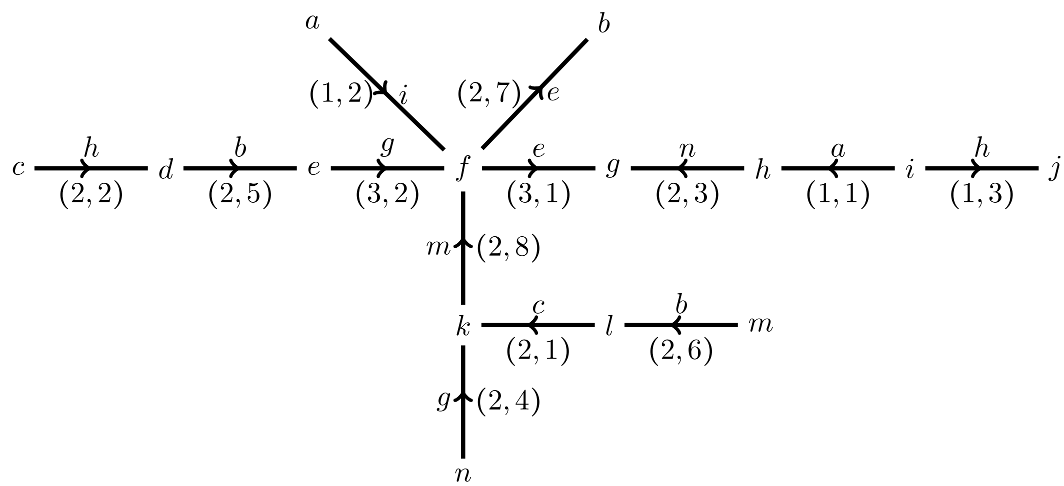
Example 5.4.10. In a similar way, but using two rows, the following LOT can be shown to be aspherical. At the end of the first row, we use the equation of b, and at the end of the second row, we use p.



Example 5.4.11. In this example we also use two rows to show that the LOT is aspherical. At the end of the first row, we use the equation of d, and at the end of the second row, we use b.



Example 5.4.12. Using three rows, the following LOT can be shown to be aspherical. At the end of the first row, we use the equation of j, at the end of the second row, we use m, and at the end of the third row, we use g.



The following notion, together with the UPP condition, will be used to ensure that the coefficients of the edges of a LOT can all be proved to be 0.

Definition 5.4.13. Let Γ be a reduced LOT. A *good enumeration* of Γ is an enumeration of the edges of Γ in consecutive rows

$$
\begin{aligned}
&a_{11}, a_{12}, \ldots, a_{1k_1}\\
&a_{21}, a_{22}, \ldots, a_{2k_2}\\
&\qquad\cdots\\
&a_{r1}, a_{r2}, \ldots, a_{rk_r},
\end{aligned}
$$

such that the following conditions are satisfied.

1. A new edge can be enumerated a_{ij} if it has one endpoint v_{ij} such that all the remaining incident edges, and the edges labeled by v_{ij} were enumerated a_{kl} with $k < i$ or $k = i, l < j$, or if it starts a new row (i.e. if $j = 1$).

2. For every $1 \le s \le r$ there exists a vertex w_s different from all the v_{ij} such that all the edges incident to w_s and all the edges labeled by w_s are enumerated a_{kl} with $k < s$.

3. For every $1 \le s \le r$, the path in Γ between w_s and the edge a_{s1} is a sequence of edges a_{sj}. The vertices of the path have all their incident and labeled edges enumerated with $kl, k \le s$.

Recall that a group G satisfies UPP if and only if for any two non-empty finite subsets X, Y in G there is an element $g \in G$ such that $gX \cap Y$ has precisely one element. Recall also that it is not known whether all LOT groups satisfy UPP or not.

Theorem 5.4.14. *Let Γ be a reduced LOT which admits a good enumeration*

$$
\begin{aligned}
&a_{11}, a_{12}, \ldots, a_{1k_1}\\
&a_{21}, a_{22}, \ldots, a_{2k_2}\\
&\qquad\cdots\\
&a_{r1}, a_{r2}, \ldots, a_{rk_r}.
\end{aligned}
$$

If, in addition, $G(\Gamma)$ is UPP, then Γ is aspherical.

