

## DIMENSION FUNCTIONS OF CANTOR SETS

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ABSTRACT. We estimate the packing measure of Cantor sets associated to non-increasing sequences through their decay. This result, dual to one obtained by Besicovitch and Taylor, allows us to characterize the dimension functions recently found by Cabrelli *et al* for these sets.

### 1. INTRODUCTION

A Cantor set is a compact perfect and totally disconnected subset of the real line. In this article we consider Cantor sets of Lebesgue measure zero. Different kinds of these sets appear in many areas of mathematics, such as number theory and dynamical systems. They are also interesting in themselves as theoretical examples and counterexamples. A classical way to understand them quantitatively is through the Hausdorff measure and dimension.

A function  $h : (0, \lambda_h] \rightarrow (0, \infty]$ , where  $\lambda_h > 0$ , is said to be a *dimension function* if it is continuous, nondecreasing and  $h(x) \rightarrow 0$  as  $x \rightarrow 0$ . We denote by  $\mathcal{D}$  the set of dimension functions.

Given  $E \subset \mathbb{R}$  and  $h \in \mathcal{D}$ , we set  $h(E) = h(|E|)$ , where  $|E|$  is the diameter of the set  $E$ .

Recall that a  $\delta$ -covering of a given set  $E$  is a countable family of subsets of  $\mathbb{R}$  covering  $E$  whose diameters are less than  $\delta$ . The  $h$ -Hausdorff measure of  $E$  is defined as

$$(1.1) \quad \mathcal{H}^h(E) = \lim_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} h(U_i) : \{U_i\} \text{ is a } \delta\text{-covering of } E \right\}.$$

We say that  $E$  is an  $h$ -set if  $0 < \mathcal{H}^h(E) < +\infty$ .

When the dimension function is  $g_s(x) = x^s$ , for  $s \geq 0$ , we set  $\mathcal{H}^s := \mathcal{H}^{g_s}$  ( $\mathcal{H}^0$  is the counting measure). The Hausdorff dimension of the set  $E$ , denoted by  $\dim E$ , is the unique value  $t$  for which  $\mathcal{H}^s(E) = 0$  if  $s > t$  and  $\mathcal{H}^s(E) = +\infty$  if  $s < t$  (see Proposition 2.2). This property allows us to obtain an intuitive classification of how *thin* a subset of  $\mathbb{R}$  of Lebesgue measure zero is.

A set  $E$  is said to be *dimensional* if there is at least one  $h \in \mathcal{D}$  that makes  $E$  an  $h$ -set. Not all sets are dimensional (cf. [Bes39]); in fact, there are open problems about the dimensionality of certain sets, for instance, the set of Liouville numbers, which

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has Hausdorff dimension zero (see for example [Ols03]). Nevertheless, Cabrelli *et al* [CMMS04] showed that every Cantor set associated to a nonincreasing sequence  $a$  is dimensional; that is, they constructed a function  $h_a \in \mathcal{D}$  for which  $C_a$  is an  $h_a$ -set. Moreover, they show that if the sequence  $a$  behaves like  $n^{-1/s}$ , then  $h_a \equiv g_s$  (see definition below) and therefore  $C_a$  is an  $s$ -set. But in other cases the behavior of these functions is not so clear. For example, there exists a sequence  $a$  such that  $C_a$  is an  $\alpha$ -set but  $h_a \not\equiv g_\alpha$  ([CHM02]). So these functions could be too general in order to give a satisfactory idea about the size of the set. To understand this situation we study the packing premeasure of these sets, which is defined as follows. A  $\delta$ -packing of a given set  $E$  is a disjoint family of open balls centered at  $E$  with diameters less than  $\delta$ . The  $h$ -packing premeasure of  $E$  is defined as

$$P_0^h(E) = \overline{\lim}_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} h(B_i) : \{B_i\}_i \text{ is a } \delta\text{-packing of } E \right\}.$$

As a consequence of our main result, which is dual to the one obtained in [BT54] and will be dealt with in Section 4, we are able to characterize completely when a dimension function is equivalent to a power function (Theorem 4.4). That is, for a nonincreasing sequence  $a$  and  $h \in \mathcal{D}$  we obtain that

$$h \equiv g_s \iff 0 < \mathcal{H}^s(C_a) \leq P_0^s(C_a) < +\infty.$$

Thus, to have that  $g_\alpha \equiv h_a$ , it is not only necessary that  $C_a$  is an  $\alpha$ -set but also that  $P_0^\alpha(C_a) < +\infty$ .

## 2. SOME REMARKS AND DEFINITIONS

By the definition of  $P_0^h$ , it is clear that it is monotone but it is not a measure because it is not  $\sigma$ -additive; the  $h$ -packing measure  $\mathcal{P}^h$  is obtained by a standard argument,  $\mathcal{P}^h(E) = \inf \{ \sum_{i=1}^{\infty} P_0^h(E_i) : E = \bigcup_i E_i \}$ .

As with Hausdorff measures, given a set  $E$  there exists a critical value  $\dim_P E$ , the *packing dimension* of  $E$ , such that  $\mathcal{P}^s(E) = 0$  if  $s > \dim_P E$  and  $\mathcal{P}^s(E) = +\infty$  if  $s < \dim_P E$ . Analogously for the prepacking measure family  $\{P_0^s\}$  we call  $\Delta E$  its critical value. In [Tri82] it is shown that  $\Delta E$  coincides with the upper Box dimension of  $E$ , which we now define.

