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# A minimum problem with free boundary in Orlicz spaces ☆

Sandra Martínez *, Noemi Wolanski

*Departamento de Matemática, FCEyN, UBA (1428) Buenos Aires, Argentina*

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**Abstract**

We consider the optimization problem of minimizing $\int_\Omega G(|\nabla u|) + \lambda\chi_{\{u>0\}}\,dx$ in the class of functions $W^{1,G}(\Omega)$ with $u - \varphi_0 \in W_0^{1,G}(\Omega)$, for a given $\varphi_0 \geqslant 0$ and bounded. $W^{1,G}(\Omega)$ is the class of weakly differentiable functions with $\int_\Omega G(|\nabla u|)\,dx < \infty$. The conditions on the function $G$ allow for a different behavior at 0 and at $\infty$. We prove that every solution $u$ is locally Lipschitz continuous, that it is a solution to a free boundary problem and that the free boundary, $\Omega \cap \partial\{u > 0\}$, is a regular surface. Also, we introduce the notion of weak solution to the free boundary problem solved by the minimizers and prove the Lipschitz regularity of the weak solutions and the $C^{1,\alpha}$ regularity of their free boundaries near "flat" free boundary points.


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* Corresponding author.

*E-mail addresses:* smartin@dm.uba.ar (S. Martínez), wolanski@dm.uba.ar (N. Wolanski).

*URL:* http://mate.dm.uba.ar/~wolanski (N. Wolanski).

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## 1. Introduction

In this paper we study the following minimization problem. For $\Omega$ a smooth bounded domain in $\mathbb{R}^N$ and $\varphi_0$ a nonnegative function with $\varphi_0 \in L^\infty(\Omega)$ and $\int_\Omega G(|\nabla\varphi_0|)\,dx < \infty$, we consider the problem of minimizing the functional,

$$\mathcal{J}(u) = \int_\Omega G\big(|\nabla u|\big) + \lambda\chi_{\{u>0\}}\,dx \tag{1.1}$$

in the class of functions

$$\mathcal{K} = \left\{ v \in L^1(\Omega)\colon \int_\Omega G\big(|\nabla v|\big)\,dx < \infty,\ v = \varphi_0 \text{ on } \partial\Omega \right\}.$$

This kind of optimization problem has been widely studied for different functions $G$. In fact, the first paper in which this problem was studied is [4]. The authors considered the case $G(t) = t^2$. They proved that minimizers are weak solutions to the free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ u = 0,\ |\nabla u| = \lambda & \text{on } \partial\{u > 0\} \end{cases} \tag{1.2}$$

and proved the Lipschitz regularity of the solutions and the $C^{1,\alpha}$ regularity of the free boundaries.

This free boundary problem appears in several applications. A very important one is that of fluid flow. In that context, the free boundary condition is known as Bernoulli's condition.

The results of [4] have been generalized to several cases. For instance, in [5] the authors consider problem (1.1) for a convex function $G$ such that $ct < G'(t) < Ct$ for some positive constants $c$ and $C$. Recently, in the article [7] the authors considered the case $G(t) = t^p$ with $1 < p < \infty$. In these two papers only minimizers are studied. Minimizers satisfy very good properties like nondegeneracy at the free boundary and uniform positive density of the set $\{u = 0\}$ at free boundary points. On the other hand, the free boundary problem (1.2) and its counterpart for different choices of functions $G$ appears in different contexts. For instance, as limits of singular perturbation problems of interest in combustion theory (see for instance, [6,13]). The study of weak solutions to (1.2) also appears when considering some optimization problems with a volume constrain (see for instance, [2,3,10–12,16]). Thus, the study of the regularity of weak solutions and their free boundaries, while including the case of minimizers, it is of a wider interest.

Thus, one of the goals of this paper is to return to the ideas of [4] and study weak solutions. Nevertheless, our main goal is to get these results under the natural conditions on $G$ introduced by Lieberman (see [15]) for the study of the regularity of weak solutions to the elliptic equation (possibly degenerate or singular)

$$\mathcal{L}u = \operatorname{div}\left( g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|} \right) \tag{1.3}$$

where $g(t) = G'(t)$.

These conditions ensure that Eq. (1.3) is equivalent to a uniformly elliptic equation in nondivergence form with ellipticity constants independent of the solution $u$ on sets where $\nabla u \neq 0$.

Moreover, these conditions do not imply any kind of homogeneity on the function $G$ and moreover, they allow for a different behavior of the function $g$ when $|\nabla u|$ is close to zero or infinity. Namely, we assume that $g$ satisfies

$$0 < \delta \leqslant \frac{tg'(t)}{g(t)} \leqslant g_0 \quad \forall t > 0, \tag{1.4}$$

for certain constants $\delta$ and $g_0$.

Observe that $\delta = g_0 = p - 1$ when $G(t) = t^p$, and conversely, if $\delta = g_0$ then $G$ is a power. A different example consists of a function $G$ such that $g(t) = t^a \log(bt + c)$ with $a, b, c > 0$ that satisfies (1.4) with $\delta = a$ and $g_0 = a + 1$. Another interesting case is that of a function $G$ with $g \in C^1([0,\infty))$, $g(t) = c_1 t^{a_1}$ for $t \leqslant s$, $g(t) = c_2 t^{a_2} + d$ for $t \geqslant s$. In this case $g$ satisfies (1.4) with $\delta = \min(a_1, a_2)$ and $g_0 = \max(a_1, a_2)$.

Moreover, any linear combination with positive coefficients of functions satisfying (1.4) also satisfies (1.4). Also, if $g_1$ and $g_2$ satisfy condition (1.4) with constants $\delta^i$ and $g_0^i$, $i = 1, 2$, the function $g = g_1 g_2$ satisfies (1.4) with $\delta = \delta^1 + \delta^2$ and $g_0 = g_0^1 + g_0^2$, and the function $g(t) = g_1(g_2(t))$ satisfies (1.4) with $\delta = \delta^1\delta^2$ and $g_0 = g_0^1 g_0^2$.

This observation shows that there is a wide range of functions $G$ under the hypothesis of this paper.

The main results in this article are:

**Theorem 1.1.** *If $g$ satisfies* (1.4)*, there exists a minimizer of $\mathcal{J}$ in $\mathcal{K}$ and any minimizer $u$ is non-negative and belongs to $C^{0,1}_{\mathrm{loc}}(\Omega)$. Moreover, for any domain $D \Subset \Omega$ containing a free boundary point, the Lipschitz constant of $u$ in $D$ is controlled in terms of $N, g_0, \delta, \mathrm{dist}(D, \partial\Omega)$ and $\lambda$.*

We also prove that $\mathcal{L}u = 0$ in the set $\{u > 0\}$ and that $\{u > 0\}$ has finite perimeter locally in $\Omega$. As usual, we define the reduced boundary by $\partial_{\mathrm{red}}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\} \mid |\nu_u(x)| = 1\}$, where $\nu_u(x)$ is the unit outer normal in the measure theoretic sense (see [9]), when it exists, and $\nu_u(x) = 0$ otherwise. Then, we prove that $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}) = 0$.

We also prove that minimizers have an asymptotic development near any point in their reduced free boundary. Namely,

**Theorem 1.2.** *Let $u$ be a minimizer, then for every $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$,*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|) \quad \text{as } x \to x_0, \tag{1.5}$$

*where $\lambda^*$ is such that $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$. (Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$ and $v^- = -\min(v, 0)$).*

So that, in a weak sense minimizers satisfy

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \{u > 0\}, \\ u = 0, \quad |\nabla u| = \lambda^* & \text{on } \Omega \cap \partial\{u > 0\}. \end{cases} \tag{1.6}$$

These results suggest that we consider weak solutions of the problem (1.6). We give two different definitions of weak solution (Definitions 8.1 and 8.2). Minimizers of the functional $\mathcal{J}$ verify both definitions of weak solution. The main difference between these two definitions is that for functions satisfying Definition 8.1 we have that $\mathcal{H}^{N-1}(\partial\{u>0\}\setminus\partial_{\text{red}}\{u>0\})=0$, whereas for functions satisfying Definition 8.2 we may have $\partial_{\text{red}}\{u>0\}=\emptyset$. Definition 8.2 is more suitable for limits of singular perturbation problems.

Hypotheses (1), (2) and (3) of Definition 8.1 are similar to the ones in the definition of weak solution to the problem studied in [4]. In our case, we add hypothesis (4) in order to prove that weak solutions satisfying Definition 8.1 also have the asymptotic development (1.5) at $\mathcal{H}^{N-1}$ almost every point of the reduced free boundary. Condition (4) is also used in the proof of the regularity of the free boundary. We prove the following theorem.

**Theorem 1.3.** *Let $u$ be a weak solution. Then, $\mathcal{H}^{N-1}$ almost every point in the reduced free boundary $\partial_{\text{red}}\{u>0\}$ has a neighborhood where the free boundary is a $C^{1,\alpha}$ surface. Moreover, if $u$ is a weak solution according to Definition* 8.1*, the remainder of the free boundary has $\mathcal{H}^{N-1}$-measure zero.*

We point out that we prove that, if $u$ is a weak solution, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of every point where $u$ has the asymptotic development (1.5) for some unit vector $\nu$. We prove that this is the case for every point in the reduced free boundary when $u$ is a minimizer (see Theorem 7.1). So that, if $u$ is a minimizer the reduced free boundary is an open $C^{1,\alpha}$ surface and the remainder of the free boundary has $\mathcal{H}^{N-1}$-measure zero.

**Outline of the paper and technical comments.** In Section 2 we give some properties of the function $g$ and define some spaces that we use to prove existence of minimizers. Then, we prove some properties of solutions and subsolution of $\mathcal{L}v=0$. We also state some real analytic properties for functions with finite $\int_\Omega G(|\nabla u|)\,dx$ and we prove a Cacciopoli type inequality valid for these functions. We also prove an inequality (Theorem 2.3) that will be used several times in this work.

In Section 3 we prove the existence of minimizers and that they are subsolutions of $\mathcal{L}v=0$. We also prove a maximum principle and the positivity of the minimizers. The existence of minimizers, while standard in its form, makes strong use of the Orlicz spaces and the second inequality in condition (1.4).

In Section 4 we prove that any local minimizer $u$ is Hölder continuous (Theorem 4.1), $\mathcal{L}u=0$ in $\{u>0\}$ (Lemma 4.1) and finally we prove the local Lipschitz continuity (Theorem 4.2). The proof of the Hölder continuity of the minimizers is a key step in our analysis. Although, the proof follows closely the one for the case of the $p$-Laplacian [7], here we have to use all the properties of the function $G$ which mainly come into play through the inequality in Theorem 2.3.

In Section 5 we prove that minimizers satisfy a nondegeneracy property near the free boundary $\Omega\cap\partial\{u>0\}$. We also prove that the sets $\{u>0\}$ and $\{u=0\}$ have locally uniform positive density at the free boundary (Theorem 5.1). In this theorem we make strong use of the properties of $G$ and the corresponding Orlicz space.

In Section 6 we prove that the free boundary has Hausdorff dimension $N-1$ and we obtain a representation theorem for minimizers (Theorem 6.3). This implies that $\{u>0\}$ has locally finite perimeter in $\Omega$. Finally we prove that $\mathcal{H}^{N-1}(\partial\{u>0\}\setminus\partial_{\text{red}}\{u>0\})=0$.

In Section 7 we give some properties of blow up sequences of minimizers. We prove that any limit of a blow up sequence of minimizers is again a minimizer (Lemma 7.2) and we finally prove the asymptotic development of minimizers at every point in their reduced free boundary (Theorem 7.1).

In Section 8 we give the definition of weak solution (Definitions 8.1 and 8.2). We show that most of the properties that we proved for minimizers also hold for weak solutions according to Definition 8.1, and we mention the differences between the two definitions (Remarks 8.2 and 8.3).

In Section 9 we prove the regularity of the free boundary of weak solutions near "flat" free boundary points (Theorem 9.3) and then, we deduce the regularity of the free boundary of weak solutions near almost every point in their reduced free boundary and, in the case of minimizers, the regularity of the whole reduced free boundary (Theorem 9.4). While most of the steps of the proof of the regularity of the free boundary of weak solutions are very similar to the corresponding ones for minimizers in the uniformly elliptic case considered in [5] and in the case $G(t) = t^p$ considered in [7], there are some steps that need a new proof since weak solutions do not verify the locally uniform positive density of $\{u = 0\}$ at the free boundary (see Lemmas 9.1 and 9.5 and Theorem 9.3).

## 2. Properties of the function $G$

In this section we state and prove some properties of the function $G$ and its derivative $g$ that are used throughout the paper. We also state some real analytic properties for functions with finite $\int_\Omega G(|\nabla u|)\,dx$ like a form of Poincaré Inequality, a Cacciopoli type inequality, the Hölder continuity of functions in a kind of Morrey type space, properties of weak solutions to $\mathcal{L}u = 0$ and a comparison principle for sub and supersolutions. We also prove an important inequality (Theorem 2.3). All these properties will be thoroughly used throughout the paper. Some of them have been proved in [15]. We only write down the proof of statements not contained in [15].

**Lemma 2.1.** *The function $g$ satisfies the following properties:*

(g1) $\min\{s^\delta, s^{g_0}\}g(t) \leqslant g(st) \leqslant \max\{s^\delta, s^{g_0}\}g(t)$,
(g2) *$G$ is convex and $C^2$,*
(g3) $\frac{tg(t)}{1+g_0} \leqslant G(t) \leqslant tg(t)\ \forall t \geqslant 0$.

**Proof.** For the proofs of (g1)–(g3) see [15]. □

**Remark 2.1.** By (g1) and (g3) we have a similar inequality for $G$,

(G1) $$\min\{s^{\delta+1}, s^{g_0+1}\}\frac{G(t)}{1+g_0} \leqslant G(st) \leqslant (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t)$$

and then, using the convexity of $G$ and this last inequality we have

(G2) $$G(a+b) \leqslant 2^{g_0}(1+g_0)\big(G(a)+G(b)\big) \quad \forall a, b > 0.$$

As $g$ is strictly increasing we can define $g^{-1}$. Now we prove that $g^{-1}$ satisfies a condition similar to (1.4). That is,

**Lemma 2.2.** *The function $g^{-1}$ satisfies the inequalities*

$$\frac{1}{g_0} \leqslant \frac{t(g^{-1})'(t)}{g^{-1}(t)} \leqslant \frac{1}{\delta} \quad \forall t > 0. \tag{2.1}$$

*Moreover, $g^{-1}$ satisfies*

$$(\tilde{g}1) \qquad \min\{s^{1/\delta}, s^{1/g_0}\} g^{-1}(t) \leqslant g^{-1}(st) \leqslant \max\{s^{1/\delta}, s^{1/g_0}\} g^{-1}(t)$$

*and if $\widetilde{G}$ is such that $\widetilde{G}'(t) = g^{-1}(t)$ then*

$$(\tilde{g}2) \qquad \frac{\delta t g^{-1}(t)}{1+\delta} \leqslant \widetilde{G}(t) \leqslant t g^{-1}(t) \quad \forall t \geqslant 0,$$

$$(\widetilde{G}1) \qquad \frac{(1+\delta)}{\delta} \min\{s^{1+1/\delta}, s^{1+1/g_0}\} \widetilde{G}(t) \leqslant \widetilde{G}(st) \leqslant \frac{\delta}{1+\delta} \max\{s^{1+1/\delta}, s^{1+1/g_0}\} \widetilde{G}(t),$$

$$(\tilde{g}3) \qquad ab \leqslant \varepsilon G(a) + C(\varepsilon) \widetilde{G}(b) \quad \forall a, b > 0 \text{ and } \varepsilon > 0,$$

$$(\tilde{g}4) \qquad \widetilde{G}\big(g(t)\big) \leqslant g_0 G(t).$$

**Proof.** Let $s = g^{-1}(t)$, then

$$\frac{t(g^{-1})'(t)}{g^{-1}(t)} = \frac{g(s)}{g'(s)s}$$

and using (1.4) we have the desired inequalities.

Now $(\tilde{g}1)$ follows by property (g1) applied to $g^{-1}$, and $(\tilde{g}2)$ by property (g3). $(\widetilde{G}1)$ follows by $\tilde{g}_1$ and $\tilde{g}_2$.

By Young's inequality we have that $ab \leqslant G(a) + \widetilde{G}(b)$ and then, for $0 < \varepsilon' < 1$ such that $\varepsilon = (1+g_0)\varepsilon'^{(1+\delta)}$,

$$\varepsilon' a \frac{b}{\varepsilon'} \leqslant G(\varepsilon' a) + \widetilde{G}\left(\frac{b}{\varepsilon'}\right) \leqslant \varepsilon G(a) + C(\varepsilon) \widetilde{G}(b).$$

In the last inequality we have used (G1) and $(\widetilde{G}1)$. Thus $(\tilde{g}3)$ follows.

As $g$ is strictly increasing we have that $\widetilde{G}(g(t)) + G(t) = tg(t)$ (see Eq. (5), Section 8.2 in [1]) and applying (g3), we get

$$\widetilde{G}\big(g(t)\big) = tg(t) - G(t) \leqslant g_0 G(t).$$

Thus, $(\tilde{g}4)$ follows. □

In order to prove the existence of minimizers we will use some compact embedding results. To this end, we have to define some Orlicz and Orlicz–Sobolev spaces. We recall that the functional

$$\|u\|_G = \inf\left\{k > 0\colon \int_\Omega G\left(\frac{|u(x)|}{k}\right) dx \leqslant 1\right\}$$

is a norm in the Orlicz space $L^G(\Omega)$ which is the linear hull of the Orlicz class

$$K_G(\Omega) = \left\{u \text{ measurable: } \int_\Omega G(|u|)\, dx < \infty\right\},$$

observe that this set is convex, since $G$ is also convex (property (g2)). The Orlicz–Sobolev space $W^{1,G}(\Omega)$ consists of those functions in $L^G(\Omega)$ whose distributional derivatives $\nabla u$ also belong to $L^G(\Omega)$. And we have that $\|u\|_{W^{1,G}} = \max\{\|u\|_G, \|\nabla u\|_G\}$ is a norm for this space.

**Lemma 2.3.** *There exists a constant $C = C(g_0, \delta)$ such that*

$$\|u\|_G \leqslant C \max\left\{\left(\int_\Omega G(|u|)\, dx\right)^{1/(\delta+1)}, \left(\int_\Omega G(|u|)\, dx\right)^{1/(g_0+1)}\right\}.$$

**Proof.** If $\int_\Omega G(|u|)\, dx = 0$ then $u = 0$ a.e. and the result follows. If $\int_\Omega G(|u|)\, dx \neq 0$, take $k = \max\{(2(1+g_0)\int_\Omega G(|u|)\, dx)^{1/(\delta+1)}, (2(1+g_0)\int_\Omega G(|u|)\, dx)^{1/(g_0+1)}\}$, by (G1) we have

$$\int_\Omega G\left(\frac{|u|}{k}\right) dx \leqslant (1+g_0) \max\left\{\frac{1}{k^{\delta+1}}, \frac{1}{k^{g_0+1}}\right\} \int_\Omega G(|u|)\, dx \leqslant 1$$

therefore $\|u\|_G \leqslant k$ and the result follows. □

**Theorem 2.1.** *$L^{\widetilde{G}}(\Omega)$ is the dual of $L^G(\Omega)$. Moreover, $L^G(\Omega)$ and $W^{1,G}(\Omega)$ are reflexive.*

**Proof.** As $G$ satisfies property (G1) and $\widetilde{G}$ property ($\widetilde{G}$1), we have that both pairs $(G, \Omega)$ and $(\widetilde{G}, \Omega)$ are $\Delta$-regular (see 8.7 in [1]). Therefore we are in the hypothesis of Theorems 8.19 and 8.28 at [1], and the result follows. □

**Theorem 2.2.** *$L^G(\Omega) \hookrightarrow L^{1+\delta}(\Omega)$ continuously.*

**Proof.** By Theorem 8.12 of [1] we only have to prove that $G$ dominates $t^{1+\delta}$ near infinity. That is, there exits constants $k$, $t_0$ such that $t^{1+\delta} \leqslant G(kt)$ $\forall t \geqslant t_0$. But this is true by property (G1). So the result follows. □

The following result is a Poincaré type inequality.

**Lemma 2.4.** *If $u \in W^{1,1}(\Omega)$ with $u = 0$ on $\partial\Omega$ and $\int_\Omega G(|\nabla u|)\,dx$ is finite, then*

$$\int_\Omega G\left(\frac{|u|}{R}\right)dx \leqslant \int_\Omega G\big(|\nabla u|\big)\,dx \quad \text{for } R = \operatorname{diam}\Omega.$$

**Proof.** See Lemma 2.2 of [15]. □

Now we state a generalization of Morrey's Theorem. Let

$$[u]_{0,\alpha,\Omega} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

We have the following result.

**Lemma 2.5.** *Let $u \in L^\infty(\Omega)$ such that for some $0 < \alpha < 1$ and $r_0 > 0$,*

$$\int_{B_r} G\big(|\nabla u|\big)\,dx \leqslant C r^{N+\alpha-1} \quad \forall 0 < r \leqslant r_0,$$

*with $B_r \subset \Omega$. Then, $u \in C^\alpha(\Omega)$ and there exists a constant $C_1 = C_1(C, \alpha, N, g_0, G(1))$ such that $[u]_{0,\alpha,\Omega} \leqslant C_1$.*

**Proof.** The proof of this lemma is included in the proof of Theorem 1.7 (p. 346) in [15]. □

Now, we will give some properties of subsolutions and solutions of $\mathcal{L}v = \operatorname{div}(A(\nabla v)) = 0$, where $A(p) = g(|p|)\frac{p}{|p|}$. First, let us observe that if $a_{ij} = \frac{\partial A_i}{\partial p_j}$ by using (1.4), we get

$$\min\{\delta, 1\}\frac{g(|p|)}{|p|}|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \max\{g_0, 1\}\frac{g(|p|)}{|p|}|\xi|^2, \tag{2.2}$$

which means that the equation $\mathcal{L}v = 0$ is uniformly elliptic for $\frac{g(|p|)}{|p|}$ bounded and bounded away from zero.

The next lemma is a Cacciopoli type inequality for subsolutions of $\mathcal{L}v = 0$.

