Congruencias entre formas modulares modulo potencias de primos

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Resumen

A lo largo de esta tesis hemos trabajado fundamentalmente en el problema de congruencias entre formas modulares módulo potencias de primos. La pregunta disparadora del trabajo realizado fue la siguiente: dada una forma modular f, autoforma para los operadores de Hecke, y una potencia de un número primo p^n, ¿existe una autoforma g, distinta de f, de modo tal que f y g sean congruentes módulo p^n?.

El enfoque utilizado para dar respuesta a esta pregunta fue la adecuación de las ideas de los trabajos [Ram99] y [Ram02], en los que por métodos algebraicos se intenta levantar representaciones de Galois con imagen en anillos de torsión a anillos de característica 0. Mediante la adaptación de estos métodos a las representaciones asociadas a f módulo p^n se logra dar una respuesta exhaustiva a la pregunta inicial en la mayoría casos.

La presente tesis se divide en dos capítulos. En el primero se estudia el problema correspondiente al caso en el que el anillo generado por los coeficientes de la forma f es no ramificado en el primo p. En este caso las ideas de [Ram99] y [Ram02] se logran adaptar sin mayores inconvenientes. En el segundo capítulo se aborda el caso en el que p ramifica en el anillo de coeficientes de f. Este escenario plantea un problema técnico que solo pudo ser resuelto cuando la forma f es ordinaria. Si bien ambos capítulos giran en torno a la misma idea central, los problemas técnicos que aparecen en cada uno de los casos requieren emplear estrategias esencialmente distintas para su resolución.

Palabras claves: Formas modulares, representaciones de Galois, modularidad, subida y bajada de nivel, tipos locales.

Congruences between modular forms modulo prime powers

Abstract

Along this thesis we have worked fundamentally on the problem of congruences between modular forms modulo prime powers. The motivating problem of this work was the following one: given a modular form f, eigenform for the Hecke operators, and a prime power p^n, does another eigenform g, different from f, such that f and g are congruent modulo p^n exist?.

Our approach used to solve this problem is to adapt the ideas of [Ram99] and [Ram02], where Galois representations with image in torsion rings are lifted to rings of characteristic 0 applying strictly algebraic methods. The adaptation of the methods employed in these works to the representations attached to f modulo p^n enable us to exhaustively answer our initial question in most of the cases.

The present thesis is divided into two chapters. In the first one we study the problem corresponding to the case in which the ring generated by the coefficients of the form f is unramified at p. In this case we are able to adapt the ideas of [Ram99] and [Ram02] to our setting. In the second chapter we study the case in which p ramifies in the ring generated by the coefficients of f. In this scenario we face some technical problems that we can only solve when f is ordinary. While both chapters revolve around the same main idea, the technical problems appearing in each of them require strategies that are essentially different.

Keywords: Modular forms, Galois representations, modularity, level lowering and level raising , local types.

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Introducción

A lo largo de esta tesis hemos trabajado en diferentes aspecto del problema de congruencias entre formas modulares. La aparición de congruencias entre diferentes objetos de la teoría de números sucede frecuentemente. Quizás uno de los resultados más famosos relacionados con congruencias (y ciertamente uno de los más cercanos a los contenidos de esta tesis) son los teoremas de subida y bajada de nivel de Ribet. Estos teoremas dan condiciones bajo las cuales hay congruencias módulo p entre formas modulares, con un control explícito del peso y el nivel. Brevemente, el Teorema de bajada de nivel de Ribet establece que dada una forma nueva f de nivel qN, con $q \nmid N$ y un primo p tal que la representación módulo p asociada a f es no ramificada en q entonces existe otra forma nueva g, de nivel N, que es congruente con f módulo p. Este teorema fue trascendental para la primera prueba del Último Teorema de Fermat y los teoremas de subida y bajada de nivel son parte fundamental de una serie de resultados que sirven para resolver distintos tipos de ecuaciones diofánticas.

Los resultados de Ribet se prueban a través del estudio de los anillos de Hecke, demostrando que la representación módulo p asociada a f aparece como un cociente del álgebra de Hecke de nivel N, implicando la existencia de la forma g. Una pregunta natural es si estos métodos se aplican para conseguir congruencias de orden mas alto que 1. En esta dirección, existen contraejemplos a la formulación natural de una generalización a congruencias módulo p^n. En [Dum05] el siguiente ejemplo es presentado: la curva elíptica $329a1$ corresponde a una forma nueva de peso 2 y nivel 329 y su representación 3-ádica es no ramificada en 7 al ser reducida módulo 9. Sin embargo, no existe ninguna forma nueva de peso 2 y nivel 47 congruente con $329a1$ módulo 9.

Por otro lado, se pueden aplicar las ideas de Ribet para conseguir algunos resultados relacionados con congruencias módulo p^n. Dada una forma modular f de nivel qN, tal que la reducción módulo p^n de su representación p-ádica es no ramificada en q se puede probar que esta representación aparece como un cociente del álgebra de Hecke de nivel N. Sin embargo, esto no es suficiente para probar la existencia de un levantado a característica 0 con nivel N, y en general este levantado no tiene por qué existir (como muestra el ejemplo de Dummigan). Este acercamiento, junto con ciertas definiciones básicas acerca de congruencias módulo p^n entre formas modulares (cuya definición no es obvia cuando los anillos de coeficientes de las formas en cuestión son distintos) se puede encontrar en [CW13] y [Tsa09] entre otros.

Considerando estos resultados previos, nuestro trabajo comenzó con un problema sencillo: dada una forma modular f, que sea autofunción para los operadores de Hecke, y una potencia de primo p^n, ¿existe una forma g, diferente de f, tal que f y g sean congruentes módulo p^n?. Y en caso positivo, ¿qué se puede decir acerca de una tal g?. La diferencia entre esta pregunta y las anteriores es que en este caso no estamos fijando el nivel de g, nos interesa probar la existencia de cualquier forma modular congruente a f y luego estudiar que podemos decir acerca de ella.

La estrategia para atacar este problema es considerar la representación de dimensión 2, ρ_f, asociada a f ([Eic54], [Shi71], [Del71], [DS74]), reducirla módulo p^n para obtener una representación ρ_n y estudiar el problema de levantar ρ_n nuevamente a característica 0. Si encontramos tal levantado, entonces debemos probar que es modular, asociado a una forma f. Debemos explicar un poco más en este momento, tiene sentido reducir la representación asociada a f módulo p^n siempre y cuando p no ramifique en el cuerpo de coeficientes de f. En caso contrario, la imagen de la representación se encuentra dentro de un anillo de enteros $\mathcal{O}$ de una extensión ramificada de $\mathbb{Q}_p$ y es mucho más razonable hablar de congruencias módulo π^n, donde π es el uniformizador local de $\mathcal{O}$. Estudiaremos ambos casos en este trabajo.

Para llevar a cabo esta estrategia aplicamos la misma idea principal que Ramakrishna introdujo en [Ram99] y [Ram02]. En estos trabajos, se prueba que una representación de dimensión 2 con coeficientes en un cuerpo finito $\mathbb{F}$ admite un levantado al anillo de vectores de Witt de $\mathbb{F}$, $W(\mathbb{F})$, bajo ciertas hipótesis, si se permite la aparición de ramificación extra. El argumento de Ramakrishna es puramente algebraico y de naturaleza inductiva. Consiste en estudiar las obstrucciones a levantar representaciones de $W(\mathbb{F})/p^n$ a $W(\mathbb{F})/p^{n+1}$, donde p es la característica de $\mathbb{F}$, para cada $n \in \mathbb{N}$. La naturaleza inductiva del argumento es lo que nos permite adaptarlo a nuestro entorno. Ya no comenzamos con una representación con coeficientes en un cuerpo finito $\mathbb{F}$ sino que con una representación con coeficientes en $W(\mathbb{F})/p^n$ (en el caso no ramificado) o en $\mathcal{O}/\pi^n$ (en el caso ramificado). En ambos casos, nada nos impide estudiar las mismas obstrucciones paso-a-paso que Ramakrishna estudia.

El argumento inductivo del trabajo de Ramakrishna se divide esencialmente en dos partes. La idea principal del trabajo es reducir el problema de levantar una representación global a característica 0 a una serie de problemas menores de tipo local. Naturalmente, esto involucra un argumento global que nos permita transformar un problema en el otro. Una vez conseguido esto uno debe resolver los problemas locales. Mientras que el orden lógico del argumento es el que mencionamos previamente, matemáticamente es más natural estructurar la demostración en el orden contrario. A continuación describimos brevemente ambas partes de la demostración.

Por un lado, el argumento local corresponde a resolver el siguiente problema: dada la representación ρ_n, para cada $\ell \in S$ se debe construir una familia C_ℓ de representaciones con coeficientes en característica 0, que sean levantados de $\rho_n|_{G_\ell}$, y un subespacio $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ de codimensión igual a $\dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$ tal que N_ℓ preserva a C_ℓ en el siguiente sentido, cada vez que se tiene una representación ρ que es la reducción módulo p^m de un elemento de C_ℓ y un elemento $u \in N_\ell$ se tiene que $(1 + p^{m-1}u)\rho$ también es la reducción de un miembro de C_ℓ.

Una vez que el problema local esta resuelto, y equipados con la colección de conjuntos C_ℓ y subespacios N_ℓ, se desarrolla un argumento global que provee el pasaje del escenario global al escenario local. El objetivo final de este argumento es encontrar un conjunto de primos auxiliares Q tal que el morfismo

$$\mathrm{H}^2(G_{S\cup Q}, Ad^0\bar{\rho}) \to \oplus_{\ell\in S\cup Q} \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$$

es inyectivo y el morfismo

$$\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho}) \to \oplus_{\ell\in S\cup Q} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

es sobreyectivo. Seremos más claros sobre esto a lo largo de las lineas de este trabajo, pero obtener estas dos propiedades equivale a resolver el problema dado que uno puede utilizar el siguiente argumento. Sea ρ_m una representación con coeficientes en $W(\mathbb{F})/p^m$ tal que $\rho_m|_{G_\ell}$ es la reducción de algún elemento de C_ℓ para todo $\ell \in S \cup Q$, y supongamos que queremos levantar dicha representación a $W(\mathbb{F})/p^{m+1}$ manteniendo esta propiedad. Uno puede utilizar el siguiente argumento en dos pasos:

- La obstrucción a levantar ρ_m es un elemento de $\mathrm{H}^2(G_S, Ad^0\bar{\rho})$, cuando este elemento es 0, la representación tiene un levantado. La inyectividad del primer morfismo nos dice que si la obstrucción para levantar $\rho_m|_{G_\ell}$ es trivial para todo $\ell \in S \cup Q$ entonces la obstrucción para levantar ρ_m es trivial y por lo tanto hay un levantado. Pero ya sabemos que $\rho_m|_{G_\ell}$ se levanta a característica 0 para todo $\ell \in S \cup Q$, puesto que es la reducción de un elemento de C_ℓ. Por lo tanto, existe un levantado $\tilde{\rho}_{m+1}$ de ρ_m a $W(\mathbb{F})/p^{m+1}$.

- El levantado obtenido no necesariamente tiene la propiedad deseada respecto a los conjuntos C_ℓ, pero sabemos que para cada $\ell \in S \cup Q$ existe un elemento en $z_\ell \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ tal que $(\mathrm{Id} + p^m z_\ell)\tilde{\rho}_{m+1}$ es la reducción de un elemento en C_ℓ. Más aun, cualquier elemento en la misma coclase que z_ℓ con respecto a N_ℓ también funciona, dado que N_ℓ preserva a C_ℓ.

  Ahora, usando que el segundo morfismo es sobreyectivo, debe haber un elemento z en $\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho})$ cuya imagen en $\oplus_{\ell \in S\cup Q} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$ es la clase $(z_\ell)_{\ell\in S\cup Q}$. Finalmente, $\rho_{m+1} = (\mathrm{Id} + p^m z)\tilde{\rho}_{m+1}$ es un levantado de ρ_m como el buscado.

Esto completa el paso inductivo. Debemos remarcar que a pesar de que hemos bosquejado la demostración original de Ramakrishna, que solo funciona para levantar representaciones a $W(\mathbb{F})$, la estructura de esta demostración también funciona en el caso en el que queremos levantar una representación con coeficientes en la reducción módulo π^n del anillo de enteros de una extensión ramificada de $\mathbb{Q}_p$. Otra observación que vale la pena realizar es que este argumento otorga cierto control sobre el comportamiento local de los levantados construidos en los primos de $S \cup Q$. Como en cada paso se tiene que $\rho_m|_{G_\ell}$ es la reducción de un elemento de C_ℓ, el levantado que se obtiene al pasar al límite debe tener la misma propiedad. Esta observación no solo nos permitirá producir un teorema de levantamiento con control de tipos locales (similar al resultado obtenido en el Teorema 3.2.2 de [BD] para representaciones módulo p) sino que al aplicarse al primo p, será esencial para permitirnos aplicar teoremas de levantamiento de modularidad (que requieren hipótesis cobre el comportamiento de la representación en el primo p) y finalmente pasar del problema algebraico de congruencia entre representaciones al problema de congruencia entre objetos modulares.

Esta tesis se encuentra dividida en cuatro capítulos. El **Capítulo 1** es una breve descripción de una serie de resultados teóricos preliminares, que son necesarios para una mejor comprensión de los siguientes tres capítulos.

En el **Capítulo 2** presentamos una clasificación de deformaciones con coeficientes en cuerpos finitos y en anillos de característica 0 y el estudio de los posibles tipos de reducción entre ellas. También se calculan las dimensiones de todos los grupos locales de cohomología para deformaciones módulo p. Estos resultados son esenciales para los capítulos 3 y 4, dado que la resolución de los problemas locales se apoya fuertemente en esta clasificación.

En el **Capítulo 3** se trata el problema de levantar una representación módulo p en el mismo contexto que el trabajo original de Ramakrishna, esto es, cuando el cuerpo de coeficientes al que queremos levantar la representación es una extensión no ramificada de $\mathbb{Q}$. Para conseguir este objetivo, se deben adaptar tanto el argumento global como el argumento local descriptos previamente. En la parte local esto implica imponer una condición extra a la colección de pares (C_ℓ, N_ℓ), el hecho de que todos los elementos de C_ℓ reducen a $\rho_n|_{G_\ell}$ módulo p^n. Por lo general esto se hace manualmente. Clasificaremos todos los posibles pares $(\rho, \overline{\rho})$, donde $\overline{\rho}$ es la reducción módulo p de ρ_n y ρ es un levantado local a característica 0, y para cada posible par construimos la familia C_ℓ y el subespacio N_ℓ explícitamente. El argumento global parece más sencillo de adaptar, pero involucra una serie de detalles técnicos que deben ser tenidos en cuenta. Terminamos el capítulo construyendo un ejemplo explícito de subida de nivel siguiendo la demostración del teorema principal en un caso particular. Para ello consideramos la curva E_{17a1} y hallamos una forma modular de peso 2 y nivel $17 \cdot 113$ que es congruente a ella módulo 25. Este problema es computacionalmente no trivial, dado que la aplicación directa del principal argumento del capítulo involucra calcular extensiones abelianas sobre una extensión $L/\mathbb{Q}$ de grado 120, lo que esta fuera del alcance de cualquier software matemático actual. Sin embargo, una serie de argumentos de teoría de Galois permiten reducir el grado de las extensiones con las que se debe trabajar y encontrar el nivel correcto en el que buscar la forma congruente con E_{17a1}.

Finalmente, el **Capítulo 4** aborda el caso en el que el cuerpo de coeficientes de la forma modular en cuestión es ramificado en p. Nuevamente, esto involucra revisitar ambas partes del argumento de Ramakrishna. Para resolver el problema local se sigue un argumento similar que para el caso no ramificado, solo que se deben tratar nuevas combinaciones $(\rho, \overline{\rho})$ que presentan ciertas dificultades extra. En cuanto al costado global del argumento, un obstáculo que no se presentaba en el caso no ramificado aparece. Sucede que el morfismo

$$\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho}) \to \oplus_{\ell\in S\cup Q}\,\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

que se desea suryectivo tiene núcleo no trivial para cualquier elección de Q y los subespacios N_ℓ siempre que estos se quieran escoger con las propiedades deseadas. También se sabe que el dominio y el codominio de dicho morfismo tienen la misma dimensión por lo que resulta imposible elegir el conjunto auxiliar Q para que este morfismo resulte sobreyectivo. Más aun, es conjeturalmente imposible construir subespacios N_ℓ de dimensión mayor con las propiedades deseadas para primos $\ell \neq p$. Este problema aparece cuando el cuerpo de coeficientes de la representación es ramificada puesto que en este caso hay un elemento $f \in \mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho})$ asociado a la reducción módulo π^2 de ρ_n que satisface que $f|_{G_\ell} \in N_\ell$ siempre que ρ_n es la reducción de los elementos de C_ℓ. La innovación clave para resolver este problema es relajar la condición impuesta en el primo p (i.e. agrandar el conjunto C_p). No sabemos como hacer esto en general, pero sí cuando la representación ρ_n es **ordinaria** en el primo p. Siguiendo el argumento original bajo estas nuevas condiciones no solo obtenemos un levantado de ρ_n a característica 0 pero una familia de ellos, parametrizada por el anillo $W(\mathbb{F})[[T]]$. El lado negativo de este argumento es que todos los levantados pertenecientes a esta familia satisfacen una condición más débil en el primo p, que en principio impide que estos sean modulares. Sin embargo, hacia el final del capítulo probamos la existencia de un levantado modular dentro de la familia, que es de nivel acotado pero peso desconocido.

Introduction

In this thesis we have worked around different aspects of the problem of congruences between modular forms. Congruences between different objects of number theory happen frequently and come in different shapes and flavors. Maybe the most famous results concerning congruences (and certainly one of the closest to the contents of this thesis) are Ribet's Theorems about level lowering and level raising of modular forms (see [Rib85] and [Rib90]). These theorems give conditions under which there are congruences modulo p between modular forms, with an explicit control of the level and weight. Roughly speaking, Ribet's Level Lowering Theorem states that given a newform f of level qN with $q \nmid N$ and a prime p such that the mod p representation attached to f is unramified at q there is another newform g of level N which is congruent to f modulo p. This theorem has been transcendental to the first proof of Fermat's Last Theorem and level lowering results are a key part of a range of machinery that serves to solve different Diophantine equations.

Ribet's results are proven through the study of Hecke rings, proving that the modulo p representation attached to such a f appears as a quotient of the Hecke algebra of level N, directly implying the existence of the form g. A natural question to ask is if these methods apply in order to get higher congruences. In this direction, there are counterexamples to the natural formulation of a generalization to the modulo p^n setting. In [Dum05], the following example is presented: the elliptic curve $329a1$ corresponds to a newform of weight 2 and level 329 and its mod 9 representation is unramified at 7. However, there is no newform of level 47 congruent to it modulo 9.

On the other hand, one can apply the ideas of Ribet in order to get some results concerning mod p^n congruences. Given a modular form f of level qN, such that its mod p^n representation is unramified at q it can be proved, using the same kind of ideas as in Ribet's work, that this representation appears as a quotient of the Hecke algebra of level N. However, this is not enough to prove the existence of a lift to characteristic 0 with level N, and in general this lift may not exist (as in Dummigan's example). This approach, together with the corresponding definitions of congruences modulo p^n between modular forms (which is not obvious as rings of coefficients of different forms may be different), is taken in [CW13] and [Tsa09] among others.

Taking in consideration these previous results, our work started with one simple problem: given a modular form f, which is an eigenform for Hecke operators, and a prime power p^n, is there an eigenform g, different from f, such that f and g are congruent modulo p^n? And given that, what can we say about such g?. The main difference between this question and the previous ones is that we are not fixing the level of g, we first want to prove the existence of any g and then concern about what can we say about it.

The strategy to attack this question is to take the two-dimensional representation ρ_f attached to f ([Eic54], [Shi71], [Del71], [DS74]), reduce it modulo p^n to get a representation

ρ_n an study the problem of lifting ρ_n back to characteristic 0. If such a lift is found, we then shall prove that it is modular, attached to a form g. Observe that it makes sense to reduce the representation attached to f modulo p^n whenever p does not ramify at the field of coefficients of f. Otherwise, the image of this representation lies inside the integer ring $\mathcal{O}$ of a ramified extension of $\mathbb{Q}_p$ and it is much more reasonable to talk about congruences modulo π^n, where π is the local uniformizer of $\mathcal{O}$. We will study both cases along these pages.

To carry out this strategy we employ the same main idea that Ramakrishna introduced in [Ram99] and [Ram02]. In these works, it is proven that a two-dimensional representation with coefficients in a finite field $\mathbb{F}$ has a lift to the ring of Witt vectors of $\mathbb{F}$ (from now on $W(\mathbb{F})$) under certain hypotheses, if enough extra ramification is allowed. The argument of Ramakrishna is purely algebraic and of inductive nature. It consists on studying the obstructions for lifting from $W(\mathbb{F})/p^n$ to $W(\mathbb{F})/p^{n+1}$, where p is the characteristic of $\mathbb{F}$, for each $n \in \mathbb{N}$. This inductive nature is what allows us to adapt it to our setting. We are no longer starting with a representation with coefficients in a finite field $\mathbb{F}$ but with a representation with coefficients in $W(\mathbb{F})/p^n$ (in the unramified case) or $\mathcal{O}/\pi^n$ (in the ramified case). In both cases, nothing stops us from studying the same step-by-step obstructions than Ramakrishna does.

The inductive step in Ramakrishna's work splits essentially into two parts. The main idea of the work is to reduce the problem of lifting a global representation to characteristic 0 into a series of minor and more easy to solve problems of local nature. Naturally, this involves some kind of global argument that let us pass from the global setting to the local one. Once there, one must of course solve this series of local problems. While the logical order of the argument is the one mentioned above, mathematically is more natural to structure the proof the opposite way around. We briefly describe both parts of the proof.

On one side, the local argument corresponds to solving the following problem: for each $\ell \in S$, construct a family C_ℓ of representations with coefficients in characteristic 0 rings and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ of codimension equal to $\dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$ such that N_ℓ preserves the family C_ℓ in the following way, each time that we have a representation ρ which is the mod p^m reduction of some element in C_ℓ and any element $u \in N_\ell$ we have that $(1 + p^{m-1}u)\rho$ is also the reduction of some representation of the family C_ℓ.

Once the local problem is solved, and equipped with the collection of sets C_ℓ and subspaces N_ℓ, there is a global argument that provides the passage from the global setting to the local one. The final objective of this argument is to find a set of auxiliary primes Q such that the map

$$\mathrm{H}^2(G_{S\cup Q}, Ad^0\bar{\rho}) \to \oplus_{\ell\in S\cup Q}\, \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$$

is injective, and the map

$$\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho}) \to \oplus_{\ell\in S\cup Q}\, \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

is surjective. We will go into the details through the thesis, but getting these two properties amounts to solving the problem because one can use the following argument. Take a representation ρ_m with coefficients in $W(\mathbb{F})/p^m$ such that $\rho_m|_{G_\ell}$ is the reduction of some element of C_ℓ for every $\ell \in S \cup Q$, and suppose that we want to lift it to $W(\mathbb{F})/p^{m+1}$ with the same property. One can use this two step argument:

- The obstruction to lifting ρ_m is an element of $\mathrm{H}^2(G_S, Ad^0\bar{\rho})$, when this element is 0, the representation has a lift. The injectivity of the first morphism tells us that if the obstruction to lifting $\rho_m|_{G_\ell}$ is trivial for every $\ell \in S \cup Q$ then the obstruction to lifting ρ_m is trivial and therefore there is a lift. But we already know that $\rho_m|_{G_\ell}$ lifts to characteristic 0 for every $\ell \in S \cup Q$ implying that the local obstructions are all 0. Thence, we have a lift $\tilde{\rho}_{m+1}$ of ρ_m to $W(\mathbb{F})/p^{m+1}$.

- The lift obtained does not necessarily have the desired property about the sets C_ℓ, but we know that for each $\ell \in S \cup Q$ there is an element $z_\ell \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ such that $(\mathrm{Id} + p^m z_\ell)\tilde{\rho}_{m+1}$ is the reduction of some element in C_ℓ. Moreover, every other element in the same coset than z_ℓ with respect to N_ℓ also works, as N_ℓ preserves C_ℓ.

  Now, using the surjectivity of the second morphism, there must be an element z in $\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho})$ whose image in $\oplus_{\ell \in S\cup Q} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$ is the class of $(z_\ell)_{\ell\in S\cup Q}$. Finally, $\rho_{m+1} = (\mathrm{Id} + p^m z)\tilde{\rho}_{m+1}$ is a lift of ρ_m with the desired property.

This completes the inductive step. Let us recall that while we have sketched the proof of Ramakrishna's original work, which only works for lifting representations to $W(\mathbb{F})$, this macro structure of the proof also applies to the case where we want to lift a representation with coefficients in the mod π^n reduction of some ring of integers of a ramified extension of $\mathbb{Q}_p$. Another observation worth doing is that we have control of the local behavior of the lift produced at the primes in $S \cup Q$. As at every step we have that $\rho_m|_{G_\ell}$ is the reduction of some element in C_ℓ, the lift obtained after passing to the limit will satisfy the same property. This apparently naive observation will not only let us produce a lifting theorem with control of local types (similar to what is done in Theorem 3.2.2 of [BD] for modulo p representations) but when applied to the prime p will be essential for the use of modularity lifting theorems, which are needed for our original problem of finding congruences between modular objects.

The thesis is structured in four different chapters. **Chapter 1** is a brief description of a series of preliminary theoretic results, which will be needed for the better understanding of the next three chapters.

In **Chapter 2** we present a classification of modulo p and characteristic 0 deformations, and study the possible types of reduction between them. We also compute the dimensions of all the local cohomology groups of modulo p deformations. This will be essential for both Chapter 3 and 4 as the solution of the local problem relies on this classification.

In **Chapter 3** we deal with the problem of lifting a mod p^n representation in the same setting as Ramakrishna's original work, that is, when the coefficient field that we want to lift to is the ring of integers of an unramified extension of $\mathbb{Q}_p$. To accomplish this task, we need to adapt both the local and the global arguments described above. In the local part, this amounts to impose an extra condition to the collection of pairs (C_ℓ, N_ℓ), which is to be sure that $\rho_n|_{G_\ell}$ appears as the reduction of some element in C_ℓ. Generally this is done in a quite manual fashion. We classify all the possible pairs $(\rho, \overline{\rho})$, where $\overline{\rho}$ is the mod p reduction of ρ_n and ρ is a lift to characteristic 0, and for each of them we construct the family C_ℓ of lifts of $\overline{\rho}$ such that $\rho \in C_\ell$ and a subspace N_ℓ explicitly. The global side seems to be easier to adapt, but involves a series of technical details to be taken care of. We end the chapter by constructing an explicit example of level raising by following the method. We take the elliptic curve E_{17a1} and construct a modular form of weight 2 and level $17 \cdot 113$ which is congruent to it modulo 25. This problem is computationally non-trivial, as the straightforward application of the theoretical side of this work would involve computing abelian extensions over an extension $L/\mathbb{Q}$ of degree 120, which is out of the reach of any current mathematical software. However, by using some Galois theory and a series of tricks to lower the degree of the Galois extensions we have to work with, we manage to find the correct level in which to look for the form congruent to E_{17a1}.

Chapter 4 treats the case where the field of coefficients is ramified. Again, this involves revisiting both parts of Ramakrishna's main work, but this time the local side is the one which is easier to adapt. The computations and main ideas to solve the local problem in this case are essentially the same as in the unramified case, only that new possible combinations $(\rho, \overline{\rho})$ appear. The challenging part of this problem appears when trying to solve the global side it presents. The main obstacle here appears when trying to make the morphism

$$\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho}) \to \oplus_{\ell\in S\cup Q}\,\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

surjective. Given the subspaces N_ℓ with the same properties as in the unramified case, both the domain and codomain of this morphism have the same dimension as vector space over $\mathbb{F}$. Moreover, it should not be possible (conjecturally) to build bigger subspaces N_ℓ with the desired properties for $\ell \neq p$. The issue appears as when the field of coefficients of ρ_n is ramified there is an element $f \in \mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho})$ attached to the reduction mod π^2 of ρ_n which satisfies that $f|_{G_\ell} \in N_\ell$ whenever we have that ρ_m is the reduction of some element in C_ℓ. The key innovation to deal with this problem is to relax the condition imposed at the prime p (i.e. to enlarge the set C_p). We do not know how to to this in general, but it works well when the representation ρ_n is **ordinary** at p. Following the original argument in this way, we do not only lift ρ_n but produce a family of lifts parametrized by the ring $W(\mathbb{F})[[T]]$. The downside of this argument is that all the lifts belonging to this family satisfy a weaker condition at p, which a priori prevents them from being modular. However, by the end of the chapter and adding some extra restrictions to our deformation, we prove the existence of a modular lift inside the family, which is of controlled level but unknown weight.

Chapter 1

Preliminaries

1.1 Notation

We present a list, as comprehensive as possible, of the notation that will be used throughout the thesis.

- **Primes and finite fields:** p will always denote a rational prime and $\mathbb{F}$ will be a finite field of characteristic p. Auxiliary primes will be denoted q or ℓ.
- **Galois groups:** $G_{\mathbb{Q}}$ will be $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group of $\mathbb{Q}$. In general, given a field F, G_F stands for $\mathrm{Gal}(F^s/F)$, where F^s is a separable closure of F. In almost all of our cases $F^s = \overline{F}$ an algebraic closure of F. Given a prime ℓ, G_ℓ will be a shorthand for $G_{\mathbb{Q}_\ell}$, and σ and τ will always refer to a Frobenius element and a tame inertia generator inside this absolute Galois group respectively (we leave aside the reference to the prime ℓ to avoid excess of notation). For a set of primes S, we denote by $\mathbb{Q}_S$ the maximal extension of $\mathbb{Q}$ which is unramified outside S and by G_S the Galois group $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.
- **Rings and fields of coefficients:** Given a finite field $\mathbb{F}$, we will denote its ring of Witt vectors by $W(\mathbb{F})$. By $\mathcal{O}$ we will always denote the ring of integers of a ramified finite extension $K/\mathbb{Q}_p$ with residue field $\mathbb{F}$. Its local uniformizer will be named π and e will be its ramification degree. Finally, v stands for the valuation in $\mathcal{O}$ such that $v(\pi) = 1$.
- **Characters and representations:** In general, $\overline{\rho}$, ρ_n and ρ will always be continuous representations with coefficients in $\mathbb{F}$, $W(\mathbb{F})/p^n$ or $\mathcal{O}/\pi^n$ and $W(\mathbb{F})$ or $\mathcal{O}$ respectively. We will denote the cyclotomic character by χ and given a character ω with coefficients in $\mathbb{F}^\times$, $\tilde{\omega}$ will be its Teichmuller lift. Given a representation $\rho : G \to \mathrm{GL}_2(K)$, $Ad(\rho)$ stands for its adjoint representation acting on $M_2(K)$ and $Ad^0(\rho)$ stands for the same adjoint action on the subspace of trace 0 matrices. We also denote by $\mathbb{Q}(\rho)$ the field fixed by the kernel of ρ.
- **Cohomology:** We denote
$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
which form a basis for the space of 2×2 matrices with trace 0. Also, given a prime q, which will be clear from context, d_i will stand for $\dim \mathrm{H}^i(G_q, Ad^0\overline{\rho})$ for $i = 0, 1, 2$.

1.2 Some deformation theory

The main tool of this thesis is the deformation theory of Galois representations introduced by Mazur. We will briefly recall the main results on this topic in the following sections. For a more detailed exposition, we refer to the original work of Mazur [Maz89], to the expository text [Gou04] and to [Boc05].

1.2.1 Basic definitions

Let $\mathbb{F}$ a finite field, Π a profinite group (which in our case will be $G_{\mathbb{Q}}$, G_S or G_ℓ) and $\overline{\rho} : \Pi \to \mathrm{GL}_2(\mathbb{F})$ a continuous representation. Deformation theory was born when trying to study all possible lifts of $\overline{\rho}$ to certain coefficient rings that have $\mathbb{F}$ as a quotient. We give the main definitions on the subject.

Definition. A coefficient ring is a local, noetherian, complete $W(\mathbb{F})$-algebra. A morphism of coefficient rings is a morphism of local $W(\mathbb{F})$-algebras.

We want to understand the lifts of $\overline{\rho}$ to coefficient rings. However, the problem of studying all the possible representations $\rho_R : \Pi \to \mathrm{GL}_n(R)$ such that the reduction of ρ_R modulo the maximal ideal of R is $\overline{\rho}$ is wild. In order to handle the problem, we introduce the concept of deformation.

Definition. Let Π be a profinite group, $\overline{\rho} : \Pi \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, R be a coefficient ring and $\mathcal{M}_R$ its maximal ideal.

- We say that a representation $\rho : \Pi \to \mathrm{GL}_2(R)$ is a lift of $\overline{\rho}$ to R if ρ modulo $\mathcal{M}_R$ is isomorphic to $\overline{\rho}$.
- Let ρ_1 and ρ_2 be lifts of $\overline{\rho}$ to R. We say that ρ_1 and ρ_2 are equivalent, if there exists a matrix $M \in \mathrm{GL}_2(R)$ which is congruent to the identity modulo $\mathcal{M}_R$ such that $M\rho_1 M^{-1} = \rho_2$.
- A deformation of $\overline{\rho}$ to a coefficient ring R is an equivalence class of lifts of $\overline{\rho}$ to R for the previously defined relation.

The problem of understanding deformations of $\overline{\rho}$ to different coefficient rings is much more manageable. In fact, under some mild hypothesis, it can be proved that there is an universal object that factors every possible deformation of $\overline{\rho}$. We introduce this technical condition.

Definition. We say that a profinite group Π satisfies the property Φ_p if for every open subgroup of finite index $\Pi_0 \subset \Pi$ there exist only a finite number of continuous morphisms $\Pi_0 \to \mathbb{F}_p$.

Among our Galois groups, it can be seen that both G_S and G_ℓ satisfy the property Φ_p. However, for $G_{\mathbb{Q}}$ the property does not hold and we will not be able to study representations of this group with this theory. This is not a great issue, as in general the representations that are relevant for our main problems ramify inside a finite set of primes, and therefore factor through a suitable group G_S.

The main theorem in the topic was first proved by Mazur in [Maz89] and Ramakrishna in [Ram93] pointed out that some hypotheses could be relaxed.

Theorem 1.2.1. *Let Π be a profinite group that satisfies the property Φ_p, $\overline{\rho} : \Pi \to \mathrm{GL}_n(\mathbb{F})$ a continuous representation and Λ a coefficient ring.*

Assume that $\overline{\rho}$ has no automorphisms. Then there exists a complete, noetherian, local, Λ-algebra R_u and a deformation $\rho_u : \Pi \to \mathrm{GL}_n(R_u)$ such that for every deformation $\rho : \Pi \to \mathrm{GL}_n(R)$, with R a Λ-algebra, there is an unique morphism of Λ-algebras $\psi : R_u \to R$ such that $\rho = \psi \circ \rho_u$. In this case R_u is called an universal deformation ring for $\overline{\rho}$.

In the case where $\overline{\rho}$ has automorphisms, the same holds but the morphism ψ does not need to be unique. In this case R_u is called versal ring.

Understanding the ring R_u essentially solves the problem of classifying the possible lifts of $\overline{\rho}$. However, it is often the case that R_u is singular enough to make this understanding not reachable. We will present here some of the basic results in the direction of decrypting the ring R_u.

1.2.2 Obstructions and cohomology

One key ingredient of the structure of R_u are the so called *obstructions*. Suppose that we have a deformation $\rho_1 : \Pi \to \mathrm{GL}_n(R_1)$ and a morphism $\psi : R_2 \to R_1$ with kernel $I \subseteq R_2$. Also assume that $\mathcal{M}_{R_2} \cdot I = 0$ (this is a technical assumption). It is natural to wonder if the deformation ρ_1 lifts to R_2, i.e. if there is a deformation $\rho_2 : \Pi \to \mathrm{GL}_n(R_2)$ such that $\rho_1 = \psi \circ \rho_2$.

To answer this question, we construct the following function:

- Take any function (not necessarily a morphism) $\gamma : \Pi \to \mathrm{GL}_n(R_2)$ that lifts ρ_1 to R_2.
- Consider $c(g, h) = \gamma(gh)\gamma(h)^{-1}\gamma(g)^{-1}$. As ρ_1 is a morphism and $\mathrm{Ker}(\psi) = I$, the element c has image inside $(\mathrm{Id} + I \otimes M_n(\mathbb{F}), \cdot) \simeq (I \otimes M_n(\mathbb{F}), +)$.

It can be seen that c is a 2-cocycle for the action of Π on $I \otimes M_n(\mathbb{F})$ by conjugation on $M_n(\mathbb{F})$ (through $\overline{\rho}$) and therefore it defines an element inside $\mathrm{H}^2(\Pi, \mathrm{Ad}\,\overline{\rho}) \otimes I$ (see Section 3). It is easy to check that the class of c in that group does not depend on the lift γ we picked, for different γ, the elements c obtained always differ by a 2-coboundary. The class of c is called the obstruction for lifting ρ_1 to R_2 and it holds that ρ_1 has a lift to R_2 if and only if the class of c is 0.