Proof. Let $\sum_{x\in E, g\in G} n_g^x g\overline{x}$ be an element of $\pi_2(\Gamma)$. We shall first prove that the coefficients of the edges of the first row are 0, $n_g^{a_{1j}} = 0$ for all $g \in G$.

For each a_{1j}, $j > 1$, we can use the equation of v_{1j} to express $n_g^{a_{1j}}$ in terms of $n_g^{a_{11}}$. Then we replace this in the equation of w_1 to obtain an equation involving only the coefficients $n_g^{a_{11}}$. If this equation is not a trivial equation $0 = 0$, since the group is UPP, then $n_g^{a_{11}} = 0$ for all g: suppose that there is some non-trivial coefficient, and take $X = \{g \in G : n_g^{a_{11}} \neq 0\}$, and Y the set of subscripts appearing in the equation. There is an element $g \in G$ such that $gX \cap Y$ consists of exactly one element, say $\tilde{g}$. This implies that the equation turns out to be $n_{\tilde{g}}^{a_{11}} = 0$, which is a contradiction.

To prove that the equation is not $0 = 0$, we will show that the coefficient of the first edge in the path from w_1 to a_{11} is expressed in terms of $n_g^{a_{11}}$ with an odd number of terms. The coefficients of the rest of the edges incident to w_1 are expressed with an even number of terms. The remaining edges involved in the equation of w_1 are the ones labeled by w_1, and their coefficients appear in pairs.

Let us call $e_1, \ldots, e_k = a_{11}$ the edges in the path from w_1 to a_{11}, and $w_1 = u_1, \ldots, u_{k+1}$ the vertices.

$$w_1 \text{ --- } u_2 \text{ --- } u_3 \text{ --- } \cdots \text{ --- } u_{k-1} \text{ --- } u_k \overset{a_{11}}{\text{ --- }} u_{k+1}$$

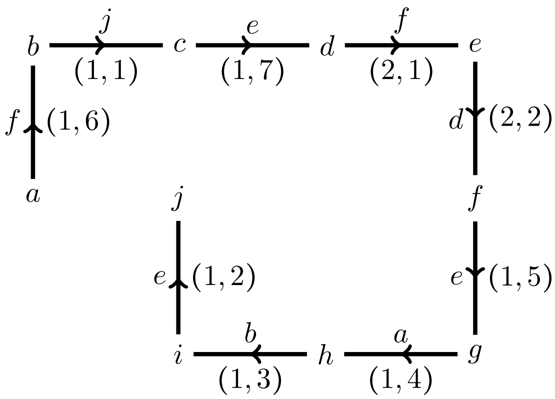
Consider the equation of each u_j, and express $n^{e_{j-1}}$ in terms of n^{e_j} and the coefficients of the edges labeled by u_j. Since the latter come in pairs, it is clear that in each step, the number of terms is odd.

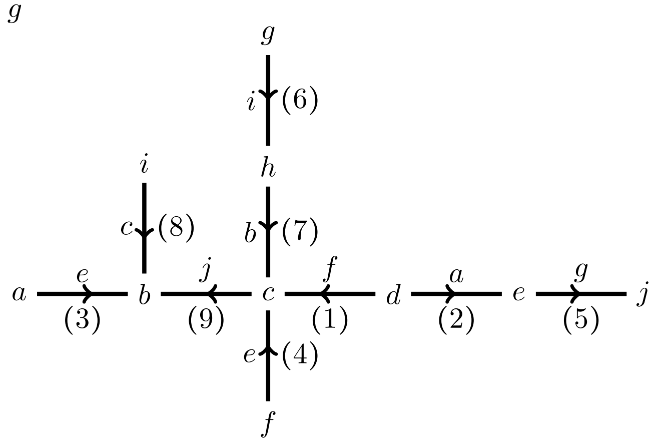
Let us now see that the terms in the expression of the remaining edges is even. If we omit the edge e_1, Γ separates in two components. We will show by induction that the coefficients of all the edges in the component of w_1 are expressed with an even number of terms.

Consider the first (i.e. with the lesser subscript) edge of the component, say a_{1j}. Since it is the first one in this component, v_{1j} must be a leaf. Thus, $n^{a_{1j}}$ is expressed in terms of the coefficients of the edges labeled by v_{1j}, which come in pairs. The coefficients of the following edges are first expressed in terms of those of the edges of this components, which by induction have an even number of terms, and of those of the edges of the other component, which come in pairs, since they are involved for their label.