**Lemma 2.6.** *Let $v$ be a nonnegative weak subsolution of $\mathcal{L}v = 0$. That is,*

$$0 \geqslant \int_\Omega g\big(|\nabla v|\big)\frac{\nabla v}{|\nabla v|}\nabla\phi\,dx \quad \forall \phi \in C_0^\infty(\Omega) \text{ such that } \phi \geqslant 0. \tag{2.3}$$

*Then, there exists $C = C(N, \delta, g_0) > 0$ such that*

$$\int_{B_r} G\big(|\nabla v|\big)\,dx \leqslant C \int_{B_{\frac{3}{2}r}} G\left(\frac{|v|}{r}\right)dx$$

*for all $r > 0$, such that $B_{\frac{3}{2}r} \subset \Omega$.*

**Proof.** Let $\phi = v\eta^{g_0+1}$, where $0 \leqslant \eta \in C_0^1(B_{\frac{3}{2}r})$, with $|\nabla\eta| \leqslant \frac{C}{r}$, $\eta \leqslant 1$, $\eta \equiv 1$ in $B_r$. Then, $\nabla\phi = \eta^{g_0+1}\nabla v + v\nabla\eta(g_0+1)\eta^{g_0}$ and replacing in (2.3) we have

$$0 \geqslant \int_{B_{\frac{3}{2}r}} g\big(|\nabla v|\big)|\nabla v|\eta^{g_0+1}\,dx + (g_0+1)\int_{B_{\frac{3}{2}r}} g\big(|\nabla v|\big)\frac{\nabla v}{|\nabla v|}\nabla\eta\, v\,\eta^{g_0}\,dx.$$

Then,

$$\int_{B_{\frac{3}{2}r}} g\big(|\nabla v|\big)|\nabla v|\eta^{g_0+1}\,dx \leqslant (g_0+1)\int_{B_{\frac{3}{2}r}} g\big(|\nabla v|\big)|\nabla\eta||v|\eta^{g_0}\,dx.$$

By property ($\tilde{g}3$) we have

$$g\big(|\nabla v|\big)|\nabla\eta||v|\eta^{g_0} \leqslant \varepsilon\widetilde{G}\big(g\big(|\nabla v|\big)\eta^{g_0}\big) + C(\varepsilon)G\big(|\nabla\eta||v|\big).$$

Then, by property ($\widetilde{G}1$) and as $\eta \leqslant 1$, we have

$$\widetilde{G}\big(g\big(|\nabla v|\big)\eta^{g_0}\big) \leqslant C\eta^{g_0(1+\frac{1}{g_0})}\widetilde{G}\big(g\big(|\nabla v|\big)\big) \leqslant C\eta^{1+g_0}G\big(|\nabla v|\big),$$

where the last inequality holds by ($\tilde{g}4$). Summing up, and using property (g3), we obtain

$$\int_{B_{\frac{3}{2}r}} G\big(|\nabla v|\big)\eta^{g_0+1}\,dx \leqslant C\varepsilon\int_{B_{\frac{3}{2}r}} G\big(|\nabla v|\big)\eta^{g_0+1}\,dx + C(\varepsilon)\int_{B_{\frac{3}{2}r}} G\big(|\nabla\eta||v|\big)\,dx,$$

and if we take $\varepsilon$ small and use the bound for $|\nabla\eta|$ we have

$$\int_{B_{\frac{3}{2}r}} G\big(|\nabla v|\big)\eta^{g_0+1}\,dx \leqslant C\int_{B_{\frac{3}{2}r}} G\big(|\nabla\eta||v|\big)\,dx \leqslant C\int_{B_{\frac{3}{2}r}} G\left(\frac{|v|}{r}\right)dx.$$

Finally, if we use that $\eta \equiv 1$ in $B_r$ the result follows. □

**Lemma 2.7.** *Let $v$ be a weak solution of $\mathcal{L}v = 0$, that is*

$$\int_{\Omega} g\big(|\nabla v|\big)\frac{\nabla v}{|\nabla v|}\nabla\phi\,dx = 0 \quad \forall\phi \in C_0^\infty(\Omega).$$

*Then $v \in C^{1,\alpha}(\Omega)$. Moreover, there exists $C = C(N,\delta,g_0) > 0$ such that for every ball $B_r \subset \Omega$,*

$$\sup_{B_{r/2}} G\big(|\nabla v|\big) \leqslant \frac{C}{r^N}\int_{B_{\frac{2}{3}r}} G\big(|\nabla v|\big)\,dx, \tag{1}$$

$$\sup_{B_{r/2}} |\nabla v| \leqslant \frac{C}{r}\sup_{B_r}|v|. \tag{2}$$

*For every $\beta \in (0, N)$, there exists $C = C(N, \beta, \delta, g_0, \|v\|_{L^\infty(\frac{2}{3}r)}) > 0$ such that*

$$\int_{B_{r/2}} G\big(|\nabla v|\big) \leqslant C r^\beta. \tag{3}$$

**Proof.** For the proof of (1) see Lemma 5.1 of [15] and for the proof of (3) see (5.9) page 346 of [15]. Let us prove (2). By using (1) and then Lemma 2.6 we have

$$\sup_{B_{r/2}} G\big(|\nabla v|\big) \leqslant \frac{C}{r^N} \int_{B_{\frac{2}{3}r}} G\big(|\nabla v|\big)\, dx \leqslant \frac{C}{r^N} \int_{B_r} G\left(\frac{|v|}{r}\right) dx \leqslant G\left(\frac{C}{r} \|v\|_{L^\infty(B_r)}\right).$$

Then

$$\big|\nabla v(y_0)\big| \leqslant \frac{C}{r} \|v\|_{L^\infty(B_r)} \quad \forall y_0 \in B_{r/2},$$

and the result follows. □

**Lemma 2.8.** *Let $U$ be an open subset, $u$ a weak subsolution and $w$ a weak supersolution of $\mathcal{L}u = 0$ in $U$. If $w \geqslant u$ on $\partial U$, then $w \geqslant u$ in $U$. If $w$ is a solution to $\mathcal{L}w = 0$ and $w = u$ on $\partial U$, then $w$ is uniquely determined.*

**Proof.**

$$\begin{aligned}
0 &\geqslant \int_U \left( g\big(|\nabla u|\big) \frac{\nabla u}{|\nabla u|} - g\big(|\nabla w|\big) \frac{\nabla w}{|\nabla w|} \right) . \nabla (u - w)^+ \, dx \\
&= \int_{U \cap \{u > w\}} \left( g\big(|\nabla u|\big) \frac{\nabla u}{|\nabla u|} - g\big(|\nabla w|\big) \frac{\nabla w}{|\nabla w|} \right) . \nabla (u - w)\, dx \\
&= \int_{U \cap \{u > w\}} \int_0^1 a_{ij}\big(\nabla u + (1-t)(\nabla w - \nabla u)\big)(u_{x_i} - w_{x_i})(u_{x_j} - w_{x_j})\, dt\, dx.
\end{aligned}$$

And using (2.2) we have that the right-hand side is grater or equal than

$$C \int_{U \cap \{u > w\}} \int_0^1 F\big(\big|\nabla u + (1-t)(\nabla w - \nabla u)\big|\big) |\nabla w - \nabla u|^2 \, dt\, dx,$$

where $F(t) = \frac{g(t)}{t}$. Now, we take the following subsets of $U$

$$S_1 = \big\{x \in U\colon\ |\nabla u - \nabla w| \leqslant 2|\nabla u|\big\}, \qquad S_2 = \big\{x \in U\colon\ |\nabla u - \nabla w| > 2|\nabla u|\big\}.$$

Then $S_1 \cup S_2 = U$ and

$$\frac{1}{2}|\nabla u| \leqslant \big|\nabla u + (1-t)(\nabla w - \nabla u)\big| \leqslant 3|\nabla u| \quad \text{in } S_1 \text{ for } t \geqslant \frac{3}{4}, \tag{2.4}$$

$$\frac{1}{4}|\nabla u - \nabla w| \leqslant \big|\nabla u + (1-t)(\nabla w - \nabla u)\big| \leqslant 3|\nabla u - \nabla w| \quad \text{in } S_2 \text{ for } t \leqslant \frac{1}{4}. \tag{2.5}$$

In $S_1$, and for $t \geqslant 3/4$ we have using (2.4), that

$$F\big(\big|\nabla u + (1-t)(\nabla w - \nabla u)\big|\big) = \frac{g(|\nabla u + (1-t)(\nabla w - \nabla u)|)}{|\nabla u + (1-t)(\nabla w - \nabla u)|} \geqslant \frac{g(\frac{1}{2}|\nabla u|)}{3|\nabla u|} \geqslant \frac{1}{2^{g_0}3} F\big(|\nabla u|\big)$$

where in the last inequality we have used (g1).

In $S_2$, and for $t \leqslant 1/4$ we have using (g3) and then (2.5) that

$$\begin{aligned} F\big(\big|\nabla u + (1-t)(\nabla w - \nabla u)\big|\big)|\nabla u - \nabla w|^2 &\geqslant \frac{G(|\nabla u + (1-t)(\nabla w - \nabla u)|)}{|\nabla u + (1-t)(\nabla w - \nabla u)|^2}|\nabla u - \nabla w|^2 \\ &\geqslant \frac{G(\frac{1}{4}|\nabla u - \nabla w|)}{9|\nabla u - \nabla w|^2}|\nabla u - \nabla w|^2 \\ &\geqslant \frac{G(|\nabla u - \nabla w|)}{4^{g_0+1}9(1+g_0)} \end{aligned}$$

where in the last inequality we have used (G1).

Therefore, we have that

$$0 \geqslant C\bigg(\int_{S_1} F\big(|\nabla u|\big)\big|\nabla(u-w)^+\big|^2\,dx + \int_{S_2} G\big(\big|\nabla(u-w)^+\big|\big)\,dx\bigg).$$

Hence $\nabla(u-w)^+ = 0$ in $S_2$ and $\nabla(u-w)^+ = 0$, or $F(|\nabla u|) = 0$ in $S_1$ in which case $\nabla u = 0$ and, by the definition of $S_1$, this implies that $\nabla(u-w) = 0$ in $S_1$. Therefore, $\nabla(u-w)^+ = 0$ in $U$, then $(u-w)^+ = 0$, which implies $u \leqslant w$. □

The following inequality will be a key tool in the proof of the Hölder continuity of minimizers. As an observation, we mention that the following result is a generalization of well-known integral inequalities for the $p$-Laplacian (see, for example, p. 4 in [7]). Here the difference is that we obtain a unique inequality for any $\delta$ and $g_0$ (for the $p$-Laplacian the inequalities were separated in two cases $p \geqslant 2$ and $1 < p < 2$).

**Theorem 2.3.** *Let $u \in W^{1,G}(\Omega)$, $B_r \Subset \Omega$ and $v$ be a solution of*

$$\mathcal{L}v = 0 \quad \text{in } B_r, \qquad v - u \in W_0^{1,G}(B_r),$$

*then*

$$\int_{B_r} \big(G\big(|\nabla u|\big) - G\big(|\nabla v|\big)\big)\,dx \geqslant C\bigg(\int_{A_2} G\big(|\nabla u - \nabla v|\big)\,dx + \int_{A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx\bigg),$$

*where* $F(t) = g(t)/t$,

$$A_1 = \{x \in B_r\colon\ |\nabla u - \nabla v| \leqslant 2|\nabla u|\} \quad \textit{and} \quad A_2 = \{x \in B_r\colon\ |\nabla u - \nabla v| > 2|\nabla u|\}$$

*and* $C = C(g_0, \delta)$.

**Proof.** Let $u^s = su + (1-s)v$. Using the integral form of the mean value theorem and the fact that $v$ is an $\mathcal{L}$-solution, we have

$$\begin{aligned}\mathcal{I} := \int_{B_r} \big(G(|\nabla u|) - G(|\nabla v|)\big)\,dx &= \int_0^1 \int_{B_r} g(|\nabla u^s|)\frac{\nabla u^s}{|\nabla u^s|}.\nabla(u-v)\,dx\,ds \\ &= \int_0^1 \frac{1}{s}\int_{B_r}\left(g(|\nabla u^s|)\frac{\nabla u^s}{|\nabla u^s|} - g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\right).\nabla\big(u^s - v\big)\,dx\,ds \\ &= \int_0^1 \frac{1}{s}\int_{B_r}\int_0^1 a_{ij}\big(\nabla u^s + (1-t)(\nabla v - \nabla u^s)\big)\big(u^s_{x_i} - v_{x_i}\big)\big(u^s_{x_j} - v_{x_j}\big)\,dt\,dx\,ds.\end{aligned}$$

And, by (2.2) we have that the right-hand side is grater than or equal to

$$C\int_0^1 \frac{1}{s}\int_{B_r}\int_0^1 F\big(\big|\nabla u^s + (1-t)(\nabla v - \nabla u^s)\big|\big)\big|\nabla v - \nabla u^s\big|^2\,dt\,dx\,ds,$$

where $F$ was defined in Lemma 2.8 and $C = C(\delta)$.

Now, we take the following subsets of $B_r$

$$S_1 = \{x \in B_r\colon\ |\nabla u^s - \nabla v| \leqslant 2|\nabla u^s|\}, \qquad S_2 = \{x \in B_r\colon\ |\nabla u^s - \nabla v| > 2|\nabla u^s|\}.$$

Then $S_1 \cup S_2 = B_r$ and

$$\frac{1}{2}|\nabla u^s| \leqslant \big|\nabla u^s + (1-t)(\nabla v - \nabla u^s)\big| \leqslant 3|\nabla u^s| \quad \text{on } S_1 \text{ for } t \geqslant \frac{3}{4}, \tag{2.6}$$

$$\frac{1}{4}|\nabla u^s - \nabla v| \leqslant \big|\nabla u^s + (1-t)(\nabla v - \nabla u^s)\big| \leqslant 3|\nabla u^s - \nabla v| \quad \text{on } S_2 \text{ for } t \leqslant \frac{1}{4}. \tag{2.7}$$

Proceeding as in Lemma 2.8, we get

$$F\big(\big|\nabla u^s + (1-t)(\nabla v - \nabla u^s)\big|\big) \geqslant \frac{1}{2^{g_0}3}F(|\nabla u^s|)$$

in $S_1$ and

$$F\big(\big|\nabla u^s + (1-t)(\nabla v - \nabla u^s)\big|\big)\big|\nabla u^s - \nabla v\big|^2 \geqslant \frac{G(|\nabla u^s - \nabla v|)}{4^{g_0+1}9(1+g_0)}$$

in $S_2$.

Therefore, we have that

$$\mathcal{I} \geqslant C\left( \int_0^1 \frac{1}{s} \int_{S_1} F\left(\left|\nabla u^s\right|\right)\left|\nabla v - \nabla u^s\right|^2 dx\, ds + \int_0^1 \frac{1}{s} \int_{S_2} G\left(\left|\nabla u^s - \nabla v\right|\right) dx\, ds \right).$$

Now, let

$$A_1 = \left\{x \in B_r\colon\ |\nabla u - \nabla v| \leqslant 2|\nabla u|\right\}, \qquad A_2 = \left\{x \in B_r\colon\ |\nabla u - \nabla v| > 2|\nabla u|\right\},$$

then $B_r = A_1 \cup A_2$, and

$$\frac{1}{2}|\nabla u| \leqslant \left|\nabla u^s\right| \leqslant 3|\nabla u| \quad \text{on } A_1 \text{ for } s \geqslant \frac{3}{4}, \tag{2.8}$$

$$\frac{1}{4}|\nabla u - \nabla v| \leqslant \left|\nabla u^s\right| \leqslant 3|\nabla u - \nabla v| \quad \text{on } A_2 \text{ for } s \leqslant \frac{1}{4}. \tag{2.9}$$

Therefore

$$\begin{aligned}
\mathcal{I} \geqslant C\Bigg( &\int_0^{1/4} \frac{1}{s} \int_{S_1 \cap A_2} F\left(\left|\nabla u^s\right|\right)\left|\nabla v - \nabla u^s\right|^2 dx\, ds + \int_{3/4}^1 \frac{1}{s} \int_{S_1 \cap A_1} F\left(\left|\nabla u^s\right|\right)\left|\nabla v - \nabla u^s\right|^2 dx\, ds \\
&+ \int_0^{1/4} \frac{1}{s} \int_{S_2 \cap A_2} G\left(\left|\nabla u^s - \nabla v\right|\right) dx\, ds + \int_{3/4}^1 \frac{1}{s} \int_{S_2 \cap A_1} G\left(\left|\nabla u^s - \nabla v\right|\right) dx\, ds \Bigg) \\
= {} & I + II + III + IV.
\end{aligned}$$

Let us estimate these four terms.

In $S_1 \cap A_2$, for $s \leqslant 1/4$ we have by (2.9) and (g1), that

$$F\left(\left|\nabla u^s\right|\right) \geqslant \frac{1}{4^{g_0} 3} F\left(|\nabla u - \nabla v|\right).$$

Therefore,

$$\begin{aligned}
I &\geqslant C \int_0^{1/4} \frac{1}{s} \int_{S_1 \cap A_2} F\left(|\nabla u - \nabla v|\right)\left|\nabla v - \nabla u^s\right|^2 dx\, ds \\
&= C \int_0^{1/4} s \int_{S_1 \cap A_2} F\left(|\nabla u - \nabla v|\right)|\nabla v - \nabla u|^2 dx\, ds \\
&\geqslant C \int_0^{1/4} s \int_{S_1 \cap A_2} G\left(|\nabla u - \nabla v|\right) dx\, ds
\end{aligned}$$

where in the last inequality we are using (g3).

In $S_1 \cap A_1$, for $s \geqslant 3/4$ we have by (2.8) and (g1), that

$$F\big(\big|\nabla u^s\big|\big) \geqslant \frac{1}{2^{g_0}3} F\big(|\nabla u|\big).$$

Therefore,

$$II \geqslant C \int_{3/4}^{1} s \int_{S_1\cap A_1} F\big(|\nabla u|\big)|\nabla v - \nabla u|^2\,dx\,ds \geqslant C \int_{3/4}^{1} \int_{S_1\cap A_1} F\big(|\nabla u|\big)|\nabla v - \nabla u|^2\,dx\,ds.$$

In $S_2 \cap A_2$, for $s \leqslant 1/4$ we have by definition of $S_2$, by (2.9) and (G1), that

$$G\big(\big|\nabla u^s - \nabla v\big|\big) \geqslant \frac{1}{2^{g_0+1}(g_0+1)} G\big(|\nabla u - \nabla v|\big),$$

therefore

$$III \geqslant C \int_{0}^{1/4} \frac{1}{s} \int_{S_2\cap A_2} G\big(|\nabla u - \nabla v|\big)\,dx\,ds \geqslant C \int_{0}^{1/4} s \int_{S_2\cap A_2} G\big(|\nabla u - \nabla v|\big)\,dx\,ds.$$

In $S_2 \cap A_1$, for $s \geqslant 3/4$ we have, by definition of $S_2$ and by (2.8)

$$\big|\nabla u^s - \nabla v\big| > 2\big|\nabla u^s\big| \geqslant |\nabla u|. \tag{2.10}$$

By (g3), using (2.10) and the definition of $A_1$ we have

$$\begin{aligned} G\big(\big|\nabla u^s - \nabla v\big|\big) &\geqslant \frac{1}{g_0+1} g\big(\big|\nabla u^s - \nabla v\big|\big)\big|\nabla u^s - \nabla v\big| \geqslant \frac{1}{g_0+1} g\big(|\nabla u|\big)\big|\nabla u^s - \nabla v\big| \\ &= \frac{1}{g_0+1} F\big(|\nabla u|\big) s|\nabla u - \nabla v||\nabla u| \geqslant \frac{s}{2(g_0+1)} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2. \end{aligned}$$

Therefore,

$$IV \geqslant C \int_{3/4}^{1} \int_{S_2\cap A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx\,ds.$$

If we sum $I + III$, we obtain

$$\begin{aligned} I + III &\geqslant \int_{0}^{1/4} Cs\bigg(\int_{S_1\cap A_2} G\big(|\nabla u - \nabla v|\big)\,dx + \int_{S_2\cap A_2} G\big(|\nabla u - \nabla v|\big)\,dx\,ds\bigg) \\ &= C \int_{0}^{1/4} s \int_{A_2} G\big(|\nabla u - \nabla v|\big)\,dx\,ds = C \int_{A_2} G\big(|\nabla u - \nabla v|\big)\,dx \end{aligned}$$

and if we sum $II + IV$, we obtain

$$II + IV \geqslant C \int_1^{3/4} \bigg( \int_{S_1 \cap A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx + \int_{S_2 \cap A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx \bigg)\,ds$$
$$= C \int_{A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx.$$

Therefore,

$$\mathcal{I} \geqslant C\bigg( \int_{A_2} G\big(|\nabla u - \nabla v|\big)\,dx + \int_{A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx \bigg), \tag{2.11}$$

where $C = C(g_0, \delta)$. □

In Section 4 we will need an explicit family of subsolutions and supersolutions in an annulus. We state here the required lemma.