Remark. Observe that whenever $\mathrm{H}^2(\Pi, \mathrm{Ad}\,\overline{\rho}) \simeq 0$, every deformation has a lift to any coefficient ring we desire. In this case we say that the problem is *unobstructed*, and the lifts of $\overline{\rho}$ are much easier to understand.

The group $\mathrm{H}^1(\Pi, \mathrm{Ad}\,\overline{\rho})$ also plays an important role in this theory. Take the same deformation ρ_1 as before and assume that its kernel is principal, i.e. $I = \langle \epsilon \rangle$.

Proposition 1.2.2. *If ρ_2 and ρ_2' are two lifts of ρ_1 to R_2 then there is an element $u \in \mathrm{H}^1(\Pi, Ad\,\overline{\rho}) \otimes I$ such that*

$$(Id + \epsilon u)\rho_2 = \rho_2'$$

as deformations.

Proof. The proof of this fact is simple, and it accounts to checking that the function u defined by $\rho_2'(g)\rho_2(g)^{-1} = \mathrm{Id} + \epsilon u(g)$ is a 1-cocycle. This follows from the fact that both ρ_2 and ρ_2' are morphisms. □

This result enable us to give a different description of $\mathrm{H}^1(\Pi, \mathrm{Ad}\,\overline{\rho})$. Recall the definition of the ring of dual numbers

$$\mathbb{F}[\varepsilon] \simeq \mathbb{F}[x]/\langle x^2 \rangle.$$

We have the following identification between elements of $\mathrm{H}^1(\Pi, \mathrm{Ad}\,\overline{\rho})$ and deformations

$$\mathrm{H}^1(\Pi, \mathrm{Ad}\,\overline{\rho}) \longleftrightarrow \{\rho : \Pi \to \mathrm{GL}_n(\mathbb{F}[\varepsilon])\}$$

that comes from identifying an element $u \in \mathrm{H}^1(\Pi, \mathrm{Ad}\,\overline{\rho})$ with the deformation $(\mathrm{Id} + u\varepsilon)\overline{\rho}$, where $\overline{\rho}$ is obtained from the inclusion $\mathbb{F} \subset \mathbb{F}[\varepsilon]$. This space is called the *tangent space* of $\overline{\rho}$.

The main theorem relating the tangent space and obstructions with the structure of the universal ring is the following.

Theorem 1.2.3. *Let $\overline{\rho}$ and Λ as in Theorem 1.2.1. Assume that $\overline{\rho}$ does not have automorphisms. Let*

$$d_1 = \dim \mathrm{H}^1(\Pi, Ad\,\overline{\rho}) \quad \text{and} \quad d_2 = \dim \mathrm{H}^2(\Pi, Ad\,\overline{\rho})$$

Let R_u be the universal ring, we have that

$$\dim_{Krull} R_u/\mathcal{M}_\Lambda R_u \geq d_1 - d_2.$$

Moreover, if $d_2 = 0$, the equality holds and $R_u \simeq \Lambda[[X_1, \ldots, X_{d_1}]]$.

Observe that as we mentioned before, in the unobstructed case ($d_2 = 0$) the ring R_u is fairly easy to understand.

1.2.3 Local conditions and universal rings

Until now, for a given residual representation $\overline{\rho} : \Pi \to \mathrm{GL}_n(\mathbb{F})$ the ring R_u constructed has the property of factoring **every** deformation of ρ to a coefficient rings. Sometimes it is the case that we do not want to consider all the deformations of ρ and we rather study only the ones that satisfy certain properties. This is the case when trying to prove modularity of a given representation, we usually only want to consider the deformations that have the same properties as the modular ones. This restriction of the problem is even useful if one wants to have a better understanding of R_u, studying different families of deformations of $\overline{\rho}$ helps to capture some properties of the structure of the whole picture.

For this, we introduce the concept of *deformation condition*. Roughly speaking, assume that the representation $\overline{\rho}$ has a property, and we want to study all the deformations of $\overline{\rho}$ that keep this property. If the desired property is well-chosen (we will be more specific in a moment) one can prove the existence of a universal deformation for it, i.e. a ring through which every deformation of $\overline{\rho}$ with the desired property factors.

There are many different ways to approach this problem. We present a classical definition that can be found on [Gou04]. We remark that although we are going to deal with local properties all the time through the exposition, most of the time we will not use this approach explicitly, and therefore it is not essential to understand the math of this thesis. However, the spirit of these results is what drives this work, and the knowledge of these topics is a bridge to a better understanding of the results presented.

Definition. Let $\overline{\rho}$ be a residual representation of dimension n. A *deformation condition* on deformations of $\overline{\rho}$ is a property $\mathcal{P}$ on n-dimensional representations of Π defined over coefficient rings which satisfies.

- The residual representation $\overline{\rho}$ has the property $\mathcal{P}$.
- Given a deformation $\rho : \Pi \to \mathrm{GL}_n(A)$ and a morphism $\alpha : A \to A_1$, if ρ has the property $\mathcal{P}$ then so does $\alpha \circ \rho$.
- Let $A \times_C B$ be a pullback of coefficient rings induced by a pair of morphisms $\alpha : A \to C$ and $\beta : B \to C$. Let $p : A \times_C B \to A$ and $q : A \times_C B \to B$ be the projections. Given a deformation $\rho : \Pi \to \mathrm{GL}_n(A \times_C B)$ then ρ has the property $\mathcal{P}$ if and only if $p \circ \rho$ and $q \circ \rho$ have it.
- Given a deformation $\rho : \Pi \to \mathrm{GL}_n(A)$ and an injective morphism $\alpha : A \to A_1$, then ρ has the property $\mathcal{P}$ if $\alpha \circ \rho$ does.

As we said before, deformation conditions admit universal rings under hypotheses similar to the general case.

Theorem 1.2.4. *Let $\overline{\rho}$ as above and $\mathcal{P}$ be a deformation condition. Then there is a versal ring $R_u^{\mathcal{P}}$ through which every deformation with the property $\mathcal{P}$ factors. Moreover, if $\overline{\rho}$ does not have automorphisms the ring $R_u^{\mathcal{P}}$ is universal, i.e. every deformation with the property $\mathcal{P}$ factors uniquely through $R^{\mathcal{P}}$.*

Proof. This result is Theorem 6.1 in [Gou04]. □

Finally, we introduce the concept of tangent space.

Definition. Let $\mathcal{P}$ be a deformation condition. We define $\mathrm{H}^1_{\mathcal{P}}(\Pi, \mathrm{Ad}\,\overline{\rho})$ as the subspace of $\mathrm{H}^1(\Pi, \mathrm{Ad}\,\overline{\rho})$ given by the deformations of $\overline{\rho}$ to $\mathbb{F}[\varepsilon]$ which have the property $\mathcal{P}$.

An example: deformations with fixed determinant. We illustrate the previous definitions with an example of a deformation condition that will be used through all the present work. One of the most simple condition that one can impose to a deformation is to fix its determinant. We need to take a little care here as different deformations have coefficients in different rings and we have to be clear about what is the meaning of fixing the determinant of deformations with different coefficient rings.

Definition. Let Λ be a coefficient ring. Assume that we are interested in studying deformations to Λ-algebras. Let δ be a continuous homomorphism $\delta : \Pi \to \Lambda^\times$. For every coefficient Λ-algebra R we define the composition

$$\delta_R : \Pi \to \Lambda^\times \to R^\times.$$

We say that a deformation $\rho : \Pi \to \mathrm{GL}_n(R)$ has determinant δ if $\det(\rho) = \delta_R$.

In this case, it is quite easy to describe both the tangent space and the universal ring for this condition, under the hypothesis that $p \nmid n$ (here p is the characteristic of $\mathbb{F}$).

Proposition 1.2.5. *Let δ as in the definition and $\mathcal{P}$ the property of having fixed determinant δ. One has:*

- *$\mathcal{P}$ is a deformation condition.*

- *If $p \nmid n$ the tangent space $\mathrm{H}^1_{\mathcal{P}}(\Pi, Ad\,\overline{\rho})$ is given by*

$$\mathrm{H}^1_{\mathcal{P}}(\Pi, Ad\,\overline{\rho}) = \mathrm{H}^1(\Pi, Ad^0\overline{\rho}) \subseteq \mathrm{H}^1(\Pi, Ad\,\overline{\rho})$$

  where $Ad^0\overline{\rho}$ refers to the action of Π on the subspace of trace 0 matrices. Moreover if $p \mid n$ then $\mathrm{H}^1(\Pi, Ad^0\overline{\rho})$ is no longer a subspace of $\mathrm{H}^1(\Pi, Ad\,\overline{\rho})$ but we can identify

$$\mathrm{H}^1_{\mathcal{P}}(\Pi, Ad\,\overline{\rho}) = \mathrm{Im}(\mathrm{H}^1(\Pi, Ad^0\overline{\rho}) \to \mathrm{H}^1(\Pi, Ad\,\overline{\rho})).$$

- *The universal rings R_u and $R_u^{\mathcal{P}}$ satisfy*

$$R_u \simeq R_u^{\mathcal{P}} \widehat{\otimes}_{\Lambda} \Lambda[[T]]$$

  where $\widehat{\otimes}_{\Lambda}$ stands for completed tensor product.

Proof. This result can be found in Chapter 6 of [Gou04]. Some of the results there are left to the reader. □

Collections of local conditions. We end this subsection with the example of deformation condition of most interest for us. Take $\Pi = G_S$ for S a finite set of primes of $\mathbb{Q}$. Let $\overline{\rho} : G_S \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation. For each prime $\ell \in S$ let $\mathcal{P}_\ell$ be a deformation condition for deformations of $\overline{\rho}|_{G_\ell}$. We say that a deformation $\rho : G_S \to \mathrm{GL}_n(R)$ has the property $\mathcal{P}$ if $\rho|_{G_\ell}$ has the property $\mathcal{P}_\ell$ for every $\ell \in S$.

Proposition 1.2.6. *The property $\mathcal{P}$ defined above is a deformation condition and is tangent space satisfies that the diagram*

$$\begin{tikzcd}
\mathrm{H}^1_{\mathcal{P}}(G_S, Ad\,\overline{\rho}) \arrow[r] \arrow[d] & \mathrm{H}^1(G_S, Ad\,\overline{\rho}) \arrow[d] \\
\bigoplus_{\ell \in S} \mathrm{H}^1_{\mathcal{P}_\ell}(G_\ell, Ad\,\overline{\rho}) \arrow[r] & \bigoplus_{\ell \in S} \mathrm{H}^1(G_\ell, Ad\,\overline{\rho})
\end{tikzcd}$$

is a pullback, i.e. the elements of the tangent space of $\mathcal{P}$ are those in $\mathrm{H}^1(G_S, Ad\,\overline{\rho})$ which lie in the tangent space of $\mathcal{P}_\ell$ when restricted to G_ℓ, for every $\ell \in S$.

1.3 Galois cohomology

The other key ingredient of this thesis is Galois cohomology. Most of our calculations rely on results of this area. We introduce the basic concepts as well as the main results we are going to use. A good reference for this topic is [NSW00].

1.3.1 Basic tools in group cohomology

In this section we will not enter into the definition of group cohomology, as it is not the objective of this thesis. Instead, we will recall its principal properties.

Let G be a group and M an abelian G-module. For each $i \in \mathbb{Z}_{\geq 0}$ we have a cohomology group $\mathrm{H}^i(G, M)$, with some salient features:

- $\mathrm{H}^i(G, M)$ is functorial on both G (contravariantly) and M (covariantly).

- **(Long exact sequence)** For a short exact sequence of G-modules

$$0 \to X \to Y \to Z \to 0$$

  there is a long exact sequence

$$\ldots \to \mathrm{H}^i(G,X) \to \mathrm{H}^i(G,Y) \to \mathrm{H}^i(G,Z) \to \mathrm{H}^{i+1}(G,X) \to \ldots.$$

- **(Inflation-restriction exact sequence)** For a normal subgroup $H \subseteq G$ one has the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(G/H, M^H) \to \mathrm{H}^1(G,M) \to \mathrm{H}^1(H,M)^{G/H} \to \mathrm{H}^2(G/H,M^H) \to \mathrm{H}^2(G,M)$$

  The maps induced by the projection $G \to G/H$ are named inflation maps and the maps induced by the inclusion $H \subseteq G$ are named restriction maps.

- **(Cup product)** For M, M' and N G-modules, if $c : M \times M' \to N$ is a pairing that preserves the action of G, then c induces the cup product

$$\cup : \mathrm{H}^m(G,M) \times \mathrm{H}^n(G,M') \to \mathrm{H}^{m+n}(G,N).$$

We will only make use of these groups of cohomology for degree 0, 1 and 2. For these, there is a quite explicit description. We give it for $i = 0$ and 1.

- $\mathrm{H}^0(G,M) = M^G$ which is the subgroup of elements of M fixed by G.
- $\mathrm{H}^1(G,M) = Z(M)/B(M)$. Here $Z(M)$ is the space of 1-cocycles, which are functions $u : G \to M$ satisfying the *cocycle condition*:

$$u(gh) = g \cdot u(h) + u(g)$$

  and $B(M)$ is the space of 1-coboundaries, which are the cocycles that have the form

$$u(g) = g \cdot m - m$$

  for a fixed $m \in M$. All the groups $\mathrm{H}^i(G,M)$ have a description in terms of i-cocycles and i-coboundaries, which are functions that depend on i variables. However we will only use this explicit description for $i = 1$.

1.3.2 Some results in Galois cohomology

In this section we turn to Galois cohomology. We employ the term Galois cohomology to refer to group cohomology when the group G involved is a Galois group.

The most natural action involving Galois groups is the action of $\mathrm{Gal}(L/K)$ on L for a Galois extension L/K. It follows from the properties of cohomology groups studied above that $\mathrm{H}^0(\mathrm{Gal}(L/K), L) = K$ and $\mathrm{H}^0(\mathrm{Gal}(L/K), L^\times) = K^\times$.

For the first cohomology groups, we have the well known Hilbert's Theorem 90.

Theorem 1.3.1 (Hilbert's Theorem 90)**.** *Let L/K be a Galois extension. Then the groups $\mathrm{H}^1(\mathrm{Gal}(L/K), L^\times)$ and $\mathrm{H}^1(\mathrm{Gal}(L/K), L)$ are both trivial.*

This describes H^0 and H^1 for the canonical action of $\mathrm{Gal}(L/K)$ for general Galois extensions L/K. For higher cohomology groups, we turn our attention to some specific fields.

Definition. A **local field** is a locally compact topological field, with respect to a non-discrete topology. A **global field** is either a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}[[T]]$.

The local fields we are going to deal with include the fields $\mathbb{Q}_\ell$ for ℓ prime, their extensions, $\mathbb{R}$ and $\mathbb{C}$. The global fields of our interest are the finite extensions of $\mathbb{Q}$.

For a field K, one can define the *Brauer group* of K as $\mathrm{Br}(K) = \mathrm{H}^2(G_K, (K^s)^\times)$. When K is a local field one has an isomorphism

$$inv_K : \mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}$$

which is called the Hasse invariant. The following is a deep result in the area.

Theorem 1.3.2. *Let F be a global field. For each place v of F consider the Hasse invariant isomorphism $inv_v : Br(F_v) \to \mathbb{Q}/\mathbb{Z}$. There is an exact sequence*

$$0 \to Br(F) \to \bigoplus_v Br(F_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

where the first arrow is induced by restriction and the second one by the morphisms inv_v.

We end this subsection with the most used tool along the lines of this thesis. These are local and global duality and product formulas for both local and global settings.

Definition. Let K be a local field. For a G_K-modulo M we define the dual M^* as $M^* = \mathrm{Hom}(M, \mu)$ where μ is the group of roots of unity in K^s.

Local duality and product formula: Let K be an extension of $\mathbb{Q}_\ell$, with ℓ prime. For any G_K-modulo M there is a pairing $M \times M^* \to \mu$ and the cup product gives a pairing

$$\mathrm{H}^i(G_K, M) \times \mathrm{H}^{2-i}(G_K, M^*) \to H^2(G_K, \mu) \simeq \mathbb{Q}/\mathbb{Z}.$$

Theorem 1.3.3 (Local Tate duality)**.** *This pairing is perfect and makes $\mathrm{H}^i(G_K, M)$ and $\mathrm{H}^{2-i}(G_K, M^*)$ dual of one another.*

Proof. For a proof of this result see Section 2 on Chapter 7 of [NSW00]. □

The other important result is a product formula for the dimensions of the cohomology groups. Define the Euler-Poincaré characteristic of a G-module as

$$\Xi(M) = \frac{\#\,\mathrm{H}^0(G_K, M)\#\,\mathrm{H}^2(G_K, M)}{\#\,\mathrm{H}^1(G_K, M)}.$$

Theorem 1.3.4. *If the order of M is prime to the characteristic of K we have that $\Xi(M) = \|\#M\|_K$, where $\|\cdot\|_K$ is the normalized absolute value in K.*

Proof. For a proof see Chapter 7 Section 2 of [NSW00]. □

Global duality: In the global case, duality is a little more involved. We introduce some special groups that appear when considering the restriction of a global representation so finite places. Here F is a global field.

Definition. Let M be a finite G_F-module and S a finite set of primes of $\mathbb{Q}$. For $i = 1, 2$ we define the Shafarevich group $\text{Ш}^i_S(M)$ as the kernel of the restriction morphism from F to F_ℓ for $\ell \in S$. In other words

$$\text{Ш}^i_S(M) = \text{Ker}\left(\mathrm{H}^i(G_F, M) \to \bigoplus_{\ell \in S} \mathrm{H}^i(G_\ell, M)\right).$$

For global fields, we have a duality result relating Shafarevich groups for M and M^*. This result is known as Poitou-Tate duality and is obtained as a part of a 9-term exact sequence.

Theorem 1.3.5 (Poitou-Tate duality)**.** *Let M be a finite G_F-module. There is a perfect pairing*

$$\text{Ш}^1_S(M^*) \times \text{Ш}^2_S(M) \to \mathbb{Q}/\mathbb{Z}$$

which makes $\text{Ш}^1_S(M^*)$ and $\text{Ш}^2_S(M)$ dual of one another.

A reference for this result is Section 6 of Chapter 8 of [NSW00]. We remark that from the Poitou-Tate exact sequence, a global Euler-Poincaré characteristic can be deduced. For the interested reader we refer to Section 8 of Chapter 7 of [NSW00].

Besides giving the powerful Poitou-Tate duality, the Poitou-Tate 9-term exact sequence can be used to prove another valuable result to this work. Let $\mathcal{P} = (\mathcal{P}_\ell)_{\ell \in S}$ be a collection of local deformation conditions with tangent space $\mathrm{H}^1_{\mathcal{P}}(G_S, Ad^0\bar{\rho})$ given by the collection of local tangent spaces

$$L_\ell = \mathrm{H}^1_{\mathcal{P}_\ell}(G_\ell, Ad^0\bar{\rho}) \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho}).$$

For each $\ell \in S$, let $L^\perp_{\mathcal{P}_\ell} \subseteq \mathrm{H}^1(G_S, (Ad^0\bar{\rho})^*)$ be the annihilator of $L_{\mathcal{P}_\ell}$ with respect to the Tate duality pairing. This collection of local subspaces defines a subspace of the global cohomology group

$$\mathrm{H}^1_{\mathcal{P}^\perp}(G_S, (Ad^0\bar{\rho})^*) = \text{Ker}(\mathrm{H}^1(G_S, (Ad^0\bar{\rho})^*) \to \bigoplus_{\ell \in S} \mathrm{H}^1(G_\ell, (Ad^0\bar{\rho})^*))/L^\perp_{\mathcal{P}_\ell}$$

which we will call the dual condition of $\mathcal{P}$. A result of Wiles (Proposition 1.6 of [Wil95]) relates the dimensions of $\mathrm{H}^1_{\mathcal{P}}(G_S, Ad^0\bar{\rho})$ and $\mathrm{H}^1_{\mathcal{P}^\perp}(G_S, (Ad^0\bar{\rho})^*)$ with a series of local contributions.

Proposition 1.3.6 (Wiles product formula)**.** *Let M be a finite dimensional $\mathbb{F}$-vector space on which $G_{\mathbb{Q}}$ acts, S a set of primes containing the set of ramification of M and for each $\ell \in S$ let $L_\ell \subseteq \mathrm{H}^1(G_S, Ad^0\bar{\rho})$ be a subspace. Denote by $L^\perp_\ell \subseteq \mathrm{H}^1(G_S, M^*)$ the annihilator of L_ℓ with respect to the Tate duality pairing and by $\mathrm{H}^1_L(G_S, M) \subseteq \mathrm{H}^1(G_S, M)$ and $\mathrm{H}^1_{L^\perp}(G_S, M^*) \subseteq \mathrm{H}^1(G_S, M^*)$ the kernels of the restrictions to $\oplus_{\ell \in S} \mathrm{H}^1(G_\ell, M)/L_\ell$ and $\oplus_{\ell \in S} \mathrm{H}^1(G_\ell, M^*)/L^\perp_\ell$ respectively. Then we have that*

$$\begin{aligned}\dim \mathrm{H}^1_L(G_S, M) - \dim \mathrm{H}^1_{L^\perp}(G_S, M^*) = \\ \dim \mathrm{H}^0(G_S, M) - \dim \mathrm{H}^0(G_S, M^*) + \sum_{\ell \in S} \dim L_\ell - \dim \mathrm{H}^0(G_\ell, M).\end{aligned}$$

1.4 Hida Theory

The last ingredient of this thesis is the theory of Hida families. This is a rich theory that crosses through the theory of p-adic modular forms. A full exposition on the subject would take too long and therefore we only make a concise statement of the main definitions and results needed in the work.

In [Ser73] Serre introduces the concept of p-adic modular form. Later on, Katz generalized this concept but we present Serre's version here as it is more than enough for developing the tools we need.

Definition. A p-adic modular form is a formal power series $\sum a_n q^n \in \overline{\mathbb{Q}_p}[[q]]$ such that there exists a sequence (f_n) of classical modular forms such that $v_p(f_n - f)$ tends to 0 as n tends to infinity.

Together with this definition, Serre also introduced the notion of weight for such a form. This is an element of a weight space which we define below.

Definition. We define Serre's weight space as the space $\mathcal{X} = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$. The integer weights form a dense subspace of $\mathcal{X}$ identifying an element $k \in \mathbb{Z}$ with the morphism $x \mapsto x^k$.

The following proposition, also from [Ser73], ensures that a p-adic modular form has a well defined weight inside Serre's weight space.

Proposition 1.4.1. *Let f be a p-adic modular form and f_n a sequence of classical modular forms that converges to f. Let k_n be the weight of f_n. Then the sequence k_n converges in $\mathcal{X}$, and the limit does not depend on the sequence f_n.*

Finally, a salient feature of p-adic modular forms, that will make them appear in our more algebraic setting, is that we can define Hecke operators on them, and whenever a form is an eigenform for them, as classical modular forms, it has a Galois representation attached. We cite the precise result below (a proof by Hida can be found in [Hid89]).

Theorem 1.4.2. *Let $f = \sum a_n q^n$ be a p-adic modular form of weight $(\alpha, \kappa) \in \mathbb{Z}_p/(p-1)\mathbb{Z}_p \times Z_p$ and level N which is an eigenform for the Hecke operators. Then there exists a continuous Galois representation*

$$\rho_{f,p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$$

such that $Tr(Frob_q) = a_q$ for every $q \nmid Np$ and $det(\rho_{f,p}) = \psi\chi^\kappa$ where ψ is a character of finite order.

To end this chapter we present the theory of Hida families. Hida proved that classical modular forms satisfying certain properties come in families of p-adic modular forms. Moreover, in the sight of the previous results this proves the existence of families of Galois representations. We introduce the main definitions and results.

Definition. A modular form $f \in S_k(N)$ is said to be ordinary at a prime p if $p \nmid N$ and $p \nmid a_p$.

Hida's main result states that every ordinary modular form is part of a family of p-adic forms known as its Hida family. We introduce the concept of Hida family and state the main result of Hida (proved in [Hid86a], [Hid86b]) to end this chapter.

Definition. A Hida family is a formal power series

$$f_\infty = \sum a_n q^n$$

where the coefficients a_n are analytical functions defined on an open subset $U \subseteq \mathcal{X}$ and such that whenever $k \in U \cap \mathbb{Z}_{\geq 2}$ is a classical weight, then the series

$$f_k = \sum a_n(k)q^n$$

is a classical normalized ordinary eigenform of weight k and level N.

Theorem 1.4.3. *Let $k \geq 2$ be an integer, $f \in S_k(N)$ and p be a prime such that f is ordinary at p. Then there exists an unique Hida family f_∞ such that f is the weight k specialization of f_∞.*

Clasificación de representaciones de Galois locales y tipos de reducción

Introducción

El objetivo de este capítulo es la obtención de resultados que serán necesarios para el desarrollo de los próximos dos capítulos. Como fue mencionado en la introducción, ambos capítulos se basan en el argumento inductivo de [Ram99] y [Ram02], que esencialmente involucra dos pasos, primero la resolución de un problema local y luego otro de naturaleza global.

Los contenidos del presente capítulo incluyen las herramientas básicas necesarias para atacar el problema local. Recordemos que este problema involucra encontrar, para una deformación $\rho_n : G_S \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$, y para cada primo ℓ en un conjunto P que contiene a S, una familia C_ℓ de representaciones con coeficientes en $\mathcal{O}$ que sean levantados de ρ_n y contengan a una representación dada ρ_ℓ, y un subespacio $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\overline{\rho})$ que preserve a C_ℓ en el siguiente sentido: cada vez que se tiene una representación ρ, que es la reducción módulo π^m de algún elemento de C_ℓ y un elemento $u \in N_\ell$ se tiene que $(1+\pi^{m-1}u)\rho$ también es reducción de un elemento de C_ℓ.

Dado que las representaciones ρ_n y ρ_ℓ no tienen ninguna propiedad particular y el conjunto de levantados de una deformación dada $\overline{\rho}$ varía según la deformación elegida, necesitamos comenzar nuestro argumento con una clasificación de deformaciones locales. Observemos que la solución al problema que estamos considerando solo depende de la deformación residual $\overline{\rho} = \rho_n \pmod{\pi}$ y de la deformación a característica cero ρ_ℓ, basta resolver el problema para cada posible par $(\overline{\rho}|_{G_\ell}, \rho_\ell)$. Más aun, si consideramos otro par $(\overline{\rho}'|_{G_\ell}, \rho'_\ell)$ tal que $\overline{\rho}'|_{G_\ell}$ es isomorfa a $\overline{\rho}|_{G_\ell}$ y ρ'_ℓ es isomorfa a ρ_ℓ sobre $\mathcal{O}$ entonces una solución para $(\overline{\rho}'|_{G_\ell}, \rho'_\ell)$ se puede obtener a partir de una para $(\overline{\rho}|_{G_\ell}, \rho_\ell)$ componiendo con los isomorfismos en cuestión. Esto nos dice que es suficiente resolver el problema local para pares formados por representantes de las clases de isomorfismo de deformaciones con coeficientes en $\mathbb{F}$ y representantes de las clases de isomorfismo de deformaciones con coeficientes en $\mathcal{O}$ módulo equivalencia sobre $\mathcal{O}$.

Tomando esto en consideración, el objetivo de este capítulo es obtener una clasificación de deformaciones con coeficientes en $\mathbb{F}$ módulo isomorfismo y una clasificación de deformaciones con coeficientes en $\mathcal{O}$ módulo isomorfismo sobre $\mathcal{O}$, y estudiar que miembros de la segunda pueden reducir a que miembros de la primera. Varios de estos resultados son conocidos pero la intención es incluir un estudio breve pero autocontenido del problema. Finalmente, sobre el final del capítulo, calculamos las dimensiones de los grupos de cohomología involucrados en el problema local para cada una de las posibles clases de isomorfismo para $Ad^0\bar{\rho}$ previamente calculadas.

Chapter 2

Classification of local Galois representations and types of reduction

2.1 Introduction

This chapter deals with some results that are needed for the development of the next two chapters. As mentioned in the introduction, both rely on the inductive argument of [Ram99] and [Ram02], which essentially involves two main steps, one that amounts for solving a local problem and another where this problem is of global nature.

The contents of this chapter are the basic tools needed for attacking the local problem. For a given deformation $\rho_n : G_S \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$, the local problem involves finding, for each prime ℓ in a given set P containing S, a family C_ℓ of representations with coefficients in characteristic 0 rings, which are lifts of the restriction of our modulo π^n representation and contain a given representation ρ_ℓ, and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\overline{\rho})$ of certain dimension that preserves the family C_ℓ in the following way: for any representation ρ which is the mod π^m reduction of some element in C_ℓ and any element $u \in N_\ell$ we have that $(1 + \pi^{m-1}u)\rho$ is also the reduction of some representation of the family C_ℓ.

As we do not have any information about the representation ρ_n or the local lift ρ_ℓ and the set of possible lifts of a given deformation $\overline{\rho}$ vary heavily according to the deformation considered, we need to start our approach to this problem with a classification of local deformations. Observe that the solution to the problem we are considering only depends on the modulo π deformation $\overline{\rho}$ and the deformation to characteristic 0 ρ_ℓ. If we solve the problem for each possible pair $(\overline{\rho}|_{G_\ell}, \rho_\ell)$ we are done. Moreover, if we consider another pair $(\overline{\rho}'|_{G_\ell}, \rho'_\ell)$ such that $\overline{\rho}'|_{G_\ell}$ is isomorphic to $\overline{\rho}|_{G_\ell}$ and ρ'_ℓ is isomorphic to ρ_ℓ over $\mathcal{O}$ then one can obtain a set and subspace for the new pair by composing with the isomorphism. Therefore, it will be enough to solve the local problem for pairs formed by representatives of the isomorphism classes of modulo π representations and representatives of the isomorphism classes for representations with coefficients in $\mathcal{O}$, under the equivalence of isomorphism over $\mathcal{O}$.

Taking all this into consideration, the objective of this chapter is to get a classification of deformations with coefficients in $\mathbb{F}$ up to isomorphism, a classification of deformations with coefficients in $\mathcal{O}$ modulo $\mathcal{O}$-equivalence, and to study which members of the second one can reduce to which members of the first one. Many of these results are well known but we include here a self contained study. Moreover, in the end of the chapter, we compute dimension of the cohomology groups involved for each possible adjoint representation $Ad^0\bar{\rho}$ considered.

The results of this chapter will be essential for the development of the arguments in chapters 3 and 4.

2.2 Classification of residual representations

We start this chapter by recalling the classification of mod π representations of G_ℓ, when $\ell \neq p$ and $\ell \neq 2$ (see for example [DS05], Section 2).

Proposition 2.2.1. *Let $\ell \neq 2$, be a prime number, with $\ell \neq p$. Then every representation $\overline{\rho} : G_\ell \to \mathrm{GL}_2(\mathbb{F})$, up to twist by a character of finite order, belongs to one of the following three types:*

- **Principal Series:** $\overline{\rho} \simeq \left(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *or* $\overline{\rho} \simeq \left(\begin{smallmatrix} 1 & \psi \\ 0 & 1 \end{smallmatrix}\right)$.
- **Steinberg:** $\overline{\rho} \simeq \left(\begin{smallmatrix} \chi & \mu \\ 0 & 1 \end{smallmatrix}\right)$, *where* $\mu \in \mathrm{H}^1(G_\ell, \mathbb{F}(\chi))$ *and* $\mu|_{I_\ell} \neq 0$.
- **Induced:** $\overline{\rho} \simeq \mathrm{Ind}_{G_M}^{G_\ell}(\xi)$, *where $M/\mathbb{Q}_\ell$ is a quadratic extension and $\xi : G_M \to \mathbb{F}^\times$ is a character not equal to its conjugate under the action of* $\mathrm{Gal}(M/\mathbb{Q}_\ell)$.

Here $\phi : G_\ell \to \mathbb{F}^\times$ is a multiplicative character and $\psi : G_\ell \to \mathbb{F}$ is an unramified additive character.

Remark. Any unramified representation is Principal Series, and can be of the form $\overline{\rho} \simeq \left(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with ϕ unramified or of the form $\overline{\rho} \simeq \left(\begin{smallmatrix} 1 & \psi \\ 0 & 1 \end{smallmatrix}\right)$, with $\psi : G_\ell \to \mathbb{F}$ an additive unramified character.

The same classification applies for representations $\tilde{\rho} : G_\ell \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$, but since we need to study reductions modulo powers of a prime, we need to look at representations with integer coefficients modulo $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ equivalence. Let L be the coefficient field of ρ, $\mathcal{O}_L$ its ring of integers, and π be a local uniformizer. Also let $\mu \in \mathrm{H}^1(G_\ell, \mathbb{Z}_p(\chi))$ be a generator of such $\mathbb{Z}_p$-module.

Proposition 2.2.2. *Let $\rho : G_\ell \to \mathrm{GL}_2(\overline{\mathbb{Z}_p})$ be a continuous representation. Then up to twist (by a finite order character times powers of the cyclotomic one) and $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ equivalence we have:*

- **Principal Series:** $\rho \simeq \begin{pmatrix} \phi & \pi^r(\phi-1) \\ 0 & 1 \end{pmatrix}$, *with $r \in \mathbb{Z}_{\leq 0}$ satisfying $\pi^r(\phi - 1) \in \overline{\mathbb{Z}_p}$ or* $\rho \simeq \left(\begin{smallmatrix} 1 & \psi \\ 0 & 1 \end{smallmatrix}\right)$.
- **Steinberg:** $\rho \simeq \left(\begin{smallmatrix} \chi & \pi^r\mu \\ 0 & 1 \end{smallmatrix}\right)$, *with* $r \in \mathbb{Z}_{\geq 0}$.
- **Induced:** *There exists a quadratic extension $M/\mathbb{Q}_\ell$ and a character $\xi : G_M \to \overline{\mathbb{Z}_p}^\times$ not equal to its conjugate under the action of $\mathrm{Gal}(M/\mathbb{Q}_\ell)$ such that $\rho \simeq \langle v_1, v_2 \rangle_{\mathcal{O}_L}$, where for α a generator of $\mathrm{Gal}(M/\mathbb{Q}_\ell)$ and $\beta \in G_M$, the action is given by*

$$\beta(v_1) = \xi(\beta)v_1, \quad \beta(v_2) = \xi^\alpha(\beta)v_2, \quad \alpha(v_1) = v_2 \quad \textit{and} \quad \alpha(v_2) = \xi(\alpha^2)v_1,$$

or

$$\rho(\beta) = \begin{pmatrix} \xi(\beta) & \frac{\xi(\beta)-\xi^\alpha(\beta)}{\pi^r} \\ 0 & \xi^\alpha(\beta) \end{pmatrix} \qquad \textit{and} \qquad \rho(\alpha) = \begin{pmatrix} -a & \frac{\xi(\alpha^2)-a^2}{\pi^r} \\ \pi^r & a \end{pmatrix}$$

where ξ^α is the character of G_M defined by $\xi^\alpha(g) = \xi(\alpha g \alpha^{-1})$ and $a \in \mathcal{O}_L^\times$. Observe that when $M/\mathbb{Q}_\ell$ is ramified we can take α and β to be a Frobenius element and a generator of the tame inertia of G_ℓ respectively.

Proof. We first consider the case where ρ is irreducible over $\overline{\mathbb{Q}_p}$. In this case the representation is induced, and in the coefficient field L, the canonical basis is $\{v_1, v_2\}$, where $v_2 = \alpha(v_1)$ for α a generator of $\mathrm{Gal}(M/\mathbb{Q}_\ell)$. Let T be an invariant lattice for ρ. There exists a least $s \in \mathbb{Z}$ such that $w_1 = \pi^s v_1 \in T$. Rescaling T we can assume that $s = 0$ (rescaling the lattice does not affect the representation). Since $\alpha(T) \subseteq T$, $v_2 = \alpha(v_1) \in T$. Since $\alpha(v_2) = \xi(\alpha^2)v_1$, with $\xi(\alpha^2) \in \mathcal{O}_L^\times$, 0 is also the least integer such that $\pi^s v_2 \in T$, and therefore $\langle v_1, v_2\rangle_{\mathcal{O}_L} \subseteq T$. If this inclusion is an equality we are in the first case of our classification.

Otherwise, as the least $s \in \mathbb{Z}$ such that $\pi^s v_1 \in T$ is 0, we can extend v_1 to a basis of T by adding a vector $w \in T$ such that $w \notin \langle v_1, v_2\rangle_{\mathcal{O}_L}$. We can write this element as $w = \lambda_1 v_1 + \lambda_2 v_2$. Notice that necessarily $v_\pi(\lambda_1) = v_\pi(\lambda_2) < 0$. Changing v_1 and v_2 by a unit we can assume that $w = \pi^{-r}(-av_1 + v_2)$, with $r < 0$. Using $\alpha(v_1) = v_2$ and $\alpha(v_2) = \xi(\alpha^2)v_1$ we can compute the matrix of α in the basis v_1, w and we get

$$\rho(\alpha) = \begin{pmatrix} -a & \pi^{-r}(\xi(\alpha^2) - a^2) \\ \pi^r & a \end{pmatrix}.$$

The action of β follows from a similar computation.