So we have proven that the coefficients of the edges of the first row are 0. For the following rows, the procedure is similar. We use the fact that the coefficients of the edges of the previous rows are already shown to be 0, therefore they contribute always an even number of terms. □

Examples 5.4.15. The following LOTs can be proved to be aspherical with the described method, if their fundamental group were known to satisfy UPP. The numbers show the order of the edges.





Remark 5.4.16. In the first of these two LOTs, which is actually a LOI, the equation of $w_1 = b$ fails to have a unique subscript with greater (or lesser) weight than the others, but the maximal weight is reached only twice in the set of subscripts. Let us see how the UPP condition can be replaced in this example by requiring $G = \pi_1(\Gamma)$ to be torsion-free.

The equations arising from the first row are the following (we have already rescaled them).

$$\begin{aligned}
&(v_{12} = j) \quad n_l^{a_{12}} = n_{leb^{-1}}^{a_{11}} - n_{le}^{a_{11}} \quad \forall l \in G,\\
&(v_{13} = i) \quad n_l^{a_{13}} = n_{lb}^{a_{12}} = n_{lbeb^{-1}}^{a_{11}} - n_{lbe}^{a_{11}} \quad \forall l \in G,\\
&(v_{14} = h) \quad n_l^{a_{14}} = n_{la}^{a_{13}} = n_{labeb^{-1}}^{a_{11}} - n_{labe}^{a_{11}} \quad \forall l \in G,\\
&(v_{15} = g) \quad n_l^{a_{15}} = n_{le}^{a_{14}} = n_{leabeb^{-1}}^{a_{11}} - n_{leabe}^{a_{11}} \quad \forall l \in G,\\
&(v_{16} = a) \quad n_l^{a_{16}} = n_l^{a_{15}} - n_{lg^{-1}}^{a_{15}} = n_{labeb^{-1}}^{a_{11}} - n_{labe}^{a_{11}} - n_{lg^{-1}abeb^{-1}}^{a_{11}} + n_{lg^{-1}abe}^{a_{11}} \quad \forall l \in G,\\
&(v_{17} = c) \quad n_l^{a_{17}} = n_{lj^{-1}}^{a_{11}} \quad \forall l \in G.
\end{aligned}$$

The equation for w_1 is

$$(w_1 = b) \quad -n_{lb^{-1}}^{a_{11}} + n_{lb^{-1}f^{-1}}^{a_{16}} + n_{lb^{-1}}^{a_{13}} - n_{li^{-1}b^{-1}}^{a_{13}} = 0 \quad \forall l \in G.$$

Replacing $n_l^{a_{1j}}$, we have

$$- n^{a_{11}}_{lb^{-1}} + n^{a_{11}}_{lb^{-1}f^{-1}abeb^{-1}} - n^{a_{11}}_{lb^{-1}f^{-1}abe} - n^{a_{11}}_{lb^{-1}f^{-1}g^{-1}abeb^{-1}} + n^{a_{11}}_{lb^{-1}f^{-1}g^{-1}abe} + \\ + n^{a_{11}}_{leb^{-1}} - n^{a_{11}}_{le} - n^{a_{11}}_{li^{-1}eb^{-1}} + n^{a_{11}}_{li^{-1}e} = 0 \quad \forall l \in G.$$

The maximal weight is 1 and it is reached in the 3rd and 7th terms. Now, let us call $u = b^{-1}f^{-1}abe$ and $v = e$, the subscripts with maximal weight, and $w = u^{-1}v$. Note that if w is the trivial element of the group G, this maximal weight is actually reached only once, and with the usual argument we have $\pi_2(L) = 0$. So we may assume $w \neq 0$. Note also that w has weight 0. If we assume that $\pi_2(L) \neq 0$, there must be an element $s \in G$ such that $n_s^{a_{11}} \neq 0$, and such that s has minimal weight.

Consider the equation above when l takes the value $l = su^{-1}$. Most of the terms must be 0, since they have a subscript with less weight than s. Only the third and seventh terms survive.

$$-n_s^{a_{11}} - n_{sw}^{a_{11}} = 0.$$

This implies that $n_{sw}^{a_{11}} \neq 0$. Similarly, replacing in the same equation $l = swu^{-1}$, we have

$$-n_{sw}^{a_{11}} - n_{sw^2}^{a_{11}} = 0,$$

which implies that $n_{sw^2}^{a_{11}} \neq 0$. Inductively we prove that $n_{sw^k}^{a_{11}} \neq 0$ for all $k \geq 1$, which is a contradiction if G is torsion-free. This argument can be applied in any case where the maximal (or minimal) weight is reached twice in the set of subscripts of the equation for w_i.