Given  $0 < \varepsilon < \infty$  and a nonempty bounded set  $E \subset \mathbb{R}^d$ , let  $N(E, \varepsilon)$  be the smallest number of balls of radius  $\varepsilon$  needed to cover  $E$ . The *lower* and *upper Box dimensions* of  $E$  are given by

$$\underline{\dim}_B E = \liminf_{\varepsilon \rightarrow 0} \frac{\log N(E, \varepsilon)}{\log 1/\varepsilon} \quad \text{and} \quad \overline{\dim}_B E = \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\log N(E, \varepsilon)}{\log 1/\varepsilon}$$

respectively. When the lower and upper limits coincide, the common value is the *Box dimension* of  $E$  and we denote it by  $\dim_B E$ .

Note that  $\mathcal{H}^h(E) \leq \mathcal{P}^h(E)$  when  $h$  is a doubling function ([TT85]). Moreover,  $\dim E \leq \dim_P E \leq \Delta E \leq 1$  and  $\dim E \leq \underline{\dim}_B E \leq \overline{\dim}_B E$ ; all these inequalities can be strict ([Tri82]).

We observe that  $\mathcal{H}^h$  is Borel-regular, which is also true if we only require the right continuity of  $h \in \mathcal{D}$  [Rog98]. On the other hand,  $\mathcal{P}^h$  and  $P_0^h$  are also Borel-regular, but in this case one has to require that  $h \in \mathcal{D}$  be left continuous [TT85, Lemma 3.2]. Since in this paper we are concerned with all of these measures, we require that  $h \in \mathcal{D}$  should be continuous.

Now we define a partial order in  $\mathcal{D}$ , which will be our way of comparing the elements of  $\mathcal{D}$ .

**Definition 2.1.** Let  $f$  and  $h$  be in  $\mathcal{D}$ .

- $f$  is smaller than  $h$ , denoted by  $f \prec h$ , if

$$\lim_{x \rightarrow 0} h(x)/f(x) = 0.$$

- $f$  is equivalent to  $h$ , denoted by  $f \equiv h$ , if

$$0 < c_1 = \liminf_{x \rightarrow 0} \frac{h(x)}{f(x)} \leq \limsup_{x \rightarrow 0} \frac{h(x)}{f(x)} = c_2 < +\infty.$$

We set  $f \preceq g$  when  $f \prec g$  or  $f = g$ . We say that  $f$  and  $g$  are not comparable if none of the relations  $f \prec g$ ,  $g \prec f$  or  $f \equiv g$  holds.

Notice that this definition is consistent with the usual law of the power functions:  $x^s \prec x^t$  iff  $s < t$ .

**Proposition 2.2.** If  $\nu^f$  is  $\mathcal{H}^f$  or  $\mathcal{P}_0^f$  or  $\mathcal{P}^f$ , then we have the following.

- i) If  $f \prec h$ , then:  $\nu^f(E) < \infty \implies \nu^h(E) = 0$ .
- ii) If  $f \equiv h$ , then:
  - a)  $\nu^f(E) < \infty \iff \nu^h(E) < \infty$ ;
  - b)  $0 < \nu^f(E) \iff 0 < \nu^h(E)$ ;
  - c) in particular,  $E$  is an  $f$ -set if and only if  $E$  is an  $h$ -set.

The proof of this proposition for the Hausdorff measure case can be found in [Rog98]; the packing cases are analogous.

### 3. CANTOR SETS ASSOCIATED TO NONINCREASING SEQUENCES

**Definition 3.1.** Let  $a = \{a_k\}$  be a positive, nonincreasing and summable sequence. Let  $I_a$  be a closed interval of length  $\sum_{k=1}^\infty a_k$ . We define  $\mathcal{C}_a$  to be the family of all closed sets  $E$  contained in  $I_a$  that are of the form  $E = I_a \setminus \bigcup_{j \geq 1} U_j$ , where  $\{U_j\}$  is a disjoint family of open intervals contained in  $I_a$  such that  $|U_k| = a_k \forall k$ .

With this definition, every element of  $\mathcal{C}_a$  has Lebesgue measure zero. From this family, we consider the Cantor set  $C_a$  associated to the sequence  $a$  constructed as follows: In the first step, we remove from  $I_a$  an open interval of length  $a_1$ , resulting in two closed intervals  $I_1^1$  and  $I_2^1$ . Having constructed the  $k$ -th step, we get the closed intervals  $I_1^k, \dots, I_{2^k}^k$  contained in  $I_a$ . The next step consists in removing from  $I_j^k$  an open interval of length  $a_{2^k+j-1}$ , obtaining the closed intervals  $I_{2j-1}^{k+1}$  and  $I_{2j}^{k+1}$ . Then we define  $C_a := \bigcap_{k=1}^\infty \bigcup_{j=1}^{2^k} I_j^k$ . Note that in this construction there is a unique form of removing open intervals at each step; also, note that for this construction it is not necessary for the sequence to be nonincreasing.

*Remark 3.2.* Since  $a$  is nonincreasing, the sequence  $\{|I_j^k|\}_{(k,j)}$ , with  $1 \leq j \leq 2^k$  and  $k \geq 1$ , is (lexicographically) nonincreasing.