On the other hand, if ρ is reducible over $\overline{\mathbb{Q}_p}$, we can chose an eigenvector inside our lattice, and extend it to a basis so that our representation is of the form (up to twist)

$$\rho \simeq \left(\begin{smallmatrix} \phi & * \\ 0 & 1 \end{smallmatrix}\right).$$

If ϕ is trivial, then $*$ is an additive character, and we are in the first case. Otherwise, if ρ is principal series, it is equivalent (modulo $\mathrm{GL}_2(L)$) to $\left(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, hence is of the form $\left(\begin{smallmatrix} \phi & u(\phi-1) \\ 0 & 1 \end{smallmatrix}\right)$. Since we want our representation to have integral coefficients we get the stated result. Finally, in the Steinberg case, our representation is $\mathrm{GL}_2(L)$-equivalent to $\left(\begin{smallmatrix} \chi & \mu \\ 0 & 1 \end{smallmatrix}\right)$, but an easy computation shows that such a representation is of the desired form as well. $\square$

Remark. In the Principal Series case, if we put $r = 0$ we get $\rho \simeq \left(\begin{smallmatrix} \phi & \phi-1 \\ 0 & 1 \end{smallmatrix}\right)$, which is equivalent to $\left(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we will make repeated use of this last representative for this class.

2.3 Types of reduction

In this section, we want to study which are the possible reductions from classes of $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$-equivalent representations to classes of representations with coefficients in $\mathrm{GL}_2(\overline{\mathbb{F}_p})$. Although this is well known to experts, and most of the claims are in [Car89], the change of types are not explicitly described in that article, so we just give a short self contained description.

Recall the condition for a character to lose ramification:

Lemma 2.3.1. *Let $\xi : G_\ell \to \overline{\mathbb{Q}_p}^\times$ a character and $\overline{\xi}$ its mod p reduction. If $\mathrm{Ker}(\xi|_{I_\ell}) \subsetneq \mathrm{Ker}(\overline{\xi}|_{I_\ell})$ then $\ell \equiv 1$ (mod p).*

Proof. Local class field theory identifies:

$$\begin{tikzcd} I_\ell^{ab} \arrow[r, "\sim"] \arrow[d] & \mathbb{Z}_\ell^\times \arrow[d] \\ G_\ell^{ab} \arrow[r, "\sim"] & \mathbb{Q}_\ell^\times \end{tikzcd}$$

So, if $L/\mathbb{Q}_p$ is a finite extension such that $\mathrm{Im}(\xi) \subseteq L^\times$ and $\pi \in \mathcal{O}_L$ is an uniformizer, we have

$$\begin{tikzcd} \operatorname{Ker}\overline{\xi} \arrow[r, "\xi|"] \arrow[d] & 1+\pi\mathcal{O}_L \arrow[d] \\ \mathbb{Z}_\ell^\times \arrow[r, "\xi"] \arrow[dr, "\overline{\xi}"'] & \mathcal{O}_L^\times \arrow[d] \\ & \mathbb{F}_L^\times \end{tikzcd}$$

The condition $\mathrm{Ker}(\xi|_{I_\ell}) \subsetneq \mathrm{Ker}(\overline{\xi}|_{I_\ell})$ is equivalent to $\xi|$ being non-trivial. As $1+\pi\mathcal{O}_L$ is a pro-p-group, $\xi|$ must vanish inside $1+\ell\mathbb{Z}_\ell$, which is a pro-ℓ-group, so $\xi|$ induces a non-trivial morphism:

$$\frac{\ker\overline{\xi}}{\ker\overline{\xi}\cap(1+\ell\mathbb{Z}_\ell)} \longrightarrow 1+\pi\mathcal{O}_L.$$

As $1+\pi\mathcal{O}_L$ is a pro-p-group, and $\ker\overline{\xi}/\ker\overline{\xi}\cap(1+\ell\mathbb{Z}_\ell) \subseteq \mathbb{Z}_\ell^\times/(1+\ell\mathbb{Z}_\ell) \simeq \mathbb{F}_\ell^\times$ such a non-trivial morphism exists if and only if $p|\ell-1$. □

Remark. Whenever an element $g \in I_\ell$ satisfies that $\xi(g) \neq 1$ and $\overline{\xi}(g) = 1$ we necessarily have $\xi(g)^{\ell-1} = 1$.

Proposition 2.3.2. *Let ρ be as above, then we have the following types of reduction:*

- *If ρ is Principal Series, then $\overline{\rho}$ is Principal Series or Steinberg, and the latter occurs only when $\ell \equiv 1 \pmod p$.*
- *If ρ is Steinberg, then $\overline{\rho}$ is Steinberg or Principal Series, and the latter occurs only when $\overline{\rho}$ is unramified.*
- *If ρ is Induced, then $\overline{\rho}$ is Induced, Steinberg or an unramified Principal Series. For the last two cases we must have $\ell \equiv -1 \pmod p$.*

Proof. If ρ is reducible, its reduction cannot be irreducible, which already excludes the case of a Principal Series or a Steinberg reducing to an Induced one. Besides this trivial observation, we study each case in detail:

- ρ *Principal Series:* in this case $\rho \simeq \left(\begin{smallmatrix} \phi & \lambda(\phi-1) \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & \phi \\ 0 & 1 \end{smallmatrix}\right)$. If $\tilde{\rho} \simeq \left(\begin{smallmatrix} \phi & \lambda(\phi-1) \\ 0 & 1 \end{smallmatrix}\right)$, the uniqueness of the semisimplification of the reduction implies that $\overline{\rho}^{ss} \simeq \left(\begin{smallmatrix} \overline{\phi} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If the reduction is of Steinberg type we need to have $\overline{\phi} = \chi$, so a character is losing ramification and this implies (by Lemma 2.3.1) that $\ell \equiv 1 \pmod p$.

  If $\rho \simeq \left(\begin{smallmatrix} 1 & \phi \\ 0 & 1 \end{smallmatrix}\right)$ then it is unramified and so is its reduction, implying that it can only be Principal Series.

- ρ *Steinberg:* in this case $\rho \simeq \left(\begin{smallmatrix} \chi & \lambda u \\ 0 & 1 \end{smallmatrix}\right)$ where $u \in \mathrm{H}^1(G_\ell, \mathbb{Z}_p(\chi))$ is the generator of the group. Its semisimplification is $\left(\begin{smallmatrix} \chi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which implies that if $\overline{\rho}$ is Principal Series then it is unramified.
- ρ *Induced:* in this case $\rho = \mathrm{Ind}_{G_M}^{G_{\mathbb{Q}_\ell}}(\xi)$, where $M/\mathbb{Q}_\ell$ is a quadratic extension and ξ is a character of G_M that does not descend to $G_{\mathbb{Q}_\ell}$. If the character $\overline{\xi}$ does not descend, then $\overline{\rho}$ is also irreducible hence Induced.

  Now suppose that $\overline{\xi}$ does descend and, for a moment, that $\overline{\rho}$ ramifies (which implies, by assumption, that $Ad^0\overline{\rho}$ ramifies). In this case the type of ρ changes when reducing. The semisimplification of the reduction we are considering is therefore
  $$\overline{\rho}^{ss} \simeq \begin{pmatrix} \overline{\xi}\epsilon & 0 \\ 0 & \overline{\xi} \end{pmatrix} = \overline{\xi} \otimes \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix},$$
  where ϵ is the quadratic character associated to $M/\mathbb{Q}_\ell$.

  Now, if $\overline{\rho}$ is Principal Series, then ϵ has to be ramified, as we are assuming that $Ad^0\overline{\rho}$ is ramified at ℓ, so $M/\mathbb{Q}_\ell$ is ramified. We claim (and will prove in the next Lemma) that this case cannot happen, i.e. if $M/\mathbb{Q}_\ell$ is ramified, any character $\xi : G_M \to \mathbb{Z}_p^\times$ that does not extend to G_ℓ then its reduction does not extend to G_ℓ either. Then the only possibility left to study is when $\overline{\rho}$ is Steinberg. Observe that if this is the case, looking at the semisimplifications we see that $\epsilon = \chi$, which only happens when $M/\mathbb{Q}_\ell$ is unramified and $\ell = -1 \pmod p$. This finishes the case where $\overline{\rho}$ is ramified.

  If $\overline{\rho}$ is unramified then ϵ has to be unramified as well, hence $M/\mathbb{Q}_\ell$ is an unramified extension. In this case, using the same argument as in Lemma 2.3.1, we conclude that $\ell^2 \equiv 1 \pmod p$. It is easy to prove that if $\ell \equiv 1 \pmod p$ then the character ξ extends to G_ℓ, therefore we necessarily have $\ell \equiv -1 \pmod p$.

$\square$

Lemma 2.3.3. *Let $M/\mathbb{Q}_\ell$ be a quadratic ramified extension and $\xi : G_M \to \overline{\mathbb{Z}_p}^\times$ a character and $\overline{\xi}$ its reduction. If $\overline{\xi}$ extends to G_ℓ then ξ does as well.*

Proof. Let $L/\mathbb{Q}_p$ be a finite extension that contains the image of ξ, and π an uniformizer of this extension. Let $\alpha \in G_\ell$ be an element not in G_M and define $\xi^\alpha(x) = \xi(\alpha x \alpha^{-1})$. We know that ξ extends to G_ℓ if and only if $\xi = \xi^\alpha$.

Via local class field theory, the character ξ corresponds to a character ψ defined over $M^\times$ and ξ^α corresponds to $\psi^\alpha(x) = \psi(\alpha(x))$, so ξ extends to G_ℓ if and only if ψ factors through the norm map $N_{M/\mathbb{Q}_\ell} : M^\times \to \mathbb{Q}_\ell^\times$. Recall that by hypotheses $\psi = \psi^\alpha \pmod \pi$ and we want to prove that $\psi = \psi^\alpha$. Let $\overline{\phi}$ be the factorization of $\overline{\psi}$ through the norm map.

If we restrict to the inertia subgroup we have the following picture:

$$\begin{tikzcd}
\operatorname{Ker}\overline{\psi} \arrow[rr, "\psi|"] \arrow[d, "N"'] \arrow[dr] & & 1+\pi\mathcal{O}_L \arrow[d] \\
\operatorname{Ker}\overline{\phi} \arrow[urr, dashed, "\phi|"] \arrow[dr] & O_M^\times \arrow[r, "\psi"] \arrow[d, "N"'] \arrow[dr, "\overline{\psi}"'] & \mathcal{O}_L^\times \arrow[d] \\
& \mathbb{Z}_\ell^\times \arrow[r, "\overline{\phi}"'] & \mathbb{F}_L^\times
\end{tikzcd}$$

We are going to construct the dashed arrow $\phi|$ of the diagram above. Observe that $\psi|$ factors through $\operatorname{Ker}\overline{\psi}/(\operatorname{Ker}\overline{\psi}\cap(1+\ell\mathbb{Z}_\ell))\subseteq\mathbb{F}_\ell^\times$ (since $1+\pi\mathcal{O}_L$ is a pro-p-group) so we have

$$\begin{tikzcd}
\operatorname{Ker}\overline{\psi} \arrow[r] \arrow[d, "N"'] & \frac{\operatorname{Ker}\overline{\psi}}{\operatorname{Ker}\overline{\psi}\cap(1+\ell\mathbb{Z}_\ell)} \arrow[r, "\psi|"] \arrow[d, "f"'] & 1+\pi O_L \\
\operatorname{Ker}\overline{\phi} \arrow[r] & \frac{\operatorname{Ker}\overline{\phi}}{\operatorname{Ker}\overline{\phi}\cap(1+\ell\mathbb{Z}_\ell)} \arrow[ur, dashed, "\phi|"'] &
\end{tikzcd}$$

where the down arrow f is $f(x)=x^2$ (since $M/\mathbb{Q}_\ell$ is ramified). So we can define the dashed arrow $\phi|$ as $\phi|(x)=\sqrt{\psi|(x)}$ where $\sqrt{\ }:1+\pi\mathcal{O}_L\to 1+\pi\mathcal{O}_L$ is the morphism that assigns to every $x\in 1+\pi\mathcal{O}_L$ its square root in $1+\pi\mathcal{O}_L$ (which exists and is unique by Hensel's Lemma). This makes the diagram commutative and proves that ϕ can be extended in $\operatorname{Ker}\overline{\phi}$.

Now we want to prove that ψ factors through the norm map. Define $\iota(x)=\psi^\alpha\psi^{-1}$. We know that $\iota:\mathcal{O}_M^\times\to 1+\pi\mathcal{O}_L$ and that $\iota(\operatorname{Ker}\overline{\xi})=1$. So it factors through $\overline{\iota}:\mathcal{O}_M^\times/\operatorname{Ker}\overline{\psi}\to 1+\pi\mathcal{O}_L$, but $\mathcal{O}_M^\times/\operatorname{Ker}\overline{\psi}\subseteq\mathbb{F}_L^\times$ and the only element of order p^n-1 inside $1+\pi\mathcal{O}_L$ is 1, so ι must be trivial and therefore $\psi=\psi^\alpha$ when restricted to $\mathcal{O}_M^\times$. In order to deduce $\psi=\psi^\alpha$ from this, we only need to check it for the uniformizer, which is $\sqrt{\delta\ell}$ with δ equal to 1 or to a non-square in $\mathbb{Q}_\ell$. We have:

$$\psi^\alpha(\sqrt{\delta\ell})=\psi(\alpha(\sqrt{\delta\ell}))=\psi(-\sqrt{\delta\ell})=\psi(-1)\psi(\sqrt{\delta\ell})=\psi(\sqrt{\delta\ell}).$$

The last equality follows from $\psi(-1)=\phi(N(-1))=\phi(1)=1$, because $-1\in\mathcal{O}_M^\times$. We have proved that ξ extends to G_ℓ. □

2.4 Local cohomological dimensions

To apply Ramakrishna's inductive method in our situation we need to compute the dimensions $d_i=\dim\mathrm{H}^i(G_\ell,Ad^0\bar{\rho})$ for $i=1,2$. The strategy in each case is as follows: we first compute d_0 and d_0^* (where $d_i^*=\dim\mathrm{H}^i(G_\ell,(Ad^0\bar{\rho})^*)$). By local Tate duality $d_2=d_0^*$ and then we can derive d_1 from the local Euler-Poincaré characteristic (which is zero as we are considering primes $\ell\neq p$). We end this chapter by doing this computation in each case of the classification of mod π representations by choosing a good basis for each space.

Ramified Principal Series case:

In this case we have $\bar{\rho}=\left(\begin{smallmatrix}\phi&0\\0&1\end{smallmatrix}\right)$ with ϕ a ramified multiplicative character. It easily follows that $Ad^0\bar{\rho}\simeq\mathbb{F}(1)\oplus\mathbb{F}(\phi)\oplus\mathbb{F}(\phi^{-1})$. As ϕ is ramified, $\mathbb{F}(\phi)$ (resp. $\mathbb{F}(\phi^{-1})$) is not isomorphic to $\mathbb{F}(1)$ nor $\mathbb{F}(\chi)$. So we have two cases:

(1) $\ell\equiv 1\pmod p$ then $d_0=1$, $d_2=1$ and therefore $d_1=2$.

(2) $\ell\not\equiv 1\pmod p$ then $d_0=1$, $d_2=0$ and therefore $d_1=1$.

Steinberg case:

In this case we need to do the computations by hand. Considering the basis $\{e_{01},e_{10},e_{00}+e_{11}\}$ of the space of matrices with trace zero and explicitly computing the action of $Ad^0\bar{\rho}$ on them, we derive the values of the numbers d_i, which are:

(1) If $\ell \equiv 1 \pmod p$ then $d_0 = 1$, $d_2 = 1$ and therefore $d_1 = 2$.

(2) If $\ell \equiv -1 \pmod p$ then $d_0 = 0$, $d_2 = 1$ and therefore $d_1 = 1$.

(3) If $\ell \not\equiv \pm 1 \pmod p$ then $d_0 = 0$, $d_2 = 0$ and therefore $d_1 = 0$.

Induced case:

Recall the following Lemma (see [Ram02], Lemma 4)

Lemma 2.4.1. *Let $M/\mathbb{Q}_\ell$ a quadratic extension and $\overline{\rho} : G_\ell \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}_p})$ be twist-equivalent to $\mathrm{Ind}_{G_M}^{G_\ell} \xi$, with ξ a character of G_M which is not equal to its conjugate under the action of* $\mathrm{Gal}(M/\mathbb{Q}_\ell)$.

Then $Ad^0\overline{\rho} \simeq A_1 \oplus A_2$, with A_i an absolutely irreducible G_ℓ-module of dimension i and $\mathrm{H}^0(G_\ell, Ad^0\overline{\rho}) = 0$. Moreover $\mathrm{H}^2(G_\ell, Ad^0\overline{\rho}) = 0$ unless $M/\mathbb{Q}_\ell$ is not ramified and $\ell \equiv -1 \pmod p$ in which case it is one dimensional.

So for the Induced case we have two possibilities:

(1) If $\ell \equiv -1 \pmod p$ and $M/\mathbb{Q}_\ell$ is unramified then $d_0 = 0$, $d_2 = 1$ and therefore $d_1 = 1$.

(2) If $\ell \not\equiv -1 \pmod p$ or $M/\mathbb{Q}_\ell$ is ramified then $d_0 = 0$, $d_2 = 0$ and therefore $d_1 = 0$.

Unramified case:

if $\overline{\rho}$ is unramified, we consider the following three cases according to the image of Frobenius:

(1) $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In this case $Ad^0\overline{\rho} \simeq \mathbb{F}^3$ thence we have two possibilities:

- $\ell \equiv 1 \pmod p$ then $d_0 = 3$, $d_2 = 3$ and therefore $d_1 = 6$.
- $\ell \not\equiv 1 \pmod p$ then $d_0 = 3$, $d_2 = 0$ and therefore $d_1 = 3$.

(2) $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \not\equiv 1 \pmod p$.

We have that $Ad^0\overline{\rho} \simeq \mathbb{F} \oplus \mathbb{F}(\phi) \oplus \mathbb{F}(\phi^{-1})$, with $\phi \neq 1$ and $\phi = \chi$ only if $\alpha \equiv \ell \pmod p$. Again, we need to distinguish between cases:

- $\ell \equiv -1 \pmod p$ and $\ell \equiv \alpha, \alpha^{-1} \pmod p$ then $d_0 = 1$, $d_2 = 2$ and therefore $d_1 = 3$.
- $\ell \equiv -1 \pmod p$ and $\ell \not\equiv \alpha, \alpha^{-1} \pmod p$ then $d_0 = 1$, $d_2 = 0$ and therefore $d_1 = 1$.
- $\ell \not\equiv -1 \pmod p$ and $\ell \equiv \alpha, \alpha^{-1}$ or $1 \pmod p$ then $d_0 = 1$, $d_2 = 1$ and therefore $d_1 = 2$.
- $\ell \not\equiv -1 \pmod p$ and $\ell \not\equiv \alpha, \alpha^{-1}$ or $1 \pmod p$ then $d_0 = 1$, $d_2 = 0$ and therefore $d_1 = 1$.

(c) $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Here we do the computations by hand and establish that:

- If $\ell \equiv 1 \pmod p$ then $d_0 = d_2 = 1$ and therefore $d_1 = 2$.
- If $\ell \not\equiv 1 \pmod p$ then $d_0 = 1$, $d_2 = 0$ and therefore $d_1 = 1$.

Levantando representaciones de Galois: el caso no ramificado

Introducción

El objetivo del presente capítulo es trabajar con congruencias entre formas modulares (y más en general, representaciones abstractas) modulo potencias de un primo p cuando dicho p no ramifica en el cuerpo de coeficientes. La estrategia es adaptar los argumentos de [Ram99] y [Ram02] a este nuevo contexto. Uno de los resultados principales del capítulo es el siguiente

Teorema A. *Sea $\mathbb{F}$ un cuerpo finito de característica $p \geq 5$. Sea $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ una representación continua ramificada en un conjunto finito de primos S satisfaciendo las siguientes propiedades:*

- *La imagen de $\overline{\rho_n}$ es grande, i.e. $\mathrm{SL}_2(\mathbb{F}) \subseteq \mathrm{Im}(\overline{\rho_n})$ y $\mathrm{GL}_2(\mathbb{F}) = \mathrm{Im}(\overline{\rho_n})$ si $p = 5$.*
- *ρ_n es impar.*
- *La restricción $\overline{\rho_n}|_{G_p}$ no es un twist de la representación trivial ni de la representación indecomponible no ramificada dada por $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.*
- *ρ_n no ramifica en 2.*

Sea P un conjunto de primos conteniendo a S y para cada $\ell \in P$, $\ell \neq p$, fijemos una deformación $\rho_\ell : G_\ell \to \mathrm{GL}_2(W(\mathbb{F}))$ de $\rho_n|_{G_\ell}$. En el primo p, sea ρ_p una deformación de $\rho_n|_{G_p}$ que es ordinaria o cristalina con pesos de Hodge-Tate $\{0, k\}$ para $2 \leq k \leq p-1$. Entonces existe un conjunto finito Q de primos auxiliares $q \not\equiv \pm 1 \pmod{p}$ y una representación modular

$$\rho : G_{P \cup Q} \to \mathrm{GL}_2(W(\mathbb{F})),$$

tal que:

- *La reducción módulo p^n de ρ es ρ_n.*
- *$\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$ para todo $\ell \in P$.*
- *$\rho|_{G_q}$ es ramificada y de tipo Steinberg para todo $q \in Q$.*

Este resultado, a diferencia de los resultados de Ramakrishna, es solo acerca de representaciones impares (y por lo tanto modulares en virtud de la Conjetura de Serre). En el caso par las mismas ideas junto con algunas hipótesis extra (como en [Ram02]) dan un resultado análogo para representaciones abstractas.

Observación. El Teorema A es un resultado en el mismo espíritu que el Teorema 3.2.2 de [BD]. En dicho trabajo se prueba que existe un levantado de una representación módulo p con propiedades prescritas en un conjunto finito de primos si se deja crecer el anillo de coeficientes. La ventaja del resultado citado es que no requiere la aparición de ramificación adicional, pero este fenómeno solo sucede al considerar levantados de una representación modulo p y no módulo p^n. Por ejemplo la curva elíptica $329a1$ es no ramificada en 7 módulo 9 pero no hay formas modulares nuevas de nivel 47 que sean congruentes a ella módulo 9 (ver [Dum05]).

Para $f \in S_k(\Gamma_0(N), \epsilon)$ $(k \geq 2)$ una forma nueva con cuerpo de coeficientes K_f, llamemos $\mathcal{O}_f$ al anillo de enteros de K_f. Si $p \in \mathbb{Z}$ es un primo, notemos por $\mathfrak{p}$ a un ideal de $\mathcal{O}_f$ dividiendo a p, por K_p la completación de K_f en $\mathfrak{p}$ y por $\mathcal{O}_\mathfrak{p}$ su anillo de enteros. Finalmente, sea $\rho_{f,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_\mathfrak{p})$ la representación p-ádica asociada a f. Para un entero positivo n sea

$$\rho_n : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$$

su reducción. Aplicando el Teorema A a esta representación obtenemos el otro resultado principal de este capítulo.

Teorema B. *En las hipótesis anteriores, sea $n > 0$ un entero y $p > max(k, 3)$ un primo tal que*

- *$p \nmid N$ o f es ordinaria en p,*
- *$\mathrm{SL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) \subseteq \mathrm{Im}(\overline{\rho_{f,p}})$ y $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) = \mathrm{Im}(\overline{\rho_{f,p}})$ si $p = 5$,*
- *p no ramifica en K_f,*
- *ρ_n no ramifica en 2.*

Sea R el conjunto de ramificación de ρ_n. Si $N' = \prod_{\ell \in R} \ell^{v_\ell(N)}$ entonces existe un entero r, un conjunto $\{q_1, \ldots, q_r\}$ de primos auxiliares coprimos con N satisfaciendo $q_i \not\equiv 1 \pmod{p}$ y una forma nueva g, diferente de f, de peso k y nivel $N'q_1 \ldots q_r$ tal que f y g son congruentes modulo p^n. Más aun, la forma g puede ser elegida con la misma restricción al subgrupo de inercia que f para todos los primos de R.

Manteniendo la notación del Teorema B, obtenemos las siguientes consecuencias.

Corolario (Bajada de nivel)**.** *Sea $f \in S_k(\Gamma_0(M), \epsilon)$ una forma nueva, $\mathfrak{p}$ un primo de $\mathcal{O}_f$ sobre $p \in \mathbb{Q}$ y $\rho_n : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_f/\mathfrak{p}^n)$ la reducción módulo p de la representación $\mathfrak{p}$-ádica asociada a f. Supongamos que:*

- *$p \geq 5$.*
- *$2 \leq k \leq p - 1$.*
- *$\mathrm{SL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) \subseteq \mathrm{Im}(\overline{\rho_{f,p}})$ y $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) = \mathrm{Im}(\overline{\rho_{f,p}})$ si $p = 5$.*
- *p no ramifica en K_f.*
- *ρ_n no ramifica en 2.*

Si $\ell \mid M$ es tal que ρ_n es no ramificada en ℓ, entonces el correspondiente mapa de Hecke se factoriza a través de $\mathbb{T}_k^{\ell-old}(M, \ell)$.

Demostración. La demostración consiste en combinar el Teorema B con el resultado para primos $\ell \not\equiv 1 \pmod{p}$ (probado en [Dum05], Teorema 1). Específicamente, el Teorema B aplicado a ρ_n entrega una forma nueva g, congruente módulo p^n con f, en la que los primos que pierden nivel al reducir módulo $\mathfrak{p}^n$ no son congruentes con 1 módulo p. Dicha forma entra en las hipótesis del Teorema de Dummigan, lo que da el resultado. □

Corolario. *Sea $k \geq 2$, N impar y $f \in S_k(\Gamma_0(N), \epsilon)$ una forma nueva sin multiplicación compleja ni twists internos. Entonces para todo primo p salvo finitos, y para todo entero positivo n, existe una forma nueva g, de peso k, diferente de f y congruente con f módulo p^n.*

Demostración. Al suponer que f no tiene multiplicación compleja ni twists internos, la imagen de $\rho_{f,p}$ es grande modulo p para todo p salvo finitos. Si descartamos los primos en los que la imagen no es grande, junto con los primos menores a k, los primos que ramifican en K_f, los primos en el nivel de f y los primos 2 y 3, nos encontramos en las hipótesis del Teorema B y el resultado sigue de su aplicación. □

En cuanto al trabajo realizado para la demostración de los resultados principales de este capítulo, el mismo se centró en adaptar las ideas de [Ram02] a nuestra situación. Esto significa que el argumento principal se encuentra dividido en dos partes. Por un lado, un argumento global para probar la existencia de primos auxiliares que permiten convertir el problema de levantar ρ_n a característica 0 en una serie de problemas de levantamiento similares, pero de naturaleza local (para restricciones $\rho_n|_{G_\ell}$). Por otro lado se deben resolver dichos problemas locales. Siguiendo la estructura lógica del argumento, en la que el argumento global depende de la resolución de los problemas locales, el capítulo comienza estudiando familias de representaciones de G_ℓ.

En una primera sección presentaremos la clasificación de deformaciones para grupos de Galois locales, tanto en característica 0 como módulo p, y sus posibles tipos de reducción.

En las siguientes dos secciones abordaremos el problema de construir familias C_ℓ de levantados de $\rho_n|_{G_\ell}$ de cierta dimensión, que nos permiten resolver el problema local. En dicha construcción se utiliza la clasificación obtenida en la sección anterior y se construye una familia C_ℓ para cada posible par de clases de isomorfismo de deformaciones en característica 0 y característica p. De este modo se cubren todos los posibles casos que pueden aparecer al tener que resolver el problema local para $\rho_n|_{G_\ell}$.

Las secciones 5 y 6 trataremos el problema global. En la primera de ellas, utilizando las familias C_ℓ construidas en la sección 4 se adapta el argumento global de [Ram02] a deformaciones módulo p^n. Se prueba la existencia de primos auxiliares que permiten resolver las posibles obstrucciones para levantar ρ_n. En la segunda, se resuelve un problema técnico que aparece en la construcción de las familias C_ℓ, en ciertos casos resulta necesario emplear otro método para superar una cantidad finita de potencias de p, antes de poder utilizar el argumento global de [Ram02]. Esto se logra con ideas similares a las de [KLR05], donde se dan argumentos por los que es posible superar las obstrucciones que aparecen al intentar pasar de módulo p^m a módulo p^{m+1} al costo de agregar finitos primos auxiliares en cada paso. Cabe destacar que este argumento no se puede utilizar infinitamente para obtener el levantado a característica 0 deseado, puesto que el levantado obtenido de este modo podría ramificar en infinitos primos.

Finalmente, en la sección 7 probaremos los teoremas principales, utilizando las herramientas construidas previamente. El capítulo termina con un ejemplo explícito en la sección

8, donde se sigue el argumento de la demostración del Teorema A para obtener un ejemplo de subida de nivel.

Chapter 3

Lifting Galois representations: the unramified case

3.1 Introduction

The aim of the present chapter is to deal with congruences between modular forms (and more generally, abstract representations) modulo powers of a prime p when p does not ramify in the coefficient field of those. The main strategy is to adapt the arguments of [Ram99] and [Ram02] to this new setting, which is harder due to semisimplification problems. One of the main results of this chapter is the following.

Theorem A. *Let $\mathbb{F}$ be a finite field of characteristic $p \geq 5$. Let $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ a continuous representation ramified at a finite set of primes S satisfying the following properties:*

- *The image of $\overline{\rho_n}$ is big, i.e. $\mathrm{SL}_2(\mathbb{F}) \subseteq \mathrm{Im}(\overline{\rho_n})$ and $\mathrm{Im}(\overline{\rho_n}) = \mathrm{GL}_2(\mathbb{F})$ if $p = 5$.*
- *ρ_n is odd.*
- *The restriction $\overline{\rho_n}|_{G_p}$ is not twist equivalent to the trivial representation nor the indecomposable unramified representation given by $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.*
- *ρ_n does not ramify at 2.*

Let P be a finite set of primes containing S, and for every $\ell \in P$, $\ell \neq p$, fix a deformation $\rho_\ell : G_\ell \to W(\mathbb{F})$ of $\rho_n|_{G_\ell}$. At the prime p, let ρ_p be a deformation of $\rho_n|_{G_p}$ which is ordinary or crystalline with Hodge-Tate weights $\{0, k\}$, with $2 \leq k \leq p-1$.

Then there is a finite set Q of auxiliary primes $q \not\equiv \pm 1 \pmod{p}$ and a modular representation

$$\rho : G_{P \cup Q} \longrightarrow \mathrm{GL}_2(W(\mathbb{F})),$$

such that:

- *the reduction modulo p^n of ρ is ρ_n,*
- *$\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$ for every $\ell \in P$,*
- *$\rho|_{G_q}$ is a ramified representation of Steinberg type for every $q \in Q$.*

This result, contrary to the results of Ramakrishna, is only about odd representations (and hence modular by Serre's conjectures). In the even case, the exact same ideas plus some extra hypotheses (as in [Ram99]) give a result for any abstract representation with big image.

Remark. Theorem A is in the same spirit as Theorem 3.2.2 of [BD], where they only consider residual representations, and allow the coefficient field to grow. The advantage of their method is that it does not require to add extra ramification (so $Q = \emptyset$), but this phenomena only works while working modulo a prime. For example, the elliptic curve $329a1$ is unramified at 7 modulo 9, but there are no newforms of level 47 congruent to it modulo 9 (see [Dum05]).

Let $f \in S_k(\Gamma_0(N), \epsilon)$ ($k \geq 2$) be a newform with coefficient field K_f and denote by $\mathcal{O}_f$ the ring of integers of K_f. If p is a prime number, let $\mathfrak{p}$ denote a prime ideal in $\mathcal{O}_f$ dividing p, $K_\mathfrak{p}$ the completion at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ its ring of integers. Finally let $\rho_{f,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_\mathfrak{p})$ denote its associated p-adic Galois representation. If n is a positive integer, let

$$\rho_n : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n)$$

be its reduction. Applying Theorem A to this representation, we are able to derive the other main result of this chapter.

Theorem B. *In the above hypothesis, let $n > 0$ be an integer and $p > max(k, 3)$ be a prime such that:*

- *$p \nmid N$ or f is ordinary at p,*
- *$\mathrm{SL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) \subseteq \mathrm{Im}(\overline{\rho_{f,p}})$, and $\mathrm{Im}(\overline{\rho_{f,p}}) = \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ if $p = 5$.*
- *p does not ramify in the field of coefficients of f.*
- *ρ_n does not ramify at 2.*

Let R be the set of ramified primes of ρ_n. If $N' = \prod_{\ell \in R} \ell^{v_\ell(N)}$, then there exist an integer r, a set $\{q_1, \ldots, q_r\}$ of auxiliary primes prime to N satisfying $q_i \not\equiv 1 \pmod p$ and a newform g, different from f, of weight k and level $N'q_1 \ldots q_r$ such that f and g are congruent modulo p^n. Furthermore, the form g can be chosen with the same restriction to inertia as that of f at the primes of R.

Keeping the same notation as in Theorem B, we get the following consequences.

Corollary 3.1.1 (Lowering the level)**.** *Let $f \in S_k(\Gamma_0(M), \varepsilon)$ be a newform, $\mathfrak{p}$ a prime of $\mathcal{O}_f$ above $p \in \mathbb{Q}$ and $\rho_n : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_f/\mathfrak{p}^n)$ be a modulo $\mathfrak{p}^n$ reduction of its $\mathfrak{p}$-adic representation. Suppose that:*

- *$p \geq 5$.*
- *$2 \leq k \leq p - 1$.*
- *$\mathrm{SL}_2(\mathcal{O}_f/\mathfrak{p}) \subseteq \mathrm{Im}(\overline{\rho_n})$ and $\mathrm{Im}(\overline{\rho_n}) = \mathrm{GL}_2(\mathcal{O}_f/\mathfrak{p})$ if $p = 5$.*
- *p does not ramify in $\mathcal{O}_f$.*
- *ρ_n does not ramify at 2.*

If $\ell \mid M$ is such that ρ_n is unramified at ℓ, then the Hecke map factors through the ℓ-old quotient $\mathbb{T}_k^{\ell\text{-}old}(M,\ell)$.

Proof. The proof consists on combining the result for primes $\ell \not\equiv 1 \pmod p$ (which was proved in [Dum05], Theorem 1), with Theorem B that allows us to move the ramified primes to a situation where we get more control on the extra Steinberg ramification. Specifically, if $\ell \equiv 1 \pmod p$, then by Theorem B, we can find a form g with the same ramification as f, but without ℓ in the level at the cost of adding many Steinberg primes $q \not\equiv 1 \pmod p$. But these extra primes in the level of the form g satisfy the hypotheses of Dummigan's Theorem, so we can remove them as well.

As it has been observed in the work of Dummigan (see section 9 of [Dum05]) Theorem 1 of that work needs a Galois representation as an input and returns a Hecke map where the level has one prime removed as an output. This implies that it is not possible to make repeated use of this theorem to remove the many auxiliary primes that appear after applying Theorem B. However, it is possible to modify Theorem 1 of [Dum05] in order for it to take a Hecke map as an input (again, see section 9 of [Dum05]). We thank Professor Dummigan for noticing the issue and suggesting the solution. □

Corollary 3.1.2. *Let $k \geq 2$, N odd and $f \in S_k(\Gamma_0(N),\epsilon)$ be a newform which has no complex multiplication nor inner twists. Then for all but finitely many prime numbers p, and for all positive integers n, there exists a weight k newform g (depending on p and n) different from f, which is congruent to f modulo p^n.*

Proof. Since our form does not have complex multiplication or inner twists, by Ribet's result ([Rib85], Theorem 3.1) the image is big modulo p for all but finitely many primes p. We avoid the primes without big image as well as those smaller than the weight. We also discard the primes p that ramify in the field of coefficients of f and the ones in the level (or the non-ordinary ones), and we are in the hypotheses of the previous Theorem. □

The proof of Theorem A follows the ideas of [Ram02]. This means that it is divided into two parts. On the one hand we need to add auxiliary primes that allow us to convert the problem of lifting a global representation into the one of lifting many local ones. On the other hand, we need to solve the local problems. Following the logical structure of [Ram02], we deal with the local considerations first.

In this case, we essentially have to prove Proposition 1.6 of [Ram02] in our setting. For every prime $\ell \in P$ we need to find a set C_ℓ of deformations of $\rho_n|_{G_\ell}$ to $W(\mathbb{F})$ containing ρ_ℓ and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ of certain dimension such that its elements preserve the reductions of N_ℓ, i.e. such that whenever ρ_m is the reduction of some $\rho \in C_\ell$ modulo p^m and $u \in N_\ell$ then $(1+p^{m-1}u)\rho_m$ is the reduction of some other $\rho' \in C_\ell$. In order to get the full statement of our Theorem A we also need all the deformations in C_ℓ to be isomorphic when restricted to I_ℓ.

To achieve this, we proceed in the following way:

- The hypotheses of Theorem A provide us with a mod p^n representation ρ_n and local representation ρ_ℓ lifting $\rho_n|_{G_\ell}$.

- For each pair of the isomorphism classes for ρ_ℓ and $\overline{\rho_n}$ given by the classification obtained in Chapter 2 we construct a set C_ℓ of deformations with coefficients in $W(\mathbb{F})$ which are congruent to ρ_ℓ mod p^n and the corresponding subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ preserving it. It is the isomorphism class of ρ_ℓ who determines which set C_ℓ we pick, and the isomorphism class of $\overline{\rho_n}|_{G_\ell}$ who determines the group $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ we have to work with.