5.5 LOTs of maximal complexity

In this section we show how our method can be applied to prove the asphericity of the LOTs of maximal complexity presented by Rosebrock in [Ros10], where the notion of complexity of LOTs was introduced.

Recall that the complexity of a LOT Γ is the smallest number k such that Γ is generated by a set of k vertices.

Rosebrock proved that LOTs of complexity 2 are aspherical and that, in the case of LOIs (that is, LOTs for which the underlying graph is an interval), the complexity is bounded by $(n+1)/2$, where n is the number of vertices. Recently, Christmann and De Wolff [CdW14] proved that this upper bound for complexity extends to all LOTs. They also proved that if a LOT has maximal complexity, then it is a union of sub-LOTs of complexity 2 and therefore aspherical.

The following LOIs were exhibited by Rosebrock as examples of maximal complexity. The first two examples are of 3 and 5 vertices respectively, with complexity 2 and 3. The numbers on the edges indicate a good enumeration. We omit the orientation in these LOIs, since it is irrelevant for both the complexity and the enumeration.

$$a_1 \overset{a_2}{\underset{(1,2)}{\rule{2em}{1pt}}} b_1 \overset{a_1}{\underset{(1,1)}{\rule{2em}{1pt}}} a_2 \qquad\qquad a_1 \overset{a_2}{\underset{(1,2)}{\rule{2em}{1pt}}} b_1 \overset{a_1}{\underset{(1,1)}{\rule{2em}{1pt}}} a_2 \overset{a_3}{\underset{(2,2)}{\rule{2em}{1pt}}} b_2 \overset{a_2}{\underset{(2,1)}{\rule{2em}{1pt}}} a_3$$

The first LOI is enumerated in one row, starting with the edge labeled a_1. The vertex a_1 is $v_{1,2}$ and the vertex b_1 is w_1. Using the equation of $v_{1,2}$, we solve for $n_g^{e_{1,2}}$ as a function of $n_g^{e_{1,1}}$, using precisely two terms with consecutive subscript-weights. Thus, in the equation of b_1, which envolves one term of $n_g^{e_{1,2}}$ and one of $n_g^{e_{1,1}}$, we must have that the maximal or the minimal weight is attained only once.

The second LOI is enumerated in two rows. The first row starts with the edge of endpoints b_1, a_2, enumerated $e_{1,1}$. The second edge of the row, $e_{1,2}$, is the one with endpoints a_1, b_1, using $v_{1,2} = a_1$, for which there are no other incident edges, and the only edge labeled by it is $e_{1,1}$. The vertex $w_1 = b_1$ does not appear as an edge label, and the edges incident at it are enumerated in the first row. The second row is similar, with $v_{2,2} = a_2$ and $w_2 = b_2$. As in the previous example, in the equations of b_i the maximal or minimal weight-subscript is attained only once.

For the LOI with 9 vertices (complexity 5), the enumeration is in 4 rows.

$$a_1 \ \frac{a_2}{(1,2)}\ b_1\ \frac{a_1}{(1,1)}\ a_2\ \frac{a_3}{(2,2)}\ b_2\ \frac{a_2}{(2,1)}\ a_3\ \frac{a_4}{(3,2)}\ b_3\ \frac{a_3}{(3,1)}\ a_4\ \frac{a_5}{(4,2)}\ b_4\ \frac{a_4}{(4,1)}\ a_5$$

Similarly, this procedure can be applied to show the asphericity of every one of these LOIs, with $m-1$ rows (where m is the complexity and $2m-1$ the number of vertices).