We associate to the sequence  $a$  the summable sequences  $\underline{a}$  and  $\bar{a}$  defined as

$$\underline{a}_n = a_{2^{r+1}-1} \quad \text{and} \quad \bar{a}_n = a_{2^r},$$

with  $2^r \leq n < 2^{r+1}$ ; thus  $\underline{a}_n \leq a_n \leq \bar{a}_n \forall n$ . The sets  $C_{\underline{a}}$  and  $C_{\bar{a}}$  are uniform, which means that for each  $k \geq 1$ , the closed intervals of the  $k$ -th step have all the

same length. Observe that for each uniform Cantor set the Hausdorff and lower box dimensions coincide (cf. [CHM97]).

To study the Hausdorff measure and dimension of these sets, Besicovitch and Taylor in [BT54] studied the decay of the sequence  $b_n = r_n/n$ , where  $r_n = \sum_{j \geq n} a_j$ . They introduced the number

$$\alpha(a) = \underline{\lim}_{n \rightarrow \infty} \alpha_n, \text{ where } nb_n^{\alpha_n} = 1 \ \forall n,$$

and showed that  $\dim E \leq \alpha(a)$  for all  $E \in \mathcal{C}_a$ . In fact,  $\dim C_a = \alpha(a)$  (see, for example, [CMMS04]). Also, as a consequence of another result in [BT54] (see Proposition 4.1 below),

$$(3.1) \quad \dim C_a = \inf\{s > 0 : \underline{\lim} nb_n^s < +\infty\}.$$

In this paper (Theorem 4.2) we obtain the symmetrical result for the packing case:

$$(3.2) \quad \Delta C_a = \inf\{s > 0 : \overline{\lim} nb_n^s < +\infty\}.$$

On the other hand, the box dimensions of the sets in  $\mathcal{C}_a$  are related to the constants

$$(3.3) \quad \gamma(a) = \underline{\lim}_{k \rightarrow \infty} \frac{\log 1/k}{\log a_k} \text{ and } \beta(a) = \overline{\lim}_{k \rightarrow \infty} \frac{\log 1/k}{\log a_k}.$$

In fact, by Propositions 3.6 and 3.7 of Falconer’s book [Fal97], the box dimension of  $E \in \mathcal{C}_a$  exists if and only if  $\gamma(a) = \beta(a)$ , and in this case  $\dim_B E = \beta(a)$ . Moreover, as Falconer remarks, every  $E \in \mathcal{C}_a$  has the same upper box dimension and the same lower box dimension. The former equals  $\beta(a)$ , which in [Tri95] is shown to be equal to  $\overline{\lim}_{n \rightarrow \infty} \alpha_n$ . In the following proposition we obtain the symmetrical result, that is, the latter is  $\underline{\lim}_{n \rightarrow \infty} \alpha_n$ , but observe that  $\gamma(a) \leq \underline{\dim}_B C_a$  and the inequality can be strict (cf. Propositions 3 and 5 in [CMMS04]).

**Proposition 3.3.** *For every nonincreasing sequence  $a$  we have that  $\underline{\dim}_B C_a = \dim C_a = \alpha(a)$ .*

*Proof.* First note that  $\underline{\lim} nb_n^s \sim \underline{\lim} n\bar{b}_n^s$ . In fact, to see the nontrivial inequality, if  $\{n_j\}$  is a subsequence of the natural numbers let  $2^{l_j-1} \leq n_j < 2^{l_j}$ , so  $2^{l_j} b_{2^{l_j}}^s \leq 2n_j b_{n_j}^s$  and therefore  $\underline{\lim}_{k \rightarrow \infty} 2^k b_{2^k}^s \leq 2 \underline{\lim}_{n \rightarrow \infty} nb_n^s$ . Now for  $2^{i_n} \leq n < 2^{i_n+1}$ ,

$$\begin{aligned} \bar{r}_n &\leq \sum_{k \geq 2^{i_n}} \bar{a}_k = \sum_{k=0}^{\infty} 2^{i_n+k} a_{2^{i_n+k}} \\ &\leq 2 \sum_{k=0}^{\infty} 2^{i_n+k-1} a_{2^{i_n+k-1}} = 2 r_{2^{i_n-1}} \leq 2 r_{2^{i_n-1}}; \end{aligned}$$

hence  $\underline{\lim}_{n \rightarrow \infty} n\bar{b}_n^s \leq 4 \underline{\lim}_{j \rightarrow \infty} 2^j b_{2^j}^s \leq 8 \underline{\lim}_{n \rightarrow \infty} nb_n^s$ .

Then by (3.1)  $\dim C_a = \dim C_{\bar{a}}$ , and by Proposition 3.1 of [CHM97] we have that  $\dim C_a = \underline{\dim}_B C_{\bar{a}}$ . On the other hand,  $N(\varepsilon, C_a) \leq N(\varepsilon, C_{\bar{a}})$  ( $C_a$  can be mapped to  $C_{\bar{a}}$  by a bijection which preserves the order; then, if two points of  $C_{\bar{a}}$  are contained in an open set  $U$ , the corresponding points of  $C_a$  will be contained in an open set of the same diameter as  $U$ ), so  $\underline{\dim}_B C_a \leq \underline{\dim}_B C_{\bar{a}}$ .  $\square$

This proposition in particular asserts that  $\underline{\dim}_B E = \inf\{s > 0 : \underline{\lim} nb_n^s < +\infty\}$  for any  $E \in \mathcal{C}_a$ . On the other hand, since the critical exponent associated to

the packing premeasure of a given set is the upper box dimension, it is clear that  $\overline{\dim}_B E = \inf\{s > 0 : \overline{\lim} nb_n^s < +\infty\}$ .