In this way, we get the desired sets for every prime $\ell \in P$. Observe that we are constructing a pair (C_ℓ, N_ℓ) for each pair of isomorphism classes $(\rho_\ell, \overline{\rho}_n|_{G_\ell})$ and the deformation ρ_n does not seem to appear in this construction. This is just an illusion as we are taking C_ℓ as a set of deformations which are congruent to ρ_ℓ mod p^n, and $\rho_n|_{G_\ell}$ is precisely the mod p^n reduction of ρ_ℓ. By taking this approach we are bypassing the difficult problem of classifying mod p^n representations.

When trying to come along with this construction one flaw appears. There is one pair $\rho_n|_{G_\ell}$ for which one cannot find a pair (C_ℓ, N_ℓ) such that N_ℓ preserves the modulo p^m reductions of the elements of C_ℓ for every $m > n$ but only manage to find a pair that satisfies the desired property for all exponents $m > n_0 > n$. We overcome this problem by applying a different argument for lifting ρ_n to $W(\mathbb{F})/p^{n_0+1}$ with the desired local behavior and then we continue with the main argument of [Ram02]. This is addressed in Section 6.

Once we picked these local deformations classes and solved the issue (possibly) appearing in the first exponents, we need to construct two auxiliary sets of primes, Q_1 and Q_2 (these are Ramakrishna's Q and T) together with their respective sets C_q and subspaces N_q as for the primes in P, that satisfy the following conditions:

- The set Q_1 morally has two main properties (see Fact 16 [Ram02]): it kills the global obstructions, i.e. is such that $\text{III}^1_{S\cup Q_1}((Ad^0\bar{\rho})^*) = 0$ and therefore $\text{III}^2_{S\cup Q_1}(Ad^0\bar{\rho}) = 0$ by global duality, and the inflation map
$$\mathrm{H}^2(G_S, Ad^0\bar{\rho}) \to \mathrm{H}^2(G_{S\cup Q_1}, Ad^0\bar{\rho}),$$
is an isomorphism.

- The set Q_2 gives an isomorphism
$$\mathrm{H}^1(G_{S\cup Q_1\cup Q_2}, Ad^0\bar{\rho}) \to \bigoplus_{\ell\in S\cup Q_1\cup Q_2} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell.$$
without adding global obstructions, i.e. $\text{III}^2_{S\cup Q_1\cup Q_2} = 0$.

These auxiliary primes are essentially the same as in [Ram02], we use the same sets C_q and subspace N_q. We only need to have a little extra care when proving that $\rho_n|_{G_q}$ is the reduction of some $\rho \in C_q$ for every $q \in Q_1 \cup Q_2$.

Once we have solved the local problems and found the auxiliary primes, the inductive method starts to work. The key observation here is that this inductive step only depends on hypotheses about the mod p reduction of our representation, which tells us that no matter at which power of p we start lifting, it will work perfectly.

The inductive argument works as follows: in virtue of $\text{III}^2_{S\cup Q_1}(Ad^0\bar{\rho}) = 0$, a global deformation to $W(\mathbb{F})/p^m$ lifts to $W(\mathbb{F})/p^{m+1}$ if and only if its restrictions to the primes of $P \cup Q_1 \cup Q_2$ lift to $W(\mathbb{F})/p^{m+1}$. For $m = n$ the local condition is automatic so there exists a lift ρ_{n+1} of ρ_n to $W(\mathbb{F})/p^{n+1}$. The problem is that ρ_{n+1} may not lift again, as it can be locally obstructed. In order to remove these local obstructions we use the fact that any local deformation for primes $\ell \in P \cup Q_1 \cup Q_2$ can be modified by some element not in N_ℓ in order to be a reduction of some element of C_ℓ and therefore unobstructed. We will often refer to this as *adjusting a local deformation*. As we have an isomorphism between the global first cohomology group and the coproduct of the local first cohomology groups modulo N_ℓ, we can find an element $u \in \mathrm{H}^1(G_\mathbb{Q}, Ad^0\overline{\rho})$ that adjusts ρ_{n+1} locally for every prime in $P \cup Q_1 \cup Q_2$ making $(1+p^n u)\rho_{n+1}$ an unobstructed lift of ρ_n. From here we can repeat the process of lifting and adjusting indefinitely, finally getting a lift to $W(\mathbb{F})$.

Finally, to get Theorem A we need to prove modularity for the constructed representation, this follows from the appropriate modularity lifting theorem, using the conditions we chose for the representation at p.

Theorem B is an immediate consequence of Theorem A. The fact $f \neq g$ will follow from the fact that both forms have different levels, as the auxiliary primes involved necessarily ramify. If there is no need for auxiliary primes, we add an auxiliary prime into the set P.

Conventions: throughout this chapter, if $\det \overline{\rho} = \omega \overline{\chi}^k$, with ω unramified at p, we will consider only deformations with determinant $\tilde{\omega}\chi^k$. Given $\overline{\rho}$, after twisting it by a character of finite order we may, and will, suppose that $\overline{\rho}$ and $Ad^0\overline{\rho}$ ramify at the same set of primes S.

3.2 Types of reduction in the unramified case

We start this chapter by refining the study about classification of representations and types of reduction done in the previous one. The fact that we are working with representations with coefficients in an unramified field puts a big restriction on the possible deformation that may appear and rules out many reduction types. We summarize this in the following proposition.

Proposition 3.2.1. *Let $p \geq 5$ and $\rho : G_\ell \to \mathrm{GL}_2(W(\mathbb{F}))$ be a continuous representation.*

- *If ρ has type a ramified Principal Series then $\overline{\rho}^{ss}$ is ramified.*
- *If ρ has type an Induced representation then $\overline{\rho}^{ss}$ is ramified.*

Proof. For the first case, assume that $\overline{\rho}^{ss}$ is unramified, and ρ is Principal Series with character ϕ. Then $\overline{\phi} = 1$ which by the remark following Lemma 2.3.1 implies that $\ell \equiv 1 \pmod p$ and $\phi(\tau)$ has order a power of p. Therefore the eigenvalues of $\rho(\tau)$ generate a totally ramified extension of $\mathbb{Q}_p$ of degree at least $p-1$, which is clearly impossible as they also have to satisfy a polynomial of degree 2 over some unramified extension of $\mathbb{Q}_p$ and $p > 3$.

For the second case, assume that $\overline{\rho}^{ss}$ is unramified and ρ is induced with character ξ. Then necessarily $\overline{\xi} = \overline{\xi^\alpha}$ for $\alpha \notin G_M$, implying that the character $\psi = \xi/\xi^\alpha$ loses all of its ramification when reduced. Again by the remark following Lemma 2.3.1 this implies that $\psi(\tau)$ has order a power of p implying that it generates a totally ramified extension of degree at least $p-1 > 2$. But $\psi(\tau)$ is the quotient of the eigenvalues of $\overline{\rho}(\tau)$, so it lies in an extension of degree 2 of some unramified extension of $\mathbb{Q}_p$ which is absurd. ☐

3.3 The local picture

Having a more detailed description of the possible reduction types for unramified coefficient fields we are in a position to proceed with the local side of our argument. In order to apply

Ramakrishna's method we need to define for each prime $\ell \in P$ a set C_ℓ of deformations of $\rho_n|_{G_\ell}$ (containing ρ_ℓ) and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ of dimension $d_1 - d_2$ such that $\rho_n|_{G_\ell}$ can be successively deformed to an element of C_ℓ by deforming from $W(\mathbb{F})/p^m$ to $W(\mathbb{F})/p^{m+1}$ with adjustments at each step made only by a multiple of an element $h \notin N_\ell$. In order to get the full statement of our theorem, we have to take the extra care of picking the set C_ℓ such that all its elements agree up to isomorphism in the inertia group with ρ_ℓ.

Notice that as we mentioned in the introduction it is enough to do this for each of possible pair of $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ and $\mathrm{GL}_2(\overline{\mathbb{F}})$-isomorphism classes for ρ_ℓ and $\bar{\rho}$ and construct the set C_ℓ containing ρ_ℓ in such a way that all its members are congruent modulo p^n. This guarantees that the desired conditions hold.

The only extra care we need to take is making sure that whenever we pick a set C_ℓ, the deformations that belong to it have all coefficients in $W(\mathbb{F})$ and not in a bigger (and potentially ramified) extension of $\mathbb{Q}_p$. The potential issue that this may bring is that sometimes we cannot use the representatives of $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$-equivalence classes defined in Chapter 2, as they can have coefficients in a bigger extension. As we will see, this will force us to do some extra calculations in the Principal Series case. For the Steinberg case this is not an issue as the representatives have coefficients in $\mathbb{Z}_p$ so the required base change can be fulfilled over $W(\mathbb{F})$. Finally, in the Induced case the cohomological dimensions involved imply that the definition of C_ℓ and N_ℓ is trivial and does not depend on the representative chosen.

We classify the selection of the sets C_ℓ according to the type of $\bar{\rho}$, considering for each one, all the possible types for ρ_ℓ. The main difference between these and the sets C_ℓ constructed in [Ram02] is that in one of our cases, we are going to construct a pair (C_ℓ, N_ℓ) that satisfies the desired property only for exponents higher than a certain n_0 that may be bigger than n. The implication of this issue is that we will be able to replicate the global argument of [Ram02] only from n_0 and will have to apply another tools for lifting ρ_n between p^n and p^{n_0}.

Remark. Observe that whenever $d_2 = 0$ or $d_2 = d_1$ the problem of constructing C_ℓ and N_ℓ of the right dimension becomes trivial.

In the first case we need $\dim(N_\ell) = d_1 - d_2 = d_1$, so the only possible choice is $N_\ell = \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$. With this subspace we cannot adjust at all (as we have to take an element not in N_ℓ) but this is not a problem as $d_2 = 0$ implies that all the deformations of $\bar{\rho}$ are unobstructed and we can take C_ℓ as the set of all possible deformations of ρ_n to $W(\mathbb{F})$. Observe that in this case we still have to check that these deformations agree when restricted to inertia.

In the second case, we need $\dim(N_\ell) = d_1 - d_2 = 0$, hence $N_\ell = \{0\}$. This means that we have the whole group $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ available to adjust at every step. Taking this into account we can take any set C_ℓ and the N_ℓ-preserving-C_ℓ condition will automatically hold. We take $C_\ell = \{\rho_\ell\}$.

Case 1: $\bar{\rho}$ is ramified Principal Series. When $\bar{\rho}$ is ramified Principal Series, we have seen that ρ_ℓ can only be Principal Series. Nevertheless, the cohomology groups are different depending on whether $\ell \equiv 1 \pmod{p}$ or not. Recall that the representatives for the equivalence classes were (up to twist) $\rho_\ell \simeq \left(\begin{smallmatrix} \phi & \pi^r(\phi-1) \\ 0 & 1 \end{smallmatrix}\right)$ with $r \leq 0$ such that $\pi^r(\phi - 1)$ lies in $\overline{\mathbb{Z}_p}$. Observe that if $r \neq 0$, then $\pi \mid (\phi - 1)$ and therefore its reduction is not ramified Principal Series (the residual case $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ is unramified or Steinberg according to our classification). Then up to twist $\rho_\ell \simeq \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$ over $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ which implies that $\rho_\ell \simeq \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$ over $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ and we have the following cases:

(1) If $\ell \not\equiv 1 \mod p$, $d_0 = d_1 = 1$ and $d_2 = 0$ so we must take $N_\ell = \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ the full cohomology group so there is no possible choice at each step and C_ℓ must be the full set of deformations to characteristic zero. Notice that this is the only possible choice whenever $d_2 = 0$ and $\ell \neq p$ and in this case we have to check that any lift of $\overline{\rho}$ to $W(\mathbb{F})/p^s$ is the reduction of a characteristic zero one, but this is automatic as $d_2 = 0$ so the problem is unobstructed.

In order to check that all the elements of C_ℓ agree up to isomorphism when restricted to I_ℓ, we need to describe the set C_ℓ. If we define a morphism $\eta : G_\ell \to G_\ell/I_\ell \simeq \hat{\mathbb{Z}} \to \mathbb{Z}/p\mathbb{Z}$, then the element

$$h(g) = \begin{pmatrix} \eta(g) & 0 \\ 0 & -\eta(g) \end{pmatrix}$$

generates $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ and this implies that every lift is Principal Series, as the set $\lambda h \cdot \psi_s$, where ψ is the Teichmuller lift of $\overline{\rho}$ and λ is a scalar, exhausts all the possible reductions. In particular, the restriction to inertia is the same for all of them.

(2) If $\ell \equiv 1 \pmod p$ the picture is slightly different since $d_0 = 1$, $d_1 = 2$ and $d_2 = 1$, so we need to choose a one dimensional subspace N_ℓ and a set of deformations C_ℓ to $W(\mathbb{F})$. Observe that the isomorphism between ρ_ℓ and the representative of its $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$-equivalence class may not be realized over $W(\mathbb{F})$.

If the image of ψ_1 lies in $W(\mathbb{F})$, then the isomorphism is realized over $W(\mathbb{F})$. In that case, observe that the element h defined above lies inside $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$. Let $N_\ell = \langle h \rangle$, and $C_\ell = \left\{ \begin{pmatrix} \psi_1\gamma & 0 \\ 0 & \psi_2\gamma^{-1} \end{pmatrix} : \gamma \text{ unramified character} \right\}$.

We claim that this choice verifies the desired hypotheses. Clearly $\rho_\ell \in C_\ell$, and given any $h' \notin N_\ell$, the full $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ is generated by h and h'. Then for any mod p^m deformation ρ_m of $\overline{\rho}$ there is an element $\lambda_1 h + \lambda_2 h' \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ such that $(\lambda_1 h + \lambda_2 h')\rho_m$ lies in C_ℓ. But the action of any multiple of h preserves the elements of C_ℓ, so $\lambda_2 h' \rho_m$ already lies in C_ℓ. Note that as in the previous case, all the elements in C_ℓ agree when restricted to the inertia subgroup.

If the image of ψ_1 does not lie in $W(\mathbb{F})$ then ρ_ℓ is not isomorphic to $\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix}$ over $W(\mathbb{F})$ and we cannot use the previous choice. Instead, we need to use a canonical form for ρ_ℓ over $W(\mathbb{F})$. Assume that $\psi_1(\sigma) = \alpha$ and $\psi_2(\sigma) = \beta$, then the matrix $C = \begin{pmatrix} -\beta & -\alpha \\ 1 & 1 \end{pmatrix}$ conjugates $\begin{pmatrix} \psi_1(\sigma_\ell) & 0 \\ 0 & \psi_2(\sigma_\ell) \end{pmatrix}$ into $\begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta \end{pmatrix} \in \mathrm{GL}_2(W(\mathbb{F}))$. Therefore we can assume (applying a change of basis) that $\rho_\ell(\sigma) = \begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta \end{pmatrix}$. Then we can essentially use the same sets and subspaces as in the previous case but conjugated by C.

Let $N_\ell = \langle (\alpha - \beta) ChC^{-1} \rangle$, where h is the element defined before, and C_ℓ the set of deformations to $W(\mathbb{F})$ of the form $C \begin{pmatrix} \psi_1\gamma & 0 \\ 0 & \psi_2\gamma^{-1} \end{pmatrix} C^{-1}$ with $\gamma : G_\ell \to \overline{\mathbb{Z}_p}$ an unramified character. The factor $\alpha - \beta$ forces the element generating N_ℓ to have coefficients in $W(\mathbb{F})$. It can be easily checked that whenever ρ_m is the reduction of some element in C_ℓ and $u \in N_\ell$ then $(1+p^{m-1}u)\rho_m$ is again the reduction of an element of C_ℓ. Therefore the same reasoning as before shows that N_ℓ and C_ℓ satisfy our hypotheses.

Remark. Whenever we construct a set C_ℓ and subspace N_ℓ such that N_ℓ preserves the reductions of C_ℓ (i.e. whenever ρ_m is the reduction of some element of C_ℓ and $u \in N_\ell$, $u \cdot \rho_m$

is reduction of some element of C_ℓ as well) the same reasoning as in the first part of (2) applies and the pair (C_ℓ, N_ℓ) satisfies the desired lifting condition. In the next cases the same phenomena will occur, and we will often limit ourselves to check that N_ℓ preserves C_ℓ in the previous sense.

Case 2: $\overline{\rho}$ is Steinberg. If $\overline{\rho}$ is of Steinberg type then Proposition 2.3.2 and Proposition 3.2.1 imply that ρ_ℓ can only be Steinberg.

(1) If $\ell \not\equiv \pm 1 \pmod p$, by the previous section results, $d_0 = d_1 = d_2 = 0$, implying there is only one deformation at each p^n. We take $C_\ell = \{\rho_\ell\}$ which is the only deformation of $\overline{\rho}$ to $W(\mathbb{F})$.

(2) If $\ell \equiv -1 \pmod p$, by the previous section results, $d_1 = d_2 = 1$ and $d_0 = 0$, so $N_\ell = \{0\}$ and we have the full $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ available to adjust at every step. Then we take $C_\ell = \{\rho_\ell\}$.

(3) If $\ell \equiv 1 \pmod p$, we take the element $j \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ given by 0 at the wild inertia subgroup and by
$$j(\sigma) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \; j(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
Let $N_\ell = \langle j \rangle$ and C_ℓ the set of lifts ρ satisfying
$$\rho(\sigma) = \begin{pmatrix} \ell & * \\ 0 & 1 \end{pmatrix} \text{ and } \rho(\tau) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$
This set is formed by deformations which are isomorphic when restricted to inertia, and N_ℓ preserves its reductions.

Case 3: $\overline{\rho}$ is Induced. If $\overline{\rho}$ is Induced then the only possibility for ρ_ℓ is also being of Induced type.

(1) If $\ell \equiv -1 \pmod p$ and $M/\mathbb{Q}_\ell$ is unramified, $d_0 = 0$, $d_1 = d_2 = 1$ so N_ℓ is of codimension 1 inside a space of dimension 1, hence $N_\ell = \{0\}$. We take $C_\ell = \{\rho_\ell\}$. Since we can adjust at every step by a multiple of a given element $h \notin \{0\}$, and $d_1 = 1$, we can adjust at each step by any element of $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ to modify ρ_m as we want.

(2) If $\ell \not\equiv -1 \pmod p$ or $M/\mathbb{Q}_\ell$ is ramified, $d_0 = d_1 = d_2 = 0$, so there is only one lift at every step. This lift must be the reduction of ρ_ℓ, so there is nothing to adjust. We take $C_\ell = \{\rho_\ell\}$.

Case 4: $\overline{\rho}$ is unramified. If ρ_ℓ is also unramified, we simply take C_ℓ to be all the unramified lifts of $\overline{\rho}$ and N_ℓ the unramified part of $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$. It can be easily checked that N_ℓ has the correct dimension.

It remains to define the sets C_ℓ for the primes at which ρ_ℓ ramifies and $\overline{\rho}$ does not. By Proposition 3.2.1 this can only happen when ρ_ℓ is Steinberg.

We have that $\rho_\ell = \left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)$, with $*|_{I_\ell} \neq 0 \pmod{p^n}$. The sets C_ℓ we will pick depend on the image of σ. Recall that the eigenvalues of $\overline{\rho}(\sigma)$ are 1 and ℓ.

(1) If $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, necessarily $\ell \equiv 1 \pmod p$ implying $d_1 = 6$ and $d_2 = 3$ and therefore we need a subspace of dimension 3, preserving a family of deformations C_ℓ. In the previous cases, we have built sets C_ℓ of deformations of ρ_n that depend on $d_2 - d_1$ parameters, which in this case does not seem to be possible. However, as pointed to us by Ravi Ramakrishna, one can construct elements which are not cohomologically trivial for the residual representation, but give isomorphic lifts modulo big powers of p, as in Section 4 of [HR08]. However, this trivial primes do not come for free, in this case, the subspace N_ℓ we build will not preserve the reductions of C_ℓ for any exponent m but for all the exponent bigger than a certain one, that depends on the lift ρ_ℓ.

Let C_ℓ be the set of deformations of ρ_n satisfying:

$$\rho(\sigma) = \begin{pmatrix} \ell & * \\ 0 & 1 \end{pmatrix} \text{ and } \rho(\tau) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Observe that this family depends on two parameters and is clearly preserved by the elements $u_1, u_2 \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ given by

$$u_1(\sigma) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \ u_1(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$u_2(\sigma) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad , \ u_2(\tau) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We still need one more element of $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ to preserve C_ℓ. Recall that ρ_ℓ satisfies

$$\rho_\ell(\sigma) = \begin{pmatrix} \ell & x \\ 0 & 1 \end{pmatrix} \text{ and } \rho_\ell(\tau) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

with $y \neq 0$. Let $n_0 = \min(v(x), v(y), v(\ell - 1))$. We claim that there exists an element $\nu \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ which is not in the span of u_1 and u_2 and satisfies that whenever $m > n_0$ and ρ_m is the reduction modulo p^m of some element in C_ℓ then $(1 + p^{m-1}\nu)\rho_m$ is the same deformation as ρ_m. The element ν will depend on the valuations of x, y and $\ell - 1$.

Lemma 3.3.1. *There exists an element $\nu \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ not in $\langle u_1, u_2 \rangle$ such that whenever ρ_m is the reduction modulo p^m of some element in C_ℓ, with $m \geq n_0 + 1$, then $(1 + p^{m-1}\nu)\rho_m$ is the same deformation as ρ_m.*

Proof. The proof is divided into several cases, we first define $g_1, g_2, g_3 \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ as

$$g_1(\sigma) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \ g_1(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$g_2(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ g_2(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$g_3(\sigma) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ , \ g_3(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now enumerate a list of cases (depending on the valuations of x, y and $\ell - 1$) and for each of them specify an element v and a matrix C congruent to the identity modulo p such that $C^{-1}\rho_m C = (1 + p^{m-1}\nu)\rho_m$. Write $C = \left(\begin{smallmatrix} 1+p\alpha & p\beta \\ p\gamma & 1+p\delta \end{smallmatrix}\right)$. In each case we will give the values of α, β, γ and δ and left to the reader to check that $C^{-1}\rho_m C = (1 + p^{m-1})\rho_m$ in each of them.

- If $v(y) < v(x)$ and $v(y) < v(\ell - 1)$: take $\nu = g_3$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma y = p^{m-2} \pmod{p^{m-1}}$ and $\gamma x = \gamma(\ell - 1) = 0 \pmod{p^{m-1}}$.
- If $v(x) < v(y)$ and $v(x) < v(\ell - 1)$: take $\nu = g_2$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma x = p^{m-2} \pmod{p^{m-1}}$ and $\gamma y = \gamma(\ell - 1) = 0 \pmod{p^{m-1}}$.
- If $v(\ell - 1) < v(x)$ and $v(\ell - 1) < v(x)$: take $\nu = g_1$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma(\ell - 1) = -p^{m-2} \pmod{p^{m-1}}$ and $\gamma x = \gamma y = 0 \pmod{p^{m-1}}$.
- If $v(y) = v(\ell - 1)$ and $v(y) < v(x)$: then $y = \lambda(\ell - 1)$. Take $\nu = g_1 - \lambda g_3$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma(\ell - 1) = -p^{m-1} \pmod{p^{m-1}}$ and $\gamma x = 0 \pmod{p^{m-1}}$.
- If $v(y) = v(x)$ and $v(y) < v(\ell - 1)$: then $y = \lambda x$. Take $\nu = g_2 + \lambda g_3$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma x = p^{m-2} \pmod{p^{m-1}}$ and $\gamma(\ell - 1) = 0 \pmod{p^{m-1}}$.
- If $v(x) = v(\ell - 1)$ and $v(x) < v(y)$: then $x = \lambda(\ell - 1)$. Take $\nu = g_1 - \lambda g_2$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma(\ell - 1) = -p^{m-2} \pmod{p^{m-1}}$ and $\gamma y = 0 \pmod{p^{m-1}}$.
- If $v(x) = v(\ell - 1) = v(y)$: then $x = \lambda_1(\ell - 1)$ and $y = \lambda_2(\ell - 1)$. Take $\nu = g_1 - \lambda_1 g_2 - \lambda_2 g_3$ and C satisfying $\alpha = \delta$, $\beta = 0$, $\gamma(\ell - 1) = -p^{m-2} \pmod{p^{m-1}}$.

□

We end this case by taking C_ℓ as above and $N_\ell = \langle u_1, u_2, \nu \rangle$, for the element ν of Lemma 3.3.1. Observe that this argument fails when we are not considering exponents $m > n_0$. For those, the reduction of ρ_ℓ modulo p^m is trivial, and as the trivial deformation does not have any equivalent deformation other than itself, it is impossible to find an element ν as before in those cases.

(2) If $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with $\alpha \neq 1$, necessarily $\ell \equiv \alpha \pmod{p}$ so $d_1 = 3$ and $d_2 = 2$ if $\ell \equiv -1 \pmod{p}$ and $d_1 = 2$ and $d_2 = 1$ otherwise. In both cases, let $u \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ defined by $u(\sigma) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $u(\tau) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and take $N_\ell = \langle u \rangle$. Define the set C_ℓ of deformations ρ that satisfy

$$\rho(\sigma) = \rho_\ell(\sigma) \quad \text{and} \quad \rho(\tau) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Clearly N_ℓ preserves C_ℓ.

(3) If $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, necessarily $\ell \equiv 1 \pmod p$, so $d_1 = 2$ and $d_2 = 1$. Let $u \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ by $u(\sigma) = 0$ and $u(\tau) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and take $N_\ell = \langle u \rangle$. This subspace preserves the set C_ℓ of deformations ρ satisfying

$$\rho(\sigma) = \rho_\ell(\sigma) \quad \text{and} \quad \rho(\tau) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Remark. If we allow ramification in the coefficient field then the cases ruled out by Proposition 3.2.1 may happen. Most of them correspond to cases like the first unramified case, where a trick like in [HR08] needs to be used. It is worth pointing out that in most of those cases we can construct the corresponding sets C_ℓ and subspaces N_ℓ but the global arguments below do not adapt well to that situation. See the remark after Lemma 3.4.10. We will address this problem in the next chapter.

3.3.1 The case $\ell = p$

In this case we will pick C_p exactly as in [Ram02] (*local at p considerations*), with the observation that in the supersingular case, it follows from the work done in [Ram93] that the lifts picked have the same Hodge-Tate weights than ρ_p (which lie in the interval $[0, p-1]$) and are crystalline. Note that in each case considered by Ramakrishna, ρ_p is always trivially contained in C_p.

3.4 Auxiliary primes

For constructing the sets Q_1 and Q_2 mentioned in the introduction we will work with primes $q \not\equiv \pm 1 \pmod p$ such that $\overline{\rho}$ is not ramified at q and $\overline{\rho}(\sigma)$ has different eigenvalues of ratio q, i.e. $\bar{\rho}(\sigma) = \left(\begin{smallmatrix} qx & 0 \\ 0 & x \end{smallmatrix}\right)$ and $\bar{\rho}(\tau) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. From now on, the primes satisfying this condition will be called auxiliary primes. Moreover, when such a prime also satisfies that $\rho_n(\sigma)$ is equivalent to a diagonal matrix with diagonal values of ratio q we will call it an auxiliary prime for ρ_n. For these primes the cohomological dimensions are $\dim \mathrm{H}^0(G_q, Ad^0\bar{\rho}) = 1$, $\dim \mathrm{H}^1(G_q, Ad^0\bar{\rho}) = 2$ and $\dim \mathrm{H}^2(G_q, Ad^0\bar{\rho}) = 1$.

In this case, the set C_q is formed by the deformations ρ such that

$$\rho(\tau) = \begin{pmatrix} 1 & px \\ 0 & 1 \end{pmatrix} \;\text{ and }\; \rho(\sigma) = \begin{pmatrix} q & py \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

These two conditions define a tamely ramified deformation of $\overline{\rho}$. The set C_q is preserved by a subspace $N_q \subseteq \mathrm{H}^1(G_q, Ad^0\bar{\rho})$ of codimension 1 generated by the cocycle j given by $j(\sigma) = 0$ and $j(\tau) = e_2$.

There are two main goals we want to achieve in this section. Firstly, we would like to prove that auxiliary primes do exist for representations with coefficients in $W(\mathbb{F})/p^n$. Observe that the inductive step depends only on the reduction modulo p of ρ_n, so we only need to check that once we set the deformation set C_q, whenever we add an auxiliary prime q together with its subspace N_q, the representation $\rho_n|_{G_q}$ is the reduction of some element in C_q, i.e. we want to prove that there are primes q such that $\rho_n|_{G_q}$ sends Frobenius and a generator of the tame inertia to the matrices defined in (3.1) modulo p^n.

Secondly, we need to reprove the properties of the auxiliary primes that we are going to use in our context, although they look similar to the arguments in [Ram02].

3.4.1 Working modulo p^n

We need to prove that there exist infinitely many auxiliary primes for ρ_n, that is primes q such that $q \not\equiv \pm 1 \pmod{p}$, ρ_n is unramified at q and $\rho_n(\sigma)$ is diagonal $(q, 1)$.

Let μ_p be a primitive p-th root of unity, $D = \mathbb{Q}(Ad^0\bar{\rho}) \cap \mathbb{Q}(\mu_p)$, $K = \mathbb{Q}(Ad^0\bar{\rho})\mathbb{Q}(\mu_p)$, $D' = \mathbb{Q}(Ad^0\rho_n) \cap \mathbb{Q}(\mu_p)$ and $K' = \mathbb{Q}(Ad^0\rho_n)\mathbb{Q}(\mu_p)$, which fit in the following diagram:

$$\begin{tikzcd}[column sep=small]
 & K' \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(Ad^0\rho_n) \arrow[dr, dash] & & \mathbb{Q}(\mu_p) \arrow[dl, dash] \\
 & D' \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & K \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(Ad^0\bar{\rho}) \arrow[dr, dash] & & \mathbb{Q}(\mu_p) \arrow[dl, dash] \\
 & D \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

Observe that we can translate the conditions on q into the following:

- the condition $q \not\equiv \pm 1 \pmod{p}$ is equivalent to Frob_q not being the identity nor conjugation in $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$.
- q being an auxiliary prime for ρ_n is equivalent to being unramified in $\mathbb{Q}(Ad^0\rho_n)$, $q \not\equiv \pm 1 \pmod{p}$ and Frob_q lies in the conjugacy class of an element $\overline{M} \in \mathrm{Im}(Ad^0\rho_n)$, where M is a diagonal matrix with elements of ratio q in the diagonal.

Therefore, if we prove that there is an element $c \in \mathrm{Gal}(K'/\mathbb{Q})$ such that $c|_{\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})} = t \neq \pm 1$ and $c|_{\mathrm{Gal}(\mathbb{Q}(Ad^0\rho_n)/\mathbb{Q})} = \overline{M}$ where M is diagonal with elements of ratio t in its diagonal, then we are done using Chebotarev's Theorem. That is the content of the next proposition.

Proposition 3.4.1. *There is an element $c = a \times b \in \mathrm{Gal}(K'/\mathbb{Q}) \subseteq \mathrm{Gal}(\mathbb{Q}(Ad^0\rho_n)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ such that a comes from an element $M \in \mathrm{Im}(\rho_n) \simeq \mathrm{Gal}(\mathbb{Q}(\rho_n)/\mathbb{Q})$ which has different elements in its diagonal with ratio $b \in \mathbb{F}_p^\times \simeq \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, $b \neq \pm 1$.*

The proof is in the spirit of the arguments given in [Ram99] for finding such elements. Precisely, it is a slightly modified version of the proof of Theorem 2. We will reproduce it in our setting. Recall the following lemma (Lemma 3, IV-23 in [Ser89] [1]).

Lemma 3.4.2. *Let $p \geq 5$ and $\mathbb{F}$ a finite field of characteristic p. Let $H \subseteq \mathrm{GL}_2(W(\mathbb{F}))$ a closed subgroup and $\overline{H}$ its projection to $\mathrm{GL}_2(\mathbb{F})$. If $\mathrm{SL}_2(\mathbb{F}) \subseteq \overline{H}$ then $\mathrm{SL}_2(W(\mathbb{F})) \subseteq H$.*

This has the following easy consequences:

Corollary 3.4.3. *If $\mathrm{SL}_2(\mathbb{F}) \subseteq \mathrm{Im}(\overline{\rho})$ then $\mathrm{SL}_2(W(\mathbb{F})/p^n) \subseteq \mathrm{Im}(\rho_n)$.*

Proof. Denote by $\pi : W(\mathbb{F}) \to W(\mathbb{F})/p^n$ the projection, then this follows applying the above lemma with $H = \pi^{-1}(\mathrm{Im}(\rho_n)) \subseteq W(\mathbb{F})$ which is closed as $G_{\mathbb{Q}}$ is compact . □

The following lemma gives the existence of the element c.

Lemma 3.4.4. *For D' the field defined above, $[D' : \mathbb{Q}] \leq 2$. Moreover we have that $\mathrm{PSL}_2(\mathbb{F}) \subseteq \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/D')$.*

[1] Actually, Lemma 3 is stated and proved in [Ser89] for $\mathbb{F} = \mathbb{F}_p$ but the same proof holds for an arbitrary finite field of characteristic p.

Proof. Observe that $[\mathbb{Q}(Ad^0\rho_n) : \mathbb{Q}(Ad^0\bar{\rho})]$ is a power of p, which is coprime with the degree $[\mathbb{Q}(\mu_p) : \mathbb{Q}]$. This implies that $D' = \mathbb{Q}(Ad^0\rho_n) \cap \mathbb{Q}(\mu_p) = \mathbb{Q}(Ad^0\bar{\rho}) \cap \mathbb{Q}(\mu_p)$. In Lemma 18 of [Ram99] it is proved that if $\bar{\rho}$ is a deformation in our hypothesis the field $D = \mathbb{Q}(Ad^0\bar{\rho})\cap\mathbb{Q}(\mu_p)$ satisfies that $[D : \mathbb{Q}] = 2$ and $\mathrm{PSL}_2(\mathbb{F}) \subseteq \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/D)$, concluding our proof. $\square$

Proof of Proposition 3.4.1: If $\mathbb{F} \neq \mathbb{F}_5$, let $x \in \mathbb{F}^\times$ be any element such that $x^2 \in \mathbb{F}_p^\times$ and $x^2 \neq \pm 1$ (observe that this exists for any $\mathbb{F} \neq \mathbb{F}_5$). Let $\tilde{x} \in W(\mathbb{F})/p^n$ be a lift of x, $b \in \{1, \cdots, p-1\} \subseteq W(\mathbb{F})/p^n$ be congruent to x^2 modulo p and $M = \begin{pmatrix} \tilde{x} & 0 \\ 0 & \tilde{x}^{-1} \end{pmatrix} \in \mathrm{SL}_2(W(\mathbb{F})/p^n) \subseteq \mathrm{Im}(\rho_n)$. Then $c = (\overline{M}, b) \in \mathrm{Gal}(\mathbb{Q}(Ad^0\rho_n)/D') \times \mathrm{Gal}(\mathbb{Q}(\mu_p)/D')$ is such an element.

Observe that this proof fails for $\mathbb{F}_5$ as there is no element $x \in \mathbb{F}_5^\times$ such that $x^2 \neq \pm 1$. For $p = 5$ we are assuming that $\bar{\rho}$ is surjective. We have two different scenarios:

- First, if $D' = \mathbb{Q}$ then $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ and we can find the element c by taking a pair $(\overline{M}, b)$ where $M = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(W(\mathbb{F})/p^n) = \mathrm{Im}(\rho_n)$ and $b \equiv q \neq \pm 1 \pmod 5$
- If $D' \neq \mathbb{Q}$ then $[D' : \mathbb{Q}] = 2$. Lemma 3.4.4 tells us that $\mathrm{PSL}_2(\mathbb{F}_5) \subseteq \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/D')$. As $\mathrm{PSL}_2(\mathbb{F}_5)$ has index 2 in $\mathrm{PGL}_2(\mathbb{F}_5)$ we have that $\mathrm{PSL}_2(\mathbb{F}_5) = \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/D')$. On the other hand, we know that $\mathrm{Gal}(\mathbb{Q}(\mu_p)/D') \subseteq \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \simeq \mathbb{F}_5^\times$ has index 2, implying that $\mathrm{Gal}(\mathbb{Q}(\mu_p)/D') \simeq \{\pm 1\}$. With this information we know that the pair $(\overline{M}, b)$, for $M = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = 3$ defines an element in $\mathrm{Gal}(K'/\mathbb{Q})$, as both elements coincide when restricted to D' (both act non trivially). $\square$

Remark. The element c constructed in Proposition 3.4.1 is not the same as the one in [Ram99] for $p > 5$. In fact they live in different Galois groups, the first one lying in $\mathrm{Gal}(K'/\mathbb{Q})$ and the second one in $\mathrm{Gal}(K/\mathbb{Q})$. However, it is true that the projection of the element constructed in this work through the map $\mathrm{Gal}(K'/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$ is an element like the one defined by Ramakrishna. In particular, both elements act in the same way on $Ad^0\bar{\rho}$ (as the action of our c is through this projection). To avoid confusion we denote the projection by $\tilde{c}$.

Any prime q not ramified in K' such that Frob_q lies in the conjugacy class of c can be taken as an auxiliary prime for ρ_n. In the next subsection we are going to impose extra conditions at the auxiliary primes regarding their interaction with elements of $\mathrm{H}^1(G_\mathbb{Q}, Ad^0\bar{\rho})$ and $\mathrm{H}^2(G_\mathbb{Q}, Ad^0\bar{\rho})$.