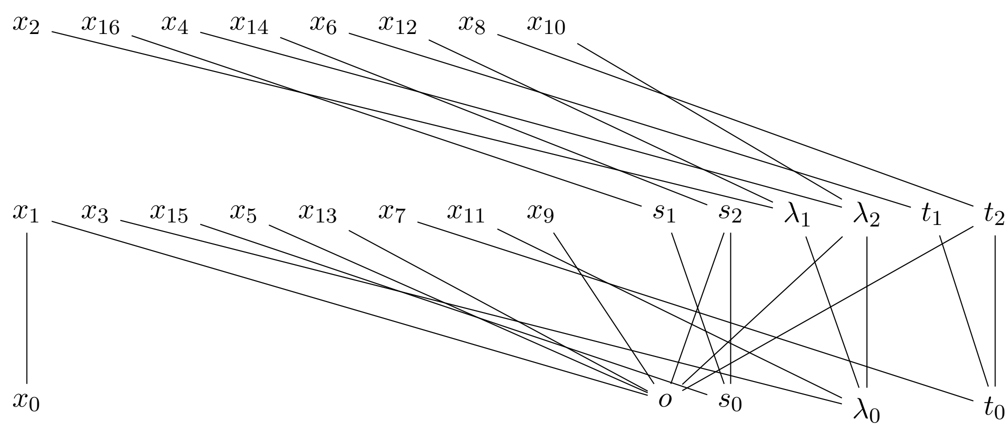
$$a_1 \ \frac{a_2}{(1,2)}\ b_1\ \frac{a_1}{(1,1)}\ a_2\ \frac{a_3}{(2,2)}\ b_2\ \frac{a_2}{(2,1)}\ a_3 \quad \cdots \quad a_{m-1}\ \frac{a_m}{(m-1,2)}\ b_{m-1}\ \frac{a_{m-1}}{(m-1,1)}\ a_m$$

The j-th row consists of the edge with endpoints b_j, a_{j+1} and label a_j, enumerated $e_{j,1}$, and the edge with endpoints a_j, b_j and label a_{j+1}, enumerated $e_{j,2}$. The vertices used in this row are $v_{j,2} = a_j$ and $w_j = b_j$.

5.6 Another simply connected subdiagram

As we mentioned before, different subdiagrams of the Hasse diagram of a poset may yield different presentations of its fundamental group and different descriptions of its second homotopy group. We present here an alternative subdiagram D for any LOT poset $\mathcal{X}(\Gamma)$, and the associated computations of $\pi_1(\mathcal{X}(\Gamma)), \pi_2(\mathcal{X}(\Gamma))$.

For every vertex u of Γ, the diagram has the edges between o and u_2, u_0 and u_1, u_0 and u_2. For every 1-cell x of Γ, D contains only the edges joining $x_i, i > 1$ with the corresponding vertex elements, and not the edges between them. Additionally, it contains the edge between x_0 and x_1.



It is easy to see that the poset represented by the diagram D is contractible, first removing, for every edge x, x_0 and then the other elements x_i. In this way we obtain a diagram of height one which clearly does not have cycles.

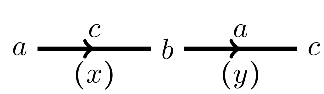
The presentation of the fundamental group obtained by this diagram is again equivalent to the LOT presentation. The second homotopy group has, as before, generators in correspondence with the edges of Γ, and equations indexed by the elements of the group and the vertices of Γ.

$$-\sum_{s_x=v} n^x_{g\lambda_x t_x^{-1}} + \sum_{t_x=v} n^x_{gv^{-1}} + \sum_{\lambda_x=v}\left[n^x_g \quad - \quad n^x_{gt_x^{-1}}\right] = 0 \qquad \forall g \in G$$

Therefore the second homotopy group of Γ can be described as follows.

$$\pi_2(L) = \{ \sum_{x\in L^1, g\in G} n^x_g g\overline{x} : \; n^x_g \in \mathbb{Z} : \quad \forall v \in L^0, \forall g \in G,$$
$$-\sum_{s_x=v} n^x_{g\lambda_x t_x^{-1}} + \sum_{t_x=v} n^x_{gv^{-1}} + \sum_{\lambda_x=v}\left[n^x_g \quad - \quad n^x_{gt_x^{-1}}\right] = 0\}$$

Example 5.6.1. Consider the following LOT Γ.