**Lemma 2.9.** *Let $w_\mu = \varepsilon e^{-\mu|x|^2}$, for $\varepsilon > 0$, $r_1 > r_2 > 0$ then there exists $\mu > 0$ such that*

$$\mathcal{L}w_\mu > 0 \quad \text{in } B_{r_1} - B_{r_2}$$

*and $\mu$ depends only on $r_2, g_0, \delta$ and $N$.*

**Proof.** First, note that

$$\mathcal{L}w = \frac{g(|\nabla w|)}{|\nabla w|^3}\bigg\{\bigg(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\bigg)\sum_{i,j} w_{x_i} w_{x_j} w_{x_i x_j} + \triangle w|\nabla w|^2\bigg\}.$$

Computing, we have

$$w_{x_i} = -2\varepsilon\mu x_i e^{-\mu|x|^2}, \qquad w_{x_i x_j} = \varepsilon\big(4\mu^2 x_i x_j - 2\mu\delta_{ij}\big)e^{-\mu|x|^2},$$
$$|\nabla w| = 2\varepsilon\mu|x|e^{-\mu|x|^2}, \tag{2.12}$$

therefore using (2.12) and (1.4) we obtain

$$e^{3\mu|x|^2}\mathcal{L}w = \varepsilon^3 \frac{g(|\nabla w|)}{|\nabla w|^3}\bigg\{\bigg(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\bigg)\big(16\mu^4|x|^4 - 8\mu^3|x|^2\big)$$
$$+ \big(4\mu^2|x|^2 - 2\mu N\big)4\mu^2|x|^2\bigg\}$$
$$= \varepsilon^3 \frac{g(|\nabla w|)}{|\nabla w|^3} 4\mu^3|x|^2\bigg\{\bigg(\frac{g'(|\nabla w|)}{g(|\nabla w|)}|\nabla w| - 1\bigg)\big(4\mu|x|^2 - 2\big) + \big(4\mu|x|^2 - 2N\big)\bigg\}$$

$$= \varepsilon^3 \frac{g(|\nabla w|)}{|\nabla w|^3} 4\mu^3 |x|^2 \left\{ \left( \frac{g'(|\nabla w|)}{g(|\nabla w|)} |\nabla w| \right) 4\mu |x|^2 - \left( \frac{g'(|\nabla w|)}{g(|\nabla w|)} |\nabla w| - 1 \right) 2 - 2N \right\}$$
$$\geqslant \varepsilon^3 \frac{g(|\nabla w|)}{|\nabla w|^3} 4\mu^3 |x|^2 \left(4\mu^2 |x|^2 \delta - K\right) \geqslant \varepsilon^3 \frac{g(|\nabla w|)}{|\nabla w|^3} 4\mu^3 {r_2}^2 \left(4\mu^2 {r_2}^2 \delta - K\right)$$

where $K = 2N$ if $g_0 < 1$ and $K = 2(g_0 - 1) + 2N$ if $g_0 > 1$. Therefore if $\mu$ is big enough, depending only on $\delta, g_0, r_2$ and $N$, we have $\mathcal{L}w > 0$. $\square$

## 3. The minimization problem

In this section we look for minimizers of the functional $\mathcal{J}$. We begin by discussing the existence of extremals. Next, we prove that any minimizer is a subsolution to the equation $\mathcal{L}u = 0$ and finally, we prove that $0 \leqslant u \leqslant \sup \varphi_0$.

**Theorem 3.1.** *If $\mathcal{J}(\varphi_0) < \infty$, then there exists a minimizer of $\mathcal{J}$.*

**Proof.** The proof of existence is standard. We write it here for the reader's convenience and in order to show how the Orlicz spaces and the condition (1.4) on the function $G$ come into play.

Take a minimizing sequence $(u_n) \subset \mathcal{K}$, then $\mathcal{J}(u_n)$ is bounded, so $\int_\Omega G(|\nabla u_n|)$ and $|\{u_n > 0\}|$ are bounded. As $u_n = \varphi_0$ in $\partial\Omega$, we have by Lemma 2.3 that $\|\nabla u_n - \nabla\varphi_0\|_G \leqslant C$ and by Lemma 2.4 we also have $\|u_n - \varphi_0\|_G \leqslant C$. Therefore, by Theorem 2.1 there exists a subsequence (that we still call $u_n$) and a function $u_0 \in W^{1,G}(\Omega)$ such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,G}(\Omega),$$

and by Theorem 2.2

$$u_n \rightharpoonup u_0 \quad \text{weakly in } W^{1,\delta+1}(\Omega),$$

and by the compactness of the immersions $W^{1,\delta+1}(\Omega) \hookrightarrow L^{\delta+1}(\Omega)$ and $W^{1,\delta+1}(\Omega) \hookrightarrow L^{\delta+1}(\partial\Omega)$ we have that

$$u_n \to u_0 \quad \text{a.e. } \Omega.$$
$$u_0 = \varphi_0 \quad \text{on } \partial\Omega,$$

Thus,

$$\left|\{u_0 > 0\}\right| \leqslant \liminf_{n\to\infty} \left|\{u_n > 0\}\right| \quad \text{and} \quad \int_\Omega G\left(|\nabla u_0|\right) dx \leqslant \liminf_{n\to\infty} \int_\Omega G\left(|\nabla u_n|\right) dx.$$

In fact,

$$\int_\Omega G\left(|\nabla u_n|\right) dx \geqslant \int_\Omega G\left(|\nabla u_0|\right) dx + \int_\Omega g\left(|\nabla u_0|\right) \frac{\nabla u_0}{|\nabla u_0|} \cdot (\nabla u_n - \nabla u_0)\, dx. \tag{3.1}$$

Recall that $\nabla u_n$ converges weakly to $\nabla u_0$ in $L^G$. Now, since by property $(\widetilde{g}4)$

$$\widetilde{G}\big(g\big(|\nabla u_0|\big)\big) \leqslant CG\big(|\nabla u_0|\big),$$

there holds that $g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|} \in L^{\widetilde{G}}$ so that, by Theorem 2.1 and passing to the limit in (3.1) we get

$$\liminf_{n\to\infty} \int_\Omega G\big(|\nabla u_n|\big)\,dx \geqslant \int_\Omega G\big(|\nabla u_0|\big)\,dx.$$

Hence $u_0 \in \mathcal{K}$ and

$$\mathcal{J}(u_0) \leqslant \liminf_{n\to\infty} \mathcal{J}(u_n) = \inf_{v\in\mathcal{K}} \mathcal{J}(v).$$

Therefore, $u_0$ is a minimizer of $\mathcal{J}$ in $\mathcal{K}$. □

**Lemma 3.1.** *Let $u$ be a minimizer of $\mathcal{J}$. Then, $u$ is an $\mathcal{L}$-subsolution.*

**Proof.** Let $\varepsilon > 0$ and $0 \leqslant \xi \in C_0^\infty$. Using the minimality of $u$ and the convexity of $G$ we have

$$0 \leqslant \frac{1}{\varepsilon}\big(\mathcal{J}(u-\varepsilon\xi) - \mathcal{J}(u)\big) \leqslant \frac{1}{\varepsilon}\int_\Omega G\big(|\nabla u - \varepsilon\nabla\xi|\big) - G\big(|\nabla u|\big)\,dx$$
$$\leqslant \int_\Omega -g\big(|\nabla u - \varepsilon\nabla\xi|\big)\frac{\nabla u - \varepsilon\nabla\xi}{|\nabla u - \varepsilon\nabla\xi|}\nabla\xi\,dx$$

and if we take $\varepsilon \to 0$ we obtain

$$0 \leqslant \int_\Omega -g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\nabla\xi\,dx. \qquad \square$$

**Lemma 3.2.** *Let $u$ be a minimizer of $\mathcal{J}$. Then $0 \leqslant u \leqslant \sup_\Omega \varphi_0$.*

**Proof.** Let $M = \sup\varphi_0$, $\varepsilon > 0$ and $v = \min(M-u, 0)$, then

$$0 \leqslant \frac{1}{\varepsilon}\big(\mathcal{J}(u+\varepsilon v) - \mathcal{J}(u)\big) = \frac{1}{\varepsilon}\bigg(\int_\Omega G\big(|\nabla u + \varepsilon\nabla v|\big) - G\big(|\nabla u|\big) + \lambda\chi_{\{u+\varepsilon v>0\}} - \lambda\chi_{\{u>0\}}\,dx\bigg)$$
$$\leqslant \frac{1}{\varepsilon}\bigg(\int_\Omega \big(G\big(|\nabla u + \varepsilon\nabla v|\big) - G\big(|\nabla u|\big)\big)\,dx\bigg) \leqslant \int_\Omega g\big(|\nabla u + \varepsilon\nabla v|\big)\frac{\nabla u + \varepsilon\nabla v}{|\nabla u + \varepsilon\nabla v|}\nabla v\,dx$$

where in the last inequality we are using the convexity of $G$.

Now, taking $\varepsilon \to 0$, using the definition of $v$ and (g3) we have that

$$0 \leqslant \int_{\Omega} g(|\nabla u|) \frac{\nabla u}{|\nabla u|} \nabla v \, dx = - \int_{\{u>M\}} g(|\nabla u|)|\nabla u| \, dx \leqslant - \int_{\{u>M\}} G(|\nabla u|) \, dx$$

$$= - \int_{\{u>M\}} G(|\nabla v|) \, dx,$$

therefore $\nabla v = 0$ in $\Omega$ and as $v = 0$ on $\partial\Omega$ we have that $v = 0$ in $\Omega$ and then $u \leqslant M$.

To prove that $u \geqslant 0$ we argue in a similar way. Take $v = \min(u, 0)$, then we have that

$$0 \leqslant \frac{1}{\varepsilon} \mathcal{J}(u - \varepsilon v) - \mathcal{J}(u) \leqslant - \int_{\Omega} g(|\nabla u - \varepsilon \nabla v|) \frac{\nabla u - \varepsilon \nabla v}{|\nabla u - \varepsilon \nabla v|} \nabla v \, dx.$$

Therefore taking $\varepsilon \to 0$, using the definition of $v$ and (g3) we have that

$$0 \geqslant \int_{\Omega} G(|\nabla v|) \, dx.$$

As in the first part, we conclude that $u \geqslant 0$. □

## 4. Lipschitz continuity

In this section we study the regularity of the minimizers of $\mathcal{J}$. The main result is the local Lipschitz continuity of a minimizer. This result, together with the rescaling invariance of the minimization problem, is a key step in the analysis. Once this regularity is proven, a blow up process (passage to the limit in linear rescalings) at points of $\partial\{u > 0\}$ allows to simplify the analysis by assuming that $u$ is a plane solution.

As a first step, we prove that minimizers are Hölder continuous. We use ideas from [7], here all the properties of the function $G$ come into play.

**Theorem 4.1.** *For every $0 < \alpha < 1$, any minimizer $u$ is in $C^{\alpha}(\Omega)$ and for $\Omega' \Subset \Omega$, $\|u\|_{C^{\alpha}(\Omega')} \leqslant C$, where $C = C(g_0, \delta, \lambda, \|u\|_{\infty}, \alpha, \operatorname{dist}(\Omega', \partial\Omega), G(1))$.*

**Proof.** We will see that, for every $0 < \alpha < 1$ and $\Omega' \Subset \Omega$ there exists $\rho_0$ such that if $y \in \Omega'$, $0 < \rho < \rho_0$ we have that

$$\frac{1}{\rho^N} \int_{B_\rho(y)} G(|\nabla u|) \, dx \leqslant C \rho^{\alpha - 1},$$

for a constant $C(N, \delta, g_0, \|u\|_{L^\infty(\Omega)}, \rho_0, G(1))$.

In fact, let $r > 0$ such that $B_r(y) \subset \Omega$. We can suppose that $y = 0$. Then if $v$ is the solution of

$$\mathcal{L}v = 0 \quad \text{in } B_r, \quad v - u \in W_0^{1,G}(B_r),$$

we have, therefore by Theorem 2.3 that

$$\int_{B_r} \big(G\big(|\nabla u|\big) - G\big(|\nabla v|\big)\big)\,dx \geqslant C\bigg(\int_{A_2} G\big(|\nabla u - \nabla v|\big)\,dx + \int_{A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx\bigg), \quad (4.1)$$

where

$$A_1 = \big\{x \in B_r\colon\ |\nabla u - \nabla v| \leqslant 2|\nabla u|\big\}, \qquad A_2 = \big\{x \in B_r\colon\ |\nabla u - \nabla v| > 2|\nabla u|\big\},$$

and $C = C(g_0, \delta)$.

On the other hand, by the minimality of $u$, we have

$$\int_{B_r} \big(G\big(|\nabla u|\big) - G\big(|\nabla v|\big)\big)\,dx \leqslant \lambda\big(\big|\{v > 0 \cap B_r\}\big| - \big|\{u > 0 \cap B_r\}\big|\big) \leqslant \lambda r^N C_N. \quad (4.2)$$

Combining (4.1) and (4.2) we obtain

$$\int_{A_2} G\big(|\nabla u - \nabla v|\big)\,dx \leqslant C\lambda r^N, \quad (4.3)$$

$$\int_{A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx \leqslant C\lambda r^N. \quad (4.4)$$

Let $\varepsilon > 0$ and suppose that $r^\varepsilon \leqslant 1/2$. Then, using (g3), Hölder's inequality, the definition of $A_1$ and (4.4) we obtain

$$\begin{aligned}\int_{A_1 \cap B_{r^{1+\varepsilon}}} G\big(|\nabla u - \nabla v|\big)\,dx &\leqslant C\bigg(\int_{A_1} F\big(|\nabla u|\big)|\nabla u - \nabla v|^2\,dx\bigg)^{1/2}\bigg(\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx\bigg)^{1/2} \\ &\leqslant C\lambda^{1/2} r^{N/2}\bigg(\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx\bigg)^{1/2}. \quad (4.5)\end{aligned}$$

Therefore, by (4.3) and (4.5), we get

$$\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u - \nabla v|\big)\,dx \leqslant C\lambda^{1/2}\bigg(\lambda^{1/2} r^N + r^{N/2}\bigg(\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx\bigg)^{1/2}\bigg). \quad (4.6)$$

On the other hand by property (3) of Lemma 2.7 we have for every $\beta \in (0, N)$, that there exists a constant $C = C(\delta, g_0, N, \beta, \|v\|_{L^\infty(B_r)})$ such that

$$\int_{B_{r/2}} G\big(|\nabla v|\big)\,dx \leqslant Cr^\beta. \quad (4.7)$$

By the maximum principle we have

$$\|v\|_{L^\infty(B_r)} \leqslant \|v\|_{L^\infty(\partial B_r)} = \|u\|_{L^\infty(\partial B_r)} \leqslant \|u\|_{L^\infty(B_r)} \leqslant \|v\|_{L^\infty(B_r)} \tag{4.8}$$

where in the last inequality we are using Lemma 2.8. Then $\|v\|_{L^\infty(B_r)} = \|u\|_{L^\infty(B_r)}$. This means that the constant $C$ depends on $\delta$, $g_0$, $N$, $\beta$ and $\|u\|_{L^\infty(B_r)}$.

By (G2) we have $G(|\nabla u|) \leqslant C(G(|\nabla u - \nabla v|) + G(|\nabla v|))$. Therefore by (4.6) and (4.7), and for $r \leqslant 1$ we have

$$\begin{aligned}\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx &\leqslant C\big(r^\beta(1+\lambda) + \lambda^{1/2} r^{N/2}\big(B_{r^{1+\varepsilon}} G\big(|\nabla u|\big)\,dx\big)^{1/2}\big)\\ &\leqslant C\left(r^\beta(1+\lambda) + r^{\beta/2}(1+\lambda)^{1/2}\left(\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx\right)^{1/2}\right).\end{aligned}$$

If we call $A = \int_{B_r^{1+\varepsilon}} G(|\nabla u|)\,dx$, we have

$$\begin{aligned}A &\leqslant C\big((1+\lambda)r^\beta + (1+\lambda)^{1/2} r^{\beta/2} A^{1/2}\big) \leqslant C\big((1+\lambda)r^\beta + 2(1+\lambda)^{1/2} r^{\beta/2} A^{1/2}\big)\\ &= C\big(\big(r^{\beta/2}(1+\lambda)^{1/2} + A^{1/2}\big)^2 - A\big),\end{aligned}$$

therefore

$$\begin{aligned}&(C+1)A \leqslant C\big(r^{\beta/2}(1+\lambda)^{1/2} + A^{1/2}\big)^2\\ &\quad\Rightarrow\quad (C+1)^{1/2} A^{1/2} \leqslant C^{1/2}\big(r^{\beta/2}(1+\lambda)^{1/2} + A^{1/2}\big)\\ &\quad\Rightarrow\quad \big((C+1)^{1/2} - C^{1/2}\big) A^{1/2} \leqslant C^{1/2} r^{\beta/2}(1+\lambda)^{1/2}.\end{aligned}$$

Thus, we have the inequality

$$\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx \leqslant \big((C+1)^{1/2} + C^{1/2}\big)^2 C(1+\lambda) r^\beta. \tag{4.9}$$

Let now $0 < \alpha < 1$, and take $\varepsilon > 0$ such that $\beta := (1+\varepsilon)(N - (1-\alpha)) < N$. Take $\rho_0 = (\frac{1}{2})^{1+1/\varepsilon}$. Then, if $0 < \rho < \rho_0$, taking $r = \rho^{1/(1+\varepsilon)}$, we have that $r^\varepsilon < 1/2$. And therefore replacing in (4.9) we have

$$\int_{B_\rho} G\big(|\nabla u|\big) \leqslant \big((C+1)^{1/2} + C^{1/2}\big) C(1+\lambda) \rho^{N-(1-\alpha)} \tag{4.10}$$

and by Lemma 2.5 we conclude that for all $0 < \alpha < 1$, $u \in C^\alpha(B_\rho)$ for $0 < \rho \leqslant \rho_0$ and $\|u\|_{C^\alpha(B_\rho)} \leqslant \overline{C}$ where $\overline{C} = \overline{C}(N, \alpha, g_0, \delta, \lambda, \rho_0, \|u\|_{L^\infty(\Omega)})$. $\square$

We then have that $u$ is continuous. Therefore, $\{u > 0\}$ is open. We can prove the following property for minimizers.

**Lemma 4.1.** *Let $u$ be a minimizer of $\mathcal{J}$. Then $u$ is an $\mathcal{L}$-solution in $\{u > 0\}$.*

**Proof.** Let $B \subset \{u > 0\}$ and $v$ such that

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B, \\ v = u & \text{in } B^c. \end{cases}$$

By the comparison principle we have that $v \geqslant u$ in $B$. Thus,

$$0 \geqslant \int_{\Omega} G(|\nabla u|) - G(|\nabla v|)\,dx + \lambda\bigl|\{u > 0\}\bigr| - \lambda\bigl|\{v > 0\}\bigr| = \int_{\Omega} G(|\nabla u|) - G(|\nabla v|)\,dx$$
$$\geqslant C\biggl(\int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx + \int_{A_2} G(|\nabla u - \nabla v|)\,dx\biggr)$$

where we are using Theorem 2.3 and $A_1$, $A_2$, and $F$ are as defined therein.

Therefore

$$\int_{A_1} F(|\nabla u|)|\nabla u - \nabla v|^2\,dx = 0.$$

Thus, $F(|\nabla u|)|\nabla u - \nabla v|^2 = 0$ in $A_1$ and, by the definition of $A_1$, we conclude that $|\nabla u - \nabla v| = 0$ in this set.

On the other hand, we also have

$$\int_{A_2} G(|\nabla u - \nabla v|)\,dx = 0$$

so that $|\nabla u - \nabla v| = 0$ everywhere in $B$.

Hence, as $u = v$ on $\partial B$ we have that $u = v$. Thus, $\mathcal{L}u = 0$ in $B$. □

In order to get the Lipschitz continuity we first prove the following estimate for minimizers.

**Lemma 4.2.** *For all $x \in \Omega$, with $5d(x) < d(x, \partial\Omega)$ we have $u(x) \leqslant Cd(x)$, where $d(x) = \operatorname{dist}(x, \{u = 0\})$. The constant $C$ depends only on $N$ and $\lambda$.*

To prove Lemma 4.2 it is enough to prove the following lemma. In this proof it is essential that the class of functions $G$ satisfying condition (1.4) is closed under the rescaling

$$G_s(t) := \frac{G(st)}{sg(s)}.$$

**Lemma 4.3.** *If $u$ is a minimizer in $B_1$ with $u(0) = 0$, there exists a constant $C$ such that $\|u\|_{L^\infty(B_{1/4})} \leqslant C$, where $C$ depends only on $N$, $\lambda$, $\delta$ and $g_0$.*

**Proof.** Suppose that there exists a sequence $u_k \in \mathcal{K}$ of minimizers in $B_1(0)$ such that

$$u_k(0) = 0 \quad \text{and} \quad \max_{\overline{B}_{1/4}} u_k(x) > k.$$

Let $d_k(x) = \operatorname{dist}(x, \{u_k = 0\})$ and $\mathcal{O}_k = \{x \in B_1\colon\ d_k(x) \leqslant \frac{1-|x|}{3}\}$. Since $u_k(0) = 0$ then $\overline{B}_{1/4} \subset \mathcal{O}_k$, therefore

$$m_k := \sup_{\mathcal{O}_k}\bigl(1 - |x|\bigr)u_k(x) \geqslant \max_{\overline{B}_{1/4}}\bigl(1 - |x|\bigr)u_k(x) \geqslant \frac{3}{4}\max_{\overline{B}_{1/4}} u_k(x) > \frac{3}{4}k.$$

For each fixed $k$, $u_k$ is bounded, then $(1 - |x|)u_k(x) \to 0$ when $|x| \to 1$ which means that there exists $x_k \in \mathcal{O}_k$ such that $(1 - |x_k|)u_k(x_k) = \sup_{\mathcal{O}_k}(1 - |x|)u_k(x)$, and then

$$u_k(x_k) = \frac{m_k}{1 - |x_k|} \geqslant m_k > \frac{3}{4}k$$

as $x_k \in \mathcal{O}_k$, and $\delta_k := d_k(x_k) \leqslant \frac{1-|x_k|}{3}$. Let $y_k \in \partial\{u_k > 0\} \cap B_1$ such that $|y_k - x_k| = \delta_k$. Then,

(1) $B_{2\delta_k}(y_k) \subset B_1$,

since if $y \in B_{2\delta_k}(y_k) \Rightarrow |y| < 3\delta_k + |x_k| \leqslant 1$,

(2) $B_{\frac{\delta_k}{2}}(y_k) \subset \mathcal{O}_k$,

since if $y \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow |y| \leqslant \frac{3}{2}\delta_k + |x_k| \leqslant 1 - \frac{3}{2}\delta_k \Rightarrow d_k(y) \leqslant \frac{\delta_k}{2} \leqslant \frac{1-|y|}{3}$ and

(3) if $z \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow 1 - |z| \geqslant 1 - |x_k| - |x_k - z| \geqslant 1 - |x_k| - \frac{3}{2}\delta_k \geqslant \frac{1-|x_k|}{2}$.