4. MAIN RESULTS

The next proposition is a result that generalizes the one established in [BT54] for the functions  $g_s$  to any function  $h \in \mathcal{D}$  (cf. [CHM02]). It shows that  $\mathcal{H}^h(C_a)$  behaves like  $nh(b_n)$  when  $n \rightarrow \infty$ .

**Proposition 4.1.** *For  $h \in \mathcal{D}$ ,  $\frac{1}{4} \underline{\lim}_{n \rightarrow \infty} nh(b_n) \leq \mathcal{H}^h(C_a) \leq 4 \underline{\lim}_{n \rightarrow \infty} nh(b_n)$ .*

*Proof.* The lower inequality  $\frac{1}{4} \underline{\lim}_{n \rightarrow \infty} nh(b_n) \leq \mathcal{H}^h(C_a)$  is obtained in exactly the same way as in [BT54] replacing  $g_s$  by  $h$ . Hence, we only show here that  $\mathcal{H}^h(C_a) \leq 4 \underline{\lim} nh(b_n)$ . We begin by noting that  $|I_j^k| = \sum_{i=0}^{\infty} 2^i a_{2^k+i} = b_{2^k}$  for  $1 \leq j \leq 2^k, k > 0$ . Then, from the identities

$$\begin{aligned} |I_1^k| &= a_{2^k} + (a_{2^{k+1}} + a_{2^{k+1}+1}) + (a_{2^{k+2}} + a_{2^{k+2}+1} + a_{2^{k+2}+2} + a_{2^{k+2}+3}) + \dots, \\ |I_1^{k-1}| &= a_{2^{k-1}} + (a_{2^{k+1-1}} + a_{2^{k+1-1}+1}) + (a_{2^{k+2-1}} + a_{2^{k+2-1}+1} + a_{2^{k+2-1}+2} + a_{2^{k+2-1}+3}) + \dots \end{aligned}$$

and Remark 3.2, we have the following estimate:

$$(4.1) \quad |I_j^k| \leq |I_1^k| \leq |I_1^{k-1}| = b_{2^{k-1}} \leq b_{2^{k-1}},$$

and hence  $\sum_{j=1}^{2^k} h(I_j^k) \leq 2^k h(b_{2^{k-1}})$ . Given  $\delta > 0$  there exists  $k_\delta$  such that  $|I_1^k| < \delta$  for  $k \geq k_\delta$ ; that is, for  $k \geq k_\delta$ , the closed intervals of the  $k$ -th step form a  $\delta$ -covering of  $C_a$ , which implies  $\mathcal{H}^h(C_a) \leq 2 \underline{\lim}_{k \rightarrow \infty} 2^k h(b_{2^k})$ , and therefore  $\mathcal{H}^h(C_a) \leq 4 \underline{\lim} nh(b_n)$ .  $\square$

Our main result is the following theorem, which is in some sense dual to the previous one.

**Theorem 4.2.** *For any  $h \in \mathcal{D}$ ,  $\frac{1}{8} \overline{\lim}_{n \rightarrow \infty} nh(b_n) \leq P_0^h(C_a) \leq 8 \overline{\lim}_{n \rightarrow \infty} nh(b_n)$ .*

*Proof.* For the first inequality, suppose that  $\overline{\lim}_{n \rightarrow \infty} nh(b_n) > d$ . To prove that  $P_0^h(C_a) \geq d/8$ , for each  $\delta > 0$  it suffices to find a  $\delta$ -packing  $\{B_i\}_i$  of  $C_a$  with  $\sum_i h(B_i) > d/8$ . Observe that, since  $\{a_n\}$  is nonincreasing, then

$$(4.2) \quad h(b_{2^k}) = h \left( 2^{-k} \sum_{1 \leq i \leq 2^k} |I_i^k| \right) \leq h(I_1^k).$$

By hypothesis there exists a subsequence  $\{n_j\}_{j \geq 1}$  such that  $n_j h(b_{n_j}) > d$ . For each  $j$ , let  $k_j$  be the unique integer for which  $2^{k_j} \leq n_j < 2^{k_j+1}$ ; since  $\{b_n\}_n$  is decreasing, it follows from (4.2) that

$$(4.3) \quad d < n_j h(b_{n_j}) < 2^{k_j+1} h(b_{2^{k_j}}) \leq 2^{k_j+1} h(I_1^{k_j}).$$

Pick  $j$  large enough so that  $|I_1^{k_j}| < \delta$ ; since the diameter of this interval is smaller than the diameter of every interval of the  $k_j - 1$  step, the family of intervals  $\{B_i\}_{i=1}^{2^{k_j-2}}$ , where  $B_i$  is centered at the right endpoint of the interval  $I_{2^{i-1}}^{k_j-1}$  and  $|B_i| = |I_1^{k_j}|$ , turns out to be a  $\delta$ -packing of  $C_a$ , and by (4.3),  $\sum_i h(B_i) = 2^{k_j-2} h(I_1^{k_j}) > d/8$ .