3.4.2 Properties of auxiliary primes

We need to impose conditions on the auxiliary primes similar to the ones in Fact 16 and Lemma 14 of [Ram02]. Concretely, for non-zero elements $f \in \mathrm{H}^1(G_P, Ad^0\bar{\rho})$ and $g \in \mathrm{H}^1(G_P, (Ad^0\bar{\rho})^*)$, the auxiliary prime q should satisfy $f|_{G_q} = 0$ or $f|_{G_q} \notin N_q$ and $g|_{G_q} \neq 0$. We need to impose these conditions for many elements at the same time.

If $f \in \mathrm{H}^1(G_P, Ad^0\bar{\rho})$, then $f|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(Ad^0\bar{\rho}))}$ is a morphism, so we can associate an extension $\widetilde{L_f}/\mathbb{Q}(Ad^0\bar{\rho})$ fixed by its kernel. Also let $L_f = \widetilde{L_f}K = \widetilde{L_f}(\mu_p)$. Analogously, for $g \in \mathrm{H}^1(G_P, (Ad^0\bar{\rho})^*)$ we define $M_g/\mathbb{Q}((Ad^0\bar{\rho})^*)$ as the fixed field by the kernel of $g|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}((Ad^0\bar{\rho})^*))}$.

Notice that we can obtain information about $f|_{G_q}$ or $g|_{G_q}$ by looking at the conjugacy class of Frob_q in $\mathrm{Gal}(L_f/\mathbb{Q})$ or $\mathrm{Gal}(M_g/\mathbb{Q})$ (as these are almost the extensions associated to the adjoint representation of $\overline{\rho}(Id+\epsilon f)$).

Let $f_1,\dots,f_{r_1}$ and $g_1,\dots,g_{r_2}$ be bases for $\mathrm{H}^1(G_P, Ad^0\bar{\rho})$ and $\mathrm{H}^1(G_P,(Ad^0\bar{\rho})^*)$ respectively. Define L to be the composition of the fields L_{f_i}, M the composition of the M_{g_j}, and $F=LM$. The following lemma is a summary of results about these extensions from [Ram99].

Lemma 3.4.5. *Let f_i and g_j as above.*

(1) We have that, as $G_{\mathbb{Q}}$-modules, $\mathrm{Gal}(L_{f_i}/K)\simeq Ad^0\bar{\rho}$ *and* $\mathrm{Gal}(M_{g_j}/K)\simeq (Ad^0\bar{\rho})^*$.

(2) We also have that the composition splits as $\mathrm{Gal}(L/K)\simeq\prod \mathrm{Gal}(L_{f_i}/K)\simeq (Ad^0\bar{\rho})^{r_1}$ *and* $\mathrm{Gal}(M/K)\simeq\prod\mathrm{Gal}(M_{g_j}/K)\simeq((Ad^0\bar{\rho})^*)^{r_2}$.

Moreover $M\cap L=K$ so $\mathrm{Gal}(F/K)\simeq\mathrm{Gal}(L/K)\times\mathrm{Gal}(M/K)$.

(3) The exact sequences

$$1\longrightarrow \mathrm{Gal}(L/K)\longrightarrow\mathrm{Gal}(L/\mathbb{Q})\longrightarrow\mathrm{Gal}(K/\mathbb{Q})\longrightarrow 1,$$

and

$$1\longrightarrow \mathrm{Gal}(M/K)\longrightarrow\mathrm{Gal}(M/\mathbb{Q})\longrightarrow\mathrm{Gal}(K/\mathbb{Q})\longrightarrow 1,$$

both split, hence $\mathrm{Gal}(F/\mathbb{Q})\simeq\mathrm{Gal}(F/K)\rtimes\mathrm{Gal}(K/\mathbb{Q})$.

Proof. The first claim is Lemma 9, the second is Lemma 11 and the last one is Lemma 13 of [Ram99] with two remarks:

- In [Ram99] these results are proved for the representation $\widetilde{Ad}^0\bar{\rho}$, which is the descent of $Ad^0\bar{\rho}$ to its minimal field of definition. As we are assuming that $\mathrm{SL}_2(\mathbb{F})\subseteq\mathrm{Im}(\bar{\rho})$, we have that $Ad^0\bar{\rho}$ is already defined in its minimal field of definition, because of Lemma 17 of [Ram99].
- In [Ram99] these lemmas are proved for $P=S$ the set of ramification of $Ad^0\bar{\rho}$, but the same proofs work for any $P\supseteq S$. □

Finally, we can read properties of $f|_{G_q}\in\mathrm{H}^1(G_q,Ad^0\bar{\rho})$ from the class of Frob_q in the group $\mathrm{Gal}(L_f/\mathbb{Q})\simeq\mathrm{Gal}(L_f/K)\rtimes\mathrm{Gal}(K/\mathbb{Q})$. Observe that the element $c\in\mathrm{Gal}(K'/\mathbb{Q})$ constructed in the previous section acts on $Ad^0\bar{\rho}$ through the projection to $\mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/\mathbb{Q})$.

Proposition 3.4.6. *Let $q\in\mathbb{Q}$ be an auxiliary prime for ρ_n, $f\in\mathrm{H}^1(G_P,Ad^0\bar{\rho})$ and $g\in\mathrm{H}^1(G_P,(Ad^0\bar{\rho})^*)$. Let $c=a\times b\in\mathrm{Gal}(K'/\mathbb{Q})$ an element like in Proposition 3.4.1.*

(1) If Frob_q *lies in the conjugacy class of* $0\rtimes c\in\mathrm{Gal}(L_f/\mathbb{Q})$ *then $f|_{G_q}=0$. The same holds for g and* $\mathrm{Gal}(M_g/\mathbb{Q})$.

(2) There are nontrivial elements $\alpha\in Ad^0\bar{\rho}$ on which c acts trivially and if Frob_q *lies in the conjugacy class of* $\alpha\rtimes c\in\mathrm{Gal}(L_f/\mathbb{Q})$ *then $f|_{G_q}\notin N_q$.*

(3) There are nontrivial elements $\beta\in(Ad^0\bar{\rho})^*$ on which c acts trivially and if Frob_q *lies in the conjugacy class of* $\beta\rtimes c\in\mathrm{Gal}(M_g/\mathbb{Q})$ *then $g|_{G_q}\neq 0$.*

This proposition is proved in [Ram99] and is the product of a series of results. We reproduce them below. In our setting, the existence of non-trivial elements $\alpha \in Ad^0\bar{\rho}$ on which c acts trivially is straightforward and simply corresponds to check that $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ acts trivially on $Ad^0\bar{\rho}$ and that the diagonal matrix $(q, 1)$ acts trivially on e_1. A similar argument provides the existence of non-trivial elements of $(Ad^0\bar{\rho})^*$ on which c acts trivially. Observe that this corresponds to Lemma 14 of [Ram99], which is non-trivial because in that case the descent of $Ad^0\bar{\rho}$ to its field of definition may not be $Ad^0\bar{\rho}$ itself.

Lemma 3.4.7. *Let L_f/K be the extension corresponding to an element $f \in \mathrm{H}^1(G_P, Ad^0\bar{\rho})$ and let $\beta = \alpha \rtimes c \in \mathrm{Gal}(L_f/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ and u be a prime of $\mathbb{Q}$, unramified in L_f, whose Frobenius lies in the conjugacy class of β.*

(1) If $\alpha = 0$ then any prime above u in K splits completely from K to L_f.

(2) If α is a non-trivial element on which c acts trivially then any prime above u do not split completely from K to L_f.

Proof. This follows from observing that as in both cases c acts trivially on α, the order of $\alpha \rtimes c$ is the least common multiple of the orders of α and c. In case (1) this implies that $\alpha \rtimes c$ has the same order in $\mathrm{Gal}(L_f/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ than c in $\mathrm{Gal}(K/\mathbb{Q})$ and therefore the primes considered split completely. In case (2), this implies that these orders are different and therefore the primes considered do not split completely. □

The same result holds for $(Ad^0\bar{\rho})^*$.

Lemma 3.4.8. *Let M_g/K be the extension corresponding to an element $g \in \mathrm{H}^1(G_P, (Ad^0\bar{\rho})^*)$ and let $\delta = \gamma \rtimes c \in \mathrm{Gal}(M_g/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$ and w be a prime of $\mathbb{Q}$, unramified in M_g, whose Frobenius lies in the conjugacy class of δ.*

(1) If $\gamma = 0$ then any prime above w in K splits completely from K to M_g.

(2) If γ is a non-trivial element on which c acts trivially then any prime above w do not split completely from K to M_g.

Proof. The proof is identical to that of Lemma 3.4.7 □

From these two results, the proof of Proposition 3.4.6 is almost immediate.

Proof of Proposition 3.4.6: This follows from the previous two lemmas and from observing that $f|_{G_q} \in N_q$ if and only if $f(\mathrm{Frob}_q) = 0$. □

Corollary 3.4.9. *There exist primes q such that $\bar{\rho}(\mathrm{Frob}_q)$ has different eigenvalues of ratio q and such that for the basis elements any of the following conditions can be achieved: $f_i|_{G_q} = 0$ or $f_i|_{G_q} \notin N_q$ and $g_j|_{G_q} = 0$ or $g_j|_{G_q} \neq 0$.*

Proof. Pick an element

$$\Omega = \omega \rtimes \tilde{c} \in \mathrm{Gal}(F/\mathbb{Q}) \simeq \left(\prod_{i=1}^{r_1} \mathrm{Gal}(L_{f_i}/\mathbb{Q}) \times \prod_{j=1}^{r_2} \mathrm{Gal}(M_{g_j}/\mathbb{Q}) \right) \rtimes \mathrm{Gal}(K/\mathbb{Q}),$$

where ω has coordinates 0 or α whether we want $f_i|_{G_q}$ to be 0 or not in N_q in the first product and 0 or β whether we want $g_j|_{G_q}$ to be 0 or not 0 in the second one. Then any q such that Frob_q lies in the conjugacy class of Ω works. □

We want the same to hold for ρ_n, i.e. to find primes q satisfying the same conditions plus $\rho_n(\mathrm{Frob}_q)$ to have different eigenvalues of ratio q. As we mentioned before, any q such that $\mathrm{Frob}_q \in \mathrm{Gal}(K'/\mathbb{Q})$ lies in the conjugacy class of c satisfies this extra condition. Therefore, we only need to check that there is an element θ in $\mathrm{Gal}(K'F/\mathbb{Q})$ such that $\theta|_{K'} = c$ and $\theta|_F = \Omega$.

Observe that $\Omega|_K = \tilde{c} = c|_K$, a necessary condition. It is enough to prove that $K' \cap F = K$, as any pair of elements in $\mathrm{Gal}(K'/\mathbb{Q})$ and $\mathrm{Gal}(F/\mathbb{Q})$ that are equal when restricted to $K' \cap F$ define an element in $\mathrm{Gal}(K'F/\mathbb{Q})$. In order to prove this, we need the following lemma.

Lemma 3.4.10. *$K' \cap F = K$.*

Proof. Let $\mathcal{H} = \mathrm{Gal}(K'/K) \subseteq \mathrm{PGL}_2(W(\mathbb{F})/p^n)$ and $\pi_1 : \mathrm{PGL}_2(W(\mathbb{F})/p^n) \to \mathrm{PGL}_2(\mathbb{F})$. Observe that H consists of the classes of matrices in $\mathrm{Im}(\rho_n)$ which are trivial in $\mathrm{PGL}_2(\mathbb{F})$, i.e. $\mathcal{H} = \mathrm{Im}(Ad^0\rho_n) \cap \mathrm{Ker}(\pi_1)$. Recall that our hypotheses imply $\mathrm{PSL}_2(W(\mathbb{F})/p^n) \subseteq \mathrm{Im}(Ad^0\rho_n) \subseteq \mathrm{PGL}_2(W(\mathbb{F})/p^n)$, and therefore $\mathrm{PSL}_2(W(\mathbb{F})/p^n) \cap \mathrm{Ker}(\pi_1) \subseteq \mathcal{H} \subseteq \mathrm{Ker}(\pi_1)$. As $[\mathrm{PSL}_2(W(\mathbb{F})/p^n) : \mathrm{PGL}_2(W(\mathbb{F})/p^n)] = 2$ and $\mathrm{Ker}(\pi_1)$ is a p group we have that $\mathcal{H} = \mathrm{Ker}(\pi_1)$.

Recall from Lemma 3.4.5 that $\mathrm{Gal}(F/K) \simeq (Ad^0\bar{\rho})^r \times (Ad^0\bar{\rho}^*)^s$ as $\mathbb{Z}[G_\mathbb{Q}]$-module. Moreover, as the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbb{F})$, both $Ad^0\bar{\rho}$ and $(Ad^0\bar{\rho})^*$ are irreducible as $\mathbb{Z}[G_\mathbb{Q}]$-modules implying that this is the decomposition of $\mathrm{Gal}(F/K)$ as $\mathbb{Z}[G_\mathbb{Q}]$ simple modules. This implies that if $K' \cap F \neq K$ then $Ad^0\bar{\rho}$ or $(Ad^0\bar{\rho})^*$ appear as a quotient of $\mathrm{Gal}(K'/K)$.

Assume that $K' \cap F \neq K$ and that there is a surjective morphism $\varpi : \mathcal{H} \to Ad^0\bar{\rho}$. Let $\pi_2 : \mathrm{PGL}_2(W(\mathbb{F})/p^n) \to \mathrm{PGL}_2(W(\mathbb{F})/p^2)$ and let $\mathcal{N} = \ker(\pi_2) \subset \mathcal{H}$. We claim that $\varpi(\mathcal{N}) = 0$. For this, observe that any matrix $\mathrm{Id} + p^2M \in \mathrm{GL}_2(W(\mathbb{F})/p^n)$ is the p-th power of some matrix $\mathrm{Id} + pN \in \mathrm{GL}_2(W(\mathbb{F})/p^n)$. Therefore, if $\mathrm{Id} + p^2M \in \mathcal{N}$ we have that

$$\varpi(Id + p^2M) = \varpi((Id + pN)^p) = p\varpi(Id + pN) = 0.$$

This implies that ϖ factors through $\mathrm{Gal}(\mathbb{Q}(Ad^0\rho_2)/K)$, where $Ad^0\rho_2$ is the reduction mod p^2 of $Ad^0\rho_n$. Since $\#\,\mathrm{Gal}(\mathbb{Q}(Ad^0\rho_2)/K) = \#(\mathrm{Im}(Ad^0\rho_2) \cap \mathrm{Ker}(\pi_1)) \leq (\#\mathbb{F})^3$ and $\#Ad^0\bar{\rho} = (\#\mathbb{F})^3$ we necessarily have $\mathrm{Gal}(\mathbb{Q}(Ad^0\rho_2)/\mathbb{Q}) = \mathrm{Gal}(L_f/\mathbb{Q})$ for some $f \in \mathrm{H}^1(G_\mathbb{Q}, Ad^0\bar{\rho})$. But this cannot happen since it would imply that the image of $Ad^0\rho_2$ splits, which is impossible as it contains $\mathrm{PSL}_2(W(\mathbb{F})/p^2)$ when $p \geq 7$ or $\mathrm{PGL}_2(W(\mathbb{F})/p^2)$ when $p = 5$.

The case where there is a surjection $\pi : \mathcal{H} \to (Ad^0\bar{\rho})^*$ works the same. ☐

Remark. As we mentioned before, this global argument does not adapt to the cases when the coefficient field is ramified. Specifically, Lemma 3.4.10 above in no longer true if we allow the coefficients to ramify, as the extension corresponding to $Ad^0\rho_2$ corresponds to an element of $\mathrm{H}^1(G_\mathbb{Q}, Ad^0\bar{\rho})$. Then we cannot apply Chebotarev's Theorem to find auxiliary primes which are nontrivial in the element of the cohomology corresponding to $Ad^0\rho_2$, so we do not get an isomorphism between local and global deformations.

With this result, a version Corollary 3.4.9 for auxiliary primes for ρ_n is immediate.

Corollary 3.4.11. *Let $\{f_i\}_{i\in A} \subseteq \mathrm{H}^1(G_S, Ad^0\bar{\rho})$ and $\{g_j\}_{j\in B} \subseteq \mathrm{H}^1(G_S, (Ad^0\bar{\rho})^*)$ be two linearly independent sets. Let $I \subseteq A$ and $J \subseteq B$.*

There exist an infinite set of primes q which are auxiliary for ρ_n and satisfy $f_i|_{G_q} = 0$ for $i \in A$ and $f_i|_{G_q} \notin N_q$ for $i \in I \backslash A$, and $g_j|_{G_q} = 0$ for $j \in B$ and $g_j|_{G_q} \neq 0$ for $j \in B \backslash J$.

In other words, we can pick auxiliary primes for ρ_n which a desired behavior in a basis of $\mathrm{H}^1(G_S, Ad^0\bar{\rho})$ and $\mathrm{H}^1(G_S, (Ad^0\bar{\rho})^*)$.

We end this section with a key property about auxiliary primes that will allow us to get the desired local to global isomorphism for H^1. For an element $\tau \in \mathrm{Gal}(L/K)$ we define the Chebotarev set T_τ as the set of auxiliary primes for ρ_n such that $\mathrm{Frob}_q \in \mathrm{Gal}(K/\mathbb{Q}) \rtimes \mathrm{Gal}(L/K)$ has its second coordinate equal to τ (the first one is determined as we are asking q to be an auxiliary prime for ρ_n).

Proposition 3.4.12. *For any $\tau \in \mathrm{Gal}(L/K)$ as above we have that*

$$\mathrm{H}^1(G_{P\cup T_\tau}, Ad^0\bar{\rho}) \longrightarrow \bigoplus_{\ell\in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$$

is a surjection.

Proof. This is essentially Proposition 10 of [Ram02], up to the fact that we are asking for a condition on $\mathrm{Gal}(K'/\mathbb{Q})$ rather than $\mathrm{Gal}(K/\mathbb{Q})$ (the set T_τ is composed by primes that are auxiliary for ρ_n). Nevertheless, the same proof applies as the main argument is that for any $g \in \mathrm{H}^1(G_{P\cup T_\tau}, (Ad^0\bar{\rho})^*)$ there are primes $q \in T_\tau$ such that $g|_{G_q} \neq 0$ and this is Proposition 3.4.6. □

3.5 The small exponent case

While the previous sections concerned about creating a setting appropriate for reproducing the proof of the main theorems in [Ram02] (task which will be accomplished in the next section) the strategy has a big gap, which is that the set C_ℓ and subspace N_ℓ of Case 4, Section 4 only work for powers p^m such that ρ_ℓ is not trivial modulo p^{m-1}.

It may be the case that we have a prime ℓ such that ρ_n is trivial (not only unramified) at ℓ, but our local deformation ρ_ℓ is ramified. In this case, we have not constructed the corresponding set C_ℓ and our main argument will not work.

To bypass this obstacle, we rely on a result by Khare, Larsen and Ramakrishna (the main idea appeared first in [KLR05] but we refer to a proposition in [Ram08] which has a cleaner statement). In those works it is essentially proved that given a global mod p^n deformation ρ_n, one can lift ρ_n a finite number of powers of p, keeping a desired local shape at a finite set of primes at the cost of adding a finite number of primes to the ramification of the deformation.

Proposition 3.5.1. *Let $\rho_n : G_P \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ and $z = (z_\ell)_{\ell\in P} \in \oplus_{\ell\in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ be any element. Then one of the following holds:*

- *There is a prime q which is auxiliary for ρ_n and an element $h \in \mathrm{H}^1(G_{P\cup\{q\}}, Ad^0\bar{\rho})$ such that the image of h in $\oplus_{\ell\in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ is z and $h|_{G_q} \in N_q$.*
- *There are two primes q_1 and q_2 auxiliary for ρ_n and an element $h \in \mathrm{H}^1(G_{P\cup\{q_1,q_2\}}, Ad^0\bar{\rho})$ such that the image of h in $\oplus_{\ell\in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ is z and $h|_{G_{q_i}} \in N_{q_i}$.*

Proof. This is Proposition 3.6 of [Ram08]. It involves a previous result proving the existence of a Chebotarev set of primes q that admit an element $h \in \mathrm{H}^1(G_{P\cup\{q\}}, Ad^0\bar{\rho})$ (Proposition 3.4) that maps to z and then a global reciprocity argument ensures that if it is the case that $h|_{G_q} \notin N_q$ one can find another prime in Q to fix this. □

The application of this result to our setting is the following.

Proposition 3.5.2. *Let $\rho_n : G_S \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ and $\rho_\ell : G_\ell \to \mathrm{GL}_2(W(\mathbb{F}))$ for $\ell \in P$ as in Theorem A. Assume that $\mathrm{III}^2_P(Ad^0\bar{\rho}) = 0$. Then for any exponent $s > n$ there is a finite set of primes P' containing P and a deformation*

$$\rho_s : G_{P'} \to \mathrm{GL}_2(W(\mathbb{F})/p^s)$$

such that:

- *ρ_s lifts ρ_n.*
- *$\rho_\ell \equiv \rho_s|_{G_\ell} \pmod{p^s}$.*
- *The primes in $P'\backslash P$ are auxiliary for ρ_n and $\rho_s|_{G_q}$ is a reduction of a member of C_q.*

In other words, we can lift ρ_n a finite number of powers of p, keeping a desired local deformation at a finite number of places at the cost of adding only finitely many auxiliary primes.

Proof. We prove this by induction in s. If $s = n$ there is nothing to prove. Assume that the result holds for an exponent s. We want to prove that it is also true for $s + 1$.

Let ρ_n and ρ_ℓ for every $\ell \in P$ as in the statement of the proposition. Applying our inductive hypothesis we get a deformation $\rho_s : G_{P'} \to \mathrm{GL}_2(W(\mathbb{F})/p^s)$ lifting ρ_n and satisfying the local conditions. As $\rho_\ell \bmod p^s$ lifts to $W(\mathbb{F})/p^{s+1}$ for all $\ell \in P$ and $\rho_s|_{G_q}$ is the reduction of some member of C_q for all $q \in P'\backslash P$, the deformation ρ_s is locally unobstructed. As we are assuming $\mathrm{III}^2_P(Ad^0\bar{\rho}) = 0$ this implies that ρ_s lifts to a $\widetilde{\rho_{s+1}} : G_{P'} \to \mathrm{GL}_2(W(\mathbb{F})/p^{s+1})$.

In order to find a lift that satisfies $\rho_\ell \equiv \rho_{s+1}|_{G_\ell} \pmod{p^s}$ for all $p \in P$ we need to adjust $\widetilde{\rho_{s+1}}$. We know that there is an element $z = (z_\ell)_{\ell\in P} \in \oplus_{\ell\in P}\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ that such that

$$(\mathrm{Id} + p^s z_\ell)\widetilde{\rho_{s+1}}|_{G_\ell} = \rho_\ell \pmod{p^{s+1}}$$

for all $\ell \in P$, and

$$(\mathrm{Id} + p^s z_q)\widetilde{\rho_{s+1}}|_{G_q} \in C_q$$

for all $q \in P'\backslash P$. We invoke Proposition 3.5.1 for $\widetilde{\rho_{s+1}}$ and z. It is easy to check that if we add the one or two auxiliary primes generated by it to the set P', the deformation $(\mathrm{Id} + p^s z)\widetilde{\rho_{s+1}} : G_{P'} \to \mathrm{GL}_2(W(\mathbb{F})/p^{s+1})$ satisfies what we want. $\square$

3.6 Proof of main theorems

In this section we prove the two main theorems of this chapter. For Theorem A we need the following lemma.

Lemma 3.6.1. *Let $\rho_n : G_S \to \mathrm{GL}_2(W(\mathbb{F})/\mathfrak{p}^n)$ as in Theorem A and let $r = \dim \mathrm{III}^1_S(Ad^0\bar{\rho})$. Then there is a set $Q = \{q_1, \ldots, q_r\}$ of auxiliary primes for ρ_n such that*

- *$\mathrm{III}^1_{S\cup Q}((Ad^0\bar{\rho})^*) \simeq \mathrm{III}^2_{S\cup Q}(Ad^0\bar{\rho}) \simeq 0$.*
- *The injective inflation map $\mathrm{H}^1(G_S, Ad^0\bar{\rho}) \to \mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho})$ is an isomorphism.*

Proof. This is Fact 16 of [Ram02] with the addition of the primes being auxiliary for ρ_n. One can check that the proof given there adapts for primes auxiliary for ρ_n.

For the first bullet, it is enough to take, for a basis $\{f_1, \ldots, f_r\}$ of $\mathrm{H}^1(G_S, (Ad^0\bar{\rho})^*)$, a set of auxiliary primes for ρ_n satisfying $f_i|_{q_j} = 0$ if $i \neq j$ and $f_i|_{q_i} \neq 0$. This set exists in virtue of Corollary 3.4.11.

For the second bullet, the proof in [Ram02] uses Wiles formula and some previously computed dimensions for cohomology groups for $Ad^0\bar{\rho}$. Observe that this does not depend on the primes being auxiliary for $\overline{\rho}$ or ρ_n, as are results for $Ad^0\bar{\rho}$. $\square$

We are now in a position to prove Theorem A.

Theorem A. *Let $\mathbb{F}$ be a finite field of characteristic $p \geq 5$. Let $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ a continuous representation ramified at a finite set of primes S satisfying the following properties:*

- *The image is big, i.e. $\mathrm{SL}_2(\mathbb{F}) \subseteq \mathrm{Im}(\overline{\rho_n})$ and $\mathrm{Im}(\overline{\rho_n}) = \mathrm{GL}_2(\mathbb{F})$ if $p = 5$.*
- *ρ_n is odd.*
- *The restriction $\overline{\rho_n}|_{G_p}$ is not twist equivalent to the trivial representation nor the indecomposable unramified representation given by $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.*
- *ρ_n does not ramify at 2.*

Let P be a finite set of primes containing S, and for every $\ell \in P$, $\ell \neq p$, fix a deformation $\rho_\ell : G_\ell \to W(\mathbb{F})$ of $\rho_n|_{G_\ell}$. At the prime p, let ρ_p be a deformation of $\rho_n|_{G_p}$ which is ordinary or crystalline with Hodge-Tate weights $\{0, k\}$, with $2 \leq k \leq p-1$.

Then there is a finite set Q of auxiliary primes $q \not\equiv \pm 1 \pmod{p}$ and a modular representation

$$\rho : G_{P \cup Q} \longrightarrow \mathrm{GL}_2(W(\mathbb{F})),$$

such that:

- *the reduction modulo p^n of ρ is ρ_n,*
- *$\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$ for every $\ell \in P$,*
- *$\rho|_{G_q}$ is a ramified representation of Steinberg type for every $q \in Q$.*

Proof. The proof mimics that of Theorem 1 of [Ram02], mixed with Proposition 3.5.2. Let $r = \dim_{\mathbb{F}} Ш^2_P(Ad^0\bar{\rho}) = \dim_{\mathbb{F}} Ш^1_P((Ad^0\bar{\rho})^*)$, and let $Q_1 = \{q_1, \ldots, q_r\}$ be set of primes given by Lemma 3.6.1, so that $Ш^2_{P\cup Q_1}(Ad^0\bar{\rho}) \simeq Ш^1_{P\cup Q_1}((Ad^0\bar{\rho})^*) \simeq 0$ and the inflation map $\mathrm{H}^1(G_P, Ad^0\bar{\rho}) \to \mathrm{H}^1(G_{P\cup Q_1}, Ad^0\bar{\rho})$ is an isomorphism.

We now apply Proposition 3.5.2 in order to put ourselves in a power of p at which we have the pairs (C_ℓ, N_ℓ) for every prime ℓ. Let n_0 be the least exponent such that ρ_ℓ modulo p^{n_0} is non-trivial for every ℓ such that ρ_ℓ is ramified. From Section 4 we know that for every $\ell \in P$ there is a pair (C_ℓ, N_ℓ) such that N_ℓ preserves the modulo p^m reductions of the elements in C_ℓ for every $m > n_0$. If $n_0 \neq n$, i.e. if there is any prime in ℓ for which ρ_n is trivial and ρ_ℓ is ramified, we apply Proposition 3.5.2 to ρ_n, with local lifts ρ_ℓ for $\ell \in P$, ρ_q any deformation in C_q for $q \in Q_1$ and exponent $s = n_0 + 1$. This gives a deformation $\rho_s : G_{P'} \to \mathrm{GL}_2(W(\mathbb{F})/p^s)$ lifting ρ_n and satisfying that $\rho_s|_{G_\ell} = \rho_\ell$ modulo p^s.

From here we carry on with the main global argument of [Ram02], we need to pick a set of primes Q_2 such that the map

$$\mathrm{H}^1(G_{P'\cup Q_2}, Ad^0\bar{\rho}) \to \bigoplus_{\ell\in P'\cup Q_2} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell,$$

is an isomorphism. Recall that once we achieved $\mathrm{III}^2_{P\cup Q_1} = 0$, no set of extra primes we consider adds new global obstructions.

The way to construct such set is as follows: take a basis $\{f_1, \ldots, f_d\}$ of the preimage under the restriction map $\mathrm{H}^1(G'_P, Ad^0\bar{\rho}) \to \oplus_{l\in P'} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ of the set $\oplus_{\ell\in P'} N_\ell$. For $1 \leq i \leq d$, let α_i be an element of $\mathrm{Gal}(L/K)$ all whose entries are 0 except the i-th which is a nonzero element in which $\tilde{c}$ acts trivially. If T_i is the Chebotarev set of primes such that its Frobenius lies in the class of $c \rtimes \alpha_i$, Proposition 3.4.12 implies that the map $\mathrm{H}^1(G_{P'\cup T_i}, Ad^0\bar{\rho}) \to \oplus_{\ell\in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ is surjective. By Lemma 14 ([Ram02]), we can pick a prime $q_i \in T_i$ such that if $Q_2 = \{q_1, \ldots, q_d\}$, then the map

$$\mathrm{H}^1(G_{P'\cup Q_2}, Ad^0\bar{\rho}) \to \oplus_{\ell\in P'} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell,$$

is surjective. It is easy to see that with this set Q_2, the map

$$\mathrm{H}^1(G_{P'\cup Q_2}, Ad^0\bar{\rho}) \to \bigoplus_{\ell\in P'\cup Q_2} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell,$$

is an isomorphism. This proves the existence of the lift $\rho : G_{P\cup Q} \to \mathrm{GL}_2(W(\mathbb{F}))$.

To prove that ρ is modular, we know it is odd and has big residual image hence it is residually modular (by Serre's conjectures). The modularity is then covered by the following two modularity lifting theorems: for the ordinary case modularity follows as a consequence of Theorem 5.2 of [SW01] (we are in a situation covered by the theorem stated in the introduction); for the supersingular case we apply Theorem 3.6 of [DFG04]. Observe that ρ is crystalline by definition and meets the shortness condition because it preserves the Hodge-Tate weights of $\rho_{f,p}$, which satisfy $2 \leq k \leq p-1$. The irreducibility condition holds because of the big image hypothesis.

It remains to check the condition on the restriction to inertia of the lift. We know that for every $\ell \in P$, the lift ρ constructed satisfies that $\rho|_{G_\ell} \in C_\ell$. For the primes ℓ such that $\rho_n|_{G_\ell}$ is ramified, the condition holds automatically as the reader may easily check that all the deformations in the corresponding set C_ℓ have isomorphic restrictions to inertia.

Finally, in the case where ρ_n is unramified and ρ_ℓ is Steinberg, observe that the set C_ℓ contains one element that is not ramified, as we are only imposing that restriction to inertia is unipotent and that all the deformations are equal to ρ_n when reduced modulo p^n. It is easily seen that all the other elements of C_ℓ are isomorphic when restricted to inertia. However, as we just proved that ρ is modular, if it were not ramified at such a prime ℓ, then the eigenvalues of $\rho(\mathrm{Frob}_\ell)$ should have the same absolute value. But the deformations of C_ℓ all satisfy that Frobenius have eigenvalues q and 1, implying that ρ is ramified at ℓ and the condition holds. $\square$

Let us recall the hypothesis of our second result: let $f \in S_k(\Gamma_0(N), \epsilon)$ be a newform, with coefficient field K_f and ring of integers $\mathcal{O}_f$. Let $\mathfrak{p}$ a prime ideal in $\mathcal{O}_f$ dividing a rational prime p and $K_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ their respective completions at $\mathfrak{p}$. Let

$$\rho_n : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n),$$

the reduction modulo $\mathfrak{p}^n$ of its p-adic Galois representation.

Theorem B. *In the above hypothesis, let $n > 0$ be an integer and $p > max(k, 3)$ be a prime such that:*

- $p \nmid N$ *or* f *is ordinary at* p,
- $\mathrm{SL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p}) \subseteq \mathrm{Im}(\overline{\rho_{f,\mathfrak{p}}})$, *and* $\mathrm{Im}(\overline{\rho_{f,\mathfrak{p}}}) = \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ *if* $p = 5$.
- p *does not ramify in the field of coefficients of* f.
- ρ_n *does not ramify at* 2.

Let R be the set of ramified primes of ρ_n. If $N' = \prod_{p \in R} p^{v_p(N)}$, then there exist an integer r, a set $\{q_1, \ldots, q_r\}$ of auxiliary primes prime to N satisfying $q_i \not\equiv 1 \pmod p$ and a newform g, different from f, of weight k and level $N'q_1 \ldots q_r$ such that f and g are congruent modulo p^n. Furthermore, the form g can be chosen with the same restriction to inertia as that of f at the primes of R.

Proof. We want to apply Theorem A to the representation ρ_n, with the local deformation $\rho_{f,p}|_{I_\ell}$ at the primes dividing N'. Note that f being a modular form implies that the representation is odd, and the hypothesis $p > k$ implies that $\rho_{f,p}|_{I_p}$ satisfies the third hypothesis of such theorem. Finally, the condition $p \nmid N$ or f being ordinary at p implies that $\rho_{f,p}|_{I_p}$ can be taken as a deformation at p.

Theorem A then gives a modular representation ρ which is congruent to $\rho_{f,p}$ modulo p^n, and of conductor dividing $N'q_1 \ldots q_r$. By the choice of the inertia action, the conductor of ρ has the same valuation as the ρ_n one at the primes dividing N', so we only need to show that all the primes q_i are ramified ones. But if this is not the case, by the choice of the sets C_{q_i}, and looking at the action of Frobenius, it would contradict Weil's Conjectures, since the roots of the Frobenius' characteristic polynomial would be 1 and q, which do not have the same absolute value.

Note that when ρ_f does not lose ramification when reduced modulo p^n and $r = 0$, the newform g that Theorem A produces could be equal to f. If this is the case, we apply Theorem A with $P = S \cup \{q\}$, q being in the hypotheses of auxiliary primes and

$$\rho_q = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$$

with $*$ ramified (up to twist). □

3.7 Example

We end this chapter by constructing an explicit example of level raising by following the proof of Theorem A for a particular representation. Let $E/\mathbb{Q}$ be an elliptic curve of prime conductor $\mathfrak{q}$ and full image at $p = 5$, i.e. $\mathrm{Gal}(\mathbb{Q}(E[5])/\mathbb{Q}) \simeq \mathrm{GL}_2(\mathbb{F}_5)$. We construct a newform in $S_2(\Gamma_0(\mathfrak{q}\mathfrak{r}))$ for some prime $\mathfrak{r}$, which is congruent to E modulo 25. The choices $p = 5$ and prime conductor are used to make the cohomological dimensions as small as possible.

Let ρ_5 be the 5-adic Galois representation attached to E (by looking at the Galois action on the Tate module). The adjoint representation of its residual representation is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$ which is isomorphic to S_5, the symmetric group in 5 elements. For $S = \{5, \mathfrak{q}\}$, we need to compute $\mathrm{H}^1(G_S, Ad^0\bar{\rho})$ and $\mathrm{H}^2(G_S, Ad^0\bar{\rho})$. Recall the following results:

- If $\ell \not\equiv \pm 1 \pmod p$ then $\mathrm{H}^2(G_\ell, Ad^0\bar{\rho}) = 0$ (see Section 3, or [Ram99] Proposition 2).
- If $\overline{\rho_5}$ is flat, and $\overline{\rho_5}|_{G_5}$ is indecomposable. Then $\mathrm{H}^2(G_5, Ad^0\bar{\rho}) = 0$ (see [Ram02], Table 3).

Let $r = \dim \text{Ш}^1_S((Ad^0\bar{\rho})^*)$, and s be the number of primes for which $H^2(G_\ell, Ad^0\bar{\rho}) \neq 0$, then (see [Ram02] Lemma, page 139):

- $\dim \mathrm{H}^1(G_S, Ad^0\bar{\rho}) = r + s + 2$.
- $\dim \mathrm{H}^2(G_S, Ad^0\bar{\rho}) = r + s$.

3.7.1 Some group theory

Recall from Lemma 3.4.5 that the elements in $\mathrm{H}^1(G_S, Ad^0\bar{\rho})$ (resp. in $\mathrm{H}^1(G_S, Ad^0\bar{\rho}^*)$) give extensions M of $\mathbb{Q}(Ad^0\bar{\rho})$ (resp. $\mathbb{Q}(Ad^0\bar{\rho}^*)$) whose Galois group over $\mathbb{Q}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5) \ltimes M_2^0(\mathbb{F}_5)$ (the 2×2 matrices with zero trace). The problem is that $\mathrm{PGL}_2(\mathbb{F}_5)$ has order 120, and we cannot do Class Field Theory in such extensions. To overcome this problem we study the groups involved so as to work with smaller extensions of $\mathbb{Q}$.