The equations for $\pi_2(\Gamma)$ corresponding to the previously chosen diagram (on the left) and to the new one (on the right) are the following.

$$\begin{cases} -n^x_{ga^{-1}} + n^y_{ga^{-1}} - n^y_{gc^{-1}a^{-1}} = 0 \\ -n^y_{gb^{-1}} + n^x_{gb^{-1}c^{-1}} = 0 \\ n^y_{gc^{-1}a^{-1}} + n^x_{gc^{-1}} - n^x_{gb^{-1}c^{-1}} = 0 \end{cases} \qquad \begin{cases} -n^x_{gcb^{-1}} + n^y_g - n^y_{gc^{-1}} = 0 \\ -n^y_{gac^{-1}} + n^x_{gb^{-1}} = 0 \\ n^y_{gc^{-1}} + n^x_g - n^x_{gb^{-1}} = 0 \end{cases}$$

Resumen del capítulo 5: Asfericidad de LOTs usando coloreos

En este capítulo estudiamos la asfericidad de los complejos LOT aplicando, como herramienta principal, los resultados obtenidos por Barmak y Minian sobre el segundo grupo fundamental de un 2-complejo en términos de coloreos del diagrama de Hasse de su poset de celdas [BM14].

En la sección 5.1 se repasan los resultados principales del trabajo de Barmak y Minian respecto del grupo fundamental y del segundo grupo de homotopía de complejos de dimensión 2 [BM12a, BM14].

Un *coloreo* de un poset X es una función $c : \mathcal{E}(X) \to G$ del conjunto de aristas del diagrama de Hasse de X a un grupo G.

El resultado fundamental de [BM12a] que utilizamos en este trabajo es que uno puede obtener una presentación del grupo fundamental de cualquier espacio finito X de la siguiente manera. Sea D un subdiagrama del diagrama de Hasse de X que contiene a todos los elementos de X y tal que corresponde a un espacio finito simplemente conexo (por ejemplo, un árbol maximal del diagrama satisface estas condiciones). Entonces se tiene una presentación de $\pi_1(X)$ con un generador por cada arista del diagrama de Hasse que no está en D, y una relación por cada *simple digon*, es decir, por cada para de caminos monótonos de aristas que se tocan sólo en los extremos. Un par de este tipo,

$$\{(x_1, x_2), \dots, (x_{k-1}, x_k)\}, \quad \{(y_1, y_2), \dots, (y_{l-1}, y_l)\},$$

donde $x_1 = y_1, x_k = y_l$ y $x_i \neq y_j$ en los demás casos, induce una relación

$$\prod_{(x_i, x_{i+1}) \notin D} (x_i, x_{i+1}) = \prod_{(y_i, y_{i+1}) \notin D} (y_i, y_{i+1}).$$

Utilizando este resultado, y a partir de una caracterización que obtienen de los revestimientos de un poset en términos de coloreos, obtienen, en [BM14], una descripción del revestimiento universal de un espacio finito X. Además, en el caso en que X sea el espacio asociado a un CW-complejo regular (por ejemplo a un complejo simplicial), obtienen una descripción del segundo grupo de homología del revestimiento universal de X, o sea, de $\pi_2(X)$, utilizando los resultados de [Min12].

Concretamente, para el caso del poset asociado a un 2-complejo regular con $\pi_1(X) = G$, se obtiene la siguiente descripción de $\pi_2(X)$ como submódulo del $\mathbb{Z}[G]$-módulo libre generado por los elementos de altura 2, y con ecuaciones asociadas a los elementos de altura 1. Para un par $x > y$, $\epsilon(x, y)$ denota la *incidencia* de x en y, que toma los valores ± 1 según ciertas elecciones de orientación similares a las usuales a nivel geométrico.

$$\pi_2(X) = \left\{ \sum_{\substack{\deg x = 2 \\ g \in G}} n_g^x g x : \; n_g^x \in \mathbb{Z} \text{ y } \sum_{x > y} \epsilon(x, y) n_{gc(x,y)}^x = 0 \quad \forall y \in X, \deg y = 1 \right\}.$$

Exhibimos luego el código para SAGE de una función que, a partir de la descripción del poset X y de un subdiagrama D adecuado, computa la presentación correspondiente de $\pi_1(X)$ y las ecuaciones para $\pi_2(X)$.

En la sección 5.2 asociamos a cada LOT Γ un LOT poset $\mathcal{X}(\Gamma)$ que es un modelo finito del complejo K_Γ. De esta manera, podemos estudiar la asfericidad del LOT a través del LOT poset, al cual le podemos aplicar métodos de espacios finitos. Damos una descripción de este poset, y exponemos el código de una función que, a partir de la descripción del LOT, calcula el LOT poset asociado.