By (2) we have

$$\max_{\mathcal{O}_k}\bigl(1 - |x|\bigr)u_k(x) \geqslant \max_{B_{\frac{\delta_k}{2}}(y_k)}\bigl(1 - |x|\bigr)u_k(x) \geqslant \max_{B_{\frac{\delta_k}{2}}(y_k)} \frac{(1 - |x_k|)}{2}u_k(x),$$

where in the last inequality we are using (3). Then,

$$2u_k(x_k) \geqslant \max_{B_{\frac{\delta_k}{2}}(y_k)} u_k(x). \tag{4.11}$$

As $B_{\delta_k}(x_k) \subset \{u_k > 0\}$ then $\mathcal{L}u_k = 0$ in $B_{\delta_k}(x_k)$, and by Harnack inequality in [15] we have

$$\min_{B_{\frac{3}{4}\delta_k}(x_k)} u_k(x) \geqslant cu_k(x_k). \tag{4.12}$$

As $\overline{B_{\frac{3}{4}\delta_k}}(x_k) \cap \overline{B_{\frac{\delta_k}{4}}}(y_k) \neq \emptyset$ we have by (4.12)

$$\max_{B_{\frac{\delta_k}{4}}(y_k)} u_k(x) \geqslant cu_k(x_k). \tag{4.13}$$

Let $w_k(x) = \frac{u_k(y_k + \frac{\delta_k}{2}x)}{u_k(x_k)}$. Then, $w_k(0) = 0$ and, by (4.11) and (4.13) we have

$$\max_{\overline{B_1}} w_k \leqslant 2, \qquad \max_{\overline{B_{1/2}}} w_k \geqslant c > 0. \tag{4.14}$$

Let now

$$J_k(w) = \int_{B_1} \frac{G(|\nabla w| c_k)}{g(c_k)c_k}\,dx + \frac{\lambda}{g(c_k)c_k}\int_{B_1} \chi_{\{w>0\}}(x)\,dx$$

where $c_k = \frac{2u_k(x_k)}{\delta_k}$ so that $c_k \to \infty$.

Let us prove, that $w_k$ is a minimizer of $J_k$. In fact, for any $v \in W^{1,G}(B_1)$ with $v = w_k$ on $\partial B_1$, define $v_k(y) = v(\frac{y-y_k}{\delta_k/2})u_k(x_k)$. Thus, $v_k = u_k$ on $\partial B_{\delta_k/2}(y_k)$. Then,

$$\begin{aligned}
J_k(w_k) &= \frac{2^N}{\delta_k^N}\biggl(\int_{B_{\frac{\delta_k}{2}}(y_k)} \frac{G(|\nabla u_k|)}{g(c_k)c_k}\,dy + \frac{\lambda}{g(c_k)c_k}\int_{B_{\frac{\delta_k}{2}}} \chi_{\{u_k>0\}}(y)\,dy\biggr)\\
&\leqslant \frac{2^N}{\delta_k^N}\biggl(\int_{B_{\frac{\delta_k}{2}}(y_k)} \frac{G(|\nabla v_k|)}{g(c_k)c_k}\,dy + \frac{\lambda}{g(c_k)c_k}\int_{B_{\frac{\delta_k}{2}}(y_k)} \chi_{\{v_k>0\}}(y)\,dy\biggr)\\
&= \int_{B_1} \frac{G(|\nabla v| c_k)}{g(c_k)c_k}\,dx + \frac{\lambda}{g(c_k)c_k}\int_{B_1} \chi_{\{v>0\}}(y)\,dx = J_k(v).
\end{aligned}$$

Let $g_k(t) := \frac{g(tc_k)}{g(c_k)}$, where the primitive of $g_k$ is $G_k(t) = \frac{G(tc_k)}{g(c_k)c_k}$ and $\lambda_k = \frac{\lambda}{g(c_k)c_k} \to 0$. Then,

$$J_k(w) = \int_{B_1} G_k\bigl(|\nabla w|\bigr)\,dx + \lambda_k \int_{B_1} \chi_{\{w>0\}}(x)\,dx.$$

Observe that for all $k$, $g_k$ satisfies the inequality (1.4), with the same constants $\delta$ and $g_0$. In fact,

$$\frac{g_k'(t)t}{g_k(t)} = \frac{g'(c_k t)c_k t}{g_k(c_k t)},$$

and then by (1.4) applied to $tc_k$ we have the desirer inequality.

Let us take $v_k \in W^{1,G}(B_{3/4})$ such that

$$\mathcal{L}_k v_k = 0 \quad \text{in } B_{3/4}, \tag{4.15}$$

$$v_k = w_k \quad \text{in } \partial B_{3/4} \tag{4.16}$$

where $\mathcal{L}_k$ is the operator associated to $g_k$. By (4.6), (4.9) (with $\varepsilon = 0$ and $r = 3/4$) and the fact that $\lambda_k \to 0$, we have that

$$\int_{B_{3/4}} G_k\big(|\nabla w_k - \nabla v_k|\big)\,dx \leqslant C\lambda_k^{1/2},$$

where $C$ depends on $\delta$, $g_0$, $N$ and $\|w_k\|_{L^\infty(B_1)}$ (observe that, since $\lambda_k$ is bounded for $k$ large then the constants in (4.6) and (4.9) are independent of $\lambda_k$). We also have, by (4.14) that $C$ depends only on $\delta$, $g_0$ and $N$. On the other hand, by (G1) and (g3) we have

$$G_k(t) = \frac{G(tc_k)}{g(c_k)c_k} \geqslant \frac{G(c_k)}{(1+g_0)g(c_k)c_k}\min\big\{t^{g_0+1}, t^{\delta+1}\big\} \geqslant \frac{1}{(1+g_0)^2}\min\big\{t^{g_0+1}, t^{\delta+1}\big\}.$$

Therefore,

$$\begin{aligned} C\lambda_k^{1/2} &\geqslant \int_{B_{3/4}} G_k\big(|\nabla w_k - \nabla v_k|\big)\,dx \\ &\geqslant \int_{B_{3/4}\cap\{|\nabla w_k - \nabla v_k|<1\}} \frac{|\nabla w_k - \nabla v_k|^{g_0+1}}{(1+g_0)^2}\,dx + \int_{B_{3/4}\cap\{|\nabla w_k - \nabla v_k|\geqslant 1\}} \frac{|\nabla w_k - \nabla v_k|^{\delta+1}}{(1+g_0)^2}\,dx. \end{aligned}$$

Hence

$$\begin{aligned} A_k &:= \int_{B_{3/4}\cap\{|\nabla w_k - \nabla v_k|\geqslant 1\}} |\nabla w_k - \nabla v_k|^{\delta+1}\,dx \to 0 \quad \text{and} \\ B_k &:= \int_{B_{3/4}\cap\{|\nabla w_k - \nabla v_k|<1\}} |\nabla w_k - \nabla v_k|^{g_0+1}\,dx \to 0. \end{aligned} \tag{4.17}$$

By Hölder inequality and (4.17) we have

$$C_k := \int_{B_{3/4}\cap\{|\nabla w_k - \nabla v_k|<1\}} |\nabla w_k - \nabla v_k|^{\delta+1}\,dx \leqslant B_k^{\frac{\delta+1}{g_0+1}}|B_{3/4}|^{\frac{g_0-\delta}{g_0+\delta}} \to 0,$$

therefore,

$$\int_{B_{3/4}} |\nabla w_k - \nabla v_k|^{\delta+1}\,dx = A_k + C_k \to 0. \tag{4.18}$$

As $w_k = v_k$ on $\partial B_{3/4}$ then $p_k = w_k - v_k \in W_0^{1,\delta+1}(B_{3/4})$ and by (4.18) we have

$$p_k \to 0 \quad \text{in } W_0^{1,\delta+1}(B_{3/4}). \tag{4.19}$$

On the other hand by Theorem 4.1 we have that

$$\|w_k\|_{C^\alpha(B')} \leqslant C\big(\|w_k\|_{L^\infty(B_{3/4})}, g_0, \delta, B'\big) \leqslant C(g_0, \delta, B') \quad \forall B' \Subset B_{3/4}. \tag{4.20}$$

(Here again we may suppose that the constant $C$ dose not depend on $\lambda_k$, since $\lambda_k \to 0$. Also, recall that $\|w_k\|_{L^\infty(B_1)} \leqslant 2$.)

As $v_k$ are solutions of (4.15) by Theorem 1.7 in [15] (see Lemma 2.5), we have for $B' \Subset B_{3/4}$

$$\|v_k\|_{C^{1,\alpha}(B')} \leqslant C\big(N, \delta, g_0, G_k(1), \operatorname{dist}(B', \partial B_{3/4}), \|v_k\|_{L^\infty(B_{3/4})}\big). \tag{4.21}$$

But $G_k(1) = \frac{G(c_k)}{c_k g(c_k)} \leqslant 1$ by (g3) and $\|v_k\|_{L^\infty(B_{3/4})} \leqslant \|w_k\|_{L^\infty(\partial B_{3/4})} \leqslant 2$. Then, this constant only depends on $N$, $\delta$ and $g_0$.

Therefore by (4.20) and (4.21) we have that there exist subsequences, that we call for simplicity $v_k$ and $w_k$, and functions $w_0, v_0 \in C^\alpha(B')$ for every $B' \Subset B_{3/4}$, such that

$$\begin{aligned} w_k \to w_0 &\quad \text{uniformly in } B_{3/4}, \\ v_k \to v_0 &\quad \text{uniformly in } B', \\ \nabla v_k \to \nabla v_0 &\quad \text{uniformly in } B'. \end{aligned}$$

Then,

$$\nabla w_k \to \nabla w_0 \quad \text{weakly in } L^{\delta+1}(B_{3/4}),$$
$$p_k = w_k - v_k \to w_0 - v_0 \quad \text{uniformly in } B'.$$

But by (4.19) we have $p_k \to 0$ in $W^{1,\delta+1}(B')$. Thus, $v_0 = w_0$.

Using Harnack's inequality of [15], we have that

$$\sup_{B_{1/2}} v_k \leqslant C \inf_{B_{1/2}} v_k$$

where the constant $C$ depends only on $g_0, \delta, N$. Then, passing to the limit and using that $v_0 = w_0$ we have that

$$\sup_{B_{1/2}} w_0 \leqslant C \inf_{B_{1/2}} w_0.$$

But by (4.14), passing to the limit again, we have that $\sup_{B_{1/2}} w_0 > c > 0$ and $\inf_{B_{1/2}} w_0 = 0$ since $w_k(0) = 0$ for all $k$, this is a contradiction. □

**Proof of Lemma 4.2.** Let $x_0 \in \{u > 0\}$ with $5d(x_0) < d(x_0, \partial\Omega)$. Take $\widetilde{u}(x) = \frac{u(y_0 + 4d_0 x)}{4d_0}$, where $d_0 = \operatorname{dist}(x_0, \partial\{u > 0\}) = \operatorname{dist}(x_0, y_0)$ with $y_0 \in \partial\{u > 0\}$. If we prove that $\widetilde{u}$ is a minimizer in $B_1(0)$, as $\widetilde{u}(0) = 0$ and $\frac{|x_0 - y_0|}{4d_0} = 1/4$, by Lemma 4.3 we have

$$C \geqslant \widetilde{u}\left(\frac{x_0 - y_0}{4d_0}\right) = \frac{u(x_0)}{4d_0}$$

and the result follows.

So, let us prove that $\widetilde{u}$ is a minimizer in $B_1(0)$. As $5d(x_0) < d(x_0, \partial\Omega)$ we have, $B_{4d_0}(y_0) \subset \Omega$. Let $\widetilde{v} \in W^{1,G}(B_1(0))$ and $v$ such that $\widetilde{v}(x) = \frac{v(y_0+4d_0x)}{4d_0}$. Then, changing variables we have

$$\int_{B_1} G\big(|\nabla\widetilde{v}|\big)\,dx = \int_{B_1} G\big(\big|\nabla v(y_0+4d_0x)\big|\big)\,dx = \int_{B_{4d_0}(y_0)} \frac{G(|\nabla v(y)|)}{d_0^N 4^N}\,dy$$

and

$$\big|\{\widetilde{v} > 0 \cap B_1\}\big| = \frac{|\{\widetilde{v} > 0 \cap B_{4d_0}(y_0)\}|}{d_0^N 4^N}.$$

As $u$ is a minimizer of $\mathcal{J}$ in $B_{4d_0}(y_0)$ we have, if $\widetilde{v} = \widetilde{u}$ on $\partial B_1(0)$,

$$\begin{aligned}
&\int_{B_1(0)} G\big(\big|\nabla\widetilde{u}(x)\big|\big)\,dx + \lambda\big|\{\widetilde{u} > 0 \cap B_1(0)\}\big| \\
&\quad= \int_{B_{4d_0}(y_0)} \frac{G(|\nabla u(y)|)}{d_0^N 4^N}\,dy + \frac{\lambda|\{u > 0 \cap B_{4d_0}(y_0)\}|}{d_0^N 4^N} \\
&\quad\leqslant \int_{B_{4d_0}(y_0)} \frac{G(|\nabla v(y)|)}{d_0^N 4^N}\,dy + \frac{\lambda|\{v > 0 \cap B_{4d_0}(y_0)\}|}{d_0^N 4^N} \\
&\quad= \int_{B_1(0)} G\big(\big|\nabla\widetilde{v}(x)\big|\big)\,dx + \lambda\big|\{\widetilde{v} > 0 \cap B_1(0)\}\big|.
\end{aligned}$$

Therefore, $\widetilde{u}$ is a minimizer of $\mathcal{J}$ in $B_1(0)$. □

Now we can prove the uniform Lipschtiz continuity of minimizers of $\mathcal{J}$.

**Theorem 4.2.** *Let $u$ be a minimizer. Then $u$ is locally Lipschitz continuous in $\Omega$. Moreover, for any connected open subset $D \Subset \Omega$ containing free boundary points, the Lipschitz constant of $u$ in $D$ is estimated by a constant $C$ depending only on $N, g_0, \delta, \operatorname{dist}(D, \partial\Omega)$ and $\lambda$.*

**Proof.** First, take $x$ such that $d(x) < \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$ and $\widetilde{u}(y) = \frac{1}{d(x)}u(x + d(x)y)$ for $y \in B_1(0)$. By Lemma 4.3 we have $\widetilde{u}(0) \leqslant C$ in $B_1$, where $C$ depends only on $N, \lambda, \delta$ and $g_0$. Since $u > 0$ in $B_{d(x)}(x)$, $\mathcal{L}u = 0$ in this ball. Thus $\mathcal{L}\widetilde{u} = 0$ in $B_1(0)$ By Harnack's inequality $\widetilde{u}(y) \leqslant C$ in $B_{1/2}(0)$ where $C$ depends only on $N, \lambda, \delta$ and $g_0$. Now, by property (2) in Lemma 2.7, $|\nabla\widetilde{u}(0)| \leqslant C\|\widetilde{u}\|_{L^\infty(B_{1/2})} \leqslant C$ where $C$ depends only on $N, \lambda, \delta$ and $g_0$. Since $\nabla u(x) = \nabla\widetilde{u}(0)$, the result follows in the case $d(x) < \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$.

Let $r_1$ such that $\operatorname{dist}(x, \partial\Omega) \geqslant r_1 > 0$ $\forall x \in D$, take $D'$, satisfying $D \Subset D' \Subset \Omega$ given by

$$D' = \big\{x \in \Omega / \operatorname{dist}(x, D) < r_1/2\big\}.$$

If $d(x) \leqslant \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$ we proved that $|\nabla u(x)| \leqslant C$. If $d(x) > \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$, thus $u > 0$ in $B_{\frac{r_1}{5}}(x)$ and $B_{\frac{r_1}{5}}(x) \subset D'$ so that $|\nabla u(x)| \leqslant \frac{C}{r_1}\|u\|_{L^\infty(D')}$.

To prove the second part of the theorem, consider now any domain $D$, and $D'$ as in the previous paragraph. Let us see that $\|u\|_{L^\infty(D')}$ is bounded by a constant that depends only on $N, D, r_1, \lambda, \delta$, and $g_0$ (we argue as in [4] Theorem 4.3). Let $r_0 = \frac{r_1}{5}$, since $D'$ is connected and not contained in $\{u > 0\} \cap \Omega$, there exists $x_0, \dots, x_k \in D'$ such that $x_j \in B_{\frac{r_0}{2}}(x_{j-1})$, $j = 1, \dots, k$, $B_{r_0}(x_j) \subset \{u > 0\}$, $j = 0, \dots, k-1$ and $B_{r_0}(x_k) \nsubseteq \{u > 0\}$. By Lemma 4.3 $u(x_k) \leqslant Cr_0$ and by Harnack's inequality in [15] we have $u(x_{j+1}) \geqslant cu(x_j)$. Inductively we obtain $u(x_0) \leqslant Cr_0$ $\forall x_0 \in D'$. Therefore, the supremum of $u$ over $D'$ can be estimated by a constant depending only on $N, r_1, \lambda, \delta$, and $g_0$. □

Observe that, if we do not use Lemma 4.2, then we obtain that the Lipschitz constant depends also on $\|u\|_{L^\infty(\Omega)}$ (that is, depends also on the Dirichlet datum $\varphi_0$).

## 5. Nondegeneracy

In this section we prove the nondegeneracy of a minimizer at the free boundary and the locally uniform positive density of the sets $\{u > 0\}$ and $\{u = 0\}$.

**Lemma 5.1.** *Let $\gamma > 0$, $D \Subset \Omega$ and $C$ the constant in Theorem 4.2. Then, if $C_1 > C$, $B_r \subset \Omega$ and $u$ is a minimizer, there holds that*

$$\frac{1}{r}\left(\fint_{B_r} u^\gamma\right)^{1/\gamma} \geqslant C_1 \quad \text{implies} \quad u > 0 \quad \text{in } B_r.$$

**Proof.** If $B_r$ contains a free boundary point, as $u$ vanishes at some point $x_0 \in B_r$, and $|\nabla u(x)| \leqslant C$ in $B_r$, then $|u(x) - u(x_0)| \leqslant Cr$, that is, $u(x) \leqslant Cr$ in $B_r$ and then $\frac{1}{r}(\fint_{B_r} u^\gamma)^{1/\gamma} \leqslant C$ which is a contradiction. □

**Lemma 5.2.** *For any $\gamma > 1$ and for any $0 < \kappa < 1$ there exists a constant $c_\kappa$ such that, for any minimizer $u$ and for every $B_r \subset \Omega$, we have*

$$\frac{1}{r}\left(\fint_{B_r} u^\gamma\right)^{1/\gamma} \leqslant c_\kappa \quad \text{implies} \quad u = 0 \quad \text{in } B_{\kappa r},$$

*where $c_\kappa$ depends also on $N, \lambda, g_0, \delta$ and $\gamma$.*

**Proof.** We may suppose that $r = 1$ and that $B_r$ is centered at zero (if not, we take the rescaled function $\tilde{u} = \frac{u(x_0 + rx')}{r}$). By Theorem 1.2 in [15] we have

$$\varepsilon := \sup_{B_{\sqrt{\kappa}}} u < C\left(\fint_{B_1} u^\gamma\right)^{1/\gamma}$$

where $C = C(\kappa, \gamma)$. Now chose $v$ such that

$$v = \begin{cases} C_1\varepsilon(e^{-\mu|x|^2} - e^{-\mu\kappa^2}) & \text{in } B_{\sqrt{\kappa}} \setminus B_\kappa, \\ 0 & \text{in } B_\kappa. \end{cases}$$

Here the constants $\mu > 0$ and $C_1 < 0$ are chosen so that $\mathcal{L}v < 0$ in $B_{\sqrt{\kappa}} \setminus B_\kappa$ (see Lemma 2.9) and $v = \varepsilon$ on $\partial B_{\sqrt{\kappa}}$. Hence, $v \geqslant u$ on $\partial B_{\sqrt{\kappa}}$, and therefore if

$$w = \begin{cases} \min(u, v) & \text{in } B_{\sqrt{\kappa}}, \\ u & \text{in } \Omega \setminus B_{\sqrt{\kappa}}, \end{cases}$$

$w$ is an admissible function for the minimizing problem. Thus, using the convexity of $G$, we find that

$$\begin{aligned}
&\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda\big|B_\kappa \cap \{u > 0\}\big| \\
&\quad = \mathcal{J}(u) - \int_{\Omega\setminus B_\kappa} G(|\nabla u|)\,dx + \lambda\big|B_\kappa \cap \{u > 0\}\big| - \lambda\big|\Omega \cap \{u > 0\}\big| \\
&\quad \leqslant \mathcal{J}(w) - \int_{\Omega\setminus B_\kappa} G(|\nabla u|)\,dx + \lambda\big|B_\kappa \cap \{u > 0\}\big| - \lambda\big|\Omega \cap \{u > 0\}\big| \\
&\quad = \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} G(|\nabla w|)\,dx - \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} G(|\nabla u|)\,dx \\
&\quad \leqslant \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} g(|\nabla w|)\frac{\nabla w}{|\nabla w|}(\nabla w - \nabla u)\,dx = -\int_{B_{\sqrt{\kappa}}\setminus B_\kappa} g(|\nabla w|)\frac{\nabla w}{|\nabla w|}\nabla(u - v)^+\,dx \\
&\quad = -\int_{(B_{\sqrt{\kappa}}\setminus B_\kappa)\cap\{u>v\}} g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\nabla(u - v)^+\,dx
\end{aligned}$$

and as $v$ is a subsolution we have

$$\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda\big|B_\kappa \cap \{u > 0\}\big| \leqslant -\int_{\partial B_\kappa} g(|\nabla v|)\frac{\nabla v}{|\nabla v|} u\,\nu\,d\mathcal{H}^{N-1}.$$

And, as $|\nabla v| \leqslant C\varepsilon$ we have that

$$\int_{B_\kappa} G(|\nabla u|)\,dx + \lambda\big|B_\kappa \cap \{u > 0\}\big| \leqslant g(C\varepsilon)\int_{\partial B_\kappa} u\,d\mathcal{H}^{N-1}.$$

By Sobolev's trace inequality and by $(\tilde{g}3)$, for $\widetilde{G}(\alpha) = \lambda$ we have

$$\int\limits_{\partial B_\kappa} u \leqslant C(N,\kappa)\int\limits_{B_\kappa} |\nabla u| + u\,dx$$
$$\leqslant C(N,\kappa)\biggl(\int\limits_{B_\kappa} G\biggl(\frac{|\nabla u|}{\alpha}\biggr) + \int\limits_{B_\kappa\cap\{u>0\}} \widetilde{G}(\alpha) + \int\limits_{B_\kappa} u\,dx\biggr)$$
$$\leqslant C(N,\kappa,\lambda)(1+\varepsilon)\biggl(\int\limits_{B_\kappa} G\bigl(|\nabla u|\bigr) + \lambda\bigl|\{u>0\}\cap B_\kappa\bigr|\biggr)$$

where in the last inequality we are using that $\int_{B_\kappa} u\,dx \leqslant \varepsilon|\{u>0\}\cap B_\kappa|$. Therefore,

$$\int\limits_{B_\kappa} G\bigl(|\nabla u|\bigr)\,dx + \lambda\bigl|B_\kappa\cap\{u>0\}\bigr| \leqslant g(C\varepsilon)C(1+\varepsilon)\biggl(\int\limits_{B_\kappa} G\bigl(|\nabla u|\bigr)\,dx + \lambda\bigl|B_\kappa\cap\{u>0\}\bigr|\biggr).$$

So that, if $\varepsilon$ is small enough

$$\int\limits_{B_\kappa} G\bigl(|\nabla u|\bigr)\,dx + \lambda|B_\kappa\cap\{u>0\} = 0.$$

Then, $u=0$ in $B_\kappa$ and the result follows. □

As a corollary we have

**Corollary 5.1.** *Let $D \Subset \Omega$, $x\in D\cap\partial\{u>0\}$. Then*

$$\sup_{B_r(x)} u \geqslant cr,$$

*where $c$ is the constant in Lemma* 5.2 *corresponding to $\kappa = 1/2$ and $\gamma$ fixed.*

**Corollary 5.2.** *For any domain $D\Subset\Omega$ there exist constants $c, C$ depending on $N, g_0, \delta, D$ and $\lambda$, such that, for any minimizer $u$ and for every $B_r(x)\subset D\cap\{u>0\}$, touching the free boundary we have*

$$cr \leqslant u(x) \leqslant Cr.$$

**Proof.** It follows by Lemmas 4.2 and 5.2. □

**Theorem 5.1.** *For any domain $D\Subset\Omega$ there exists a constant $c$, with $0<c<1$ depending on $N, g_0, \delta, D$ and $\lambda$, such that, for any minimizer $u$ and for every $B_r\subset\Omega$, centered on the free boundary we have*

$$c \leqslant \frac{|B_r\cap\{u>0\}|}{|B_r|} \leqslant 1-c.$$

**Proof.** First, by Corollary 5.1 we have that there exists $y \in B_r$ such that $u(y) > cr$ and as $u$ is a subsolution we have by Theorem 1.2 in [15] that

$$\left( \fint_{B_{\kappa r}} u^\gamma \, dx \right)^{1/\gamma} \geqslant Cu(y).$$

Therefore

$$\frac{1}{\kappa r}\left( \fint_{B_{\kappa r}} u^\gamma \, dx \right)^{1/\gamma} \geqslant \frac{C}{\kappa}.$$

Now, if $\kappa$ is small enough, we have

$$\frac{1}{\kappa r}\left( \fint_{B_{\kappa r}} u^\gamma \, dx \right)^{1/\gamma} \geqslant C_1,$$

so that by Lemma 5.1, we have that $u > 0$ in $B_{\kappa r}$, where $\kappa = \kappa(C_1, c)$. Thus,

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} \geqslant \frac{|B_{\kappa r}|}{|B_r|} = \kappa^N,$$

and $\kappa = \kappa(C_1, c)$.