Lemma 3.7.1. *Let H be a subgroup of S_5 and $V \subseteq M_2^0(\mathbb{F}_5)$ an H-stable subspace. Then $H \ltimes V$ is a subgroup of $S_5 \ltimes M_2^0(\mathbb{F}_5)$. Furthermore, if $V \subseteq W$ then $H \ltimes V$ is a normal subgroup of $H \ltimes W$ if and only if H acts trivially on W/V.*

Proof. The first claim is clear from the definition of a semi-direct product. For the second claim, note that conjugation acts in the following way

$$(h, w)(g, v)(h, w)^{-1} = (hgh^{-1}, w + h \cdot v - (hgh^{-1}) \cdot w).$$

Since hgh^{-1} varies over all elements of H, the subgroup is normal if and only if $w - h \cdot w \in V$ for all $h \in H$. $\square$

Let H be the unipotent subgroup of $\mathrm{PGL}_2(\mathbb{F}_5)$ given by matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ and let $B \subset \mathrm{PGL}_2(\mathbb{F}_5)$ the Borel subgroup (of upper triangular matrices). Clearly $|H| = 5$, $|B| = 20$ and $H \triangleleft B$. For both H and B, $M_2^0(\mathbb{F}_5)$ has the following stable submodules filtration:

$$0 \subseteq U^0 \subseteq U^1 \subseteq M_2^0(\mathbb{F}_5),$$

where U^1 is the subspace of upper triangular matrices and U^0 is the subspace of strictly upper triangular matrices. The group H acts trivially on all quotients of this filtration and B acts trivially on U^1/U^0. Using Lemma 3.7.1 we get the following Hasse diagram

(3.2)

$$\begin{tikzcd}
 & & & M \arrow[dddlll, dash] \arrow[d, dash] \\
 & & & M^{H\ltimes\{0\}} \arrow[dl, dash, "5"', "\text{Galois}"] \arrow[d, dash] \\
 & & M^{H\ltimes U^0} \arrow[dl, dash, "5"', "\text{Galois}"] \arrow[d, dash] & M^{B\ltimes\{0\}} \arrow[dl, dash, "5"', "\text{non-Galois}"] \\
\mathbb{Q}(Ad^0\bar{\rho}) \arrow[d, dash, "5"'] & M^{H\ltimes U^1} \arrow[dl, dash, "5"', "\text{Galois}"] \arrow[d, dash] & M^{B\ltimes U^0} \arrow[dl, dash, "5"', "\text{Galois}"] & \\
\mathbb{Q}(Ad^0\bar{\rho})^H \arrow[d, dash, "\text{Galois}"', "4"] & M^{B\ltimes U^1} \arrow[dl, dash, "5"', "\text{non-Galois}"] & & \\
\mathbb{Q}(Ad^0\bar{\rho})^B \arrow[d, dash, "6"] & & & \\
\mathbb{Q} & & &
\end{tikzcd}$$

Remark. The Galois closure of $M^{H\ltimes U^1}$ is M, since it is easy to check that the intersection of all Galois conjugates of $H \ltimes U^1$ in $S_5 \ltimes M_2^0(\mathbb{F}_5)$ is trivial.

We also consider the subgroup $S_3 \times C_2$ (where C_n denotes the cyclic group of order n). If we identify $S_3 \times C_2 = \langle \left(\begin{smallmatrix}1&2\\2&0\end{smallmatrix}\right), \left(\begin{smallmatrix}4&2\\1&1\end{smallmatrix}\right) \rangle \times \langle \left(\begin{smallmatrix}3&2\\2&2\end{smallmatrix}\right) \rangle$ in $\mathrm{PGL}_2(\mathbb{F}_5)$, the action decomposes as $\langle \left(\begin{smallmatrix}3&1\\0&2\end{smallmatrix}\right), \left(\begin{smallmatrix}3&0\\1&2\end{smallmatrix}\right) \rangle \oplus \langle \left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right) \rangle$. The action in the 1-dimensional subspace (which can be identified with the quotient) is non-trivial. Nevertheless its restriction to the cyclic subgroup of order 6 is trivial (such group is the stabilizer of the matrix $\left(\begin{smallmatrix}4&1\\1&1\end{smallmatrix}\right)$). It is clear that the intersection of its conjugates is trivial (since A_5 is the only normal subgroup of S_5 and the action of S_5 in $M_2^0(\mathbb{F}_5)$ is irreducible).

Lemma 3.7.2. $(C_3 \times C_2) \ltimes V_2 \lhd (S_3 \times C_2) \ltimes M_2^0(\mathbb{F}_5)$.

Proof. The previous Lemma implies that $(C_3 \times C_2) \ltimes V_2 \lhd (C_3 \times C_2) \ltimes M_2^0(\mathbb{F}_5)$ but since $C_3 \lhd S_3$, the same proof gives the statement. $\square$

For such group, we get the following Hasse diagram.

$$\begin{tikzcd}
 & M \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{Q}(Ad^0\bar{\rho}) \arrow[d, dash] & M^{C_6\ltimes V_2} \arrow[dl, dash, "\text{Galois}"] \arrow[d, dash] \\
\mathbb{Q}(Ad^0\bar{\rho})^{C_6} \arrow[d, dash] & M^{(S_3\times C_2)\ltimes V_2} \arrow[dl, dash, "\text{non-Galois}"] \\
\mathbb{Q}(Ad^0\bar{\rho})^{S_3\times C_2} \arrow[d, dash] & \\
\mathbb{Q} &
\end{tikzcd}$$

To compute with the adjoint representation, we must add the 5-th roots of unity. The Hasse diagram is the following

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(Ad^0\bar{\rho}, \xi_5) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(Ad^0\bar{\rho}) \arrow[dr, dash] & & \mathbb{Q}(\xi_5) \arrow[dl, dash] \\
 & \mathbb{Q}(\sqrt{5}) \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd} \tag{3.3}
$$

Then $\mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho}^*)/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho}, \xi_5)/\mathbb{Q}) \simeq C_4 \ltimes A_5$, where the action is through the projection $C_4 \to C_2$, and the latter action is the classical isomorphism $S_5 \simeq C_2 \ltimes A_5$. This Galois group also acts on $M_2^0(\mathbb{F}_5)$, where the C_4 part acts as $\mathbb{F}_5^\times$ (which corresponds to the mod 5-cyclotomic character action), and A_5 as before. To compute the Shafarevich group $\text{Ш}^1(G_S, Ad^0\bar{\rho}^*)$, we do a similar trick as before, we consider the subgroup $C_4 \ltimes C_3$ (which also satisfies that the intersection of its conjugates is trivial), which is an extension of the previous cyclic group of order 6, and get exactly the same degree 20 extension.

3.7.2 Particular example

In this section we will use many computations that were done using [PAR13]. The script used for the computations as well as a text file with every computation made can be found in Ariel Pacetti's webpage (`mate.dm.uba.ar/~apacetti`). Consider the elliptic curve

$$E_{89b1} : y^2 + xy = x^3 + x^2 - 2$$

Let $\rho_{E,5}$ denote the representation attached to the 5-adic Tate module of E. The residual representation has full image (using [S$^+$13]), so if we look at the representation on the 5^2 torsion points, we get a representation that is in the hypothesis of Theorem A. The residual adjoint representation corresponds to a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5) \simeq S_5$ and ramified at 5 and 89. We can search for such extensions (they are the Galois closure of a degree 5 extension) in Jones-Roberts tables (see [JR13]), and get 12 such extensions, given by the polynomials:

$$
\begin{array}{rc}
x^5 - x^4 + 5x^3 - x^2 + 6x + 1, & x^5 + 10x^3 - 20x^2 + 45x - 148, \\
x^5 - 5x^3 - 5x^2 - 5x - 6, & x^5 - 30x^2 - 30x - 97, \\
x^5 - 125x^2 + 375x + 425, & x^5 + 445x - 445, \\
x^5 - 890x^2 - 4005x - 5429, & x^5 - 890x^2 + 9790x + 10591, \\
x^5 - 445x^2 + 20915x + 159132, & x^5 + 50x^3 - 125x^2 + 350x - 680, \\
x^5 - 50x^3 - 325x^2 - 375x - 5220, & x^5 + 200x^3 - 1625x^2 + 9575x - 176395, \\
x^5 - 200x^3 - 375x^2 + 22925x - 81155. &
\end{array}
$$

To know which one corresponds to our elliptic curve, we just compute the order of Frobenius at $3, 7, 11$ and 13, which are $6, 4, 3$ and 6 respectively. If we compute the inertial degree at those primes in the above extensions, we see that the only extension with those inertial degrees is the one corresponding to $x^5 + 445x - 445$.

Lemma 3.7.3. *The representation $\overline{\rho}_{E,5}$ satisfies the following properties:*

- *The extensions corresponding to its image and the adjoint image ramify at* 89.
- *If we restrict the representation to the decomposition group at* 5, *it is ordinary and indecomposable.*

Proof. The first fact can be checked by computing the field discriminant (note that the scalar matrices correspond to an extension unramified at 89). Nevertheless, this is a more general statement, since if the residual representation is unramified at 89, by Ribet's lowering the level theorem, there should exist a weight 2 and level 1 modular form, which is not the case. To prove the second statement, we know that the representation is ordinary because $a_5(E) = -2$ (it is not divisible by 5). If the restriction to inertia at 5 were decomposable, then the order of inertia would be 4, but 5 ramifies completely in the degree 5 extension computed above. $\square$

The degree 20 subextension of $\mathbb{Q}(Ad^0\bar{\rho_E})$ is given by the polynomial

$$\begin{aligned}P(x) = x^{20} &+ 45822985000x^{16} + 245086878906250x^{14} + 535483380861855000000x^{12}\\ &+ 670170049528361372070 3125x^{10} + 23236195966282229157309570 3125x^{8}+\\ &2596208525095250777917321777 3437500x^{6} - 4031899037684302260560543711938476 56250x^{4}+\\ &4640939013548409613939783894070434570312500x^{2} + 96689369817657701380917597046902374542236328125.\end{aligned}$$

Lemma 3.7.4. $\dim \mathrm{H}^2(G_{\{5,89\}}, Ad^0\bar{\rho}_{E,5}) = 0$ *and* $\dim \mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho}_{E,5}) = 2$.

Proof. Recall that $\dim \mathrm{H}^2(G_S, Ad^0\bar{\rho_E}) = r + s$ and $\dim \mathrm{H}^1(G_S, Ad^0\bar{\rho_E}) = r + s + 2$, where $r = \dim \text{III}^1_S((Ad^0\bar{\rho_E})^*)$ and s is the number of $\ell \in S$ such that $\dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho_E}) \neq 0$.

It can be checked that E has split multiplicative reduction at 89, implying that the residual representation is Principal Series at 89.

By the comments at the beginning of Section 3.7, as $89 \equiv -1 \pmod 5$, we have that $\mathrm{H}^2(G_{89}, Ad^0\bar{\rho_E}) = 0$ and since $\overline{\rho}_{E,5}|_{G_5}$ is indecomposable $\mathrm{H}^2(G_5, Ad^0\bar{\rho_E}) = 0$ and $s = 0$. On the other hand, elements of $\text{III}^1_S((Ad^0\bar{\rho_E})^*)$ give raise to unramified degree 5 abelian extensions of $\mathbb{Q}((Ad^0\bar{\rho_E})^*)$ where the primes above 5 and 89 split completely. In particular, they are unramified extensions of $\mathbb{Q}(Ad^0\bar{\rho_E})^{C_6}$. Using [PAR13] one can check that the class number of such degree 20 extension is 24, which is not divisible by 5, so Sha is trivial and $r = 0$. $\square$

Remark. The same argument proves that $\dim \mathrm{H}^1(G_{\{5\}}, Ad^0\bar{\rho_E}) = 2$, and by the inflation-restriction exact sequence, $\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E}) \simeq \mathrm{H}^1(G_{\{5\}}, Ad^0\bar{\rho_E})$ so we restrict to elements which are unramified at 89.

Remark. In our hypothesis, the local $\mathrm{H}^1(G_5, Ad^0\bar{\rho_E})$ has dimension 3, and the subspace N_5 is that of finite flat group schemes which is 1 dimensional (by Table 3 of [Ram02]).

Consider the map

$$\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E}) \mapsto \mathrm{H}^1(G_5, Ad^0\bar{\rho_E})/N_5 \times \mathrm{H}^1(G_{89}, Ad^0\bar{\rho_E})/N_{89}. \tag{3.4}$$

Recall that $N_{89} = \mathrm{H}^1(G_{89}, Ad^0\bar{\rho_E})$, so we can just discard this term. Both spaces have dimension 2, so we need to compute the kernel of the map. Elements on the left give raise to degree 5 extensions of $L = \mathbb{Q}(Ad^0\bar{\rho_E})^H$ that are unramified outside 5 and 89. A polynomial defining L is

$$\begin{gathered}
x^{24} - 9901250x^{21} - 2291149250x^{20} - 110151406250x^{19} + 38233553109375x^{18} + 23557750800468750x^{17} + \\
1161955520408009375 0x^{16} - 1941333167816406250 0x^{15} - 1254239837597580528906 25x^{14} - \\
5148803827682672656250 0000x^{13} - 1052367824109336645517382 8125x^{12} - 1061308570774787162882324 21875x^{11} \\
- 17526325566077155347275909 1796875x^{10} + 44232966417342564073908569 335937500x^{9} + \\
2260727809663301086233535775 6591796875x^{8} + 4918993595719501665872626404 05273437500x^{7} + \\
28672677663271022255971277124 0091552734375x^{6} + 612544596163856058543914638034 96704101562500x^{5} + \\
5346974474154298521538612265233 0757202148437 50x^{4} + 33302448226823892464391700813613 24882507324218 75x^{3} + \\
53735066160353981335257513593580 63694000244140 6250x^{2} + 4715974971592347401743210281496148 22492599487304 6875x + \\
1836690601447937075527179594897747 0947694778442 3828125
\end{gathered}$$

In order to replicate the proof of Theorem A we need to understand morphism (3.4). We thank Ravi Ramakrishna for the following observation.

Lemma 3.7.5. *The morphism (3.4) has one dimensional kernel.*

Proof. The domain of the morphism (3.4) is of dimension 2. We will see that its kernel is neither 0 nor 2 dimensional.

Notice that the kernel of the morphism (3.4) is the tangent space of the deformation problem corresponding to considering minimally ramified lifts of $\overline{\rho}_E$. If the morphism were injective, then the universal deformation ring should be isomorphic to $\mathbb{Z}_5$, and there should be an unique lift to any coefficient ring. However, it can be checked that there is a modular form of level 89 and weight 2 which is congruent to E modulo 5, implying that this is not the case. Therefore, the kernel of the morphism (3.4) is not trivial.

On the other hand, since $N_89 = \mathrm{H}^1(G_{89}, Ad^0\bar{\rho_E})$, the kernel of morphism (3.4) is formed by the cocycles mapping to N_5 in $\mathrm{H}^1(G_5, Ad^0\bar{\rho_E})$. As the elements in $\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E})$ are only ramified at 5, if two linearly independent cocycles map to N_5 (which is one dimensional) we can take a linear combination of them mapping to zero. In particular, it gives an unramified extension of $\mathbb{Q}(Ad^0\bar{\rho_E})$. But we already mentioned that there are no such extensions. This implies that the dimension of the kernel of morphism (3.4) is less than 2 and completes the proof. $\square$

Lemma 3.7.5 tells us that there is a cocycle in $\mathrm{H}^1(G_{5,89}, Ad^0\bar{\rho_E})$ that maps to N_5. We want to compute this extension. The following Lemma describes the extensions corresponding to cocycles that map to N_5.

Lemma 3.7.6. *A cocycle κ lies in $\mathrm{H}^1_{flat}(G_5, Ad^0\bar{\rho})$ if and only if there is a prime above* 5 *in $\mathbb{Q}(Ad^0\bar{\rho})^B$ that does not ramify in $M^{B\ltimes U^0}$.*

Proof. Let $F = \mathbb{Q}(Ad^0\bar{\rho})^B$ and $F' = M^{B\ltimes U^0}$.

Recall that for a given cocycle $\kappa \in \mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho})$ the field M is the field fixed by the subgroup $\operatorname{Ker}\kappa|_{G_{\mathbb{Q}(Ad^0\bar{\rho})}} = \kappa|^{-1}_{G_{\mathbb{Q}(Ad^0\bar{\rho})}}(0)$. Keeping in mind that F is the field fixed by $G_F = \kappa|^{-1}_{G_F}(Ad^0\bar{\rho})$ and that with the identification $\mathrm{Gal}(M/\mathbb{Q}) \simeq \mathrm{PGL}_2(\mathbb{F}_5) \ltimes M^0(\mathbb{F}_5)$ we have $\kappa(h \ltimes M) = h \cdot M$ it can be easily seen that F' is the field fixed by $\kappa|^{-1}_{G_F}(U^0)$.

Let I_5 be a inertia group for 5 in $\mathrm{Gal}(M/\mathbb{Q})$. Recall that

$$\mathrm{H}^1_{\mathrm{flat}}(G_5, Ad^0\bar{\rho}) = \operatorname{Ker}\left(\mathrm{H}^1(G_5, Ad^0\bar{\rho}) \to \mathrm{H}^1(I_5, Ad^0\bar{\rho}/U^0)\right)$$

and therefore $\kappa \in \mathrm{H}^1_{\text{flat}}(G_5, Ad^0\bar{\rho})$ if and only if there is a representative of the class such that $\kappa(I_5) \subseteq U^0$ which happens if and only if $I_5 \subseteq \kappa^{-1}(U^0)$.

We claim that $I_5 \subseteq \kappa^{-1}(U^0)$ if and only if $I_5 \cap G_F \subseteq \kappa|_{G_F}^{-1}(U^0)$ if and only if $\kappa|_{G_F}(I_5 \cap G_F) \subseteq U^0$. This follows from

- κ factors through $\mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/\mathbb{Q}) \ltimes \mathrm{Gal}(M/\mathbb{Q}(Ad^0\bar{\rho}))$
- The image of I_5 in $\mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/\mathbb{Q}) \ltimes \mathrm{Gal}(M/\mathbb{Q}(Ad^0\bar{\rho}))$ is $\bar{\rho}(I_5) \ltimes \kappa(I_5)$.
- $\kappa(I_5 \cap \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/\mathbb{Q}) \ltimes 1) = \bar{\rho}(I_5) \ltimes 1 = \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/F) \simeq B \ltimes 1$
- U^0 is stable by $\mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho})/F) \simeq B$

On the other hand, it follows from the previous discussion that $I_5 \cap G_F \subseteq \kappa|_{G_F}^{-1}(U^0)$ if and only if the prime in F above 5 corresponding to the chosen I_5 does not ramify in F'. $\square$

Remark. In order to compute the desired cocycle, we need to find the corresponding extension F'/F. Instead of looking for degree 25 extensions of F, we will do class field theory over the extension L of degree 24, looking for degree 5 unramified at a prime above 5 with ramification degree 4 in $L/\mathbb{Q}$ (such prime must exist as the unramified prime above 5 in F ramifies completely in $\mathbb{Q}(Ad^0\bar{\rho})/F$).

The bound for the exponent of the modulus $e(\mathfrak{p})$ is given by the following result.

Proposition 3.7.7. *Let L/K be an abelian extension of prime degree p and $\mathfrak{p}$ a prime ideal of K. Let $e(\mathfrak{p}|p)$ denote the ramification degree of $\mathfrak{p}$ over the rational prime p. If $\mathfrak{p}$ ramifies in L/K, then*

$$\begin{cases} e(\mathfrak{p}) = 1 & \text{if } \mathfrak{p} \nmid p \\ 2 \leq e(\mathfrak{p}) \leq \left\lfloor \frac{pe(\mathfrak{p}|p)}{p-1} \right\rfloor + 1 & \text{if } \mathfrak{p} \mid p. \end{cases}$$

Proof. See [Coh00] Proposition 3.3.21 and Proposition 3.3.22. $\square$

The prime 5 factors as $\mathfrak{p}_{5,1}^{20}\mathfrak{p}_{5,2}^{4}$ in L, where each prime ideal $\mathfrak{p}_{5,i}$ has inertial degree 1. By Remark 3.7.2 we do not need to allow ramification at the prime 89. Recall that we are looking for extensions unramified at $\mathfrak{p}_{5,2}$, Proposition 3.7.7 tells us that the modulus is $\mathfrak{p}_{5,1}^{25}\mathfrak{p}_{5,2}^{0}$. Such class group (using [PAR13]) is isomorphic to

$$C_{100} \times C_5 \times C_5 \times C_5 \times C_5 \times C_5 \times C_5 \times C_5 \times C_5.$$

From all these degree 5 extensions, we need to identify the ones that correspond to elements in $\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E})$ (which give extensions isomorphic to $M_2^0(\mathbb{F}_5)$). Let $\tilde{L}$ denote the abelian degree 5 extension $M^{H\ltimes U^1}$ of L attached to an extension M in $\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E})$.

Lemma 3.7.8. *If a rational prime p is unramified in $\mathbb{Q}(Ad^0\bar{\rho_E})$ and has a prime ideal of L over it with inertial degree 5, then it has inertial degree 5 in M.*

Proof. Let $\mathfrak{p}$ be a prime in M dividing the prime with inertial degree 5 in L. Since the maximal 5-Sylow subgroup of S_5 is cyclic of order 5, the decomposition group of p in $\mathbb{Q}(Ad^0\bar{\rho_E})$ is cyclic of order 5. Then the decomposition group $D(\mathfrak{p})$ is a subgroup $C_5 \ltimes M_2^0(\mathbb{F}_5)$. Since a cyclic group cannot be written as a semidirect product of groups whose orders are divisible by 5, $D(\mathfrak{p}) = C_5$. $\square$

Test 1: for each prime p check whether it has inertial degree 5 in $\mathbb{Q}(Ad^0\bar{\rho_E})$ or not (by looking how the degree 5 polynomial splits modulo p). If it does, search for all primes in L with inertial degree 5, and restrict to the subspace of characters in the class group which are trivial on them.

This first test lowers the dimension drastically. With primes up to 300, we find that the subspace V which passes the test has dimension 2.

Lemma 3.7.9. *Let L/K be a Galois extension, and M/L be a Galois extension of prime degree p corresponding to a character χ. Consider the vector space obtained by evaluating the Galois conjugates of χ at all different prime ideals, and let r denote its dimension (as an $\mathbb{F}_p$ vector space). Then the Galois closure of M over K has degree p^r.*

Proof. This is an easy exercise of Galois theory. □

Test 2: consider each character of V as a character on $\mathbb{Q}(Ad^0\bar{\rho_E})$ by composing with the norm map to L. To compute the action of $\mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho_E})/\mathbb{Q})$ on it, it is enough to determine its values at prime ideals which split completely in $\mathbb{Q}(Ad^0\bar{\rho_E})/\mathbb{Q}$ (they have density 1 in $\mathbb{Q}(Ad^0\bar{\rho_E})$) where the Galois action becomes simpler. To compute the conjugates of the character, we compute the values that the character takes on the conjugates of these primes. Let $\alpha_1, \ldots, \alpha_5$, be the the roots of $Q(x) = x^5 + 445 * x - 445$ (so $\mathbb{Q}(Ad^0\bar{\rho_E}) = \mathbb{Q}(\alpha_1, \ldots, \alpha_5)$) and let $L = \mathbb{Q}(\beta)$, where $\beta = P(\alpha_1, \ldots, \alpha_5)$ (in our case, we can take $P(x_1, \ldots, x_5) = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1$). Recall that any prime ideal $\mathfrak{q} \in \mathcal{O}_L$ which splits completely can be presented in the form $\mathfrak{q} = \langle \beta - a_\mathfrak{q}, q\rangle_{\mathcal{O}_L}$ where $q = \mathcal{N}(\mathfrak{q})$ and $a_\mathfrak{q} \in \mathbb{F}_q$. In particular, $a_\mathfrak{q}$ is the unique element in $\mathbb{F}_q$ which satisfies that $v_\mathfrak{q}(\beta - a_\mathfrak{q}) \geq 1$.

Note that since $Q(x)$ factors linearly modulo q (with roots $\tilde{\alpha_1}, \ldots, \tilde{\alpha_5}$), there is a match between $\{\alpha_i\}$ and $\{\tilde{\alpha_i}\}$ which makes $a_\mathfrak{q} = P(\tilde{\alpha_1}, \ldots, \tilde{\alpha_5})$ (since $\alpha_i - \tilde{\alpha_i} \in (q)$). Then if $\sigma \in \mathrm{Gal}(\mathbb{Q}(Ad^0\bar{\rho_E})/\mathbb{Q})$ (which we identify with S_5), its action on $\mathfrak{q}$ is given by sending the ideal $\mathfrak{q}$ to the unique ideal $\tilde{\mathfrak{q}}$ such that $a_{\tilde{\mathfrak{q}}}$ equals $P(\tilde{\alpha}_{\sigma(1)}, \ldots, \tilde{\alpha}_{\sigma(5)})$.

With this procedure, we loop over all characters of V (up to powers, i.e. we can think of them as elements in $\mathbb{P}^2(\mathbb{F}_5)$) and compute the number of Galois conjugates of it at a finite list of primes (the first 5 splitting primes work) discarding the ones giving a vector space of dimension greater than 3. There are only 2 elements in $\mathbb{P}^4(\mathbb{F}_5)$ whose vector space has dimension smaller than 4. One of these elements corresponds to our cocycle.

To identify it, we need to run a not so rigorous test. Recall that we are searching for extensions whose Galois group is $S_5 \ltimes M_2^0(\mathbb{F}_5)$. Since we cannot compute the Galois closure of our degree 5 extensions, we use Chebotarev density theorem. If M is such an extension, and a prime number has inertial degree 6 in $\mathbb{Q}(Ad^0\bar{\rho_E})$, then it might have inertial degree 6 or 30 in M. Furthermore, once we fixed an element in S_5 of order 6, it is easy to see that there are 100 choices (out of the 125) of elements in $S_5 \ltimes M_2^0(\mathbb{F}_5)$ of order 30 and 25 of order 6 whose projection to S_5 gives the chosen order 6 element, giving a density of 0.8.

Test 3: for the two characters, we check whether they are trivial or not at all primes with inertial degree 6 in $\mathbb{Q}(Ad^0\bar{\rho_E})$ up to a given bound, say 10.000. For the first character, we find that 156 out of 208 primes have inertial degree bigger than 6 while in the second case the same happens for 24 out of 208 primes. This implies that the first character corresponds to the extension we are looking for.

Remark. One can make the third test complete by using some explicit version of Chebotarev density theorem but the range of computation will take too long without assuming for example Artin's conjectures.

We know that the image of (3.4) has dimension 1. In particular just one extra prime is enough to get an isomorphism. We search for a prime $q \not\equiv \pm 1 \pmod 5$ and such that $a_q \equiv \pm(q+1) \pmod{25}$. The prime $q = 293$ satisfies both conditions, since $a_{293} = -6 \equiv -(293+1) \pmod{25}$.

Theorem 3.7.10. *There exists a weight* 2 *modular form of level* $89 \cdot 293$ *which is congruent modulo* 5^2 *to the modular form attached to* E_{89b1}.

Proof. In view of the previous discussion, we just need to check that 293 is the right choice for the map

$$\mathrm{H}^1(G_{\{5,89,293\}}, Ad^0\bar{\rho_E}) \mapsto \mathrm{H}^1(G_5, Ad^0\bar{\rho_E})/N_5 \times \mathrm{H}^1(G_{89}, Ad^0\bar{\rho_E})/N_{89} \times \mathrm{H}^1(G_{293}, Ad^0\bar{\rho_E})/N_{293},$$

to be an isomorphism. Since $293 \not\equiv \pm 1 \pmod 5$, $\dim \mathrm{H}^1(G_{\{5,89,293\}}, Ad^0\bar{\rho_E}) = 3$. Let κ_{293} denote a non-zero element not in $\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E})$, and let κ_1, κ_2 be a basis of $\mathrm{H}^1(G_{\{5,89\}}, Ad^0\bar{\rho_E})$, such that $\kappa_2 = \kappa$ (the cycle unramified at $\mathfrak{p}_{5,2}$). Then in the basis $\{\kappa_1, \kappa_2, \kappa_3\}$ the linear transformation matrix looks like $\left(\begin{smallmatrix} a & 0 & b \\ c & 0 & d \\ e & f & g \end{smallmatrix}\right)$. To prove it is invertible, it is enough to prove that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible, and that f is non-zero.

Since the image of $\mathrm{H}^1(G_{\{5\}}, Ad^0\bar{\rho_E})$ in $\mathrm{H}^1(G_5, Ad^0\bar{\rho_E})$ is two dimensional, if κ_{293} restricted to G_5 is not linearly independent with them, there should exist an extension of $\mathbb{Q}(Ad^0\bar{\rho_E})$ which is unramified outside 293, but using CFT one easily sees that there are no such extensions (the ray class group is isomorphic to $C_{3504} \times C_{12} \times C_2$). Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible.

To prove that $f \neq 0$, we need to check that the prime 293 does not split completely in the extension attached to the cocycle κ. Using the complete description of such cocycle (as a character of a class group) we evaluate it at the primes dividing 293 and see that it is trivial at 2 primes, and not trivial at the other 4 ones, which implies that 293 does not split completely from $\mathbb{Q}(Ad^0\bar{\rho_E})$ to M. This ends the proof. □

Remark. In this particular case, searching for the particular form is out of computational reach, as the level $89 \cdot 293$ is too big to compute the corresponding space.

Levantando representaciones de Galois: el caso ramificado

Introducción

El presente capítulo es en algún sentido una continuación del trabajo realizado en el anterior. En él construimos, para un cuerpo finito $\mathbb{F}$, levantados de representaciones $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ a $\mathrm{GL}_2(W(\mathbb{F}))$. En este capítulo mostramos como dichos resultados se pueden generalizar a representaciones cuyos coeficientes viven en extensiones finitas $K/\mathbb{Q}_p$ ramificadas.

El método empleado en el Capítulo 2 seguía las ideas de [Ram99] y [Ram02], adaptadas a representaciones módulo p^n. Como fue mencionado en dicho capítulo, estos métodos fallan cuando el cuerpo de coeficientes en el que se esta trabajando es ramificado. El problema es que la demostración del Teorema A del capítulo 2 se basa en demostrar que existen subespacios $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$, para primos ℓ de un conjunto finito M que preservan a ciertas familias de deformaciones C_ℓ que contienen a $\rho_n|_{G_\ell}$ y hacen que el morfismo de restricción local

$$\mathrm{H}^1(G_M, Ad^0\bar{\rho}) \to \bigoplus_{\ell \in M} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

sea sobreyectivo. Si se desea realizar lo mismo para una representación $\rho_n : G_M \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$, donde $\mathcal{O}$ es el anillo de enteros de una extensión finita ramificada de $\mathbb{Q}$, se debe tener en cuenta que la reducción módulo π^2 de ρ_n es una deformación de $\overline{\rho}$ al anillo $\mathbb{F}[\varepsilon]$ de números duales y como tal tiene asociada un elemento $f \in \mathrm{H}^1(G_M, Ad^0\bar{\rho})$. El problema es que siempre que se quiera definir una colección de subespacios N_ℓ que preserven las familias C_ℓ, el elemento f se encontrará en el núcleo del morfismo de restricción local, lo que hace que el mismo nunca pueda ser sobreyectivo, dado que dominio y codominio tienen la misma dimensión como $\mathbb{F}$-espacios vectoriales.

La innovación de este capítulo es la idea de relajar las condiciones locales, de forma que los subespacios N_ℓ sean de dimensión mayor y el morfismo de restricción local pueda ser sobreyectivo. Sin embargo, es de esperar que no sea posible relajar las hipótesis en los primos $\ell \neq p$, dado que el anillo universal para $\overline{\rho}|_{G_\ell}$ no debería tener un cociente suave de dimensión mayor a $\dim \mathrm{H}^1(G_\ell, Ad^0\bar{\rho}) - \dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$. Por lo tanto, la única opción es relajar la condición impuesta en el primo p. A este fin, introducimos la noción para una deformación de ser "casi ordinaria.

Definición. Decimos que una deformación es "casi ordinaria" si su restricción al subgrupo de inercia es triangular superior y su semisimplificación no es escalar, i.e. si

$$\rho|_{I_p} \simeq \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$

donde $\psi_1 \neq \psi_2$.

Utilizando esta condición local en el primo p se obtiene el siguiente teorema (Teorema 4.4.1), que es uno de los principales resultados de este capítulo.

Teorema. *Sea $p \geq 5$ y $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2/(\mathcal{O}/\pi^n)$ una representación continua que es impar, no ramificada en 2 y casi ordinaria en p. Asumamos que $\mathrm{Im}(\rho_n)$ contiene a $\mathrm{SL}_2(\mathcal{O}/p)$ si $n \geq e$ y $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ en caso contrario. Sea P un conjunto de primos de $\mathbb{Q}$ conteniendo el conjunto de ramificación de ρ_n y al primo p. Para cada $\ell \in P\backslash\{p\}$ se fija una deformación $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ que no es mala (ver Definición 4.2). Entonces existe una representación continua $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ y un conjunto finito de primos R tal que:*

- *ρ levanta a ρ_n, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.*
- *ρ es no ramificada fuera de $P \cup R$.*
- *Para todo $\ell \in P$, $\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$.*
- *ρ es casi ordinaria en P.*
- *Todos los primos de R, excepto posiblemente uno, no son congruentes con 1 módulo p.*

De hecho, el método empleado no da solo un levantado de ρ_n a $\mathcal{O}$, sino una familia de levantados a anillos de característica 0 parametrizado por un levantado de ρ_n al anillo $W(\mathbb{F})[[X]]$ (ver Teorema 4.4.15). El costado negativo de nuestra construcción es que las deformaciones de nuestra familia no son ordinarias, sino casi ordinarias, lo que implica que la mayoría de los puntos de dicha familia no son Hodge-Tate en el primo p (y por lo tanto no son modulares). La pregunta natural que aparece es acerca de la existencia de puntos modulares en dicha familia. Nuestro trabajo no da una respuesta definitiva a dicha pregunta, pero cuando la deformacion satisface ciertas condiciones, estamos en condiciones de probar la existencia de puntos modulares, resultado que se encuentra en el Teorema 4.5.1.

Teorema. *Sea $p \geq 5$ un primo, $\mathcal{O}$ el anillo de enteros de una extensión finita $K/\mathbb{Q}_p$ con grado de ramificación $e > 1$ y π su uniformizador local. Sea $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ una representación continua satisfaciendo*

- *ρ_n es impar.*
- *$\mathrm{Im}(\rho_n)$ contiene $\mathrm{SL}_2(\mathcal{O}/p)$ si $n \geq e$ y $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ en caso contrario.*
- *ρ_n es ordinaria en p.*
- *ρ_n es no ramificada en 2.*

Sea P un conjunto finito de primos conteniendo al conjunto de ramificación de ρ_n y al primo p y para cada $\ell \in P\backslash\{p\}$ sea $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ una deformación de $\rho_n|_{G_\ell}$ que no es mala (ver Definición 4.2). Entonces existe un conjunto finito de primos R y una representación $\rho : G_{P\cup R} \to \mathrm{GL}_2(\mathcal{O})$ tal que

- *ρ levanta a ρ_n, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.*

- *ρ es modular.*

- *Para todo $\ell \in P$, $\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$.*

- *Para todo $q \in R$, $\rho|_{I_q}$ es unipotente y todos los primos de R, salvo quizás uno, satisfacen $q \not\equiv 1 \pmod{p}$.*

- *ρ es ordinaria en p.*

- *ρ_n y la colección $\{\rho_\ell\}$ son una familia balanceada (ver Definición 4.5).*

La estrategia para probar ambos teoremas es, a grandes rasgos, similar a la aplicada en el capítulo 2 y en [Ram02]. Del mismo modo que en el capítulo anterior, se debe resolver un problema local y un problema global. En la primera sección se trata el problema local. Al igual que en el capítulo anterior el mismo consiste en construir familias de levantados de $\rho_n|_{G_\ell}$ de cierta dimensión. La principal diferencia es que al ser el anillo de coeficientes ramificado la cantidad de tipos de reducción posibles aumenta y hay casos en los que no es posible encontrar la familia deseada. Dichos casos son excluidos (ver Definición 4.2), y en la subsección correspondiente se muestra cual es el problema que presentan.

La sección 3, trata la condición local en el primo p. Se trata de una sección breve en la que se introduce el concepto de deformación casi ordinaria y se prueba que el espacio de dichas deformaciones de $\rho_n|_{G_p}$ posee la dimensión deseada.

La sección 4 trata el argumento global e incluye la demostración del primer teorema principal. Al igual que en el capítulo anterior, se debe tratar dos casos por separado, uno emulando los argumentos globales de [Ram99] y [Ram02], y otro para sobrellevar el problema técnico que aparece al no poder construir las familias C_ℓ para todos los exponentes. Nuevamente se deben adaptar los argumentos de [KLR05], para probar un resultado de levantamiento de una potencia de π agregando finitos primos auxiliares. Cabe destacar que los argumentos globales no son los mismos que en el caso no ramificado, debido a la presencia del elemento f, lo que hace que la extensión asociada a ρ_n tenga intersección no trivial con las extensiones asociadas a los elementos de los grupos de cohomología.