En la sección 5.3 aplicamos los resultados de Barmak y Minian [BM12a, BM14] para describir el segundo grupo de homotopía de los complejos LOT. Para eso, elegimos un subgiagrama D del diagrama de Hasse de un LOT poset dado, cuyo poset asociado siempre resulta simplemente conexo. La presentación del grupo fundamental obtenida resulta equivalente a la presentación original del LOT. Y la descripción de $\pi_2(\mathcal{X}(\Gamma))$ obtenida es la siguiente.

Teorema 5.3.1. *Sea Γ un LOT con vértices V y aristas E. Entonces*

$$\pi_2(\Gamma) = \{ \sum_{x \in E, g \in G} n_g^x g \overline{x} : \; n_g^x \in \mathbb{Z} : \quad \forall v \in V, \forall g \in G$$
$$- \sum_{s_x = v} n_{gv^{-1}}^x + \sum_{t_x = v} n_{gv^{-1}\lambda_x^{-1}}^x + \sum_{\lambda_x = v} (n_{gv^{-1}}^x - n_{gt_x^{-1}v^{-1}}^x) = 0 \}.$$

Los resultados principales de este capítlo se encuentran en la sección 5.4, donde a partir de la descripción obtenida para el segundo grupo de homotopía de los LOTs, probamos la asfericidad de una amplia clase de LOTs. Desarrollamos un método que consiste en usar las ecuaciones de los vértices para despejar los coeficientes asociados a las aristas enterminos de los de una arista en particular, y luego con una ecuación más probar que éstos son 0. Para poder enunciar estos resultados, debemos introducir un poco de notación.

Si un vértice es etiqueta de una o más aristas, lo llamaremos *vértice etiqueta*. Un LOT se dice *inyectivo* si ningún vértice es etiqueta de más de una arista. En un LOT inyectivo hay exactamente un vértice que no es etiqueta.

Si se fija un vértice a de Γ, llamado centro, entonces para cualquier otro vértice $v \in \Gamma$ hay una arista incidente que lo acerca hacia el centro (la que forma parte del único camino en Γ que los une). El resto de sus aristas incidentes lo alejan del centro. Las llamaremos, respectivamente, arista *interna* de v y aristas *externas* de v.

Un LOT se dice *reducido* si no se le pueden aplicar las operaciones elementales definidas por Howie en [How85] (ver definición 4.4.1). Estas operaciones preservan el tipo homotoópico simple del complejo asociado. Es por eso que basta considerar LOTs reducidos a la hora de investigar la conjetura de asfericidad de LOTs.

Teorema 5.4.3. *Sea Γ un LOT inyectivo y reducido y sea a, el único vértice que no es etiqueta, fijado como centro de Γ. Supongamos que hay un orden $v_1, \ldots, v_n$ de los vértices etiqueta que satisface lo siguiente.*

1. *Para cada $1 \le i \le n-1$, la arista interna de v_i (que llamaremos x_i) está etiquetada con v_{i+1}.*
2. *La arista interna de v_n, x_n está etiquetada con v_1.*
3. *Para cada $1 \le i \le n$, las aristas externas de v_i están etiquetadas por vértices previos (es decir, vértices v_k con $k < i$).*

Entonces $\pi_2(\Gamma)$ está generado, como $\mathbb{Z}[\pi_1(\Gamma)]$-módulo, por un único elemento.

Este resultado sugiere un método para el estudio de la asfericidad de LOTs inyectivos. Como se describe en la demotración, con cada vértice se despeja el coeficiente asociado a su arista interna, y todos terminan expresados en términos de n^{x_n}. Finalmente, se considera uno de los dos vértices que no se utilizaron, para obtener una ecuación en términos de n^{x_n}, y tratar de probar que éstos son 0. Mostramos luego ejemplos en los cuales esto es posible. Para hacer esto, utilizamos la siguiente observación. El *peso* de un elemento de $G(\Gamma)$ es la suma total de los exponentes de los generadores, en cualquier palabra que lo represente. Esto es un morfismo bien definido $G(\Gamma) \to \mathbb{Z}$, dado que las relaciones de una presentación LOT tienen exponente total 0.