In order to prove the other inequality, we may assume that $r = 1$. Let us suppose by contradiction that, there exists a sequence of minimizers $u_k$ in $B_1$, such that, $0 \in \partial\{u_k > 0\}$, with $|\{u_k = 0\} \cap B_1| = \varepsilon_k \to 0$. If we take $v_k \in W^{1,G}(B_{1/2})$ such that

$$\mathcal{L}v_k = 0 \qquad \text{in } B_{1/2}, \tag{5.1}$$

$$v_k = u_k \qquad \text{in } \partial B_{1/2}. \tag{5.2}$$

Let $A_1$ and $A_2$ as in Theorem 2.3, for $r = 1/2$. Then we have, by (4.2) that

$$\int_{A_2} G\big(|\nabla u_k - \nabla v_k|\big)\, dx \leqslant C\varepsilon_k \quad \text{and} \quad \int_{A_1} F\big(|\nabla u_k|\big)|\nabla u_k - \nabla v_k|^2 \, dx \leqslant C\varepsilon_k,$$

where $C = C(\delta, g_0)$. By (4.5) (with $\varepsilon = 0$ and $r = 1/2$) we have

$$\int_{A_1} G\big(|\nabla u_k - \nabla v_k|\big)\, dx \leqslant C\left( \int_{A_1} F\big(|\nabla u_k|\big)|\nabla u_k - \nabla v_k|^2 \, dx \right)^{1/2} \left( \int_{A_1} G\big(|\nabla u_k|\big) \right)^{1/2}.$$

Therefore, by (4.9), there exists $C$ independent of $k$ such that

$$\int_{B_{1/2}} G\big(|\nabla u_k - \nabla v_k|\big)\, dx \leqslant C\varepsilon_k^{1/2} \to 0.$$

As $u_k = v_k$ on $\partial B_{1/2}$, $w_k = u_k - v_k \in W_0^{1,\delta+1}(B_{1/2})$. Thus,

$$w_k \to 0 \quad \text{in } W_0^{1,\delta+1}(B_{1/2}). \tag{5.3}$$

By Theorem 4.1 and Theorem 1.7 in [15], we have

$$\|u_k\|_{C^\alpha(B_{1/2})} \leqslant C\big(N, \delta, g_0, \|u_k\|_{L^\infty(B_{1/2})}, \alpha\big) \quad (\text{for } \varepsilon_k \text{ small}),$$
$$\|v_k\|_{C^{1,\alpha}(B')} \leqslant C\big(N, \delta, g_0, G(1), \|u_k\|_{L^\infty(B_{1/2})}, \alpha\big) \quad \big(\text{see (4.8)}\big).$$

Therefore, there exist subsequences, that we call for simplicity $u_k$ and $v_k$, and functions $v_0 \in C^1(B')$, $u_0 \in C(B')$ for all $B' \Subset B_{1/2}$ such that

$$\begin{aligned}
u_k &\to u_0 && \text{uniformly in } B_{1/2},\\
v_k &\to v_0 && \text{uniformly in } B',\\
\nabla v_k &\to \nabla v_0 && \text{uniformly in } B',\\
\nabla u_k &\to \nabla u_0 && \text{weakly in } L^{\delta+1}(B_{1/2}),\\
w_k = u_k - v_k &\to 0 && \text{uniformly in } B'.
\end{aligned}$$

Thus, $v_0 = u_0$. By Lemma 5.2 we have that

$$\bigg(\fint_{B_{1/4}} u_k^\gamma\bigg)^{1/\gamma} \geqslant C > 0.$$

Therefore, passing to the limit, we have

$$\bigg(\fint_{B_{1/4}} u_0^\gamma\bigg)^{1/\gamma} \geqslant C > 0.$$

On the other hand, by Harnack inequality $\sup_{B_{1/4}} v_k \leqslant C \inf_{B_{1/4}} v_k$ and again, passing to the limit we have, $\sup_{B_{1/4}} u_0 \leqslant C \inf_{B_{1/4}} u_0$. As $u_0(0) = 0$, then $u_0 \equiv 0$ in $B_{1/4}$, which is a contradiction. □

**Remark 5.1.** Theorem 5.1 implies that the free boundary has Lebesgue measure zero. Moreover, it implies that for every $D \Subset \Omega$, the intersection $\partial\{u > 0\} \cap D$ has Hausdorff dimension less than $N$. In fact, to prove these statements, it is enough to use the left-hand side estimate in Theorem 5.1. In fact, this estimate says that the set of Lebesgue points of $\chi_{\{u>0\}}$ in $\partial\{u > 0\} \cap D$ is empty. On the other hand almost every point $x_0 \in \partial\{u > 0\} \cap D$ is a Lebesgue point, therefore $|\partial\{u > 0\} \cap D| = 0$.

## 6. The measure $\Lambda = \mathcal{L}u$

In this section we prove that $\{u > 0\} \cap \Omega$ is locally of finite perimeter. Then, we study the measure $\Lambda = \mathcal{L}u$ and prove that it is absolutely continuous with respect to the $\mathcal{H}^{N-1}$ measure on the free boundary. This result gives rice to a representation theorem for the measure $\Lambda$. Finally, we prove that almost every point in the free boundary belongs to the reduced free boundary.

**Theorem 6.1.** *For every $\varphi \in C_0^\infty(\Omega)$ such that* $\operatorname{supp}(\varphi) \subset \{u > 0\}$,

$$\int_\Omega g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\nabla\varphi = 0. \tag{6.1}$$

*Moreover, the application*

$$\Lambda(\varphi) := -\int_\Omega g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\nabla\varphi\,dx$$

*from $C_0^\infty(\Omega)$ into $\mathbb{R}$ defines a nonnegative Radon measure $\Lambda = \mathcal{L}u$ with support on $\Omega \cap \partial\{u > 0\}$.*

**Proof.** We know that $u$ is an $\mathcal{L}$-subsolution, then by the Riesz Representation Theorem, there exists a nonnegative Radon measure $\Lambda$, such that $\mathcal{L}u = \Lambda$. And as $\mathcal{L}u = 0$ in $\{u > 0\}$, then for any $\varphi \in C_0^\infty(\Omega \setminus \partial\{u > 0\})$

$$\Lambda(\varphi) = -\int_{\{u>0\}} \nabla\varphi g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\,dx = 0,$$

and the result follows. □

Now we want to prove that $\Omega \cap \partial\{u > 0\}$, has Hausdorff dimension $N - 1$. First we need the following lemma.

**Lemma 6.1.** *If $u_k$ is a sequence of minimizers in compact subsets of $B_1$, such that $u_k \to u_0$ uniformly in $B_1$, then*

(1) $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance,*
(2) $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* $L^1_{\text{loc}}(\mathbb{R}^N)$,
(3) *if* $0 \in \partial\{u_k > 0\}$, *then* $0 \in \partial\{u_0 > 0\}$.

**Proof.** Here we only have to use Lemma 5.2 and Theorem 5.1 and the fact that $u_k \to u_0$ uniformly in compacts subsets of $B_1$. To see the complete proof see pp. 19–20 in [5]. □

Now, we prove the following theorem.

**Theorem 6.2.** *For any domain $D \Subset \Omega$ there exist constants $c, C$, depending on $N, g_0, \delta, D$ and $\lambda$, such that, for any minimizer $u$ and for every $B_r \subset \Omega$, centered on the free boundary we have*

$$cr^{N-1} \leqslant \int_{B_r} d\Lambda \leqslant Cr^{N-1}.$$

**Proof.** Let $\xi \in C_0^\infty(\Omega)$, $\xi \geqslant 0$. Then,

$$\Lambda(\xi) = -\int_{\{u>0\}} \nabla\xi g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\,dx.$$

Approximating $\chi_{B_r}$ from below by a sequence $\{\xi_n\}$ such that $\xi_n = 1$ in $B_{r-\frac{1}{n}}$ and $|\nabla\xi_n| \leqslant C_N n$ and using that $u$ is Lipschitz we have that

$$\left|\int_\Omega \nabla\xi_n g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\,dx\right| \leqslant Cn|B_r \setminus B_{r-\frac{1}{n}}| \leqslant C\big(r^{N-1} + O(1/n)\big).$$

Then, as

$$\int_\Omega \xi_n\,d\Lambda \to \int_{B_r} d\Lambda,$$

the bound from above holds.

In order to prove the other inequality, we will suppose that $r = 1$. Arguing by contradiction we assume that there exists a sequence of minimizers $u_k$ in $B_1$, with $0 \in \partial\{u_k > 0\}$, and $\Lambda_k = \mathcal{L}u_k$, such that $\int_{B_1} d\Lambda_k = \varepsilon_k \to 0$. As the $u_k$'s are uniformly Lipschitz, we can assume that $u_k \to u_0$ uniformly in $B_{1/2}$. Let $h_k = g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}$. Then, there exists a subsequence and a function $h_0$ such that $h_k \rightharpoonup h_0$ $*$-weakly in $L^\infty(B_{1/2})$. We claim that $h_0 = g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$. In fact, if $B_\rho \Subset \{u_0 > 0\}$ then there exists a subsequence such that $u_k \to u_0$ strongly in $C^{1,\alpha}(B_\rho)$. So that $h_0 = g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$. If $B_\rho \subset \{u_0 = 0\}$, then by Lemma 5.2 we have that $u_k = 0$ in $B_{\rho\kappa}$ for $k \geqslant k_0(\kappa)$. Thus $h_0 = 0 = g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$ also in this case. Finally $\partial\{u_0 > 0\} \cap B_{1/2}$ has zero Lebesgue measure. In fact, by (1) in Lemma 6.1, every point $x_0 \in \partial\{u_0 > 0\} \cap B_{1/2}$ is a limit point of $x_k \in \partial\{u_k > 0\} \cap B_{1/2}$. Thus,

$$\left(\fint_{B_r(x_0)} u_0^\gamma\right)^{1/\gamma} \geqslant cr$$

for any ball $B_r(x_0) \subset B_{1/2}$. Using this fact, and the Lipschitz continuity we have that $|B_r(x_0) \cap \{u_0 > 0\}| \geqslant c|B_r(x_0)|$ with $c > 0$. This implies that $|\partial\{u_0 > 0\} \cap B_{1/2}| = 0$ (see Remark 5.1).

Therefore, for all $\xi \in C_0^\infty(B_{1/2})$, $\xi \geqslant 0$ we have

$$\int_{B_{1/2}} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi = \lim_{k\to\infty}\int_{B_{1/2}} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi.$$

On the other hand,

$$\int_{B_{1/2}} \xi\, d\Lambda_0 = \lim_{k\to\infty}\int_{B_{1/2}} \xi\, d\Lambda_k \leqslant \|\xi\|_{L^\infty(B_{1/2})}\lim_{k\to\infty}\varepsilon_k = 0.$$

Therefore $\Lambda_0 = 0$ in $B_{1/2}$. That is, $\mathcal{L}u_0 = 0$ in $B_{1/2}$. But $u_0 \geqslant 0$ and $u_0(0) = 0$, so that by the Harnack inequality we have $u_0 = 0$ in $B_{1/2}$.

On the other hand, $0 \in \partial\{u_k > 0\}$, and by the nondegeneracy, we have

$$\left(\int_{B_{1/4}} u_k^\gamma\right)^{1/\gamma} \geqslant c > 0.$$

Thus,

$$\left(\int_{B_{1/4}} u_0^\gamma\right)^{1/\gamma} \geqslant c > 0$$

which is a contradiction. □

Therefore, we have the following representation theorem.

**Theorem 6.3** *(Representation Theorem). Let u be a minimizer. Then,*

(1) $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ *for every* $D \Subset \Omega$.
(2) *There exists a Borel function* $q_u$ *such that*

$$\mathcal{L}u = q_u\mathcal{H}^{N-1}\lfloor\partial\{u > 0\},$$

*i.e.*

$$-\int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\varphi\, dx = \int_{\Omega\cap\partial\{u>0\}} \varphi q_u\, d\mathcal{H}^{N-1} \quad \forall\varphi \in C_0^\infty(\Omega).$$

(3) *For* $D \Subset \Omega$ *there are constants* $0 < c \leqslant C < \infty$ *depending on* $N, g_0, \delta, \Omega, D$ *and* $\lambda$ *such that for* $B_r(x) \subset D$ *and* $x \in \partial\{u > 0\}$,

$$c \leqslant q_u(x) \leqslant C, \qquad cr^{N-1} \leqslant \mathcal{H}^{N-1}(B_r(x) \cap \partial\{u > 0\}) \leqslant Cr^{N-1}.$$

**Proof.** It follows as in Theorem 4.5 in [4]. □

**Remark 6.1.** As $u$ satisfies the conclusions of Theorem 6.3, the set $\Omega \cap \{u > 0\}$ has finite perimeter locally in $\Omega$ (see [9, 4.5.11]). That is, $\mu_u := -\nabla \chi_{\{u>0\}}$ is a Borel measure, and the total variation $|\mu_u|$ is a Radon measure. We define the reduced boundary as in [9, 4.5.5] (see also [8]) by $\partial_{\mathrm{red}}\{u > 0\} := \{x \in \Omega \cap \partial\{u > 0\} / |\nu_u(x)| = 1\}$, where $\nu_u(x)$ is the unit vector with

$$\int_{B_r(x)} |\chi_{\{u>0\}} - \chi_{\{y/\langle y-x,\nu_u(x)\rangle<0\}}| = o(r^N) \tag{6.2}$$

for $r \to 0$, if such a vector exists, and $\nu_u(x) = 0$ otherwise. By the results in [9, Theorem 4.5.6] we have

$$\mu_u = \nu_u \mathcal{H}^{N-1} \lfloor \partial_{\mathrm{red}}\{u > 0\}.$$

**Lemma 6.2.** $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}) = 0.$

**Proof.** This is a consequence of the density property of Theorem 5.1 and Theorem 4.5.6(3) of [9]. □

## 7. Asymptotic development and identification of the function $q_u$

In this section we give some properties of blow-up sequences of minimizers, we prove that any limit of a blow-up sequence is a minimizer. We prove the asymptotic development of minimizers near points in their reduced free boundary. We finally identify the function $q_u$ for almost every point in the reduced free boundary.

We first prove some properties of blow-up sequences.

**Definition 7.1.** Let $B_{\rho_k}(x_k) \subset \Omega$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Let

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

We call $u_k$ a blow-up sequence with respect to $B_{\rho_k}(x_k)$.

Since $u$ is locally Lipschitz continuous, there exists a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that, for a subsequence,

$$u_k \to u_0 \quad \text{in } C^\alpha_{\mathrm{loc}}(\mathbb{R}^N) \text{ for every } 0 < \alpha < 1,$$
$$\nabla u_k \to \nabla u_0 \quad * \text{-weakly in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N),$$

and $u_0$ is Lipschitz in $\mathbb{R}^N$ with constant $L$.

**Lemma 7.1.** *If $u$ is a minimizer, then*

(1) $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance,*
(2) $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$,

(3) $\nabla u_k \to \nabla u_0$ *uniformly in compact subsets of* $\{u_0 > 0\}$,
(4) $\nabla u_k \to \nabla u_0$ *a.e. in* $\Omega$,
(5) *if* $x_k \in \partial\{u > 0\}$, *then* $0 \in \partial\{u_0 > 0\}$,
(6) $\mathcal{L}u_0 = 0$ *in* $\{u_0 > 0\}$.

**Proof.** (1), (2) and (5) follow as in Lemma 6.1. For the proof of (3) and (4) see pp. 19–20 in [5]. (6) follows by Lemma 4.1 and by (2) and (3). □

**Lemma 7.2.** *If* $u(x_m) = 0$, $x_m \to x_0$ *in* $\Omega$. *Then, any blow-up limit* $u_0$ *respect to* $B_{\rho_m}(x_m)$ *is a minimizer of* $\mathcal{J}$ *in any ball.*

**Proof.** Let $u_m$, $u_0$ be as is Lemma 7.1, $R > 0$ and $v$ such that $v - u_0 \in W_0^{1,G}(B_R(0))$. Let $\eta \in C_0^\infty(B_R(0))$, $0 \leqslant \eta \leqslant 1$ and $v_m = v + (1-\eta)(u_m - u_0)$ then $v_m = u_m$ in $\partial B_R(0)$. Therefore

$$\int_{B_R(0)} G\big(|\nabla u_m|\big)\,dx + \lambda\chi_{\{u_m>0\}} \leqslant \int_{B_R(0)} G\big(|\nabla v_m|\big)\,dx + \lambda\chi_{\{v_m>0\}}.$$

As $|\nabla u_m| \leqslant C$ and $\nabla u_m \to \nabla u_0$ a.e., we have

$$\int_{B_R(0)} G\big(|\nabla u_m|\big)\,dx \to \int_{B_R(0)} G\big(|\nabla u_0|\big)\,dx, \qquad \int_{B_R(0)} G\big(|\nabla v_m|\big)\,dx \to \int_{B_R(0)} G\big(|\nabla v|\big)\,dx$$

and

$$\chi_{\{v_m>0\}} \leqslant \chi_{\{v>0\}} + \chi_{\{\eta<1\}}.$$

Therefore,

$$\int_{B_R(0)} G\big(|\nabla u_0|\big)\,dx + \lambda\chi_{\{u_0>0\}} \leqslant \int_{B_R(0)} G\big(|\nabla v|\big)\,dx + \lambda\chi_{\{v>0\}} + \lambda\chi_{\{\eta<1\}}.$$

Taking $\eta$ such that $|\{\eta < 1\} \cap B_R(0)| \to 0$ we have the desired result. □

Let $\lambda^*$ be such that, $g(\lambda^*)\lambda^* - G(\lambda^*) = \lambda$. Then we have

**Lemma 7.3.** *Let* $u$ *be a minimizer in* $\mathbb{R}^N$ *such that* $u = \lambda_0\langle x, \nu_0\rangle^-$ *in* $B_{r_0}$, *with* $r_0 > 0$, $0 < \lambda_0 < \infty$ *and* $\nu_0$ *a unit vector. Then,* $\lambda_0 = \lambda^*$.

**Proof.** Let $\tau_\varepsilon(x) = x + \varepsilon\eta(x)$ with $\eta \in C_0^\infty(B_{r_0})$, and let $u_\varepsilon(\tau_\varepsilon(x)) = u(x)$. Then,

$$0 \leqslant \mathcal{J}(u_\varepsilon) - \mathcal{J}(u),$$

$$\big|B_{r_0} \cap \{u_\varepsilon > 0\}\big| = \int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} |\det D\tau_\varepsilon|\,dx = \int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \big(1 + \varepsilon \operatorname{div}\eta + o(\varepsilon)\big)\,dx$$

and

$$\int_{B_{r_0}\cap\{u_\varepsilon>0\}} G\big(|\nabla u_\varepsilon|\big)\,dy$$
$$= \int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \left(G\big(|\nabla u|\big)+\varepsilon\left(G\big(|\nabla u|\big)\operatorname{div}\eta-\frac{g(|\nabla u|)}{|\nabla u|}\nabla u D\eta\nabla u\right)\right)dx+o(\varepsilon).$$

Therefore, since $u_\varepsilon = u$ in $\mathbb{R}^N \setminus B_{r_0}$,

$$0 \leqslant \varepsilon \int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \left(\big(G\big(|\nabla u|\big)+\lambda\big)\operatorname{div}\eta-\frac{g(|\nabla u|)}{|\nabla u|}\nabla u D\eta\nabla u\right)dx+o(\varepsilon).$$

Thus,

$$\int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \left(\big(G\big(|\nabla u|\big)+\lambda\big)\operatorname{div}\eta-\frac{g(|\nabla u|)}{|\nabla u|}\nabla u D\eta\nabla u\right)dx\geqslant 0.$$

If we change $\eta$ by $-\eta$ and recall that $\nabla u = -\lambda_0\,\nu_0$ in $\{\langle x,\nu_0\rangle<0\}$ we obtain

$$\int_{B_{r_0}\cap\{\langle x,\nu_0\rangle<0\}} \Big(\big(G(\lambda_0)+\lambda\big)\operatorname{div}\eta-g(\lambda_0)\lambda_0\nu_0 D\eta\nu_0\Big)dx=0$$

for all $\eta\in C_0^\infty(B_{r_0})$.