Finalmente, en la sección 5, se prueba que bajo ciertas hipótesis, dentro de la familia de representaciones construida debe haber puntos modulares. Para eso se apela a la teoría de Hida que establece que toda forma ordinaria es parte de una familia p-ádica de formas modulares.

En cuanto al trabajo realizado, queremos observar que los resultados obtenidos en este capítulo tienen intersección con los de [KR14] (en particular, el Teorema 4.4.1 es un resultado en el mismo espíritu que el Teorema 11 de [KR14]). Sin embargo, cabe resaltar que ambos trabajos fueron realizados de forma independiente en su primeras versiones, y que la versión definitiva de este capítulo tomó algunos resultados de dicho trabajo para relajar algunas hipótesis. En particular la hipótesis extra para el Teorema 4.5.1 fue tomada de dicho trabajo y el propio teorema puede ser visto como una versión previa a los resultados de [KR14] que prueban la existencia de levantados de un peso deseado para deformaciones módulo π^n bajo nuestras hipótesis.

Chapter 4

Lifting Galois representations: the ramified case

4.1 Introduction

The present chapter is in some sense a continuation of the work done in the previous one. There we constructed, for a finite field $\mathbb{F}$, lifts of representations $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(\mathbb{F})/p^n)$ to $\mathrm{GL}_2(W(\mathbb{F}))$. Here we prove how to extend these results to representations with coefficients in finite ramified extensions $K/\mathbb{Q}_p$ of ramification degree $e > 1$.

The method used in Chapter 2 followed the ideas of [Ram99] and [Ram02], adapted to the modulo p^n setting. As noticed before (see the remark before Proposition 3.4.12) these methods do not straightforwardly generalize to representations $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ where $\mathcal{O}$ is the ring of integers of K. The obstacle is that the field extension given by the kernel of $\rho_n \pmod{\pi^2}$ (which we denote ρ_2) as an element in $\mathrm{H}^1(G_{\mathbb{Q}}, Ad^0\bar{\rho})$ is non-trivial, so if we want to define a deformation problem containing ρ_n, it does not matter which finite set of auxiliary primes M we choose, the morphism of Theorem A in Chapter 2

$$\mathrm{H}^1(G_M, Ad^0\bar{\rho}) \to \bigoplus_{\ell \in M} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell \tag{4.1}$$

will never be an isomorphism since both sides have the same dimension but the morphism has non trivial kernel (the cohomological element corresponding to ρ_2 is in the kernel).

The key innovation is to relax local conditions so that the morphism (4.1) is no longer an isomorphism, but a surjective map with one dimensional kernel. This will be enough for the lifting purpose, since it allows us to adjust globally the local adjustments at each step. We cannot relax conditions at primes $\ell \neq p$, since the local deformation ring of $\overline{\rho}|_{G_\ell}$ should not have a smooth quotient of dimension bigger than $\dim \mathrm{H}^1(G_\ell, Ad^0\bar{\rho}) - \dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$. Therefore, we need to impose a different local condition at the prime p. The condition we impose is the same as in [CM09].

Definition. We say that a deformation is "nearly ordinary" if its restriction to the inertia subgroup is upper-triangular and its semisimplification is not scalar, i.e. if

$$\rho|_{I_p} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

with $\psi_1 \neq \psi_2$.

Using this local condition at p, we are able to derive the following theorem (Theorem 4.4.1), which is one of the main results of this work.

Theorem. *Let $p \geq 5$ and $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation which is odd, unramified at 2 and nearly ordinary at p. Assume that $\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise. Let P be a set of primes of $\mathbb{Q}$ containing the ramification set of ρ_n and the prime p. For each $\ell \in P \backslash \{p\}$ fix a local deformation $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ that lifts $\rho_n|_{G_\ell}$ and is not bad (see Definition 4.2). Then there exists a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ and a finite set of primes R such that:*

- *ρ lifts ρ_n, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.*
- *ρ is unramified outside $P \cup R$.*
- *For every $\ell \in P$, $\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$.*
- *ρ is nearly ordinary at p.*
- *All the primes of R, except possibly one, are not congruent to 1 modulo p.*

In fact, the method provides us not only a lift of ρ_n to $\mathcal{O}$ but a family of lifts to characteristic 0 rings, parametrized by a lift to the coefficient ring $W(\mathbb{F})[[X]]$ (see Corollary 4.4.15). The downside is this family of representations is not ordinary but nearly ordinary, which implies that most points are not Hodge-Tate (in particular not modular). We present the natural question, are there any modular points in this family?. We cannot give an answer to this question, but under certain assumptions we can guarantee the existence of such points. That is the content of Theorem 4.5.1.

Theorem. *Let $p \geq 5$ be a prime, $\mathcal{O}$ the ring of integers of a finite extension $K/\mathbb{Q}_p$ with ramification degree $e > 1$ and π its local uniformizer. Let $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation satisfying*

- *ρ_n is odd.*
- *$\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise.*
- *ρ_n is ordinary at p.*
- *ρ_n is unramified at 2.*

Let P be a set of primes containing the ramification set of ρ_n and the prime p, and for each $\ell \in P \backslash \{p\}$ pick a local deformation $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ lifting $\rho_n|_{G_\ell}$ which is not bad (see Definition 4.2). Then there exists a finite set of primes R and a continuous representation $\rho : G_{P \cup R} \to \mathrm{GL}_2(\mathcal{O})$ such that

- *ρ lifts ρ_n, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.*
- *ρ is modular.*
- *For every $\ell \in P$, $\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$ over $\mathrm{GL}_2(\mathcal{O})$.*
- *For every $q \in R$ $\rho|_{I_q}$ is unipotent and all but possibly one prime of R satisfy $q \not\equiv \pm 1 \pmod{p}$.*

- *ρ is ordinary at p.*
- *ρ_n and the collection $\{\rho_\ell\}$ satisfy balancedness assumption (see Definition 4.5).*

The strategy to prove both theorems is similar to the one employed in Chapter 2 and [Ram02]. For every prime $\ell \in P\backslash\{p\}$ we need to find a set C_ℓ of deformations of $\rho_n|_{G_\ell}$ to $W(\mathbb{F})$ containing ρ_ℓ and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ of certain dimension such that its elements preserve the reductions of C_ℓ. In order to get the full statement of our theorem we also need all the deformations in C_ℓ to be isomorphic when restricted to I_ℓ. As in the previous chapter, we describe the argument employed to accomplish this task.

- The hypotheses of our theorem provide us with a mod π^n representation ρ_n and local representation ρ_ℓ lifting $\rho_n|_{G_\ell}$ to $\mathcal{O}$.
- We classify all the possible ρ_ℓ up to $\overline{\mathbb{Z}_p}$-isomorphism and all the possible $\overline{\rho_n}$ up to $\overline{\mathbb{F}}$-isomorphism.
- For each pair of isomorphism classes for ρ_ℓ and $\overline{\rho_n}$ we construct a set C_ℓ of deformations with coefficients in $\mathcal{O}$ which are congruent to ρ_ℓ mod π^n and the corresponding subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ preserving it. In some way the isomorphism class of ρ_ℓ determines which set C_ℓ we pick, and the isomorphism class of $\overline{\rho_n}|_{G_\ell}$ determines the group $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ we have to work with.

In this way, we get the desired sets for every prime $\ell \in P$. Again, we emphasize that although ρ_n does not explicitly appear in this argument, the set C_ℓ is formed by deformations that are congruent to ρ_ℓ modulo π^n, and the reduction of ρ_ℓ modulo π^n is precisely ρ_n. However, contrary to what happens in the unramified case, there are some combinations $(\rho_\ell, \overline{\rho_n}|_{G_\ell})$ for which it does not seem possible to find the set C_ℓ and subspace $\mathcal{N}_\ell$. We address this problem in section 2. As a final remark, we would like to point out that finding the pair (C_ℓ, N_ℓ) is exactly the same as finding a smooth quotient of dimension $d_1 - d_2$ of the universal local at ℓ deformation ring for $\bar{\rho}$ that contains ρ_n as a point.

It remains to introduce a local condition at $\ell = p$. Here we take C_p to be the set of nearly ordinary deformations, which will give a larger subspace N_p and therefore a smaller codomain for the morphism (4.1).

Given this local setting, and after some manipulation of the groups appearing in (4.1) and the ones appearing in the local-to-global map for $\mathrm{H}^2(G_M, Ad^0\bar{\rho})$:

$$\mathrm{H}^2(G_M, Ad^0\bar{\rho}) \to \bigoplus_{\ell \in M} \mathrm{H}^2(G_\ell, Ad^0\bar{\rho}) \tag{4.2}$$

we are able to make the map (4.1) surjective (with a 1-dimensional kernel) and the map (4.2) injective. This implies that, with enough local conditions, the problem of lifting $\bar{\rho}$ is unobstructed, which gives Theorem 4.4.15.

The tricky part when trying to carry on this strategy is that as happens in case 4 of section 2 of chapter 2, there are some cases where the subspaces N_ℓ preserve the reductions modulo π^n of the elements of C_ℓ, not for all n but for n bigger than certain integer α. To overcome this situation we will, following the ideas of [KLR05], lift by adding one set of auxiliary primes for each power of π, until we reach the lift modulo π^α for the main method to work. In this way we will get Theorem 4.4.1. The difference between this argument and the one of section

6 of the previous chapter is that we need to adapt these arguments to the case of a ramified coefficient field.

Theorem 4.5.1 follows from studying the possible modular points appearing in the universal ring provided by Theorem 4.4.15. Notice that we will obtain a modular lift of ρ_n each time we find a nearly ordinary lift ρ such that the characters appearing in the diagonal of $\rho|_{I_p}$ can be written as an integral power of the cyclotomic character times a character of finite order. We only manage to find such points under the quite restrictive balancedness assumption.

We would like to remark that the method employed for the proof of Theorem 4.4.1 seems to generalize well to base fields other than $\mathbb{Q}$. However, the more interesting question of getting an analog to Theorem 4.5.1 seems to be of higher difficulty (as it already is over $\mathbb{Q}$). We currently have some work in progress in this direction. As a final remark, we would like to point out that the results obtained in this work have overlap with the ones in [KR14] (in particular Theorem 4.4.1 is in the same spirit than Theorem 11 of [KR14]). Both works were independent in their first versions and after that we took some results. In particular, balancedness assumption is took from that work and our final Theorem 4.5.1 can be seen as a previous version of the more powerful result about existence of lifts of controlled weight proven in [KR14].

Section 2 concerns with the construction of the sets C_ℓ and subspaces N_ℓ for primes $\ell \neq p$. In Section 3 we do the same for the nearly ordinary condition at p. Section 4 treats global arguments for lifting and proves the first main theorem of this work. It is divided into two subsections, one for exponents at which we have C_ℓ and N_ℓ for every $\ell \in P$ and one for the ones at which we do not. In Section 5 we prove the other main theorem which concerns modularity.

4.1.1 Conventions:

As in chapter 2, all our deformations will have fixed determinant $\omega\chi^k$. This will imply that all our cohomology groups are those of $Ad^0\bar{\rho}$ and not $\mathrm{Ad}\rho$. Again we will twist $\bar{\rho}$ by a character of finite order to be able to suppose that $\bar{\rho}$ and $Ad^0\bar{\rho}$ ramify in the same set of primes.

4.2 Local deformation theory at $\ell \neq p$

Let $\ell \neq p$ a prime and let $\rho : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. We denote by $\bar{\rho}$ its reduction mod π.

Recall the notion of an element of $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ preserving a set of deformations.

Definition. Let C_ℓ be a set of deformations $\{\rho : G_\ell \to \mathrm{GL}_2(\mathcal{O})\}$ and $u \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$. We say that u preserves C_ℓ if for any $\rho_m : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\pi^m)$ which is the reduction of a deformation of C_ℓ we have that $u \cdot \rho_m$ is also the reduction of some deformation of C_ℓ.

The main objective of this section is to study the problem of constructing, given $\rho_n : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ and $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$, a deformation of ρ_n to $\mathcal{O}$, a set C_ℓ of deformations of ρ_n to $\mathcal{O}$ that contains ρ_ℓ and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ preserving it. As we mentioned in the introduction, instead of constructing these for every possible pair (ρ_n, ρ_ℓ), we will do it for every possible pair $(\overline{\rho_n}, \rho_\ell)$, as this classification covers all the possible ρ_n that lift to $\mathcal{O}$ (there may exist representations ρ_n with coefficients in $\mathcal{O}/\pi^n$ that do not lift to $\mathcal{O}$, but the input of our theorem is always a mod π^n representation that lifts locally).

As claimed in the introduction, there are some pairs $(\overline{\rho_n}, \rho_\ell)$ for which we are unable to find the set C_ℓ. We describe them below.

Definition. We say that a representation $\rho : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ is "bad" if

$$\rho \simeq \begin{pmatrix} \psi_1 & \frac{\psi_1 - \psi_2}{\pi^r} \\ 0 & \psi_2 \end{pmatrix}$$

over $\mathrm{GL}_2(\mathcal{O})$ with $r > 0$ and moreover the following holds:

- $\overline{\rho}$ is unramified and $\overline{\rho}^{ss}(\sigma)$ is scalar.
- $v(\psi_1(\sigma) - \psi_2(\sigma)) < v(\frac{\psi_1(\tau) - \psi_2(\tau)}{\pi^r})$.

We currently do not know how to find the desired family of deformations in these cases. Observe that finding the families C_ℓ and subspaces N_ℓ accounts for proving that the Balancedness Assumption of [KR14] holds (see Assumption before Definition 3). In that scenario, if we start with a mod π^n deformation ρ_n and a prime ℓ for which every lift to $\mathcal{O}$ is bad, we do not know how to prove that the Assumption holds.

The objective of this section is proving the following result:

Proposition 4.2.1. *Let $\rho : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. If ρ is not bad then there always exists a positive integer α, a set C_ℓ of deformations of the reduction of ρ modulo π^α to characteristic 0 that contains ρ (up to $\mathrm{GL}_2(\mathcal{O})$-isomorphism), and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ such that:*

- *All the elements of C_ℓ are isomorphic over $GL_2(\mathcal{O})$ when restricted to inertia.*
- *$N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ has codimension equal to $\dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$.*
- *Every $u \in N_\ell$ preserves the mod π^m reductions of elements of C_ℓ for $m \geq \alpha$.*

In other words, there exists a smooth deformation condition containing ρ of dimension equal to $\dim \mathrm{H}^1(G_\ell, Ad^0\bar{\rho}) - \dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$ if we start lifting from a big enough exponent.

In order to prove this, we will use the results about types of 2-dimensional deformations that appeared in chapter 2. We will construct the sets C_ℓ for each combination of representatives of these classifications. Proposition 2.2.1 classifies the possible modulo π deformations $\overline{\rho}$, Proposition 2.2.2 deals with isomorphism classes for ρ_ℓ and finally Proposition 2.3.2 tells us which combinations may happen.

To prove Proposition 4.2.1 we now consider all the possible pairs of $\mathrm{GL}_2(\mathbb{F})$ and $\mathrm{GL}_2(\mathcal{O})$-equivalence classes for ρ and $\overline{\rho}$ (indexing them first by the class of $\overline{\rho}$), and for each of them we define the corresponding deformation class and cohomology subspace. We feel free to enlarge the field of coefficients $\mathcal{O}$ in order to use the desired representative of the equivalence class (which may not be defined over $\mathcal{O}$).

• **Case 1: $\overline{\rho}$ is ramified Principal Series.** Proposition 2.3.2 implies that a mod π Principal Series can only come from a characteristic 0 principal series. The full study of this case is done in Case 1 of Section 4 of Chapter 2. The work there is done for $\mathcal{O}$ unramified but the same applies in our situation. In fact, the study here is simpler, as in that case the assumption about ramification prevents us from enlarging the coefficient field and an integral model for the representatives must be found.

• **Case 2: $\overline{\rho}$ is Steinberg.** When $\overline{\rho}$ is Steinberg, it can be the reduction of any of the three characteristic 0 ramified types:

· **Case 2.1: ρ_ℓ is Steinberg**. The definition of C_ℓ and N_ℓ is essentially the same as in Chapter 2, Case 2 of Section 4. Although in that scenario only the case where $\mathcal{O}/\mathbb{Z}_p$ is unramified is treated, the ramification of $\mathcal{O}$ does not affect the results.

· **Case 2.2: ρ_ℓ is Principal Series**. Proposition 2.3.2 implies that $\ell \equiv 1 \pmod p$. Let $\rho_\ell = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$. Without loss of generality, we can take $*(\tau) = 1$. We have the following lemma:

Lemma 4.2.2. *A deformation $\rho_m : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\pi^m)$ which has the form:*

$$\rho_m(\tau) = \begin{pmatrix} \psi(\tau) & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_m(\sigma) = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix},$$

has a unique lift to characteristic zero of the same form if and only if $\beta^2+\gamma(\psi(\tau)-1)\beta+\Psi \equiv 0 \pmod{\pi^m}$, where Ψ is a fixed lift determinant.

Proof. This Lemma is part of a computation made in Proposition 3.4 of [Kha06]. There, it is done for deformations with coefficients in unramified coefficient field and its mod p reductions, but the same proof works in general. □

Let $j \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ be the element defined by:

$$j(\sigma) = e_2 \ , \ j(\tau) = 0,$$

and take N_q to be the subspace it generates. Also let C_ℓ be the set of deformations to $\mathcal{O}$ which have the form given in Lemma 4.2.2. Observe that any mod π^m reduction of an element of C_ℓ satisfies the equation $\beta^2 + \gamma(\psi(\tau) - 1)\beta + \Psi = 0$, and acting by j on it does not affect this (as π divides $\psi(\tau) - 1$ so adding a multiple of π^{m-1} to γ does not change the equation modulo π^m). Then, Lemma 4.2.2 guarantees that C_ℓ and N_ℓ satisfy the property we are looking for.

· **Case 2.3: ρ_ℓ is Induced**. Proposition 2.3.2 tells us that necessarily $\ell \equiv -1 \pmod \pi$ and by results in Section 3 of Chapter 2 we have that $d_1 = d_2 = 1$. Therefore we can take $N_\ell = \{0\}$ and $C_\ell = \{\rho_\ell\}$.

• **Case 3: $\overline{\rho}$ is Induced.** By Proposition 2.3.2, when $\overline{\rho}$ is Induced, ρ_ℓ must be Induced as well. The choice of C_ℓ and N_ℓ in this case is explained in Case 3 of Section 4 of Chapter 2.

• **Case 4: $\overline{\rho}$ is unramified.** Being unramified, $\overline{\rho}$ allows lifts to any type of deformation. We must treat each of them separately as they have many subcases.

Case 4.1: ρ_ℓ is Steinberg. This case is Case 4 of Section 4 of Chapter 2 for an unramified coefficient field. The computation remains valid in the ramified setting. The only extra care we need to take is the following, the set C_ℓ as defined in chapter 2 does not satisfy that all its members are isomorphic when restricted to inertia, as when ρ_ℓ modulo π^{n_0} is non-trivial but unramified, there is one member of C_ℓ which is unramified. In order to avoid this issue, we take α as the least exponent such that ρ_ℓ modulo π^α is ramified.

There are two other cases left to study, in each of them we will distinguish between three types of equivalence classes for $\overline{\rho}$, according to the image of Frobenius. This case is the one that includes bad primes, the calculations made here show where the badness condition appears.

Case 4.2: ρ_ℓ is Principal Series. In this case we have $\rho_\ell = \begin{pmatrix} \phi & \pi^{-r}(\phi-1) \\ 0 & 1 \end{pmatrix}$ with $r \geq 0$. By Proposition 2.3.2 we necessarily have $\ell \equiv 1 \pmod{\pi}$.

· If $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} \beta & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\beta \neq 1$ we have $d_1 = 2$ and $d_2 = 1$ so we are looking for a one-dimensional subspace N_ℓ. Observe that necessarily $\rho_\ell \simeq \left(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ over $\mathcal{O}$ as $\psi(\sigma) \equiv \beta \not\equiv 1 \pmod{\pi}$ so $(\psi - 1)/\pi$ is not an integer.

Let $u \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ be the cocycle defined by $u(\sigma) = e_1$ and $u(\tau) = 0$. We can take $N_\ell = \langle u \rangle$ and C_ℓ the set of representations of the form $\rho \simeq \begin{pmatrix} \phi\psi & 0 \\ 0 & \psi^{-1} \end{pmatrix}$ for $\psi : G_\ell \to \mathcal{O}^\times$ unramified. It is easily checked that these satisfy the desired properties.

· If $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ the corresponding dimensions are $d_1 = 2$ and $d_2 = 1$. In this case we have $\rho_\ell \simeq \begin{pmatrix} \phi & \frac{\phi-1}{\pi^r} \\ 0 & 1 \end{pmatrix}$, with $v(\phi - 1) = r$. Now take the cocycle defined by $u(\sigma) = 0$ and $u(\tau) = e_2$. We take $\alpha = v(\phi(\tau) - 1) + 2$ and set $N_\ell = \langle u \rangle$ and C_ℓ the set of deformations of $\overline{\rho}$ such that $\rho \simeq \begin{pmatrix} \gamma\phi & \beta\frac{\gamma\phi-\gamma^{-1}}{\pi^r} \\ 0 & \gamma^{-1} \end{pmatrix}$ for some $\beta \in \mathcal{O}^\times$ and γ an unramified character congruent to 1 modulo π^α.

Lemma 4.2.3. *The set C_ℓ and subspace N_ℓ defined above satisfy that N_ℓ preserves C_ℓ mod π^m for all m such that ϕ is ramified mod π^m.*

Proof. Assume that $\rho_\ell(\sigma) = \begin{pmatrix} a & \frac{a-b}{\pi^r} \\ 0 & b \end{pmatrix}$ and $\rho_\ell(\tau) = \begin{pmatrix} x & \frac{x-y}{\pi^r} \\ 0 & y \end{pmatrix}$. Our assumptions on ρ_ℓ not being bad tell us that $v(a-b) \geq v((x-y)/\pi^r)$. If we have $v(a-b) > v(x-y)$ then we change σ by $\tau\sigma$ (which is another Frobenius element). In this way, we can assume that $v(a-b) \leq v(x-y)$.

We want to prove that if we have a deformation $\rho_m : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\pi^m)$ that sends

$$\rho_m(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \text{ and } \rho_m(\tau) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix},$$

with $x \neq y$, which is the reduction of some element of C_ℓ (i.e. $c = \beta(a-b)$ and $z = \beta(x-y)$ for some $\beta, a, b, c, x, y, z \in \mathcal{O}$), then $u \cdot \rho_m$ is also the reduction of some element of C_ℓ. Recall that

$$u \cdot \rho_m(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \text{ and } u \cdot \rho_m(\tau) = \begin{pmatrix} x & z + \pi^{m-1} \\ 0 & y \end{pmatrix}.$$

It is easily checked that the cocycle v that sends σ to $\lambda_1 e_1 + \lambda_2 e_2$ and τ to 0 is a coboundary for any choice of $\lambda_1, \lambda_2 \in \mathbb{F}$. So ρ_m can also be thought as

$$u \cdot \rho_m(\sigma) = \begin{pmatrix} a(1 + \lambda_1\pi^{m-1}) & c + \lambda_2\pi^{m-1} \\ 0 & b(1 + \lambda_1\pi^{m-1})^{-1} \end{pmatrix} \text{ and } u \cdot \rho_m(\tau) = \begin{pmatrix} x & z + \pi^{n-1} \\ 0 & y \end{pmatrix}$$

for any $\lambda_1, \lambda_2 \in \mathbb{F}$. Therefore, it is enough to find $\lambda_1, \lambda_2 \in \mathcal{O}$ such that

$$\frac{x-y}{z + \pi^{m-1}} = \frac{a(1 + \lambda_1\pi^{m-1}) - b(1 + \lambda_1\pi^{m-1})^{-1}}{c + \lambda_2\pi^{m-1}}.$$

given that

$$\frac{x-y}{z} = \frac{a-b}{c}.$$

Expanding this equation we find out that it is equivalent to

$$\lambda_1(z + \pi^{m-1})(a + b(1 + \lambda_1\pi^{m-1})^{-1}) + a - b = \lambda_2(x-y).$$

which has a solution given that $v(a-b) \geq v(z)$ (we solve first for λ_1 in order for both sides to have the same valuation, and then there is a λ_2 that makes the equality true). $\square$

Remark. Observe that when the condition $v(a-b) \geq v(z)$ does not hold, the last equation of the proof does not have a solution, since no matter which $\lambda_1, \lambda_2 \in \mathcal{O}$ we pick, the valuation of the left hand side is $v(a-b)$ and the valuation of the right hand side is bigger or equal than $v(x-y) \geq v(z) > v(a-b)$. Moreover, following the same type of computations we can prove that for the chosen set C_ℓ there is no non-trivial $u \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ such that the subspace $\langle u \rangle$ preserves C_ℓ.

· If $\overline{\rho}(\sigma) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have $d_1 = 6$ and $d_2 = 3$ therefore we need to find a subspace N_ℓ of dimension 3. This case is a little more involved than the other two as there are non-trivial elements of $\mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ that act trivially modulo π^m for high exponents of π. We follow the same ideas as in the study of the Steinberg-reducing-to-unramified case of previous chapter (the spirit of these ideas is taken from the approach to trivial primes followed in [HR08]).

Assume first that $\rho_\ell \simeq \left(\begin{smallmatrix} \phi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We will take C_ℓ as the set of deformations of the form

$$\rho \simeq \begin{pmatrix} \gamma\phi & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$

with $\gamma : G_\ell \to \mathcal{O}^\times$ an unramified character, that lift the reduction mod π^α of ρ_ℓ, with $\alpha = v(\phi(\tau) - 1) + 2$. Clearly, the set C_ℓ is preserved by the cocycle u_1 that sends $\sigma \mapsto e_1$ and $\tau \mapsto 0$.

We will construct two more cocycles u_2 and u_3 that act trivially on reductions modulo π^m of deformations on C_ℓ for $m \geq \alpha$. Let $\rho_m : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\pi^m)$ be the mod π^m reduction of an element in C_ℓ. Let $\rho_m(\sigma) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and $\rho_m(\tau) = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$. Let $u \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$. To prove that u acts trivially on ρ_m we need to find a matrix $C \in \mathrm{GL}_2(\mathcal{O}/\pi^m)$ such that $C \equiv Id \pmod{\pi}$ and $C\rho_m C^{-1} = (Id + \pi^{m-1}u)\rho_m$. One can find such matrix by taking $C = \left(\begin{smallmatrix} 1+\pi\alpha & \pi\beta \\ \pi\gamma & 1+\pi\delta \end{smallmatrix}\right)$ and explicitly computing $C\tilde{\rho} = (Id + \pi^{m-1}u)\rho_m C$ at σ and τ. In this way, one finds out that if $v(\phi(\tau) - 1) > v(\phi(\sigma) - 1)$ then the cocycles u_2 and u_3 sending σ to e_2 and e_3 respectively and τ to 0 act trivially on the reductions of elements of C_ℓ. The corresponding base change matrices C_i that conjugate $(Id + \pi^{m-1}u_i)\rho_m$ into ρ_m are

$$C_2 = \begin{pmatrix} 1 & \frac{-\pi^{m-2}}{a-b} \\ 0 & 1 \end{pmatrix} \text{ and } C_3 = \begin{pmatrix} 1 & 0 \\ \frac{\pi^{m-2}}{a-b} & 1 \end{pmatrix}.$$

It remains to check what happens when $v(\phi(\tau) - 1) \leq v(\phi(\sigma) - 1)$. Observe that we can always assume that $v(\phi(\tau) - 1) = v(\phi(\tau) - 1)$ in this case, by simply changing σ for $\tau\sigma$. Then we can take u_2 sending σ to e_2 and τ to λe_2 and u_3 sending σ to e_3 and τ to λe_3 for $\lambda = (x-y)/(a-b) \in \mathbb{F}$ (notice that this does not depend on ρ_m). Again, the action of both cocycles will be trivial and the base change matrices will be the same as before.

It remains to consider the case where $\rho_\ell \simeq \left(\begin{smallmatrix} \phi & (\phi-1)/\pi^r \\ 0 & 1 \end{smallmatrix}\right)$ for $r > 0$. Let $\alpha = v((\phi - 1)/\pi^r) + 2$ and let C_ℓ be the set of deformations ρ of the mod π^α reduction of ρ_ℓ such that $\rho \simeq \left(\begin{smallmatrix} \gamma\phi & \beta\frac{\gamma\phi - \gamma^{-1}}{\pi^r} \\ 0 & \gamma^{-1} \end{smallmatrix}\right)$ for some $\beta \in \mathcal{O}^\times$ and γ an unramified character congruent to 1 modulo π^α. By doing the exact same calculation as in Lemma 4.2.3 it can be proved that the cocycle u_1 that sends σ to 0 and τ to e_2 preserves the reductions of elements in C_ℓ, given that ρ_ℓ is not bad. We still need two more elements preserving C_ℓ. As in the previous case, we have two cocycles that act trivially on mod π^m reductions of elements of C_ℓ for $m \geq \alpha$. Let ρ_m be a mod π^m reduction of some element in C_ℓ given by $\rho_m(\sigma) = \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)$ and $\rho_m(\tau) = \left(\begin{smallmatrix} x & z \\ 0 & y \end{smallmatrix}\right)$. As in

Lemma 4.2.3 we can assume that $v(a-b) = v(x-y)$. Let

$$\lambda = \frac{x-y}{a-b} = \frac{z}{c}.$$

Let u_2 be the cocycle that sends σ to λe_1 and τ to e_1 and u_3 the one that sends σ to λe_2 and τ to e_2. We claim that these act trivially on ρ_m if $m \geq v(z)+2$.

It is easy to see that the base change matrices given by

$$C_2 = \begin{pmatrix} 1 & 0 \\ \frac{-\pi^{m-2}}{z} & 1 \end{pmatrix} \text{ and } C_3 = \begin{pmatrix} 1 + \frac{\pi^{m-2}}{z} & 0 \\ 0 & 1 \end{pmatrix}$$

serve to prove the trivialness of the action of u_2 and u_3 respectively, which concludes this case.

Case 4.3: ρ_ℓ is Induced. Proposition 2.3.2 says that whenever ρ_ℓ is induced and $\overline{\rho}$ is unramified, $\overline{\rho}(\sigma)$ has eigenvalues 1 and -1 (up to twist) and $q \equiv -1 \pmod p$. In this case we have that $d_1 = 3$ and $d_2 = 2$. We want to find a set C_ℓ and a subspace N_ℓ of dimension 1 preserving it. Let $C_\ell = \{\rho_\ell\}$. As in the Principal Series case, we will be able to find non trivial cocycles that act trivially on mod π^m reductions of ρ_ℓ for m big enough. We split into the two possible families of $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$-equivalence classes of induced representations given by Proposition 2.2.2.

· If $\rho_\ell(\sigma) = \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)$ and $\rho_\ell(\tau) = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, it can be checked that the cocycle u sending τ to 0 and σ to e_2 is non trivial. If $r = v(x-y)$ then u acts trivially for all $m \geq r+2$ and the base change matrix is given by

$$C = \begin{pmatrix} 1 & \pi^{m-r-1}u_1 \\ \pi^{m-r-1}u_2 & 1 \end{pmatrix}$$

where $u_1, u_2 \in \mathcal{O}$ are such that $u_1 - tu_2 \equiv \pi^{r-1} \pmod{\pi^r}$.

· If $\rho_\ell(\sigma) = \left(\begin{smallmatrix} -a & c \\ b & a \end{smallmatrix}\right)$ and $\rho_\ell(\tau) = \left(\begin{smallmatrix} x & z \\ 0 & y \end{smallmatrix}\right)$, as in the previous case, it can be checked that the cocycle u that sends σ to 0 and τ to e_2 is non trivial. Again, the action of this cocycle in the reduction modulo π^m of ρ_ℓ is trivial for $m \geq v(z)+2$. In this case, the base change matrix is given by

$$C = \begin{pmatrix} 1 + \frac{\pi^{m-2}}{z} & -\frac{c\pi^{m-2}}{2az} \\ -\frac{b\pi^{m-2}}{2az} & 1 \end{pmatrix}.$$

4.3 Local deformation theory at p

At the prime p we will impose the deformation condition of being "nearly ordinary" (as in [CM09]). This section is mainly about gathering previously done calculations, and all the deformations appearing are deformations of the local Galois group G_p.

Definition. We say that a deformation of G_p is "nearly ordinary" if its restriction to the inertia subgroup is upper-triangular and its semisimplification is not scalar, i.e. if

$$\rho|_{I_p} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

with $\psi_1 \neq \psi_2$.

We will prove the following theorem.

Theorem 4.3.1. *Let $\rho_n : G_p \to \mathcal{O}/\pi^n$ be a nearly ordinary deformation and $\overline{\rho}$ its mod π reduction. There is a family of nearly ordinary deformations C_p to characteristic 0 such that ρ_n is the reduction of a member of C_p and a subspace $N_p \subseteq \mathrm{H}^1(G_p, Ad^0\bar{\rho})$ of codimension equal to $\dim \mathrm{H}^2(G_p, Ad^0\bar{\rho}) + 1$ preserving C_p in the sense of Proposition 4.2.1.*

Proof. Let

$$\overline{\rho} \simeq \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

By twisting $\overline{\rho}$ by ψ_2^{-1} we can assume that $\psi_2 = 1$. Let U be the set of upper-triangular 2×2 matrices of trace 0. To prove the theorem we will construct for each possible $\overline{\rho}$, the corresponding set and subspace and verify that its dimensions satisfy the statement of the theorem. In most of the cases C_p will consist on all nearly ordinary deformations of $\overline{\rho}$ and N_p will be the image of $\mathrm{H}^1(G_p, U)$ in $\mathrm{H}^1(G_p, Ad^0\bar{\rho})$.

In order to see this, we simply compute the dimension of the image of $\mathrm{H}^1(G_p, U)$ in $\mathrm{H}^1(G_p, Ad^0\bar{\rho})$ and compare it with $\dim \mathrm{H}^1(G_p, Ad^0\bar{\rho}) - \dim \mathrm{H}^2(G_p, Ad^0\bar{\rho}) - 1$. We can reduce the computation of all these values to finding the dimensions of $\mathrm{H}^0(G_p, U)$, $\mathrm{H}^0(G_p, U^*)$, $\mathrm{H}^0(G_p, Ad^0\bar{\rho})$ and $\mathrm{H}^0(G_p, (Ad^0\bar{\rho})^*)$ by using local Tate duality and the formula for the Euler-Poincaré characteristic (which is equal to 2 for U and 3 for $Ad^0\bar{\rho}$). The kernel of the map $\mathrm{H}^1(G_p, U) \to \mathrm{H}^1(G_p, Ad^0\bar{\rho})$ induced by $U \subseteq Ad^0\bar{\rho}$ is contained in $\mathrm{H}^0(G_p, Ad^0\bar{\rho}/U)$, which is equal to 0 given that $\psi_1 \neq \psi_2$. With all these tools, the required dimensions are easily computed and we obtain that taking C_p as the set of all nearly ordinary deformations and N_p as the image of $\mathrm{H}^1(G_p, U)$ in $\mathrm{H}^1(G_p, Ad^0\bar{\rho})$ works for all cases but the one in which $\overline{\rho}$ is decomposable and ψ_1 is the cyclotomic character (recall we are assuming $\psi_2 = 1$).

In this case the universal ring for nearly ordinary deformations is not smooth and C_p is not preserved by N_p as there are some mod π^m nearly ordinary deformations of $\overline{\rho}$ that do not lift back to characteristic 0. We need to take a smaller set C_p in this case. In order to solve this, we claim that the universal deformation ring for nearly ordinary lifts of $\overline{\rho}$ with fixed determinant ψ is isomorphic to the universal deformation ring for **ordinary** lifts of $\overline{\rho}$ with arbitrary determinant. To see this, just observe that from any nearly ordinary deformation of $\overline{\rho}$ to a ring A we can obtain an ordinary lift of $\overline{\rho}$ by twisting by inverse of the character appearing in the place $(2, 2)$. To go the other way round, if we have an ordinary deformation $\tilde{\rho}$ of $\overline{\rho}$ to A given by

$$\tilde{\rho} = \begin{pmatrix} \omega_1 & * \\ 0 & \omega_2 \end{pmatrix}$$

where ω_2 is unramified and want to obtain a nearly ordinary deformation of $\overline{\rho}$ with determinant ψ, we need to twist by a square root of $\psi(\omega_1\omega_2)^{-1}$. This character has a square root by Hensel's lemma, as its reduction modulo the maximal ideal is 1 which has a square root. Given this identification, the wanted result follows from Proposition 6 of [KR14], where the same result is proven for the ordinary arbitrary determinant case. □

4.4 Global deformation theory

In this section we prove the one of the main results of this article.