For the second inequality, if  $\{B_i\}_{i=1}^N$  is a  $\delta$ -packing of  $C_a$ , we define

$$k_i = \min\{k : I_j^k \subset B_i \text{ for some } 1 \leq j \leq 2^k\}.$$

By the definition of  $k_i$ ,  $B_i$  is centered at a point of an interval of the  $k_i - 1$  step, but it does not contain the interval, so  $|B_i| < |I_{j_i}^{k_i-2}|$ , where  $I_{j_i}^{k_i-2}$  is the interval of the  $k_i - 2$  step which contains the center of  $B_i$ . Then, by the monotony of  $h$  and (4.1),

$$(4.4) \quad \sum_{i=1}^N h(B_i) \leq \sum_{i=1}^N h(I_{j_i}^{k_i-2}) \leq \sum_{i=1}^N h(b_{2^{k_i-3}}).$$

Furthermore, we can assume that  $|B_1| \geq \dots \geq |B_N|$ , so  $k_1 \geq \dots \geq k_N$ . Let  $l_1 > \dots > l_M$  denote those  $k_i$ 's that do not repeat themselves. Let  $\theta_m$  be the number of repeated  $l_m$ 's; i.e.,  $\theta_m$  tells us how many of the  $B_i$ 's contain an interval of the  $l_m$ -th step but none of the previous ones. Since  $\{B_i\}_{i=1}^N$  is a disjoint family,  $\theta_1$  cannot be greater than the number of intervals of step  $l_1$ , which is  $2^{l_1}$ ; each ball of the packing associated to  $l_1$  contains  $2^{l_2-l_1}$  intervals of step  $l_2$  and therefore  $\theta_2 \leq 2^{l_2} - \theta_1 2^{l_2-l_1}$ . Continuing with this process we obtain

$$\theta_M \leq 2^{l_M} - \sum_{i=1}^{M-1} \theta_i 2^{l_M-l_i} = 2^{l_M} \left( 1 - \sum_{i=1}^{M-1} \frac{\theta_i}{2^{l_i}} \right),$$

and hence  $\sum_{i=1}^M \theta_i / 2^{l_i} \leq 1$ .

Finally, choose  $\delta$  sufficiently small such that  $2^{l_1-3} \geq n_0$ , where

$$\sup_{n \geq n_0} nh(b_n) \leq \overline{\lim}_{n \rightarrow \infty} nh(b_n) + \varepsilon.$$

Then,

$$\sum_{i=1}^N h(B_i) \leq \sum_{j=1}^M \frac{\theta_j}{2^{l_j-3}} 2^{l_j-3} h(b_{2^{l_j-3}}) \leq 8(\overline{\lim}_{n \rightarrow \infty} nh(b_n) + \varepsilon),$$

and from this the theorem follows. □

We are now ready to complete the characterization promised in the introduction.

Note that the function  $h_a \in \mathcal{D}$  found in [CMMS04] is defined in such a way that  $h_a(b_n) = 1/n$  for all  $n$ . Then, by Theorem 4.2 we obtain that  $P_0^{h_a}(C_a) < +\infty$ , and therefore the Cantor sets associated to nonincreasing sequences not only are dimensional but also have a dimension function which simultaneously *regularizes* the covering and packing processes in the construction of the measures. We need the following lemma.

**Lemma 4.3.** *Let  $h, g \in \mathcal{D}$ . In addition, assume that  $0 < \mathcal{H}^h(C_a) \leq P_0^h(C_a) < +\infty$ . Then*

- a)  $h \equiv g \iff 0 < \underline{\lim}_{n \rightarrow \infty} ng(b_n) \leq \overline{\lim}_{n \rightarrow \infty} ng(b_n) < +\infty;$
- b)  $g \prec h \iff \lim_{n \rightarrow \infty} ng(b_n) = +\infty;$
- c)  $h \prec g \iff \lim_{n \rightarrow \infty} ng(b_n) = 0.$

*Proof.* By Proposition 4.1 and Theorem 4.2, since  $\mathcal{H}^h(C_a) > 0$  and  $P_0^h(C_a) < +\infty$ , there are constants  $0 < c_h$  and  $C_h < +\infty$  such that  $c_h 1/n \leq h(b_n) \leq C_h 1/n$ , and all three necessary conditions follow. On the other hand, if  $\{y_j\}$  is a sequence that decreases to 0, then there exists a subsequence  $\{n_j\}$  such that  $b_{n_j+1} \leq y_j < b_{n_j}$ . Therefore

$$\frac{g(y_j)}{h(y_j)} \leq \frac{g(b_{n_j})}{h(b_{n_j+1})} \leq 2c_h^{-1} n_j g(b_{n_j})$$

and

$$\frac{g(y_j)}{h(y_j)} \geq (2C_h)^{-1}(n_j + 1)g(b_{n_j+1}),$$

from which the sufficient conditions also follow. □

We have now the main theorem of this part.

**Theorem 4.4.** *Let  $a$  be a nonincreasing sequence, and let  $h \in \mathcal{D}$  be such that  $0 < \mathcal{H}^h(C_a) \leq P^h(C_a) < +\infty$ . Then, for  $g \in \mathcal{D}$  we have:*

- a)  $h \equiv g \iff 0 < \mathcal{H}^g(C_a) \leq P_0^g(C_a) < +\infty$ ;
- b)  $g \prec h \iff \mathcal{H}^g(C_a) = +\infty$ ;
- c)  $h \prec g \iff P_0^g(C_a) = 0$ .