Take $\eta(x)=\phi(|x|)\nu_0$ with $\operatorname{supp}\phi\subset(-r_0,r_0)$. Then,

$$\operatorname{div}\eta(x)=\frac{\phi'(|x|)}{|x|}\langle x,\nu_0\rangle,$$
$$\nu_0 D\eta\nu_0=\nu_{0i}\frac{\partial\eta_j}{\partial x_i}\nu_{0j}=\langle x,\nu_0\rangle\frac{\phi'(|x|)}{|x|}=\operatorname{div}\eta.$$

Hence

$$0=\int_{\{\langle x,\nu_0\rangle<0\}\cap B_{r_0}(0)} \big(G(\lambda_0)+\lambda-g(\lambda_0)\lambda_0\big)\operatorname{div}\eta\,dx$$
$$=\big(G(\lambda_0)+\lambda-g(\lambda_0)\lambda_0\big)\int_{\{\langle x,\nu_0\rangle=0\}\cap B_{r_0}} \eta\nu_0\,d\mathcal{H}^{N-1}(x)$$
$$=\big(G(\lambda_0)+\lambda-g(\lambda_0)\lambda_0\big)\int_{\{\langle x,\nu_0\rangle=0\}\cap B_{r_0}} \phi\big(|x|\big)\,d\mathcal{H}^{N-1}(x)$$

for all $\phi\in C_0^\infty(-r_0,r_0)$.

Therefore, $g(\lambda_0)\lambda_0-G(\lambda_0)=\lambda$. □

**Lemma 7.4.** *Let $u \in \mathcal{K}$ be a minimizer. Then, for every $x_0 \in \Omega \cap \partial\{u > 0\}$*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} \left|\nabla u(x)\right| = \lambda^*. \tag{7.1}$$

**Proof.** Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let

$$l := \limsup_{\substack{x \to x_0 \\ u(x) > 0}} \left|\nabla u(x)\right|.$$

Then there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \quad \left|\nabla u(z_k)\right| \to l.$$

Let $y_k$ be the nearest point to $z_k$ on $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$. Consider the blow-up sequence with respect to $B_{d_k}(y_k)$ with limit $u_0$, such that there exists

$$\nu := \lim_{k \to \infty} e_k,$$

where $e_k = \frac{y_k - z_k}{d_k}$, and suppose that $\nu = e_N$. Then, by Lemma 7.1(1), $0 \in \partial\{u_0 > 0\}$. By Lemma 7.1(2) and by Lemma 7.2 we have that $u_0$ satisfies Theorem 5.1. Then, $B_1(-e_N) \subset \{u_0 > 0\}$. By Lemma 7.1(3) we obtain

$$|\nabla u_0| \leqslant l \quad \text{in } \{u_0 > 0\} \quad \text{and} \quad \left|\nabla u_0(-e_N)\right| = l.$$

Then, $0 < l < \infty$ and since, by Lemma 7.1(6), we have that $u_0$ is an $\mathcal{L}$ solution in $\{u_0 > 0\}$ then, we have that $u$ is locally $C^{1,\alpha}$ there. Thus, there exists $\mu > 0$ such that $|\nabla u_0| > l/2$ in $B_\mu(-e_N)$. Let $e = \frac{\nabla u_0(-e_N)}{|\nabla u_0(-e_N)|}$. Let $v = \frac{\partial u_0}{\partial e}$, then $v$ satisfies the uniformly elliptic equation, $D_i(a_{ij} D_j v) = 0$ where

$$a_{ij} = \frac{g(|\nabla u_0|)}{|\nabla u_0|}\left[\left(\frac{g'(|\nabla u_0|)}{g(|\nabla u_0|)}|\nabla u_0| - 1\right)\frac{D_i u_0 D_j u_0}{|\nabla u_0|^2} + \delta_{ij}\right].$$

Then, by the strong maximum principle we have $D_e u_0 = l$ in $B_\mu(-e_N)$ so that, $\nabla u_0 = le$ in $B_\mu(-e_N)$. By continuation we can prove that this is true in $B_1(-e_N)$. Then, $u_0(x) = l\langle x, e\rangle + C$ in $B_1(-e_N)$. As $u_0(0) = 0$ and $u_0 > 0$ in $B_1(-e_N)$, we have $u_0(x) = l\langle x, e\rangle$ and $e = -e_N$. Therefore $u_0(x) = -lx_N$ in $B_1(-e_N)$. Using again a continuation argument we have that $u_0(x) = -lx_N$ in $\{x_N < 0\}$.

Now, we want to prove that $u_0 = 0$ in $\{0 < x_N < \varepsilon_0\}$ for some $\varepsilon_0 > 0$.

We argue by contradiction. Let

$$s := \limsup_{\substack{x_N \to 0^+ \, x' \in \mathbb{R}^{N-1} \\ u_0(x', x_N) > 0}} D_N u_0(x', x_N),$$

and suppose that $s > 0$ ($s < \infty$ since $u_0$ is uniformly Lipschitz). Let $(z_k, h_k)$ such that, $h_k \to 0^+$ and $D_N u_0(z_k, h_k) \to s$, and take a blow up sequence with respect to $B_{h_k}(z_k, 0)$ with limit $u_{00}$.

Arguing as before, we have that $u_{00} = sx_N$ for $x_N > 0$. On the other hand, we have $u_{00} = -lx_N$ for $x_N < 0$. By Lemma 7.2 $u_{00}$ is a minimizer, and as all the points of the form $(x', 0)$ belong to the free boundary, we get a contradiction to the positive density property of the set $\{u_{00} = 0\}$ (Theorem 5.1).

Therefore, $s = 0$. But this implies that $u_0(x', x_N) = o(x_N)$ as $x_N \searrow 0^+$. Thus, for all $\varepsilon > 0$, $h_0 > 0$,

$$\frac{1}{r}\biggl(\ \fint_{B_r(x_0)} u_0^\gamma\biggr)^{1/\gamma} < \varepsilon \quad \text{if } x_0 = (y_0, h_0) \text{ and } r = h_0$$

for $r$ small enough independent of $y_0$. Then, by the nondegeneracy property, we have that $u_0 = 0$ in $\{0 < x_N < \varepsilon_0\}$.

Now, by Lemmas 7.2 and 7.3 we conclude that $l = \lambda^*$, and the result follows. □

Now we prove the asymptotic development of minimizers. We will use the following fact.

**Remark 7.1.** Observe that in $\{|\nabla u| \geqslant c\}$, $u$ satisfies a linear nondivergence uniformly elliptic equation, $Tu = 0$ of the form

$$Tv = b_{ij}(\nabla u) D_{ij} v = 0 \tag{7.2}$$

where

$$b_{ij} = \delta_{ij} + \left(\frac{g'(|\nabla u|)|\nabla u|}{g(|\nabla u|)} - 1\right)\frac{D_i u D_j u}{|\nabla u|^2}, \tag{7.3}$$

and the matrix $b_{ij}(\nabla u)$ is $\beta$-elliptic in $\{|\nabla u| > c\}$, where $\beta = \max\{\max\{g_0, 1\}, \max\{1, 1/\delta\}\}$.

**Theorem 7.1.** *Let $u$ be a minimizer. Then, at every $x_0 \in \partial_{\text{red}}\{u > 0\}$, $u$ has the following asymptotic development*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|), \tag{7.4}$$

*where $\nu(x_0)$ is the outer unit normal to $\partial\{u > 0\}$ at $x_0$.*

**Proof.** Take $B_{\rho_k}(x_0)$ balls with $\rho_k \to 0$ and let $u_k$ be a blow-up sequence with respect to these balls with limit $u_0$. Suppose that $\nu_u(x_0) = e_N$.

First we prove that

$$\begin{cases} u_0 = 0 & \text{in } \{x_N \geqslant 0\}, \\ u_0 > 0 & \text{in } \{x_N < 0\}. \end{cases}$$

In fact, by Lemma 7.1, $\chi_{\{u_k > 0\}}$ converges to $\chi_{\{u_0 > 0\}}$ in $L^1_{\text{loc}}$. On the other hand, $\chi_{\{u_k > 0\}}$ converges to $\chi_{\{x_N < 0\}}$ in $L^1_{\text{loc}}$ by (6.2). It follows that $u_0 = 0$ in $\{x_N \geqslant 0\}$ and $u_0 > 0$ a.e. in $\{x_N < 0\}$.

If $u_0$ were zero somewhere in $\{x_N < 0\}$ there should exist a point $\bar{x}$ in $\{x_N < 0\} \cap \partial\{u_0 > 0\}$. But, as $u_0$ is a minimizer, for $0 < r < |\bar{x}_N|$,

$$\frac{|B_r(\bar{x}) \cap \{u_0 = 0\} \cap \{x_N < 0\}|}{|B_r(\bar{x})|} \geqslant c > 0.$$

Since this is a contradiction we conclude that $u_0 > 0$ in $\{x_N < 0\}$ and therefore $\mathcal{L}u_0 = 0$ in this set. Since $u_0 = 0$ on $\{x_N = 0\}$, we conclude that $u_0 \in C^{1,\alpha}(\{x_N \leqslant 0\})$ (see [15]). Thus, there exists $0 \leqslant \lambda_0 < \infty$ such that

$$u_0(x) = \lambda_0 x_N^- + o\big(|x|\big).$$

By the nondegeneracy of $u$ at every free boundary point (Lemma 5.2) we deduce that $\lambda_0 > 0$.

Now, let $u_{00}$ be a blow-up limit of $u_0$. This is, $u_{00}(x) = \lim \frac{u_0(r_n x)}{r_n}$ with $r_n \to 0$. Then, $u_{00} = \lambda_0 x_N^-$. Since $u_{00}$ is again a minimizer, Lemma 7.3 gives that $\lambda_0 = \lambda^*$.

Let us see that actually $u_0 = \lambda^* x_N^-$. In fact, by applying Lemma 7.4 we see that $|\nabla u_0| \leqslant \lambda^*$ and thus, $u_0 \leqslant \lambda^* x_N^-$. Since the function $w = \lambda^* x_N^-$ is a solution to

$$Tw = \sum_{i,j} b_{ij} w_{x_i x_j} = 0 \quad \text{in } \{x_N < 0\}$$

with $b_{ij}$ as in (7.3) and $u_0$ is a classical solution of the same equation in a neighborhood of any point where $|\nabla u_0| > 0$, and since $u_0 \leqslant w$ in $\{x_N < 0\}$, $u_0 = w$ in $\{x_N = 0\}$, there holds that either $u_0 \equiv w$ or $u_0 < w$. In the latter case, there exists $\delta_0 > 0$ such that

$$(w - u_0)(x) \geqslant -\delta_0 x_N + o\big(|x|\big).$$

But $(w - u_0)(x) = o(|x|)$. Thus, $u_0 \equiv w = \lambda^* x_N^-$.

Finally, since the blow-up limit $u_0$ is independent of the blow-up sequence $\rho_k$, we deduce that

$$u(x) = \lambda^* \big\langle x - x_0, \nu(x_0)\big\rangle^- + o\big(|x - x_0|\big). \qquad \square$$

**Lemma 7.5.** *For $\mathcal{H}^{N-1}$-almost every point $x_0$ in $\partial_{\text{red}}\{u > 0\}$ there holds that*

$$\int\limits_{B_r(x_0) \cap \partial\{u>0\}} \big|q_u - q_u(x_0)\big| \, d\mathcal{H}^{N-1} = o\big(r^{N-1}\big), \quad \textit{as } r \to 0.$$

**Proof.** It follows by Theorem 6.3(3) that $q_u$ is locally integrable in $\mathbb{R}^{N-1}$ and therefore almost every point is a Lebesgue point. $\square$

**Lemma 7.6.** *Let $u$ be a minimizer, then for $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial_{\text{red}}\{u > 0\}$,*

$$q_u(x_0) = g(\lambda^*).$$

**Proof.** Let $u_0$ be as in Theorem 7.1. Now let

$$\xi(x)=\min\left(2\left(1-\frac{|x_N|}{2},1\right)\right)\eta(x_1,\dots,x_{N-1})$$

where $\eta\in C_0^\infty(B_r')$ (where $B_r'$ is a ball $(N-1)$ dimensional with radius $r$) and $\eta\geqslant 0$. Proceeding as in [4, p. 121] and using Lemmas 7.1 and 7.5, we get for almost every point $x_0\in\partial_{\mathrm{red}}\{u>0\}$ and $u_0=\lim_{r\to 0}\frac{u(x_0+rx)}{r}$ that

$$-\int\limits_{B_r\cap\{x_N<0\}} g\left(|\nabla u_0|\right)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx=q_u(x_0)\int\limits_{B_r'}\xi(x',0)\,d\mathcal{H}^{N-1}\quad\forall\xi\in C_0^\infty(B_r),\tag{7.5}$$

where we have assumed that $\nu(x_0)=e_N$.

By Lemma 7.1, $u_0=\lambda^*x_N^-$. Substituting in (7.5) we get

$$g(\lambda^*)\int\limits_{B_r'}\xi(x',0)\,d\mathcal{H}^{N-1}=q_u(x_0)\int\limits_{B_r'}\xi(x',0)\,d\mathcal{H}^{N-1}\quad\forall\xi\in C_0^\infty(B_r).$$

Thus, $q_u(x_0)=g(\lambda^*)$. □

As a corollary we have

**Theorem 7.2.** *Let $u$ be a minimizer, then for $\mathcal{H}^{N-1}$ a.e $x_0\in\partial\{u>0\}$, the following properties hold:*

$$q_u(x_0)=g(\lambda^*)$$

*and*

$$u(x)=\lambda^*\left\langle x-x_0,\nu_u(x_0)\right\rangle^-+o\left(|x-x_0|\right)$$

*where $\lambda^*$ is such that $g(\lambda^*)\lambda^*-G(\lambda^*)=\lambda$.*

**Proof.** The result follows by Lemma 6.2 and by Theorem 7.1. □

## 8. Weak solutions

In this section we introduce the notion of weak solution. The idea, as in [4], is to identify the essential properties that minimizers satisfy and that may be found in applications in which minimization does not take place. For instance, in [16] we study a singular perturbation problem for the operator $\mathcal{L}$ and prove that limits of this singular perturbation problem are weak solutions in the sense of Definition 8.2. In the next section, we will prove that weak solutions have smooth free boundaries. In this way, the regularity results may be applied both to minimizers and to limits of singular perturbation problems.

With these applications in mind, we introduce two notions of weak solution. Definition 8.1 is similar to the one in [4] for the case $\mathcal{L}=\Delta$. On the other hand, as stated before, Definition 8.2 is more suitable for limits of the singular perturbation problem.

Since we want to ask as little as possible for a function $u$ to be a weak solution, some properties already proved for minimizers need a new proof. We keep these proofs as short as possible by sending the reader to the corresponding proofs for minimizers as soon as possible.

One of the main differences between these two definitions of weak solution is that for weak solutions according to Definition 8.1 almost every free boundary point is in the reduced free boundary. Instead, weak solutions according to Definition 8.2 may have an empty reduced boundary (see, for instance, Example 5.8 in [4]).

In the sequel $\lambda^*$ will be a fixed positive constant.

**Definition 8.1** *(Weak solution I).* We call $u$ a weak solution (I), if

(1) $u$ is continuous and nonnegative in $\Omega$ and $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$.
(2) For $D \Subset \Omega$ there are constants $0 < c_{\min} \leqslant C_{\max}$, $\gamma \geqslant 1$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$

$$c_{\min} \leqslant \frac{1}{r}\left( \fint_{B_r(x)} u^\gamma \, dx \right)^{1/\gamma} \leqslant C_{\max}.$$

(3)
$$\mathcal{L}u = g(\lambda^*)\mathcal{H}^{N-1} \lfloor \partial_{\mathrm{red}}\{u > 0\}.$$

i.e.

$$-\int_{\Omega} g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\nabla\varphi \, dx = \int_{\Omega \cap \partial_{\mathrm{red}}\{u>0\}} \varphi g(\lambda^*) \, d\mathcal{H}^{N-1} \quad \forall \varphi \in C_0^\infty(\Omega).$$

(4)
$$\limsup_{\substack{x \to x_0 \\ u(x)>0}} \big|\nabla u(x)\big| \leqslant \lambda^*, \quad \text{for every } x_0 \in \Omega \cap \partial\{u > 0\}.$$

**Definition 8.2** *(Weak solution II).* We call $u$ a weak solution (II), if

(1) $u$ is continuous and nonnegative in $\Omega$ and $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$.
(2) For $D \Subset \Omega$ there are constants $0 < c_{\min} \leqslant C_{\max}$, $\gamma \geqslant 1$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$

$$c_{\min} \leqslant \frac{1}{r}\left( \fint_{B_r(x)} u^\gamma \, dx \right)^{1/\gamma} \leqslant C_{\max}.$$

(3) For $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$, $u$ has the asymptotic development

$$u(x) = \lambda^*\big\langle x - x_0, \nu(x_0)\big\rangle^- + o\big(|x - x_0|\big)$$

where $\nu(x_0)$ is the unit exterior normal to $\partial\{u > 0\}$ at $x_0$ in the measure theoretic sense.
(4)

$$\limsup_{\substack{x \to x_0 \\ u(x)>0}} \big|\nabla u(x)\big| \leqslant \lambda^*, \quad \text{for every } x_0 \in \Omega \cap \partial\{u > 0\}.$$

(5) For any ball $B \subset \{u = 0\}$ touching $\Omega \cap \partial\{u > 0\}$ at $x_0$ we have

$$\limsup_{x \to x_0} \frac{u(x)}{\operatorname{dist}(x, B)} \geqslant \lambda^*.$$

**Lemma 8.1.** *If $u$ satisfies the hypothesis* (1) *of Definitions* 8.1 *and* 8.2 *then $u$ is in $W^{1,G}_{\mathrm{loc}}(\Omega)$ and $\Lambda := \mathcal{L}u$ is a nonnegative Radon measure with support in $\Omega \cap \partial\{u > 0\}$ (in particular, $u$ is an $\mathcal{L}$-subsolution in $\Omega$).*

**Proof.** Since $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$, then $u$ is in $C^{1,\alpha}$ in $\Omega \cap \{u > 0\}$. For $s > 0$, take $v = (u - s)^+$. Let $\eta \in C_0^\infty(\Omega)$ with $0 \leqslant \eta \leqslant 1$. We have

$$\begin{aligned} 0 &= \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla\big(\eta^{g_0+1} v\big)\, dx \\ &= \int_{\Omega \cap \{u>s\}} \eta^{g_0+1} g\big(|\nabla u|\big)|\nabla u| + (g_0 + 1) \int_\Omega \eta^{g_0} v \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla \eta\, dx. \end{aligned}$$

Therefore,

$$\int_{\Omega \cap \{u>s\}} \eta^{g_0+1} g\big(|\nabla u|\big)|\nabla u|\, dx \leqslant (g_0 + 1) \int_{\Omega \cap \{u>s\}} g\big(|\nabla u|\big) v |\eta|^{g_0} |\nabla \eta|\, dx \tag{8.1}$$

by ($\tilde{g}$3), ($\widetilde{G}$1) and ($\tilde{g}$4) we obtain

$$\begin{aligned} g\big(|\nabla u|\big)|\eta|^{g_0}|v||\nabla \eta| &\leqslant \varepsilon \widetilde{G}\big(g\big(|\nabla u|\big)|\eta|^{g_0}\big) + C(\varepsilon) G\big(|v||\nabla \eta|\big) \\ &\leqslant C\varepsilon \eta^{g_0+1} \widetilde{G}\big(g\big(|\nabla u|\big)\big) + C(\varepsilon) G\big(|v||\nabla \eta|\big) \\ &\leqslant C\varepsilon G\big(|\nabla u|\big)\eta^{g_0+1} + C(\varepsilon) G\big(|v||\nabla \eta|\big). \end{aligned}$$

Then, using (g3), (8.1) and choosing $\varepsilon$ small enough, we have that

$$\int_{\Omega \cap \{u>s\}} \eta^{g_0+1} G\big(|\nabla u|\big)\, dx \leqslant C \int_{\Omega \cap \{u>s\}} G\big(|v||\nabla \eta|\big)\, dx \leqslant C \int_\Omega G\big(|u||\nabla \eta|\big)\, dx.$$

Then, letting $s \to 0$ yields the first assertion.

To prove the second part, take $\xi \in C_0^\infty(\Omega)$ nonnegative, $\varepsilon > 0$ and $v = \max(\min(1, 2 - \frac{u}{\varepsilon}), 0)$. As $\mathcal{L}u = 0$ in $\{u > 0\}$, we have that

$$\begin{aligned} \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla \xi\, dx &= \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla\big(\xi(1 - v)\big)\, dx + \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla(\xi v)\, dx \\ &= \int_\Omega \frac{g(|\nabla u|)}{|\nabla u|} \nabla u \nabla(\xi v)\, dx \end{aligned}$$

$$= \int\limits_{\Omega\cap\{0<u<2\varepsilon\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \nabla(\xi v)\,dx$$

$$= \int\limits_{\Omega\cap\{\varepsilon<u<2\varepsilon\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \nabla\left(\xi\left(2-\frac{u}{\varepsilon}\right)\right)dx$$

$$+ \int\limits_{\Omega\cap\{0<u<\varepsilon\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \nabla\xi\,dx$$

$$\leqslant 2\int\limits_{\Omega\cap\{\varepsilon<u<2\varepsilon\}} g\big(|\nabla u|\big)|\nabla\xi|\,dx + \int\limits_{\Omega\cap\{0<u<\varepsilon\}} g\big(|\nabla u|\big)|\nabla\xi|\,dx$$

$$\leqslant 2\int\limits_{\Omega\cap\{0<u<2\varepsilon\}} g\big(|\nabla u|\big)|\nabla\xi|\,dx,$$

which tends to zero when $\varepsilon \to 0$ yielding the desired result. □

Now we will prove as in Theorem 5.1, the density property of the set $\{u > 0\}$ at free boundary points. It is not true in general, for weak solutions satisfying only properties (1) and (2) of Definitions 8.1 or 8.2 that the set $\{u = 0\}$ has positive density at $\mathcal{H}^{N-1}$-almost every free boundary point (see examples in [4]).