Observación 5.4.2. Sea Γ un LOT y sea $\sum_{x \in E, g \in G} n_g^x g\overline{x}$ un elemento dado de $\pi_2(\Gamma)$. Supongamos que para una arista a de Γ se tiene una ecuación en términos de los coeficientes n_g^a de la forma

$$n^a_{gu_1} + n^a_{gu_2} + \cdots + n^a_{gu_i} + \cdots + n^a_{gu_k} = 0 \quad \forall g \in G$$

y supngamos además que u_i tiene peso estrictamente mayor que $u_j, j \neq i$.

Como hay finitos coeficientes no nulos (suponiendo que no son todos nulos), podemos elegir ξ tal que $n^a_\xi \neq 0$ y tal que $w(\xi)$ es mínimo. La ecuación puede reescribirse como

$$n^a_{hu_i^{-1}u_1} + n^a_{hu_i^{-1}u_2} + \cdots + n^a_h + \cdots + n^a_{u_i^{-1}u_k} = 0 \quad \forall h \in G.$$

Esta ecuación debe valer cuando $h = \xi$, pero $w(\xi u_i^{-1} u_j) < w(\xi)$, so $n^a_{hu_i^{-1}u_j} = 0$ para todo $j \neq i$. Por lo tanto, la ecuación resulta $n^a_\xi = 0$, lo cual es una contradicción.

Similarmente, los coeficientes serán todos 0 si se tiene un u_i con peso menor estricto que el resto.

Para poder genralizar este resultado a LOTs no inyectivos, el orden de las aristas debe ser reemplazado por la siguiente hipótesis.

Existe una enumeración de las aristas de Γ en filas

$$\begin{array}{l} a_{11}, a_{12}, \dots, a_{1k_1} \\ a_{21}, a_{22}, \dots, a_{2k_2} \\ \qquad \dots \\ a_{r1}, a_{r2}, \dots, a_{rk_r}, \end{array}$$

satisfaciendo la siguiente regla. Se puede enumerar una nueva arista a_{ij} si tiene un extremo v_{ij} tal que las demás aristas incidentes en v_{ij} y las etiquetadas con v_{ij} fueron enumeradas como a_{kl} con $k < i$ o $k = i, l < j$, o si la arista comienza una nueva fila (i.e. si $j = 1$).

A partir de esta hipótesis se puede mostrar la asfericidad de muchos LOTs. Mostramos algunos ejemplos. Luego, para generalizar estos resultados, introducimos la siguiente noción.

Definición. Sea Γ un LOT reducido. Una *buena enumeración* de Γ es una enumeración de las aristas de Γ en filas

$$\begin{array}{l} a_{11}, a_{12}, \dots, a_{1k_1} \\ a_{21}, a_{22}, \dots, a_{2k_2} \\ \qquad \dots \\ a_{r1}, a_{r2}, \dots, a_{rk_r}, \end{array}$$

satisfaciendo las siguientes reglas

1. Se puede enumerar una nueva arista a_{ij} si tiene un extremo v_{ij} tal que las demás aristas incidentes en v_{ij} y las etiquetadas con v_{ij} fueron enumeradas como a_{kl} con $k < i$ o $k = i, l < j$, o si la arista comienza una nueva fila (i.e. si $j = 1$).

2. Para cada $1 \leq s \leq r$ existe un vértice w_s distinto de todos los v_{ij} tal que todas las aristas incidentes en w_s y etiquetadas con w_s fueron enumeradas como a_{kl} con $k \leq s$.

3. Para cada $1 \leq s \leq r$, el camino en Γ entre w_s y la arista a_{s1} es una sicesión de aristas a_{sj}, tal que los vértices del camino tienen todas sus aristas incidentes y etiquetadas enumeradas con $kl, k \leq s$.

Teorema 5.4.14. *Sea Γ un LOT reducido que admite una buena enumeración, y tal que $G(\Gamma)$ es UPP. Entonces Γ es asférico.*

Mostramos luego varios ejemplos donde se aplica este resultado, incluyendo LOTs no inyectivos de cualquier diámetro y complejidad.

En la sección 5.5 mostramos cómo nuestro método se aplica para probar fácilmente la asfericidad de los ejemplos presentados por Rosebrock [Ros10] como LOTs de máxima complejidad.

Por último, en la sección 5.6 calculamos la descripción del segundo grupo de homotopía de los LOT a partir de los resultados de Barmak y Minian, considerando otro subdiagrama simplemente conexo del diagrama de Hasse del LOT poset.

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