*In particular,  $h$  will be equivalent to  $x^s$  if and only if  $0 < \mathcal{H}^s(C_a) \leq P_0^s(C_a) < +\infty$ .*

*Proof.* The proof is immediate from Proposition 4.1, Theorem 4.2 and Lemma 4.3. □

**Corollary 4.5.** *Let  $\alpha = \dim C_a$  and  $\beta = \Delta C_a$ . Take  $h_a$  to be the dimension function of  $C_a$ .*

- a) *If  $s < \alpha$  and  $\beta < t$ , then  $g_s \prec h_a$  and  $h_a \prec g_t$ . If  $\mathcal{H}^\alpha(C_a) = +\infty$ , then  $g_\alpha \prec h_a$ , and if  $\mathcal{H}^\alpha(C_a) < +\infty$ , then  $g_\alpha \not\prec h_a$ . If  $P_0^\beta(C_a) = 0$ , then  $h_a \prec g_\beta$ , and if  $P_0^\beta(C_a) > 0$ , then  $h_a \not\prec g_\beta$ .*
- b) *In the case  $\alpha < \beta$ ,  $h_a \not\equiv g_s$  for no  $s \geq 0$ . Moreover, if  $\alpha < t < \beta$ , then  $h_a$  and  $g_t$  are not comparable. In the limit cases, if  $\mathcal{H}^\alpha(C_a) < +\infty$ , then  $h_a$  and  $g_\alpha$  are not comparable, and if  $P_0^\beta(C_a) > 0$ ,  $h_a$  and  $g_\beta$  are not comparable.*

*Proof.* In the case  $\mathcal{H}^\alpha(C_a) < +\infty$ , if  $g_a$  were to satisfy  $g_\alpha \prec h_a$ , Proposition 2.2 would imply that  $\mathcal{H}^{h_a}(C_a) = 0$ , which is a contradiction. Hence  $g_\alpha \not\prec h_a$ . Analogously,  $P_0^\beta(C_a) > 0$  implies that  $h_a \not\prec g_\beta$ , for if not  $P_0^{h_a}(C_a) = +\infty$ , contradicting Theorem 4.2. The rest of the claims of item a) are immediate from Theorem 4.4.

To show b), if  $h_a \equiv g_s$  for some  $s \geq 0$ , then  $0 < \mathcal{H}^s(C_a) \leq P_0^s(C_a) < +\infty$ ; therefore  $\alpha = \beta$ . Moreover, by Proposition 2.2, it follows that  $g_s \not\prec h$  when  $s > \dim C_a$ , and also  $h_a \not\prec g_s$  if  $0 \leq s < \beta$ . □

Note that, since  $\alpha = \underline{\dim}_B C_a$  and  $\beta = \overline{\dim}_B C_a$ , part b) of this corollary emphasizes that in order to have  $h_a \equiv g_s$  we need  $\underline{\dim}_B C_a = \overline{\dim}_B C_a$ . But this latter condition and the fact that  $C_a$  is an  $\alpha$ -set are not sufficient to ensure the equivalence, and thus the hypothesis of Theorem 4.4 a) cannot be weakened to *existence of box dimension and  $C_a$  being an  $\alpha$ -set*.

**Example 4.6.** For each  $0 < s < 1$  there exists a Cantor set  $C_a$  associated to a nonincreasing sequence for which  $\dim C_a = \overline{\dim}_B C_a = s$  and  $0 < \mathcal{H}^s(C_a) < +\infty$ , but  $P_0^s(C_a) = +\infty$ .

To construct this sequence we set  $\lambda_k = (\frac{1}{2})^{\frac{k}{s+\varepsilon_k}}$ , where

$$\varepsilon_k = \begin{cases} \frac{s \log l}{k}, & 2^m < k \leq 2^{m+1}, l = k - 2^m, m \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let us define  $a_j = \lambda_k$  for  $2^{k-1} \leq j < 2^k$  and  $k \geq 1$ . The uniform Cantor set  $C_a$  gives us the example. In fact, it is easy to check that  $a$  is summable, decreasing

and, using (3.3), that  $\dim C_a = \overline{\dim}_B C_a = s$ . Next we check the claims about the measures:

- a)  $0 < \underline{\lim}_{n \rightarrow \infty} nb_n^s$ : Since  $b_{2^k} = \sum_{i \geq 0} 2^i \lambda_{k+1+i}$  and  $\lambda_j \geq (\frac{1}{2})^{\frac{j}{s}} \quad \forall j$ , we have that

$$2^k b_{2^k}^s \geq 2^k \left( \sum_{i \geq 0} \left( \frac{1}{2} \right)^{\frac{k+1+i}{s} - i} \right)^s = \frac{1}{2} \left( \sum_{i \geq 0} \left( \frac{1}{2} \right)^{\left(\frac{1}{s}-1\right)i} \right)^s = c_s > 0$$

and remember that  $\underline{\lim}_{n \rightarrow \infty} nb_n^s \sim \underline{\lim}_{k \rightarrow \infty} 2^k b_{2^k}^s$ .