**Theorem 8.1.** *For any domain $D \Subset \Omega$ there exists a constant $c$, with $0 < c < 1$ depending on $N, \gamma, g_0, \delta, D$, $c_{\min}$ and $C_{\max}$, such that, for any function $u$ satisfying* (1) *and* (2) *of Definitions* 8.1 *and* 8.2 *and for every $B_r \subset D$, centered at the free boundary we have*

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} \geqslant c.$$

**Proof.** The proof follows as in Theorem 5.1, the only difference here is that, instead of using Lemmas 5.1 and 5.2, we use property (2) of Definitions 8.1 and 8.2. □

**Remark 8.1.** Now, by Remark 5.1 we have that the free boundary has Lebesgue measure zero. Moreover, for every $D \Subset \Omega$, the intersection $\partial\{u > 0\} \cap D$ has Hausdorff dimension less than $N$.

**Lemma 8.2.** *If $u$ satisfies hypotheses* (1) *and* (2) *of Definitions* 8.1 *and* 8.2 *then*

(1) *$u$ is Lipschitz and for any domain $D \Subset \Omega$, the Lipschitz constant depends only on $N, \gamma, g_0, \delta, \operatorname{dist}(D, \partial\Omega)$ and $C_{\max}$, provided $D$ contains a free boundary point.*
(2) *For any domain $D \Subset \Omega$ there exist constants $c$, $C$ depending on $N, \gamma, g_0, \delta, D$, $c_{\min}$ and $C_{\max}$, such that, for every $B_r \subset D$ centered at the free boundary we have*

$$cr^{N-1} \leqslant \int\limits_{B_r} d\Lambda \leqslant Cr^{N-1}.$$

**Proof.** The proof of (1) is similar to the one in Theorem 4.2. The only change that we have to make here is the following, instead of using Lemma 4.2 we have to use property (2) of Definitions 8.1 and 8.2. We give the proof for the readers convenience.

Let $d(x) = \operatorname{dist}(x, \Omega \cap \partial\{u > 0\})$. First, take $x$ such that $d(x) < \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$. Let $y \in \partial\{u > 0\} \cap \partial B_{d(x)}(x)$. As $u > 0$ in $B_{d(x)}(x)$, $\mathcal{L}u = 0$ in that ball and $u$ is an $\mathcal{L}$-subsolution in $B_{3d(x)}(y)$. By using the gradient estimates and Harnack's inequality of [15] (see Lemma 2.7) and property (2) of Definitions 8.1 and 8.2 we have

$$\left|\nabla u(x)\right| \leqslant C\frac{1}{d(x)}\sup_{B_{d(x)}(x)} u \leqslant C\frac{1}{d(x)}\sup_{B_{2d(x)}(y)} u \leqslant C\frac{1}{d(x)}\left(\;\fint_{B_{3d(x)}(y)} u^\gamma\,dx\right)^{1/\gamma} \leqslant CC_{\max}.$$

So, the result follows in the case $d(x) < \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$.

Let $r_1$ be such that $\operatorname{dist}(x, \partial\Omega) \geqslant r_1 > 0$ $\forall x \in D$, take $D'$, satisfying $D \Subset D' \Subset \Omega$ given by

$$D' = \left\{x \in \Omega \;\middle|\; \operatorname{dist}(x, D) < r_1/2\right\}.$$

Let $x \in D$. If $d(x) \leqslant \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$ we have proved that $|\nabla u(x)| \leqslant C$.

If $d(x) > \frac{1}{5}\operatorname{dist}(x, \partial\Omega)$, $u > 0$ in $B_{\frac{r_1}{5}}(x)$ and $B_{\frac{r_1}{5}}(x) \subset D'$ so that $|\nabla u(x)| \leqslant \frac{C}{r_1}\|u\|_{L^\infty(D')}$.

To prove the second part of (1), consider now a connected domain $D$ that contains a free boundary point and let $D'$ as in the previous paragraph. Let us see that $\|u\|_{L^\infty(D')}$ is bounded by a constant that depends only on $N, \gamma, D, r_1, \lambda, \delta$, and $g_0$. Let $r_0 = \frac{r_1}{4}$ and $x_0 \in D$. Since $D'$ is connected and not contained in $\{u > 0\} \cap \Omega$, there exists $x_1, \dots, x_k \in D'$ such that $x_j \in B_{\frac{r_0}{2}}(x_{j-1})$, $j = 1, \dots, k$, $B_{r_0}(x_j) \subset \{u > 0\}$, $j = 0, \dots, k-1$ and $B_{r_0}(x_k) \not\subset \{u > 0\}$. Let $y_0 \in \partial\{u > 0\} \cap B_{r_0}(x_k)$. As $u$ is an $\mathcal{L}$-subsolution, by Theorem 1.2 in [15] there exists $C$ depending on $N, \gamma, \delta, g_0$ such that

$$u(x_k) \leqslant C\left(\;\fint_{B_{2r_0}(y_0)} u^\gamma\,dx\right)^{1/\gamma} \leqslant CC_{\max}r_0,$$

where in the last inequality we have used property (2) of Definitions 8.1 and 8.2. By Harnack's inequality in [15] we have $u(x_{j+1}) \geqslant cu(x_j)$. Inductively we obtain $u(x_0) \leqslant Cr_0$ $\forall x_0 \in D'$. Therefore, the supremum of $u$ over $D'$ can be estimated by a constant depending only on $N, \gamma, r_1, \lambda, \delta$, and $g_0$.

In order to prove (2) we use that Lemma 6.1 holds if $u_k$ is a sequence of functions satisfying properties (1) and (2) of Definitions 8.1 and 8.2 with the same constants $c_{\min}$ and $c_{\max}$. Then, the rest of the proof follows as in Theorem 6.2. □

**Remark 8.2.** Now, we are under the conditions used in the proof of Theorem 6.3 and therefore this result applies to functions $u$ satisfying properties (1) and (2) of Definition 8.1 and 8.2. That is, $\Omega \cap \partial\{u > 0\}$ has finite perimeter and there exists a Borel function $q_u$ defined on $\Omega \cap \partial\{u > 0\}$ such that $\mathcal{L}u = q_u\mathcal{H}^{N-1}\lfloor\partial\{u > 0\}$.

As $u$ satisfies the conclusions of Theorem 6.3 then Remark 6.1 also holds. We also have that any blow-up sequence satisfies the properties of Lemma 7.1.

Moreover, we have the following result that holds at points $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ that are Lebesgue points of the function $q_u$ and are such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^{N-1}(\partial\{u > 0\} \cap B(x_0, r))}{\mathcal{H}^{N-1}(B'(x_0, r))} \leqslant 1. \tag{8.2}$$

(Here $B'(x_0, r) = \{x' \in \mathbb{R}^{N-1} \mid |x'| < r\}$.)

Recall that $\mathcal{H}^{N-1}$-a.e. point in $\partial_{\mathrm{red}}\{u > 0\}$ satisfies (8.2) (see Theorem 3.1.21 in [9]).

**Lemma 8.3.** *If $u$ is a function satisfying properties* (1), (2) *and* (3) *of Definition* 8.1 *or* 8.2 *we have that $q_u(x_0) = g(\lambda^*)$ for $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$.*

**Proof.** Clearly, we only have to prove the statement for weak solutions (II).

If $u$ satisfies (3) of Definition 8.2, take $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ such that

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|).$$

Take $\rho_k \to 0$ and $u_k(x) = \frac{1}{\rho_k} u(x_0 + \rho_k x)$. If $\xi \in C_0^\infty(\Omega)$ we have

$$-\int_{\{u>0\}} g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\xi\,dx = \int_{\partial\{u>0\}} q_u(x)\xi\,d\mathcal{H}^{N-1},$$

and if we replace $\xi$ by $\xi_k(x) = \rho_k \xi(\frac{x-x_0}{\rho_k})$ with $\xi \in C_0^\infty(B_R)$, $k \geqslant k_0$ and we change variables we obtain

$$-\int_{\{u_k>0\}} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi\,dx = \int_{\partial\{u_k>0\}} q_u(x_0 + \rho_k x)\xi\,d\mathcal{H}^{N-1}.$$

Now, recall that for a subsequence, $\chi_{\{u_k>0\}} \to \chi_{\{x_N<0\}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} \rightharpoonup g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. Thus,

$$\int_{\{u_k>0\}} g(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi\,dx \to \int_{\{x_N<0\}} g(|\nabla u_0|)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi\,dx.$$

On the other hand, $\partial\{u_k > 0\} \to \{x_N = 0\}$ locally in Hausdorff distance. Then, if $x_0$ is a Lebesgue point of $q_u$ satisfying (8.2),

$$\int_{\partial\{u_k>0\}} q_u(x_0 + \rho_k x)\xi\,d\mathcal{H}^{N-1} \to q_u(x_0)\int_{\{x_N=0\}} \xi\,d\mathcal{H}^{N-1}. \tag{8.3}$$

As $\nabla u_0 = -\lambda^* e_N \chi_{\{x_N<0\}}$, we deduce that for almost every point $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$, $q_u(x_0) = g(\lambda^*)$. □

Now we prove the asymptotic development for weak solutions satisfying Definition 8.1.

**Lemma 8.4.** *If $u$ satisfies* (1), (2), (3) *and* (4) *of Definition* 8.1*, then for $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ satisfying* (8.2)*, $u$ has the following asymptotic development*

$$u(x) = \lambda^* \langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|) \tag{8.4}$$

*where $\nu(x_0)$ is the unit outer normal to the free boundary at $x_0$.*

**Proof.** Let $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ and let $\rho_k \to 0$. Let $u_k(x) = \frac{1}{\rho_k} u(x_0 + \rho_k x)$ be a blow-up sequence (observe that $u_k$ is again a weak solution in the rescaled domain). Assume that $u_k \to u_0$ uniformly on compact subsets of $\mathbb{R}^N$. Also assume that $\nu(x_0) = e_N$. As in the proof of Theorem 7.1 we deduce that

$$u_0 \geqslant 0 \quad \text{in } \{x_N < 0\},$$
$$u_0 = 0 \quad \text{in } \{x_N \geqslant 0\}.$$

Let us see that $u_0 > 0$ in $\{x_N < 0\}$. To this end, let $D \Subset \{x_N < 0\}$ and let $\xi \in C_0^\infty(D)$. For $k$ large enough,

$$-\int_{\{u_k>0\}} g(|\nabla u_k|) \frac{\nabla u_k}{|\nabla u_k|} \nabla \xi \, dx = \int_{\partial_{\mathrm{red}}\{u_k>0\}} g(\lambda^*) \xi(x) \, d\mathcal{H}^{N-1}. \tag{8.5}$$

As in [4, p. 121], we have that for every $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ satisfying (8.2),

$$\mathcal{H}^{N-1}(\partial\{u_k > 0\} \cap D) \to 0 \quad \text{as } k \to \infty.$$

Thus, the right-hand side of (8.5) goes to zero as $k \to \infty$. Since the left-hand side goes to

$$-\int g(|\nabla u_0|) \frac{\nabla u_0}{|\nabla u_0|} \nabla \xi \, dx$$

we deduce that $\mathcal{L} u_0 = 0$ in $\{x_N < 0\}$. Thus, $u_0 > 0$ in $\{x_N < 0\}$.

As in Theorem 7.1 we have that there exists $0 < \lambda_0 < \infty$ such that

$$u_0(x) = \lambda_0 x_N^- + o(|x|).$$

By property (2) of Lemma 7.1 we have that

$$\chi_{\{u_k>0\}} \to \chi_{\{x_N<0\}} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } k \to \infty.$$

Let now $\xi \in C_0^\infty(\mathbb{R}^N)$ in (8.5). Passing to the limit as $k \to \infty$ and using Lemma 7.1(1) we get

$$-\int_{\{x_N<0\}} g(|\nabla u_0|) \frac{\nabla u_0}{|\nabla u_0|} \nabla \xi \, dx = \int_{\{x_N=0\}} g(\lambda^*) \xi(x) \, d\mathcal{H}^{N-1}.$$

Replacing $\xi$ by $r\xi(x/r)$ with $r \to 0$, using the fact that $\frac{1}{r}u_0(rx) \to \lambda_0 x_N^-$ uniformly on compact sets of $\mathbb{R}^N$, changing variables and passing to the limit we get

$$g(\lambda_0) \int\limits_{\{x_N<0\}} \xi_N \, dx = g(\lambda^*) \int\limits_{\{x_N=0\}} \xi(x) \, d\mathcal{H}^{N-1}.$$

Thus, $\lambda_0 = \lambda^*$.

At this point we proceed as in Theorem 7.1 to deduce that actually $u_0(x) = \lambda^* x_N^-$ (observe that here we are using property (4) of Definition 8.1). As the blow up limit $u_0$ is independent of the blow up sequence $\rho_k$ we conclude that $u$ has the asymptotic development (8.4). □

Now we prove the property that we mentioned in the introduction to this section. The following lemma only holds for weak solutions satisfying Definition 8.1.

**Lemma 8.5.** *If $u$ satisfies* (1), (2) *and* (3) *of Definition* 8.1*,*

(1) $\mathcal{H}^{N-1}(\partial\{u>0\} \setminus \partial_{\mathrm{red}}\{u>0\}) = 0$.
(2) $|D \cap \{u=0\}| > 0$ *for every open set $D \subset \Omega$ containing a point of* $\{u=0\}$.
(3) *For any ball $B$ in $\{u=0\}$ touching $\Omega \cap \partial\{u>0\}$ at $x_0$, there holds that*

$$\limsup_{x\to x_0} \frac{u(x)}{\operatorname{dist}(x,B)} \geqslant \lambda^*. \tag{8.6}$$

**Proof.** By [9, 4.5.6 (3)] we have

$$|\mu_u|\big(B_r(x_0)\big) = o\big(r^{N-1}\big) \quad \text{for } r \to 0 \tag{8.7}$$

for $\mathcal{H}^{N-1}$ almost all points $x_0 \in \partial\{u>0\} \setminus \partial_{\mathrm{red}}\{u>0\}$. (Recall that $\mu_u = -\nabla\chi_{\{u>0\}}$.) Assume there exists $x_0 \in \partial\{u>0\} \setminus \partial_{\mathrm{red}}\{u>0\}$ satisfying (8.7). Therefore, if $u_0$ is a blow-up limit with respect to balls $B_{\rho_k}(x_0)$, we obtain for $\xi \in C_0^\infty(B_1)$ that

$$\begin{aligned}
-\int\limits_{\mathbb{R}^N} g\big(|\nabla u_0|\big)\frac{\nabla u_0}{|\nabla u_0|}\nabla\xi \, dx \leftarrow -\int\limits_{\mathbb{R}^N} g\big(|\nabla u_k|\big)\frac{\nabla u_k}{|\nabla u_k|}\nabla\xi \, dx \\
&= \rho_k^{1-N} g(\lambda^*) \int\limits_{\partial_{\mathrm{red}}\{u>0\}\cap B_{\rho_k}(x_0)} \xi\left(\frac{y-x_0}{\rho_k}\right) d\mathcal{H}^{N-1} \\
&= \rho_k^{1-N} g(\lambda^*) \int\limits_{B_{\rho_k}(x_0)} \xi\left(\frac{y-x_0}{\rho_k}\right) d|\mu_u|(x) \\
&\leqslant C\rho_k^{1-N} |\mu_u|\big(B_{\rho_k}(x_0)\big) \to 0,
\end{aligned}$$

therefore $\mathcal{L}u_0 = 0$. Since $u_0(0) = 0$, we must have $u_0 = 0$, but this contradicts the nondegeneracy property (2) of Definition 8.1. Therefore (1) holds.

To prove (2), suppose that $\chi_{\{u>0\}} = 1$ almost everywhere in $D$, hence the reduced boundary must be outside of $D$. Then by Definition 8.1(3) the function $\mathcal{L}u = 0$ in $D$, and therefore $u$ is positive. Hence $D \cap \{u = 0\} = \emptyset$.

In order to prove (3), let $l$ be the finite limit on the left of (8.6), and $y_k \to x_0$ with $u(y_k) > 0$ and

$$\frac{u(y_k)}{d_k} \to l, \quad d_k = \operatorname{dist}(y_k, B).$$

Consider the blow-up sequence $u_k$ with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k - y_k| = d_k$, and choose a subsequence with blow-up limit $u_0$, such that

$$e := \lim_{k\to\infty} \frac{x_k - y_k}{d_k}$$

exists. Then by construction, since $l > 0$ by nondegenaracy, $u_0(-e) = l$, and $u_0(x) \leqslant -l\langle x, e\rangle$ for $x \cdot e \leqslant 0$, $u_0(x) = 0$ for $x \cdot e \geqslant 0$. Both, $u_0$ and $l\langle x, e\rangle^-$ are $\mathcal{L}$ solutions in $\{u_0 > 0\}$, and coincide in $-e$. Since $l > 0$, and $|\nabla u_0| > l/2$ in a neighborhood of $-e$, we have that $\mathcal{L}$ is uniformly elliptic there. Then we can apply the strong maximum principle to conclude that they must coincide in that neighborhood of $-e$. By a continuation argument, we have that $u_0 = l\langle x, e\rangle^-$.

By the Representation Theorem, $\forall \varphi \in C_0^\infty(B_1)$, $\varphi \geqslant 0$

$$\begin{aligned}\int_{\partial\{u_k>0\}} \varphi q_{u_k}\, d\mathcal{H}^{N-1} &= -\int_{\mathbb{R}^N} g\big(|\nabla u_k|\big)\frac{\nabla u_k}{|\nabla u_k|}\nabla\varphi\, dx \to -\int_{\mathbb{R}^N} g\big(|\nabla u_0|\big)\frac{\nabla u_0}{|\nabla u_0|}\nabla\varphi\, dx \\ &= g(l)\int_{\{\langle x,e\rangle=0\}} \varphi\, d\mathcal{H}^{N-1} \end{aligned} \tag{8.8}$$

and

$$\begin{aligned}\int_{\partial\{u_k>0\}} \varphi\, d\mathcal{H}^{N-1} &\geqslant \int_{\partial_{\text{red}}\{u_k>0\}} \varphi\langle e.\nu_{u_k}\rangle\, d\mathcal{H}^{N-1} \\ &= \int \varphi e.d\mu_{u_k} = \int_{\{u_k>0\}} \partial_e\varphi\, dx \to \int_{\{\langle x,e\rangle<0\}} \partial_e\varphi\, dx \\ &= \int_{\{\langle x,e\rangle=0\}} \varphi\, d\mathcal{H}^{N-1}. \end{aligned} \tag{8.9}$$

Therefore, for weak solutions of type (I) and (II) we have

$$g(l) \geqslant \liminf_{x\to x_0} q_u(x).$$

Now, if $u$ is a weak solution of type I we have, that $q_u(x) = g(\lambda^*)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \cap \partial\{u > 0\}$. Thus, $g(l) \geqslant g(\lambda^*)$ and $l \geqslant \lambda^*$. □

We then conclude

**Theorem 8.2.** *If $u$ satisfies* (1), (2), (3) *and* (4) *of Definition* 8.1*, then for $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial\{u > 0\}$, $u$ has the asymptotic development* (8.4).

**Proof.** It follows by Remark 8.2 and Lemmas 8.4 and 8.5. □

**Remark 8.3.** Now we have that with the additional hypothesis (4), weak solutions (I) satisfy the same properties that we proved in the previous section for minimizers (with the only difference that in (4) we have a less than or equal instead of an equal). The extra hypothesis (5), in the definition of weak solution (II) (which always holds, by Lemma 8.5, for weak solutions (I)) is used in key steps of the proof of the regularity of the free boundary. On the other hand, observe that minimizers have the asymptotic development (8.4) at every point in their reduced free boundary, but we only proved that this development holds at almost every point of $\partial_{\mathrm{red}}\{u > 0\}$ when $u$ is a weak solution.

## 9. Regularity of the free boundary

In this section we prove the regularity of the free boundary of a weak solution $u$ in a neighborhood of every "flat" free boundary point. In particular, we prove the regularity in a neighborhood of every point in $\partial_{\mathrm{red}}\{u > 0\}$ where $u$ has the asymptotic development (8.4). Then, if $u$ is a minimizer, $\partial_{\mathrm{red}}\{u > 0\}$ is smooth and the remainder of the free boundary has $\mathcal{H}^{N-1}$-measure zero.

We will recall some definitions and we will point out the only significant differences with the proofs in [7] for the case $G(t) = t^p$. The rest of the proof of the regularity then follows as Sections 6, 7, 8 and 9 of [7]. The main differences with [7] come from the fact that we do not assume the locally uniform positive density of the set $\{u \equiv 0\}$ at the free boundary. This is a property satisfied by minimizers that is not know to hold, in principle, for weak solutions that appear in a different context. This uniform density property implies, in particular, that $\mathcal{H}^{N-1}$-almost every point in the free boundary belongs to the reduced free boundary and this is a very strong assumption that we do not want to make.

**Remark 9.1.** In [7], Sections 6, 7 and 8 the authors use the fact that when $|\nabla u| \geqslant c$, $u$ satisfies a linear nondivergence uniformly elliptic equation, $Tu = 0$. In our case we have that when $|\nabla u| \geqslant c$, $u$ is a solution of the equation defined in Remark 7.1. As in those sections the authors only use the fact that this operator is linear and uniformly elliptic, then the results of those sections in [7] extend to our case without any change.

For the reader convenience, we will sketch here the proof of the regularity of the free boundary by a series of steps and we will write down the proof in those cases in which we had to make modifications.

### *9.1. Flatness and nondegeneracy of the gradient*

**Definition 9.1** *(Flat free boundary points).* Let $0 < \sigma_+, \sigma_- \leqslant 1$ and $\tau > 0$. We say that $u$ is of class

$$F(\sigma_+, \sigma_-; \tau) \quad \text{in } B_\rho = B_\rho(0)$$

if

(1) $0 \in \partial\{u > 0\}$ and

$$\begin{aligned} &u = 0 && \text{for } x_N \geqslant \sigma_+ \rho, \\ &u(x) \geqslant -\lambda^*(x_N + \sigma_- \rho) && \text{for } x_N \leqslant -\sigma_- \rho. \end{aligned}$$

(2) $|\nabla u| \leqslant \lambda^*(1+\tau)$ in $B_\rho$.