Theorem 4.4.1. *Let $p \geq 5$, $n \geq 2$ and $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation which is odd, unramified at 2 and nearly ordinary at p. Assume that $\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise. Let P be a set of primes of $\mathbb{Q}$ containing the ramification set of ρ_n and the prime p. For each $\ell \in P \backslash \{p\}$ fix a local deformation $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ that lifts $\rho_n|_{G_\ell}$ and is not bad (see Definition 4.2). Then there exists a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ and a finite set of primes R such that:*

- *ρ lifts ρ_n, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.*
- *ρ is unramified outside $P \cup R$.*
- *For every $\ell \in P$, $\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$.*
- *ρ is nearly ordinary at p.*
- *All the primes of R, except possibly one, are not congruent to 1 modulo p.*

The proof of this theorem essentially consists of finding a way to lift ρ_n to characteristic 0 one power of π at a time. We will split the proof into two sections, essentially because the local results we have so far are split in two different cases depending on whether the exponent m in each step is big enough or not. Let α be the integer obtained in the following way: for each $\ell \in P \backslash p$, Proposition 4.2.1 gives an integer α_ℓ such that there is a set C_ℓ containing ρ_ℓ and a subspace N_ℓ preserving its reductions mod π^n for $n \geq \alpha_\ell$. Let α be the maximum of the α_ℓ's for $\ell \in P \backslash \{p\}$. When lifting from π^m to π^{m+1} for $m \geq \alpha$ we are in a global setting similar to the one in [Ram02]. The existence of the sets C_ℓ and subspaces N_ℓ let us mimic the argument given there. When working modulo m for $m < \alpha$, we do not count with these sets and subspaces for all the primes of P and therefore are unable to overcome the local obstructions in the same fashion as before. In this case, we will follow the ideas from [KLR05] and will lift to $\mathcal{O}/\pi^\alpha$ by adding a finite number of auxiliary primes at each power of π.

As $\mathcal{O}$ is the ring of integers of a ramified extension we have that $\mathcal{O}/\pi^2$ is isomorphic to the dual numbers and therefore the projection mod π^2 of ρ_n defines an element in $\mathrm{H}^1(G_{\mathbb{Q}}, Ad^0\bar{\rho})$ which we will call f. Observe that our hypotheses imply that the image of the projection of ρ_n to $\mathcal{O}/\pi^2$ contains $\mathrm{SL}_2(\mathcal{O}/\pi^2)$ and thus $f \neq 0$ as an element in $\mathrm{H}^1(G_{\mathbb{Q}}, Ad^0\bar{\rho})$.

4.4.1 Getting to mod π^α

Assume that the exponent n we start with is strictly smaller than the natural number α from Proposition 4.2.1 (if this is not the case we are done). The idea is to adjust the main argument of [KLR05] to our situation. Recall the following definition.

Definition. A prime q is *nice* for $\overline{\rho}$ if it satisfies the following properties

- The prime q is not congruent to ± 1 mod p.
- The representation ρ_n is unramified at q.
- The eigenvalues of $\overline{\rho}(\sigma)$ have ratio q.

We say that q is nice for ρ_n if furthermore

- The matrix $\rho_n(\sigma)$ is equivalent to $\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

At a nice prime q we consider the set

$$C_q = \{\text{deformations } \rho \text{ of } \overline{\rho}|_{G_q} \ : \ \rho(\sigma) = \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)\},$$

and the subspace $N_q \subseteq \mathrm{H}^1(G_q, Ad^0\bar{\rho})$ generated by the cocycle u sending σ to 0 and τ to e_2. It is easy to check that u preserves the reductions of elements of C_q.

The work on [KLR05] is based on the existence of nice primes that are either zero or non zero at certain elements of both $\mathrm{H}^1(G_U, Ad^0\bar{\rho})$ and $\mathrm{H}^1(G_U, (Ad^0\bar{\rho})^*)$ for different sets of primes U. We claim that the same arguments work in this settings except for the cases when the element $f \in \mathrm{H}^1(G_P, Ad^0\bar{\rho})$ attached to $\rho_n \bmod \pi^2$ is involved. We will sort this obstacle by adding the following primes.

Definition. We say that a prime q is a *special prime* for f if

- The representation ρ_n is unramified at q.
- $\rho_n(\sigma) = \left(\begin{smallmatrix} 1 & \pi \\ 0 & 1 \end{smallmatrix}\right)$.
- The prime $q \equiv 1 \pmod{\pi^n}$.

Note that for such primes q we have that $f|_{G_q} \neq 0$ as an element in $\mathrm{H}^1(G_q, Ad^0\bar{\rho})$. We need some partial results to state and prove the main result of this case (Theorem 4.4.9). The following is the ramified version of Corollary 3.4.11.

Lemma 4.4.2. *Let $\overline{\rho}$ be the modulo π reduction of a representation ρ_n as in the hypotheses of Theorem 4.4.15 and $f, f_1, \ldots, f_r \in \mathrm{H}^1(G_{\mathbb{Q}}, Ad^0\bar{\rho})$ and $g_1, \ldots, g_s \in \mathrm{H}^1(G_{\mathbb{Q}}, (Ad^0\bar{\rho})^*)$ be linearly independent sets.*

a) Let $I \subseteq \{1, \ldots r\}$ and $J \subseteq \{1, \ldots, s\}$. There is a Chebotarev set of primes q such that

- *q is nice for ρ_n.*
- *$f_i|_{G_q} \neq 0$ if $i \in I$ and $f_i|_{G_q} = 0$ if $i \notin I$. Moreover, for the primes of I one can ask for $f_i|_{G_q} \notin N_q$.*
- *$g_j|_{G_q} \neq 0$ if $j \in J$ and $g_j|_{G_q} = 0$ if $j \in J$*

b) Also, there is a Chebotarev set of primes q' such that

- *q' is a special prime for f, henceforth $f|_{G_{q'}} \neq 0$.*
- *$f_i|_{G_{q'}} = 0$ for all $1 \leq i \leq r$.*

Remark. For special primes we can also define a set C_q of deformations to characteristic 0 and a subspace $N_q \subseteq \mathrm{H}^1(G_q, Ad^0\bar{\rho})$ of codimension $\dim \mathrm{H}^2(G_q, Ad^0\bar{\rho})$ preserving it. This is explicitly done in Lemma 3.3.1 of Chapter 2. Notice that this corresponds to the case of a trivial local $\overline{\rho}$ lifting to a Steinberg ρ. The special primes do not fall into the problematic case of Lemma 3.3.1 of Chapter 2, as ρ_n is not trivial at G_q.

Proof. This is a slight modification of Fact 5 of [KLR05]. The main problem with nice primes in ramified extensions is that if q is a nice prime then $f|_{G_q} = 0$. The use of special primes for f solves this problem, since almost by definition if q' is a special prime, $f|_{G'_q} \neq 0$. In order to check that Chebotarev conditions at the different f_i's and g_j's are disjoint from the condition of being nice for ρ_n, and that this last condition only overlaps with extension corresponding to the element f, we need to understand the Galois structure of the corresponding extensions. We will prove a ramified version of Lemma 3.4.10 of Chapter 2. Following the same notation as before, let $K = \mathbb{Q}(Ad^0\bar{\rho})\mathbb{Q}(\mu_p)$. For each f_i let L_i the extension of K given by its kernel and for each g_j let M_j be the corresponding one. Finally let $K' = K \cdot \mathbb{Q}(Ad^0\rho_n)$ and L_f be the extension of K given by f. Let $L = L_f \prod L_i$ and $M = \prod M_j$. We claim that $K' \cap LM = L_f$.

For this, let $\mathcal{H} = \mathrm{Gal}(K'/K) \subseteq \mathrm{PGL}_2(\mathcal{O}/\pi^n)$ and $\pi_1 : \mathrm{PGL}_2(\mathcal{O}/\pi^n) \to \mathrm{PGL}_2(\mathbb{F})$. Observe that $\mathcal{H}$ consists on the classes of matrices in $\mathrm{Im}(\rho_n)$ which are trivial in $\mathrm{PGL}_2(\mathbb{F})$, i.e. $\mathcal{H} = \mathrm{Im}(Ad^0\rho_n) \cap \mathrm{Ker}(\pi_1)$. Recall that our hypotheses imply $\mathrm{PSL}_2(\mathcal{O}/\pi^n) \subseteq \mathrm{Im}(Ad^0\rho_n) \subseteq \mathrm{PGL}_2(\mathcal{O}/\pi^n)$, and therefore $\mathrm{PSL}_2(\mathcal{O}/\pi^n) \cap \mathrm{Ker}(\pi_1) \subseteq \mathcal{H} \subseteq \mathrm{Ker}(\pi_1)$. As $[\mathrm{PSL}_2(\mathcal{O}/\pi^n) : \mathrm{PGL}_2(\mathcal{O}/\pi^n)] = 2$ and $\mathrm{Ker}(\pi_1)$ is a p group we have that $\mathcal{H} = \mathrm{Ker}(\pi_1)$.

Recall that $\mathrm{Gal}(F/K) \simeq (Ad^0\bar{\rho})^r \times (Ad^0\bar{\rho}^*)^s$ as $\mathbb{Z}[G_{\mathbb{Q}}]$-module and by Lemma 7 of [Ram99], this is its decomposition as $\mathbb{Z}[G_{\mathbb{Q}}]$ simple modules. This implies that $K' \cap LM$ is the direct sum of the quotients of $\mathrm{Gal}(K'/K) \simeq \mathcal{H}$ isomorphic to $Ad^0\bar{\rho}$ or $(Ad^0\bar{\rho})^*$. To prove that the only such quotient is $\mathrm{Gal}(L_f/K)$ observe that any surjective morphism $\mathcal{H} \to Ad^0\bar{\rho}$ or $(Ad^0\bar{\rho})^*$ must contain $[\mathcal{H} : \mathcal{H}]$ inside its kernel. We will prove in Lemma 4.4.3 below that such commutator is equal to the subgroup of $\mathcal{H}$ formed by the matrices congruent to the identity modulo π^2. This finishes the proof as implies that such quotient necessarily factors through $\mathrm{Gal}(L_f/K)$. □

Lemma 4.4.3. *If $H \subseteq \mathrm{SL}_2(\mathcal{O}/\pi^n)$ is the subgroup consisting of matrices congruent to the identity modulo p then its commutator subgroup $[H : H]$ is the subgroup H' of $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ formed by the matrices congruent to the identity modulo π^2.*

Proof. It is easy to check that H/H' is abelian, implying that $[H : H] \subseteq H'$. For the other inclusion, observe that H' is generated by the set of elements of the form

$$\begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1+\pi^2 z & 0 \\ 0 & (1+\pi^2 z)^{-1} \end{pmatrix}$$

for $x, y, z \in \mathcal{O}$. It is easy to verify the following identities for any $a, b \in \mathcal{O}$:

- $\left[\left(\begin{smallmatrix} 1 & \pi a \\ 0 & 1 \end{smallmatrix}\right) : \left(\begin{smallmatrix} 1+\pi & 0 \\ 0 & (1+\pi)^{-1} \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 1 & -\pi^2 a(\pi+2) \\ 0 & 1 \end{smallmatrix}\right) \in [H : H]$.
- $\left[\left(\begin{smallmatrix} 1 & 0 \\ \pi b & 1 \end{smallmatrix}\right) : \left(\begin{smallmatrix} 1+\pi & 0 \\ 0 & (1+\pi)^{-1} \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 1 & 0 \\ b\pi^2 \frac{\pi+2}{(\pi+1)^2} & 1 \end{smallmatrix}\right) \in [H : H]$.

This shows that the first two families of generators of H' lie inside $[H : H]$. In order to prove that $[H : H] = H'$ it only remains to check that $\left(\begin{smallmatrix} 1+\pi^2 z & 0 \\ 0 & (1+\pi^2 z)^{-1} \end{smallmatrix}\right) \in [H : H]$ for any $z \in \mathcal{O}$. But $[\left(\begin{smallmatrix} 1 & 0 \\ \pi & 1 \end{smallmatrix}\right) : \left(\begin{smallmatrix} 1 & \pi c \\ 0 & 1 \end{smallmatrix}\right)] = \left(\begin{smallmatrix} 1-c\pi^2 & c^2\pi^3 \\ -c\pi^3 & c^2\pi^4+c\pi^2+1 \end{smallmatrix}\right) \in [H : H]$. Multiplying this element by matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ \pi^2 x & 1 \end{smallmatrix}\right)$ we get $\left(\begin{smallmatrix} 1-c\pi^2 & c^2\pi^3 \\ 0 & (1-c\pi^2)^{-1} \end{smallmatrix}\right) \in [H : H]$ (as we can raise the power of π appearing in the place $(2, 1)$ eventually making it 0 modulo π^n). The same argument applies to matrices of the form $\left(\begin{smallmatrix} 1 & \pi^2 y \\ 0 & 1 \end{smallmatrix}\right)$ we get $\left(\begin{smallmatrix} 1-c\pi^2 & 0 \\ 0 & (1-c\pi^2)^{-1} \end{smallmatrix}\right) \in [H : H]$. □

Lemma 4.4.2 gives the existence of auxiliary primes that kill global obstructions. We introduced special primes because otherwise we would have not been able to modify the behavior of f.

Lemma 4.4.4. *Let ρ_n and P as in the hypotheses of Theorem 4.4.1. Then there exists a finite set P' consisting of nice primes for ρ_n and eventually one special prime for f such that* $\text{III}^1_{P\cup P'}(Ad^0\bar{\rho})$ *and* $\text{III}^2_{P\cup P'}(Ad^0\bar{\rho})$ *are both trivial.*

Proof. Recall that $\text{III}^2_P(Ad^0\bar{\rho}) \simeq \text{III}^1_P((Ad^0\bar{\rho})^*)$ by global duality. If $f \notin \text{III}^1_P(Ad^0\bar{\rho})$ this follows from taking basis $\{f_1,\dots,f_r\}$ and $\{g_1,\dots,g_s\}$ of $\text{III}^1_P(Ad^0\bar{\rho})$ and $\text{III}^1_P((Ad^0\bar{\rho})^*)$ respectively and choosing, by applying Lemma 4.4.2, sets of nice primes $q_1,\dots,q_r$ and $q'_1,\dots,q'_s$ such that

- $f_i|_{G_{q_j}} = 0$ if $i \neq j$ and $f_i|_{Gq_i} \neq 0$.
- $g_i|_{G_{q'_j}} = 0$ if $i \neq j$ and $g_i|_{G_{q'_i}} \neq 0$.

If, otherwise, $f \in \text{III}^1_P(Ad^0\bar{\rho})$, we do the same but taking a special prime for f instead of a nice prime. □

From the previous lemmas, we can assume that $\text{III}^1_P(Ad^0\bar{\rho})$ and $\text{III}^2_P(Ad^0\bar{\rho})$ are both trivial by enlarging P if necessary. This imply, as in [KLR05], the following two key propositions.

Proposition 4.4.5. *Let S be a finite set of primes and $\rho_m : G_S \to \mathrm{GL}_2(\mathcal{O}/\pi^m)$ a continuous representation in the hypotheses of Theorem 4.4.1 such that $\text{III}^1_S(Ad^0\bar{\rho}) \simeq \text{III}^2_S(Ad^0\bar{\rho}) \simeq 0$. Then there exists a set Q of nice primes for ρ_m such that the maps*

$$\mathrm{H}^1(G_{S\cup Q}, Ad^0\bar{\rho}) \to \bigoplus_{\ell\in S} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$$

and

$$\mathrm{H}^1(G_{S\cup Q}, (Ad^0\bar{\rho})^*) \to \bigoplus_{\ell\in Q} \mathrm{H}^1(G_\ell, (Ad^0\bar{\rho})^*)$$

are isomorphisms.

Proof. Given the existence of auxiliary primes, this is just Lemma 8 of [KLR05]. □

Proposition 4.4.6. *Let ρ_m, S and Q as in Proposition 4.4.5. For each $q_i \in Q$ pick an element $h_i \in \mathrm{H}^1(G_{q_i}, Ad^0\bar{\rho})$. Then there is a finite set T of nice primes for ρ_m and an element*

$$g \in \mathrm{H}^1(G_{S\cup Q\cup T}, Ad^0\bar{\rho})$$

satisfying

- *$g|_{G_\ell} = 0$ for $\ell \in S$.*
- *$g(\sigma_{q_i}) = h_i(\sigma_{q_i})$ for $q_i \in Q$.*
- *$g(\sigma_t) = 0$ for $t \in T$.*

This is Corollary 11 of [KLR05], which follows from Propositions 9 and 10 of that work. We both proposition below. It is not difficult to check that the arguments used to prove these results adapt to our setting, however we give a brief explanation of the tools used in the proofs.

Proposition 4.4.7. *Let ρ_m be a deformation of $\overline{\rho}$ to $\mathrm{GL}_2(\mathcal{O}/\pi^m)$ unramified outside a set S and assume that $\text{III}^1_S(Ad^0\bar{\rho})$ and $\text{III}^2_S(Ad^0\bar{\rho})$ are trivial. Let $Q = \{q_1, \dots, q_n\}$ be a set of auxiliary primes from Proposition 4.4.5 and A any set of primes disjoint from $S \cup Q$. Then for any k between 1 and n there exist an infinite set of primes T_k such that*

- *All the primes of T_k are nice for ρ_m.*
- *The kernel of the map*

$$\mathrm{H}^1(G_{S\cup Q\cup\{t_k\}}, Ad^0\bar{\rho}) \to \bigoplus_{\ell\in S} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$$

  is spanned by a single element f_{t_k}.
- *$f_{t_k}|_{G_\ell} = 0$ for all $\ell \in S \cup Q \cup A \backslash \{q_k\}$.*
- *f_{t_k} is unramified atG_{q_k} and $f_{t_k}(\sigma_{q_k}) \neq 0$.*

Proof. This is Proposition 9 of [KLR05] for a ramified coefficient field. It can be easily checked that its proof adapts to our setting as it is based on a repeated application of Wiles product formula (Proposition 1.3.6) combined with Lemmas 4.4.2 and 4.4.5 and a description of the cohomology groups for nice primes. □

Proposition 4.4.8. *Retaining the notation from Proposition 4.4.7, there is a set $\tilde{T}_k \subset T_k$ of one or two elements for which there is a linear combination f_k of the elements f_{t_k} for $t_k \in \tilde{T}_k$ such that*

- *$f_k(\sigma_{t_k}) = 0$ for all $t_k \in \tilde{T}_k$ and $f_k(\sigma_{q_k}) \neq 0$.*
- *If $j < k$ then for $t_j \in \tilde{T}_j$ we have $f_{t_k}(\sigma_{t_j}) = 0$.*

Proof. Again, this is Proposition 10 of [KLR05] for a ramified coefficient field. We can check once again that the same proof works when the field of coefficients is ramified as it is a clever use of global reciprocity (Theorem 1.3.2) together with Proposition 4.4.7 and again Wiles formula (Proposition 1.3.6), Lemmas 4.4.2 and 4.4.5 and a description of cohomology groups for nice primes. □

With this las results, it is easy to prove Proposition 4.4.6.

Proof of Proposition 4.4.6. It is easy to see that an appropriate linear combination of the elements f_k from Proposition 4.4.8 works. □

We are now able to state and prove the main theorem of this section. Recall that for q a nice prime there is a set C_q of deformations to characteristic 0 and a subspace $N_q \subseteq \mathrm{H}^1(G_q, Ad^0\bar{\rho})$ of dimension 1 preserving its reductions.

Theorem 4.4.9. *Let ρ_n and P as in Theorem 4.4.1 and let α be an integer greater or equal than n. Pick, for each $\ell \in P$ a lift*

$$\rho_{\ell,\alpha} : G_\ell \to \mathrm{GL}_2(\mathcal{O}/\pi^\alpha)$$

of $\rho_n|_{G_\ell}$. Then, there is a finite set of primes P', formed by nice primes for ρ_n except possibly one which is a special prime, and a lift

$$\rho_\alpha : G_{P\cup P'} \to \mathrm{GL}_2(\mathcal{O}/\pi^\alpha)$$

of ρ_n such that $\rho_\alpha|_{G_\ell} \simeq \rho_{\ell,\alpha}$ for every $\ell \in P$ and $\rho_\alpha|_{G_q}$ is the reduction of some member of C_q for every $q \in P'$.

Proof. We will prove the theorem by induction on α. If $\alpha = n$ the statement is trivial. Assume the theorem is true for $\alpha = m$, we want to prove that it holds for $m+1$. We start by enlarging P in order to make both $\text{Ш}^1_P(Ad^0\bar{\rho})$ and $\text{Ш}^2_P(Ad^0\bar{\rho})$ trivial. This may add a special prime to P'. Now apply the theorem for $n = m$ and the collection of local deformations given by the reductions mod π^m of the local representations $\rho_{\ell,m+1}$. Then, there is a lift $\rho_m : G_{P\cup P'} \to \mathrm{GL}_2(\mathcal{O}/\pi^m)$ such that $\rho_m|_{G_\ell} = \rho_{\ell,m+1} \pmod{\pi^m}$, where P' consists on nice primes for ρ_n. We will add two sets of nice primes in order to first get a lift of ρ_m to $\mathcal{O}/\pi^{m+1}$ and then locally adjust this lift. Since $\text{Ш}^2_P(Ad^0\bar{\rho}) = 0$, ρ_m has no global obstructions. Also observe that ρ_m is unobstructed at the primes of P, as $\rho_m|_{G_\ell}$ lifts to $\rho_{\ell,m+1}$ and at the primes of P', as $\rho_m|_{G_q}$ is the reduction of some member of C_q. Therefore ρ_m is both globally and locally unobstructed implying that it lifts to some

$$\tilde{\rho}_{m+1} : G_{P\cup P'} \to \mathrm{GL}_2(\mathcal{O}/\pi^{m+1}).$$

To complete the proof, we need to fix the local behavior of $\tilde{\rho}_{m+1}$. We will do this in two steps. First of all, pick for each $\ell \in P \cup P'$ a class $u_\ell \in \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ such that

- $(1 + \pi^m u_\ell)\tilde{\rho}_{m+1}|_{G_\ell} \simeq \rho_{\ell,m+1}$ for $\ell \in P$.
- $(1 + \pi^m u_\ell)\tilde{\rho}_{m+1}|_{G_\ell}$ is a reduction of a member of C_ℓ for $\ell \in P'$.

Now, let Q be the set of nice primes produced by applying Proposition 4.4.5 to $S = P \cup P'$. As the map

$$\mathrm{H}^1(G_{P\cup P'\cup Q}, Ad^0\bar{\rho}) \to \bigoplus_{\ell\in P\cup P'} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$$

is an isomorphism, there is a class $g_1 \in \mathrm{H}^1(G_{P\cup P'\cup Q}, Ad^0\bar{\rho})$ such that $g_1|_{G_\ell} = u_\ell$ for all $\ell \in P \cup P'$. Acting by this element on $\tilde{\rho}_{m+1}$ fixes its local behavior at the places of $P \cup P'$ but may ruin it at the newly added primes of Q. We will solve this issue by adding a second set of auxiliary primes.

We pick, for each $q_i \in Q$ a class $h_i \in \mathrm{H}^1(G_{q_i}, Ad^0\bar{\rho})$ such that

$$(1 + \pi^m(h_i + g_1))\tilde{\rho}_{m+1}(\sigma_{q_i}) = \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix}.$$

Let T and g_2 be respectively the set of nice primes and the element of $\mathrm{H}^1(G_{P\cup P'\cup Q\cup T}, Ad^0\bar{\rho})$ obtained from applying Proposition 4.4.6 with $S = P \cup P'$ and Q and h_i as above. It is easy to check that

- $(1 + \pi^m g)\tilde{\rho}_{m+1}|_{G_\ell} \simeq \rho_{\ell,m+1}$ for $\ell \in P$.
- $(1 + \pi^m g)\tilde{\rho}_{m+1}|_{G_\ell} \in C_\ell$ for $\ell \in P' \cup Q \cup T$.

It follows that $\rho_{m+1} = (1 + \pi^m g)\tilde{\rho}_{m+1}$ satisfies what we need, completing the proof. □

4.4.2 Exponent α and above

Assume we have a representation ρ_n as in Theorem 4.4.1 with $n \geq \alpha$ (since otherwise we apply Theorem 4.4.9). To ease the notation, let P denote the set $P \cup P'$ if we applied Theorem 4.4.9.

Recall from Sections 2 and 3 that for exponents bigger than α we have defined for each $\ell \in P$ a set of deformations C_ℓ of ρ_n to characteristic 0 and a subspace $N_\ell \subseteq \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})$ such that N_ℓ preserves the reductions of the elements in C_ℓ, in the sense of Proposition 4.2.1. They also satisfy that $\dim N_\ell = \dim \mathrm{H}^1(G_\ell, Ad^0\bar{\rho}) - \dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$ for $\ell \in P\backslash\{p\}$ and $\dim N_p = \dim \mathrm{H}^1(G_p, Ad^0\bar{\rho}) - \dim \mathrm{H}^2(G_p, Ad^0\bar{\rho}) + 1$.

In this setting, we can mimic the arguments of [Ram02] in our situation with some minor modifications. We start by collecting a series of results that will prove useful.

Lemma 4.4.10. *Let $r = \dim \mathrm{III}^1_P((Ad^0\bar{\rho})^*)$ and $s = \sum_{\ell \in P} \dim \mathrm{H}^2(G_\ell, Ad^0\bar{\rho})$. Then*

$$\dim \mathrm{H}^1(G_P, Ad^0\bar{\rho}) = r + s + 2.$$

Proof. This is the Lemma before Lemma 10 in [Ram02]. $\square$

Proposition 4.4.11. *Let $\{f, f_1, \ldots, f_{r+s+1}\}$ be a basis of $\mathrm{H}^1(G_P, Ad^0\bar{\rho})$, where f is the element attached to ρ_n mod π^2. There is a set $Q = \{q_1, \ldots, q_r\}$ of nice primes for ρ_n not in P such that:*

- $\mathrm{III}^1_{P\cup Q}((Ad^0\bar{\rho})^*) = 0$ *and* $\mathrm{III}^2_{P\cup Q}(Ad^0\bar{\rho}) = 0$.
- $f_i|_{G_{q_j}} = 0$ *for* $i \neq j$ *and* $f|_{q_j} = 0$ *for all* j.
- $f_i|_{G_{q_i}} \notin N_{q_i}$
- *The inflation map* $\mathrm{H}^1(G_P, Ad^0\bar{\rho}) \to \mathrm{H}^1(G_{P\cup Q}, Ad^0\bar{\rho})$ *is an isomorphism.*

Proof. This is a part of the results obtained in Proposition 4.4.2 and Lemma 4.4.4. The only observation that we must make is that in Lemma 4.4.4, the special prime is only used for getting $\mathrm{III}^1_{P\cup Q}(Ad^0\bar{\rho}) = 0$, so for this result it is enough to use nice primes. $\square$

Lemma 4.4.12. *Let $\langle f, f_1, \ldots, f_d\rangle$ be the kernel of the map*

$$\mathrm{H}^1(G_{P\cup Q}, Ad^0\bar{\rho}) \to \bigoplus_{\ell \in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell.$$

Then $r \geq d$.

Proof. This follow from computing the following dimensions: $\dim \oplus_{\ell\in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell = s + 1$ and $\dim \mathrm{H}^1(G_{P\cup Q}, Ad^0\bar{\rho}) = r + s + 2$. $\square$

Lemma 4.4.13. *There is a finite set of nice primes $\{t_{r+1}, \ldots, t_d\}$ for ρ_n such that*

- $f_i|_{t_j} = 0$ *if* $i \neq j$ *and* $f_i|_{t_i} \notin N_{t_i}$.
- *The restriction map*

$$\mathrm{H}^1(G_{P\cup Q\cup T}, Ad^0\bar{\rho}) \to \bigoplus_{\ell \in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

is surjective.

Proof. This is Lemma 14 of [Ram02], which is an almost immediate consequence of Proposition 10 of the same work. It is easy to check that the proof of Proposition 10 given there adapts to the ramified setting, as it is a combination of Poitou-Tate duality (Theorem 1.3.5) with Lemma 4.4.13. □

The set $Q \cup T$ will serve as the auxiliary set for Theorem 4.4.1. So far we have the following properties:

- For $1 \leq i \leq r$: $f_i|_{G_q} = 0$ for all $q \in Q \cup T$ except $q_i \in Q$ for which $f_i|_{G_{q_i}} \notin N_{q_i}$.
- For $r+1 \leq i \leq d$: $f_i|_{G_q} = 0$ for all $q \in Q \cup T$ except $t_i \in T$ for which $f_i|_{G_{t_i}} \notin N_{t_i}$.
- The restriction $\langle f_{d+1}, \dots f_{d+s+1} \rangle \to \oplus_{\ell \in P} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$ is an isomorphism.
- $f|_{G_\ell} \in N_\ell$ for every $\ell \in P \cup Q \cup T$.

It easily follows from these properties that

Proposition 4.4.14. *The map*

$$\mathrm{H}^1(G_{P\cup Q\cup T}, Ad^0\bar{\rho}) \to \bigoplus_{\ell \in P\cup Q\cup T} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell$$

is surjective and has one dimensional kernel generated by f.

From this, we can easily deduce Theorem 4.4.1 in the same way as Theorem 1 is proved in [Ram02]. Moreover we have the following corollary

Corollary 4.4.15. *Let $\rho_n : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ and $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ as in Theorem 4.4.1. Consider the collection $\mathcal{L}$ of deformation conditions given by the pairs (C_ℓ, N_ℓ) for $\ell \in P \cup Q \cup T$. Let R_u be the deformation problem with fixed determinant and local conditions $\mathcal{L}$. Then R_u has a quotient isomorphic to $W(\mathbb{F})[[X]]$ containing ρ_n.*

Proof. Proposition 4.4.14 tells us that

$$\mathrm{Ker}\left(\mathrm{H}^1(G_{P\cup Q\cup T}, Ad^0\bar{\rho}) \to \oplus_{\ell \in P\cup Q\cup T} \mathrm{H}^1(G_\ell, Ad^0\bar{\rho})/N_\ell\right) = \langle f \rangle.$$

As $\text{Ш}^2_{P\cup Q\cup T}(Ad^0\bar{\rho}) = 0$, we all the obstructions for the problem are local. Moreover, the fact that N_ℓ preserves C_ℓ for all $m \geq n_0$ implies that the local problems are also unobstructed above ρ_{n_0}. This implies that there is a family of lifts of ρ_n which is parametrized by a ring $R_u^n \simeq W(\mathbb{F})[[X]]$. □

4.5 Modularity

So far we have constructed, for a mod π^n representation ρ_n, which is nearly ordinary at p, a global deformation $\rho_u^n : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(\mathbb{F})[[X]])$ that lifts ρ_n and is also nearly ordinary at p. This gives a family of lifts of ρ_n to rings of dimension and characteristic 0. A natural question to ask is if this family contains any modular points. We cannot give a complete answer to this question but we can say something in a particular case. Recall the **b**alancedness assumption introduced in [KR14].

Definition (Balancedness Assumption)**.** Let ρ_n and the local deformations ρ_ℓ as in the hypotheses of our main theorems. We say that they satisfy the balancedness assumption if for each $\ell \in P$ there is a smooth quotient of the versal deformation ring for $\overline{\rho}$ containing ρ_ℓ whose tangent space has dimension $\dim \mathrm{H}^0(G_\ell, Ad^0\overline{\rho})$. In other words, the collection of deformations satisfy the balancedness assumption if we can construct the sets (C_ℓ, N_ℓ) is a way such that N_ℓ preserves C_ℓ for every power of π.

Observe that when ρ_n and the collection $\{\rho_\ell\}$ satisfy the balancedness assumption, it can be proved in the same way as in the previous Section that the universal deformation ring R_u for $\overline{\rho}$, with local conditions imposed by the local smooth quotients, is isomorphic to $W(\mathbb{F})[[X]]$. Under this hypotheses, we can guarantee that this deformation ring has modular points, but we cannot control their level.

Remark. Observe that from the thorough analysis of the local problem done in Section 4.2 one can give conditions under which the collection of deformations considered satisfy balancedness assumption.

Theorem 4.5.1. *Let $p \geq 5$ be a prime, $\mathcal{O}$ the ring of integers of a finite extension $K/\mathbb{Q}_p$ with ramification degree $e > 1$ and π its local uniformizer. Let $\rho_n : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation satisfying*

- *ρ_n is odd.*
- *$\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise.*
- *ρ_n is ordinary at p.*
- *ρ_n is unramified at 2.*

Let P be a set of primes containing the ramification set of ρ_n and the prime p, and for each $\ell \in P\backslash\{p\}$ pick a local deformation $\rho_\ell : G_\ell \to \mathrm{GL}_2(\mathcal{O})$ lifting $\rho_n|_{G_\ell}$, which is not bad. Then there exists a finite set of primes Q and a continuous representation $\rho : G_{P\cup Q} \to \mathrm{GL}_2(\mathcal{O})$ such that

- *ρ lifts ρ_n, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.*
- *ρ is modular.*
- *For every $\ell \in P$, $\rho|_{I_\ell} \simeq \rho_\ell|_{I_\ell}$ over $\mathrm{GL}_2(\mathcal{O})$.*
- *For every $q \in Q$, $\rho|_{I_q}$ is unipotent and $q \not\equiv \pm 1 \pmod{p}$ for all but possibly one $q \in Q$.*
- *ρ is ordinary at p.*
- *The collection ρ_n, ρ_ell satisfy balancedness assumption.*

Proof. As mentioned above, when balancedness assumptions holds, nearly ordinary deformations with fixed determinant $\omega\chi^k$ satisfying certain local conditions at the primes of $P \cup Q$ are parametrized by an universal ring $R_u \simeq W(\mathbb{F})[[X]]$ and a universal deformation ρ_u that satisfies

$$\rho_u|_{I_p} \simeq \begin{pmatrix} \Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix}.$$

As we are asking $\overline{\rho}$ to be modular, the work of [DT94] ensures that R_u contains a twist of characteristic zero modular. Specifically, [DT94] guarantees that $\overline{\rho}$ has an ordinary lift of classical weight and arbitrary determinant which we will call ρ_f. After twisting this lift by the corresponding power of the cyclotomic character, it lies in our family of deformations with fixed determinant. We claim that this implies that the family of representations that we have constructed is isomorphic to the Hida family of ρ_f.

To see this, let $\rho_h : G_P \to R_h \simeq W(\mathbb{F})[[X]]$ be the representation attached to the Hida family of ρ_f. We apply a twisting argument similar to that of Theorem 4.3.1. Let $\Psi = \det \rho_u$ and $\Phi = \det \rho_h$. Then we twist ρ_h by the square root of $\Psi(\Phi)^{-1}$ (which exists as this character is congruent to 1 modulo π) to obtain a deformation with coefficients in R_h that must factor through R_u (as it has the right determinant an satisfies local conditions). This gives a surjective morphism $R_u \to R_h$, which must be an isomorphism as both rings are isomorphic to $W(\mathbb{F})[[X]]$.

The existence of twists of modular points lifting ρ_n in R_u is now immediate. We sketch a proof below. Assume that ρ_n is induced by the morphism $W(\mathbb{F})[[X]] \to \mathcal{O}/\pi^n$ sending $X \to \overline{z}$. Then for any $z \in \mathcal{O}$ lifting $\overline{z}$ we have a deformation lifting ρ_n induced by the morphism sending $X \mapsto z$. Such deformation must satisfy

$$\rho_z|_{I_p} \simeq \begin{pmatrix} \omega_1\psi_1\chi^b & * \\ 0 & \omega_2\psi_2\chi^{k-b} \end{pmatrix}$$

for ω_1, ω_2 unramified characters, ψ_1 and ψ_2 of finite order and $b \in \mathcal{O}_m$.

We claim that the exponent b is an analytic function of z. To see this, let $g \in I_p$ be an element at which the cyclotomic character is not trivial and let

$$\rho_u(g) = \begin{pmatrix} \alpha(x) & * \\ 0 & \beta(x) \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{Z}_p[[x]]$. Evaluating at z we have that:

$$\alpha(z) = \psi_1(g)\chi(g)^b$$

As $\psi_1 : G_{\mathbb{Q}} \to \mathcal{O}^\times$ is a character of finite order, there is an integer M independent of ψ (depending only on $\mathcal{O}$) such that $\psi^M = 1$. Therefore, we can recover b as

$$b(z) = \frac{\log((\alpha(z))^M)}{M\chi(g)}$$

which is analytic.

Moreover, as we have seen that R_u is actually a Hida family, all the weights b must lie in Z_p. Therefore we have that $b : \mathcal{O} \to \mathbb{Z}_p$ is an analytic function, and as it is open and $\mathbb{Z} \subset \mathbb{Z}_p$ is dense, it holds that there is an element $z \in \mathcal{O}$ such that $z = \overline{z} \pmod{\pi}$ and $b(z) \in \mathbb{Z}$, finishing the proof. $\square$

Finally, we cite the main result of [KR14], which goes in the same direction as our last Theorem. There it is proved that in the same hypotheses, if one allows even more ramification, there is a modular point of any desired weight.

Corollary 4.5.2. *Let ρ_n in the same hypotheses as in Theorem 4.5.1. Then there exists a finite set of primes Q and a continuous representation $\rho : G_{P\cup Q} \to \mathrm{GL}_2(\mathcal{O})$ satisfying the consequences of Theorem 4.5.1 which also is of weight 2.*

Proof. Once we have a lift of ρ_n to characteristic 0 (which exists by Theorem 4.5.1) the corollary follows from the results of Section 5 of [KR14]. This gives another lift of ρ_n which may ramify at a bigger set of primes but is of weight 2. $\square$

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