- b)  $\overline{\lim}_{n \rightarrow \infty} nb_n^s = +\infty$ : Let  $n_j = 2^{2^j-1}$ ,  $j$  odd. Then  $b_{n_j} = \sum_{i \geq 0} 2^i \lambda_{2^j+i} > \lambda_{2^j}$ ; therefore,

$$n_j b_{n_j}^s \geq 2^{2^j-1} \lambda_{2^j}^s = 2^{\frac{2^j(j-1)\log 2}{2^j+(j-1)\log 2} - 1},$$

which increases to infinity with  $j$ .

- c)  $\underline{\lim}_{n \rightarrow \infty} nb_n^s < +\infty$ : Now we set  $n_j = 2^{2^j}$  for  $j$  odd and observe that

$$(4.5) \quad n_j b_{n_j}^s = n_j \left( \sum_{i \geq 0} 2^i \lambda_{2^j+1+i} \right)^s = \left( \sum_{i \geq j} \sum_{l=1}^{2^i} 2^{2^i+l+2^j\left(\frac{1-s}{s}\right)-1} \lambda_{2^i+l} \right)^s.$$

Each sum in  $l$  is bounded by a geometric term; more precisely, if  $j$  is sufficiently large there is a constant  $C_s$  depending only on  $s$  such that

$$\sum_{l=1}^{2^i} 2^{2^i+l+2^j\left(\frac{1-s}{s}\right)} \lambda_{2^i+l} \leq C_s \left( \frac{1}{2} \right)^{i-j},$$

or equivalently,

$$(4.6) \quad \sum_{l=1}^{2^i} 2^l \lambda_{2^i+l} \leq C_s \left( \frac{1}{2} \right)^{2^i+2^j\left(\frac{1-s}{s}\right)+i-j}.$$

This is easy to check when  $i$  is odd. For  $i$  even we obtain  $\sum_{l=1}^{2^i} 2^l \lambda_{2^i+l} < C_s \left( \frac{1}{2} \right)^{\frac{2^i}{s+\varepsilon}}$  for small  $\varepsilon$ . Thus (4.6) will hold if

$$(4.7) \quad 2^i \left( \frac{1-(s+\varepsilon)}{s+\varepsilon} \right) \geq 2^j \left( \frac{1-s}{s} \right) + i - j.$$

But notice that (4.7) is true for all  $i > j$  choosing  $\varepsilon$  sufficiently small and  $j$  large enough so that  $\varepsilon_{2^j} < \varepsilon$ .

*Remark 4.7.* Proposition 4.1 and Theorem 4.2 are not valid in general as the following arguments show.

If  $a$  is a nonincreasing sequence and  $\tilde{a}$  is any rearrangement of  $a$ , then  $r_n^a \leq r_n^{\tilde{a}}$ ; hence, by (3.1) and since  $C_{\tilde{a}} \in \mathcal{C}_a$ ,

$$\inf\{s > 0 : \underline{\lim} n(b_n^{\tilde{a}})^s < +\infty\} \geq \dim C_a \geq \dim C_{\tilde{a}},$$

and each positive and summable sequence  $a$  has a rearrangement  $z$  for which  $\dim C_z = 0$  (cf. [CMPS05]). This shows the counterexample for Proposition 4.1.



In the case of Theorem 4.2 let us see that (3.2) need not be true for a general sequence. Let  $\tilde{a}$  be any rearrangement of  $a$ . Let  $\eta(\tilde{a}) := \inf\{s > 0 : \overline{\lim} n(b_n^{\tilde{a}})^s < +\infty\}$  and consider  $\beta(\tilde{a})$  as defined in Section 3. Note that if  $t > \eta(\tilde{a})$ , then  $\tilde{a}_n < Cn^{1-1/t} \forall n$ , and hence  $\beta(\tilde{a}) \leq \frac{t}{1-t}$ , which implies that  $\beta(\tilde{a}) \leq \frac{\eta(\tilde{a})}{1-\eta(\tilde{a})}$ . Therefore  $\frac{\beta(\tilde{a})}{1+\beta(\tilde{a})} \leq \eta(\tilde{a})$ . Thus, to show that this theorem fails in general, we exhibit a nonincreasing sequence  $a$  and a rearrangement  $\tilde{a}$  of it for which

$$(4.8) \quad \beta(a) < \frac{\beta(\tilde{a})}{1 + \beta(\tilde{a})},$$

and therefore  $\Delta C_{\tilde{a}} = \Delta C_a = \beta(a) < \eta(\tilde{a})$ , where  $\Delta C_{\tilde{a}}$  is the critical exponent for  $P_0^s$ , the packing premeasure. For  $a_n = (\frac{1}{n})^p$ , we set

$$\tilde{a}_n = \begin{cases} a_{3^k}, & n = \lceil \log 3^k \rceil, n \neq 3^j \forall j, \\ a_{\lceil \log 3^k \rceil}, & n = 3^k, n \neq \lceil \log 3^j \rceil \forall j, \\ a_n, & \text{otherwise,} \end{cases}$$

where  $\lceil s \rceil$  denotes the smallest integer greater than  $s$ . Observe that this is a rearrangement of  $a$  since the sequence  $\{\lceil \log 3^k \rceil\}$  is strictly increasing and  $\lceil \log 3^l \rceil \neq 3^j$  for each  $l$  and each  $j$ . Then it is easy to check that  $\beta(a) = 1/p$  and that  $\beta(\tilde{a}) = \frac{\log 3}{p}$ . Therefore (4.8) holds for  $p > \frac{\log 3}{\log 3 - 1}$ .