If the origin is replaced by $x_0$ and the direction $e_N$ by the unit vector $\nu$ we say that $u$ is of class $F(\sigma_+, \sigma_-; \tau)$ in $B_\rho(x_0)$ in direction $\nu$.

It is in the proof of the following theorems where we strongly use the extra hypothesis (5) of weak solution (II) (which is always satisfied by weak solutions (I)). For the details see Section 6 in [7].

**Theorem 9.1.** *There exist* $\sigma_0 > 0$ *and* $C_0 > 0$ *such that*

$$u \in F(\sigma, 1; \sigma) \quad \text{in } B_1 \quad \text{implies} \quad u \in F(2\sigma, C_0\sigma; \sigma) \quad \text{in } B_{1/2}$$

*for* $0 < \sigma < \sigma_0$.

**Proof.** It follows as in the proof of Theorem 6.3 in [7] by Remark 9.1. □

**Theorem 9.2.** *For every* $\delta > 0$ *there exist* $\sigma_\delta > 0$ *and* $C_\delta > 0$ *such that*

$$u \in F(\sigma, 1; \sigma) \quad \text{in } B_1 \quad \text{implies} \quad |\nabla u| \geqslant \lambda^* - \delta \quad \text{in } B_{1/2} \cap \{x_N \leqslant -C_\delta \sigma\}$$

*for* $0 < \sigma < \sigma_\delta$.

**Proof.** It follows as in the proof of Theorem 6.4 in [7] by Remark 9.1. □

*9.2. Nonhomogeneous blow-up*

**Lemma 9.1.** *Let* $u_k \in F(\sigma_k, \sigma_k; \tau_k) \in B_{\rho_k}$ *with* $\sigma_k \to 0$, $\tau_k \sigma_k^{-2} \to 0$. *For* $y \in B_1'$, *set*

$$f_k^+(y) = \sup\bigl\{h\colon\ (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\bigr\},$$
$$f_k^-(y) = \inf\bigl\{h\colon\ (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\bigr\}.$$

*Then, for a subsequence,*

(1) $f(y) = \limsup_{\substack{z \to y \\ k \to \infty}} f_k^+(z) = \liminf_{\substack{z \to y \\ k \to \infty}} f_k^-(z)$ *for all* $y \in B_1'$.
*Further,* $f_k^+ \to f$, $f_k^- \to f$ *uniformly,* $f(0) = 0$, $|f| \leqslant 1$ *and* $f$ *is continuous.*
(2) $f$ *is subharmonic.*

**Proof.** (1) is the analogue of Lemma 5.3 in [5]. The proof is based on Theorem 6.3 and is identical to the one of Lemma 7.3 in [4].

The proof of (2) is a little bit different since here we do not have in general that $q_{u_k}(x) = g(\lambda^*)$ $\mathcal{H}^{N-1}$-a.e. point in $\partial\{u_k > 0\}$. Instead, we have that this equality holds for $\mathcal{H}^{N-1}$-a.e. point in $\partial_{\mathrm{red}}\{u_k > 0\}$.

We may assume by replacing $u_k$ by $\widetilde{u}_k = \frac{1}{\rho_k} u_k(\rho_k x)$, that $\rho_k = 1$. Let us assume, by contradiction that there is a ball $B'_\rho(y_0) \subset B'_1$ and a harmonic function $g$ in a neighborhood of this ball, such that

$$g > f \quad \text{on } \partial B'_\rho(y_0) \quad \text{and} \quad f(y_0) > g(y_0).$$

Let

$$Z^+ = \bigl\{x \in B_1 \bigm| x = (y, h),\ y \in B'_\rho(y_0),\ h > \sigma_k g(y)\bigr\},$$

and similarly $Z_0$ and $Z^-$. As in Lemma 7.5 in [4], using the same test function and the Representation Theorem 6.3 (see Remark 8.2) we arrive at

$$\int_{\{u_k>0\}\cap Z_0} g\bigl(|\nabla u_k|\bigr)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nu \, d\mathcal{H}^{N-1} = \int_{\partial\{u_k>0\}\cap Z^+} q_{u_k}(x)\, d\mathcal{H}^{N-1}. \tag{9.1}$$

As $u_k \in F(\sigma_k, \sigma_k, \tau_k)$ we have that $|\nabla u_k| \leqslant \lambda^*(1 + \tau_k)$ and, by Lemma 8.3, there holds that $q_{u_k}(x) = g(\lambda^*)$ for $\mathcal{H}^{N-1}$-a.e. point in $\partial_{\mathrm{red}}\{u_k > 0\}$. Then, by (9.1) we have

$$g(\lambda^*)\mathcal{H}^{N-1}\bigl(\partial_{\mathrm{red}}\{u_k > 0\} \cap Z^+\bigr) \leqslant g\bigl(\lambda^*(1 + \tau_k)\bigr)\mathcal{H}^{N-1}\bigl(\{u_k > 0\} \cap Z_0\bigr). \tag{9.2}$$

On the other hand, by the excess area estimate in Lemma 7.5 in [4] we have that

$$\mathcal{H}^{N-1}(\partial_{\mathrm{red}} E_k \cap Z) \geqslant \mathcal{H}^{N-1}(Z_0) + c\sigma_k^2,$$

where $Z = B'_\rho(y_0) \times \mathbb{R}$ and $E_k = \{u_k > 0\} \cup Z^-$.

We also have

$$\mathcal{H}^{N-1}(\partial_{\mathrm{red}} E_k \cap Z) \leqslant \mathcal{H}^{N-1}\bigl(Z^+ \cap \partial_{\mathrm{red}}\{u_k > 0\}\bigr) + \mathcal{H}^{N-1}\bigl(Z_0 \cap \{u_k = 0\}\bigr).$$

Using these two inequalities and the fact that $\mathcal{H}^{N-1}(Z_0 \cap \partial\{u_k > 0\}) = 0$ (if this is not true we replace $g$ by $g + c_0$ for a small constant $c_0$) we have that

$$\mathcal{H}^{N-1}\bigl(\partial_{\mathrm{red}}\{u_k > 0\} \cap Z^+\bigr) \geqslant \mathcal{H}^{N-1}\bigl(Z_0 \cap \{u_k > 0\}\bigr) + c\sigma_k^2. \tag{9.3}$$

Finally by (9.2) and (9.3) we have that

$$g(\lambda^*)\bigl[\mathcal{H}^{N-1}\bigl(\{u_k > 0\} \cap Z_0\bigr) + c\sigma_k^2\bigr] \leqslant g\bigl(\lambda^*(1 + \tau_k)\bigr)\mathcal{H}^{N-1}\bigl(\{u_k > 0\} \cap Z_0\bigr).$$

Therefore, for some positive constant $c$ we have

$$c \leqslant \frac{g(\lambda^*(1 + \tau_k)) - g(\lambda^*)}{\sigma_k^2}$$

and this contradicts the fact that $\frac{\tau_k}{\sigma_k^2} \to 0$ as $k \to \infty$. $\square$

**Lemma 9.2.** *There exists a positive constant $C = C(N)$ such that, for any $y \in B'_{r/2}$,*

$$\int_0^{1/4} \frac{1}{r^2}\biggl( \fint_{\partial B'_r(y)} f - f(y)\biggr) \leqslant C_1.$$

**Proof.** It follows as in Lemma 8.3 at [7], by Remark 9.1 and Theorem 9.2. □

With these two lemmas we have by Lemmas 7.7 and 7.8 in [4],

**Lemma 9.3.**

(1) *$f$ is Lipschitz in $\bar{B}'_{1/4}$ with Lipschitz constant depending on $C_1$ and $N$.*
(2) *There exists a constant $C = C(N) > 0$ and for $0 < \theta < 1$, there exists $c_\theta = c(\theta, N) > 0$, such that we can find a ball $B'_r$ and a vector $l \in \mathbb{R}^{N-1}$ with*

$$c_\theta \leqslant r \leqslant \theta, \quad |l| \leqslant C, \quad \text{and} \quad f(y) \leqslant l.y + \frac{\theta}{2} r \quad \text{for } |y| \leqslant r.$$

And as in Lemma 7.9 in [4] we have

**Lemma 9.4.** *Let $\theta$, $C$, $c_\theta$ as in Lemma 9.3. There exists a positive constants $\sigma_\theta$, such that*

$$u \in F(\sigma, \sigma; \tau) \quad \text{in } B_\rho \text{ in direction } \nu \tag{9.4}$$

*with $\sigma \leqslant \sigma_\theta$, $\tau \leqslant \sigma_\theta \sigma^2$, implies*

$$u \in F(\theta\sigma, 1; \tau) \quad \text{in } B_{\bar\rho} \text{ in direction } \bar\nu$$

*for some $\bar\rho$ and $\bar\nu$ with $c_\theta \rho \leqslant \bar\rho \leqslant \theta\rho$ and $|\bar\nu - \nu| \leqslant C\sigma$, where $\sigma_\theta = \sigma_\theta(\theta, N)$.*

**Lemma 9.5.** *Given $0 < \theta < 1$, there exist positive constants $\sigma_\theta$, $c_\theta$ and $C$ such that*

$$u \in F(\sigma, 1; \tau) \quad \text{in } B_\rho \text{ in direction } \nu \tag{9.5}$$

*with $\sigma \leqslant \sigma_\theta$ and $\tau \leqslant \sigma_\theta \sigma^2$, then*

$$u \in F\bigl(\theta\sigma, \theta\sigma; \theta^2\tau\bigr) \quad \text{in } B_{\bar\rho} \text{ in direction } \bar\nu$$

*for some $\bar\rho$ and $\bar\nu$ with $c_\theta \rho \leqslant \bar\rho \leqslant \frac{1}{4}\rho$ and $|\bar\nu - \nu| \leqslant C\sigma$, where $c_\theta = c_\theta(\theta, N)$, $C = C(N, \delta, g_0)$, $\sigma_\theta = \sigma_\theta(\theta, N)$.*

**Proof.** We obtain the improvement of the value $\tau$ inductively. Assume that $\rho = 1$. If $\sigma_\theta$ is small enough, we can apply Theorem 9.1 and obtain

$$u \in F(C\sigma, C\sigma; \tau) \quad \text{in } B_{1/2} \text{ in direction } \nu.$$

Then for $0 < \theta_1 \leqslant \frac{1}{2}$ we can apply Lemma 9.4, if again $\sigma_\theta$ is small, and we obtain

$$u \in F(C\theta_1\sigma, C\sigma; \tau) \quad \text{in } B_{r_1} \text{ in direction } \nu_1 \tag{9.6}$$

for some $r_1, \nu_1$ with

$$c_{\theta_1} \leqslant 2r_1 \leqslant \theta_1, \quad \text{and} \quad |\nu_1 - \nu| \leqslant C\sigma.$$

In order to improve $\tau$, we consider the functions $U_\varepsilon = (G(|\nabla u|) - G(\lambda^*) - \varepsilon)^+$ and $U_0 = (G(|\nabla u|) - G(\lambda^*))^+$ in $B_{2r_1}$. By Lemma 7.4, and (4) in Definitions 8.1 and 8.2 we know that $U_\varepsilon$ vanishes in a neighborhood of the free boundary. Since $U_\varepsilon > 0$ implies $G(|\nabla u|) > G(\lambda^*) + \varepsilon$, the closure of $\{U_\varepsilon > 0\}$ is contained in $\{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\}$. The function $u$ satisfies the linearized equation

$$Tu = b_{ij}(\nabla u)D_{ij}u = 0$$

where $b_{ij}$ is defined in (7.2), and is uniformly elliptic in $\{G(|\nabla u|) > G(\lambda^*) + \varepsilon/2\}$ with ellipticity constant $\beta$ independent of $u$.

Let $v = G(|\nabla u|)$. By Lemma 1 in [14], we have that $v$ satisfies

$$Mv = D_i\big(b_{ij}(\nabla u)D_j v\big) \geqslant 0 \quad \text{in } \big\{G\big(|\nabla u|\big) > G(\lambda^*) + \varepsilon/2\big\}.$$

Hence $U_\varepsilon$ satisfies

$$MU_\varepsilon \geqslant 0 \quad \text{in } \big\{G\big(|\nabla u|\big) > G(\lambda^*) + \varepsilon/2\big\}.$$

Extending the operator $M$ with the uniformly elliptic divergence-form operator

$$\widetilde{M}w = D_i\big(\widetilde{b}_{ij}(x)D_j w\big) \quad \text{in } B_{2r_1}$$

with measurable coefficients such that

$$\widetilde{b}_{ij}(x) = b_{ij}(\nabla u) \quad \text{in } \big\{G\big(|\nabla u|\big) > G(\lambda^*) + \varepsilon/2\big\},$$

we obtain

$$\widetilde{M}U_\varepsilon \geqslant 0 \quad \text{in } B_{2r_1}.$$

Moreover, by (9.5) we have that $U_\varepsilon \leqslant G(\lambda^*(1+\tau)) - G(\lambda^*)$ and by (9.6) $U_\varepsilon = 0$ in $B = B_{r_1/4}(\frac{r_1}{2}\nu_1)$, if $C\sigma \leqslant 1/2$.

Take now $V$ such that

$$\begin{cases} \widetilde{M}V = 0 & \text{in } B_{2r_1} \setminus \bar{B}, \\ V = G(\lambda^*(1+\tau)) - G(\lambda^*) & \text{on } \partial B_{2r_1}, \\ V = 0 & \text{on } \partial B. \end{cases}$$

Then, there exists $0 < c(N, \beta) < 1$ such that $V \leqslant c(G(\lambda^*(1+\tau)) - G(\lambda^*))$ in $B_{r_1}$. Applying the maximum principle we have that $U_\varepsilon \leqslant c(G(\lambda^*(1+\tau)) - G(\lambda^*))$ in $B_{r_1}$. Taking $\varepsilon \to 0$ we obtain

$$G\big(|\nabla u|\big) \leqslant cG\big(\lambda^*(1+\tau)\big) + G(\lambda^*)(1-c) \quad \text{in } B_{r_1}.$$

Since, $G(\lambda^*(1+\tau)) = G(\lambda^*) + g(\lambda^*)\lambda^*\tau + o(\tau)$ we have that

$$cG\big(\lambda^*(1+\tau)\big) + G(\lambda^*)(1-c) = G(\lambda^*) + cg(\lambda^*)\lambda^*\tau + o(\tau),$$

and since $G$ is strictly increasing, we have

$$\begin{aligned}|\nabla u| &\leqslant G^{-1}\big(G(\lambda^*) + cg(\lambda^*)\lambda^*\tau + o(\tau)\big)\\ &= \lambda^* + \frac{1}{g(\lambda^*)}\big(g(\lambda^*)\lambda^*\tau c + o(\tau)\big) + o(\tau)\\ &= \lambda^*\left(1 + \tau\left(c + \frac{o(\tau)}{\tau}\right)\right) \leqslant \lambda^*\left(1 + \tau\frac{(c+1)}{2}\right),\end{aligned}$$

if we choose $\tau$ small enough. And we see that if we choose $\theta_1$ small enough (depending on $N$), we have

$$u \in F\big(\theta_0\sigma, 1; \theta_0^2\tau\big) \quad \text{in } B_{r_1} \text{ in direction } \nu_1,$$

where $\theta_0 = \sqrt{\frac{c+1}{2}}$.

We can repeat this argument a finite number of times, and we obtain

$$u \in F\big(\theta_0^m\sigma, 1; \theta_0^{2m}\tau\big) \quad \text{in } B_{r_1\ldots r_m} \text{ in direction } \nu_m,$$

with

$$c_{\theta_j} \leqslant 2r_j \leqslant \theta_j, \quad \text{and} \quad |\nu_m - \nu| \leqslant \frac{C}{1-\theta_0}\sigma.$$

Finally we choose $m$ large enough and use Theorem 9.1. □

## *9.3. Smoothness of the free boundary*

**Theorem 9.3.** *Suppose that $u$ is a weak solution, and $D \Subset \Omega$. Then there exist positive constants $\bar\sigma_0$, $C$ and $\alpha$ such that if*

$$u \in F(\sigma, 1; \infty) \quad \textit{in } B_\rho(x_0) \subset D \textit{ in direction } \nu$$

*with $\sigma \leqslant \bar\sigma_0$, $\rho \leqslant \bar\rho_0(\bar\sigma_0, \sigma)$, then*

$$B_{\rho/4}(x_0) \cap \partial\{u > 0\} \textit{ is a } C^{1,\alpha} \textit{ surface,}$$

*more precisely, a graph in direction* $\nu$ *of a* $C^{1,\alpha}$ *function, and, for any* $x_1$, $x_2$ *on this surface*

$$\left|\nu(x_1) - \nu(x_2)\right| \leqslant C\sigma \left|\frac{x_1 - x_2}{\rho}\right|^{\alpha}.$$

**Proof.** By property (4) in Definitions 8.1 and 8.2 we have that, for every $\rho$-neighborhood $D_\rho$ of $D \cap \partial\{u > 0\}$,

$$\left|\nabla u(x)\right| \leqslant \lambda^* + \tau(\rho), \quad \text{for every } x \in D_\rho,$$

where $\tau(\rho) \to 0$ when $\rho \to 0$.

Therefore,

$$u \in F(\sigma, 1; \tau) \quad \text{in } B_\rho(x_0) \text{ in direction } \nu.$$

Applying Theorem 9.1 we have that

$$u \in F(C_0\sigma, C_0\sigma; \tau) \quad \text{in } B_{\rho/2}(x_0) \text{ in direction } \nu$$

if $\sigma \leqslant \sigma_0$ and $\tau \leqslant \sigma$.

Let $x_1 \in B_{\rho/2}(x_0) \cap \partial\{u > 0\}$ then

$$u \in F(C_0\sigma, 1; \tau) \quad \text{in } B_{\rho/2}(x_1) \text{ in direction } \nu$$

and applying again Theorem 9.1 we have

$$u \in F\big(C_0^2\sigma, C_0^2\sigma; \tau\big) \quad \text{in } B_{\rho/4}(x_1) \text{ in direction } \nu$$

if $C_0\sigma \leqslant \sigma_0$ and $\tau \leqslant C_0\sigma$.

Let $0 < \theta < 1$, take $\rho_0 = \rho/4$, $\nu_0 = \nu$, $C = C_0^2$, $\sigma \leqslant \frac{\sigma_\theta}{C}$ and $\tau \leqslant \sigma_\theta C^2\sigma^2$. Now, by Lemma 9.5 and iterating we get that there exist sequences $\rho_m$ and $\nu_m$ such that

$$u \in F\big(\theta^m C\sigma, \theta^m C\sigma; \theta^{2m}\tau\big) \quad \text{in } B_{\rho_m}(x_1) \text{ in direction } \nu_m$$

with $c_\theta\rho_m \leqslant \rho_{m+1} \leqslant \rho_m/4$ and $|\nu_{m+1} - \nu_m| \leqslant \theta^m C\sigma$.

Thus, we have that $|\langle x - x_1, \nu_m\rangle| \leqslant \theta^m C\sigma\rho_m$ for $x \in B_{\rho_m}(x_1) \cap \partial\{u > 0\}$.

We also have that there exists $\nu(x_1) = \lim_{m\to\infty} \nu_m$ and

$$\left|\nu(x_1) - \nu_m\right| \leqslant \frac{C\theta^m}{1-\theta}\sigma.$$

Now let $x \in B_{\rho/4}(x_1) \cap \partial\{u > 0\}$ and choose $m$ such that $\rho_{m+1} \leqslant |x - x_1| \leqslant \rho_m$. Then

$$\left|\big\langle x - x_1, \nu(x_1)\big\rangle\right| \leqslant C\theta^m\sigma\left(\frac{|x - x_1|}{1-\theta} + \rho_m\right) \leqslant C\theta^m\sigma\left(\frac{1}{1-\theta} + \frac{1}{c_\theta}\right)|x - x_1|$$

and since $|x - x_1| \leqslant c_\theta^{m+1}\rho_0$ we have

$$\theta^{m+1} \leqslant \left(\frac{|x - x_1|}{\rho_0}\right)^\alpha \quad \text{with } \alpha = \frac{\log(\theta)}{\log(c_\theta)},$$

and we conclude that

$$\left|\langle x - x_1, \nu(x_1)\rangle\right| \leqslant \frac{C\sigma}{\rho^\alpha}|x - x_1|^{1+\alpha}.$$

Finally, observe that the result follows if we take, $\bar\sigma_0 = \min\{\sigma_0, \frac{\sigma_0}{C_0}, \frac{\sigma_\theta}{C}\}$ and if we choose $\bar\rho_0$ small enough such that if $\rho \leqslant \bar\rho_0$, $\tau(\rho) \leqslant \min\{\sigma, C_0\sigma, \sigma_\theta C^2\sigma^2\}$. □

**Remark 9.2.** By Lemma 8.4, Definition 8.2 and by the nondegeneracy, we have that there exists a set $A \subset \partial_{\text{red}}\{u > 0\}$, with $\mathcal{H}^{N-1}(\partial_{\text{red}}\{u > 0\} \setminus A) = 0$, such that for $x_0 \in A$ we have that $u \in F(\sigma_\rho, 1; \infty)$ in $B_\rho(x_0)$ in direction $\nu_u(x_0)$, with $\sigma_\rho \to 0$ for $\rho \to 0$. Observe that by Theorem 7.1 when $u$ is a minimizer $A = \partial_{\text{red}}\{u > 0\}$. Hence applying Theorem 9.3 we have

**Theorem 9.4.** *If $u$ is a weak solution then there exists a subset $A \subset \partial_{\text{red}}\{u > 0\}$ with $\mathcal{H}^{N-1}(\partial_{\text{red}}\{u > 0\} \setminus A) = 0$ such that for any $x_0 \in A$ there exists $r > 0$ so that $B_r(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface. Moreover, if $u$ satisfies Definition* 8.1 *then the remainder of $\partial\{u > 0\}$ has $\mathcal{H}^{N-1}$-measure zero. Finally, if $u$ is a minimizer, $\partial_{\text{red}}\{u > 0\}$ is a $C^{1,\alpha}$ surface and $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\} = 0$.*

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