However these propositions keep on being true if we ask the Cantor set to be uniform with no further assumptions on the sequence. This can be seen in Theorem 4.2 since inequalities (4.3) and (4.4), where the decay assumption is needed, are still true for uniform Cantor sets. For Proposition 4.1 note that Lemma 4 of [BT54] does not need the decay. Moreover, to each Cantor set associated to a nonincreasing sequence corresponds a uniform Cantor set with equivalent  $h$ -Hausdorff measure and  $h$ -packing premeasure.

Finally we give an application of the results of this section.

Let  $p > 1$  and  $q \in \mathbb{R}$ . Consider the sequence  $a$  defined by  $a_n = (\log n)^q/n^p$ ,  $\forall n > 1$ . We denote by  $C_{p,q}$  the Cantor set associated to  $a$  and define the dimension function  $h_{p,q}(x) = x^{\frac{1}{p}}/(-\log x)^{\frac{q}{p}}$ .

Since  $\gamma(a) = \beta(a) = 1/p$ , we have that  $\dim C_{p,q} = \dim_B C_{p,q} = 1/p$  for any  $q$ . Moreover, if  $q = 0$ ,  $C_p = C_{p,0}$  is a  $1/p$ -set (cf. [CMPS05]). Even more, as in this case  $r_n \sim 1/n^{p-1}$ , Theorem 4.2 implies that  $P_0^{\frac{1}{p}}(C_a) < +\infty$ . The next corollary extends these results.

**Corollary 4.8.** *With the above notation,  $0 < \mathcal{H}^{h_{p,q}}(C_{p,q}) \leq P_0^{h_{p,q}}(C_{p,q}) < +\infty$ . In particular,  $C_{p,q}$  is an  $h_{p,q}$ -set.*

*Proof.* We show that  $r_n \sim (\log n)^q/n^{p-1}$  and from this it is easy to see that  $h_{p,q}(b_n) \sim 1/n$ . First suppose that  $q \geq 0$ . In this case we have that

$$r_n \geq (\log n)^q \sum_{k \geq n} k^{-p} \geq c_1 \frac{(\log n)^q}{n^{p-1}}$$

and

$$r_n \leq \int_{n-1}^{\infty} \frac{(\log t)^q}{t^p} dt,$$

so integrating by parts we get

$$\begin{aligned} r_n &\leq \frac{1}{p-1} \left( \frac{(\log(n-1))^q}{(n-1)^{p-1}} + q \int_{n-1}^{\infty} \frac{(\log t)^{q-1}}{t^p} dt \right) \\ &= c_p \frac{(\log n)^q}{n^{p-1}} + O\left(\frac{(\log n)^{q-1}}{n^{p-1}}\right) \\ &\leq c_2 \frac{(\log n)^q}{n^{p-1}}, \end{aligned}$$

where  $c_1$  and  $c_2$  are constants depending only on  $p$  and  $q$ .

Now suppose that  $q < 0$ . In this case it is easy to see that  $r_n \leq c_3(\log n)^q/n^{p-1}$ . On the other hand, integrating by parts twice and since  $q(q-1) > 0$  we obtain

$$\begin{aligned} r_n &\geq \frac{1}{p-1} \left( \frac{(\log n)^q}{n^{p-1}} + \frac{q}{p-1} \frac{(\log n)^{q-1}}{n^{p-1}} + \frac{q(q-1)}{p-1} \int_n^{\infty} \frac{(\log t)^{q-2}}{t^{p-1}} dt \right) \\ &\geq \frac{1}{p-1} \frac{(\log n)^q}{n^{p-1}} \left( 1 + \frac{q}{p-1} (\log n)^{-1} \right); \end{aligned}$$

taking  $n$  sufficiently large it follows that  $r_n \geq c_4(\log n)^q/n^{p-1}$ .  $\square$

Note that  $h_{p,q} \preceq h_{s,t}$  if and only if  $(1/p, q) \leq_l (1/s, t)$  ( $\leq_l$  denote the lexicographical order in  $(0, 1) \times \mathbb{R}$ ). Define  $\mathcal{H}^{\frac{1}{p}, q} = \mathcal{H}^{h_{p,q}}$ , so that  $\mathcal{H}^{\frac{1}{p}, 0} = \mathcal{H}^{\frac{1}{p}}$ . As a consequence of the above result we can conclude the following. If  $q < 0$ ,  $(1/p, q) <_l (1/p, 0)$ , and since  $\mathcal{H}^{p,q}(C_{p,q}) < +\infty$ , we have that  $\mathcal{H}^{\frac{1}{p}, q}(C_{p,q}) = 0$ . On the other hand, if  $q > 0$ , then  $(1/p, 0) <_l (1/p, q)$  and  $\mathcal{H}^{p,q}(C_{p,q}) > 0$  implies that  $\mathcal{H}^{\frac{1}{p}, q}(C_{p,q}) = +\infty$ . Hence, the family  $\{h_{p,q}\}$  provides a more accurate classification than the usual  $\{g_s\}$ , i.e., it distinguishes more sets; furthermore, the dimension induced by this family can be seen as the set  $((0, 1) \times \mathbb{R}) \cup \{(0, 0)\} \cup \{(1, 0)\}$  ordered with  $\leq_l$ , and the set function  $\dim$  turns out to be the restriction of this order to  $(0, 1) \times \{0\}$ .

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