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TOPOLOGICAL ANALYSIS OF CHAOTIC ORBITS: REVISITING HYPERION

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ABSTRACT

There is emerging interest in the possibility of chaotic evolution in astrophysical systems. To mention just one example, recent well-sampled ground-based observations of the Saturnian satellite Hyperion strongly suggest that it is exhibiting chaotic behavior. We present a general technique, the method of close returns, for the analysis of data from astronomical objects believed to be exhibiting chaotic motion. The method is based on the extraction of pieces of the evolution that exhibit nearly periodic behavior—episodes during which the object stays near in phase space to some unstable periodic orbit. Such orbits generally act as skeletal features, tracing the topological organization of the manifold on which the chaotic dynamics takes place. This method does not require data sets as lengthy as other nonlinear analysis techniques do and is therefore well suited to many astronomical observing programs. Well sampled data covering between twenty and forty characteristic periods of the system have been found to be sufficient for the application of this technique. Additional strengths of this method are its robustness in the presence of noise and the ability for a user to clearly distinguish between periodic, random, and chaotic behavior by inspection of the resulting two-dimensional image. As an example of its power, we analyze close returns in a numerically generated data set, based on a model for Hyperion extensively studied in the literature, corresponding to nightly observations of the satellite. We show that with a small data set, embedded unstable periodic orbits can be extracted and that these orbits can be responsible for nearly periodic behavior lasting a substantial fraction of the observing run.

Subject headings: celestial mechanics, stellar dynamics — methods: analytical — planets and satellites: individual (Hyperion, Saturn)

1. INTRODUCTION

Nonlinear dynamics and chaos have been very active fields of research in applied mathematics and physics in the last 10 years. A century ago, the study of nonlinear dynamics was strongly tied to celestial mechanics; in fact, it was the search for a solution to the gravitational three-body problem that motivated Poincaré to develop the first qualitative studies in this area (Poincaré 1892; Holmes 1990). Interest in this problem was linked to predicting observables within the solar system such as the perturbation on some bodies due to the non-negligible mass of Jupiter.

Given this historical connection, it is ironic that there exists today a gap between the recently developed theoretical dynamical methods and observational astronomy. This is largely due to the natural restriction on the lengths of most astronomical data sets. The presence of universalities in bifurcations has caught the attention of many physicists, biologists, and others (Bai-Lin 1984), but it is not possible (or desirable) to make the necessary parameter changes in such a study of the dynamical evolution of celestial bodies to uncover similar bifurcations. Fortunately, additional methods have been proposed recently to address problems in which all information must be gleaned from only one point in parameter space. These methods are better suited for analyzing celestial mechanical data sets. The most widespread approach to this problem is the study of the metric properties of the chaotic solution, such as the calculation of Lyapunov characteristic exponents and fractal dimensions (Lichtenberg & Lieberman 1982; Grassberger & Procaccia 1983). Unfortunately, this method usually involves the analysis of large amounts of data (on the order of 10^4–10^5 points), making the comparison of information obtained numerically with astronomical observations virtually impossible.

In our solar system there is an object that appears to be undergoing chaotic rotation, the Saturnian satellite Hyperion. The rotational dynamics of this highly aspherical body was theoretically predicted to be chaotic by Wisdom, Peale, & Mignard (1984). Sound evidence supporting this claim was presented recently by Klavetter (1989a, b), who performed well-sampled, precise, ground-based observations covering 3.5 orbital periods. Klavetter found no acceptable periodic fit to the data and invested a great deal of effort fitting an acceptable *chaotic* rotation curve to his observations. The properties of Hyperion used by Klavetter are listed in Table 1.

In this paper we present a recently proposed general method (Mindlin et al. 1990; Tufillaro, Solari, & Gilmore 1990) for analyzing a fairly small chaotic data set, and apply it to a simplified model for the rotational behavior of Hyperion. This

TABLE 1

PROPERTIES OF HYPERION[a]

Parameter	Value	Reference
Body axes, a, b, and c	$185 \times 140 \times 113 \pm 10$ km	1
Semimajor axis	24.55 R_{Saturn}	2
Orbital period	21.277 days	2
Eccentricity	0.1042	2

[a] From Klavetter 1989b.

REFERENCES.—(1) Thomas & Veverka 1985; (2) Woltjer 1928.

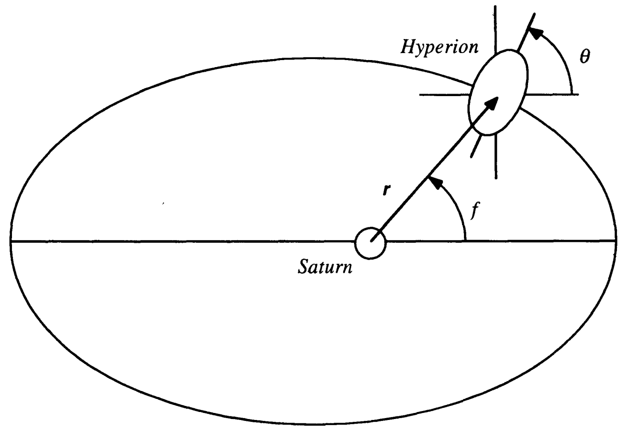

FIG. 1.—Schematic diagram showing the important angular variables in the spin-orbit coupling of Hyperion. The ellipsoid is assumed to be oriented in the orbital plane such that the two important dynamical variables are θ and θ'. The periodicity of f provides the forcing for the system.

method can, in principle, be applied to various other astronomical time series as well. The technique is known as "the method of close returns"; it seeks to identify closed orbits of low period (typically on the order of a few times the driving period) and to determine how these orbits are organized topologically among themselves. This organization provides information about the mechanisms responsible for generating chaotic dynamics. The method has been used successfully to refine theoretical models when applied to dissipative dynamical systems (Mindlin & Gilmore 1992). The paper is organized as follows. In § 2 we review the model studied by Wisdom et al. (1984). This model is used to generate a numerical time series mimicking the light curve of Hyperion. In § 3 we show how to extract the unstable periodic orbits, to which the trajectory stays near, from a chaotic data set. We apply this method to our numerical data set, generated using Klavetter's measurements of the body parameters, with various initial conditions. In § 4 we present our results in the form of two-dimensional *close returns images*, in which stretches of nearly periodic behavior are evident for long segments of the time series. In § 5 we discuss the strengths and weaknesses of the method and state our conclusions.

2. THE MODEL

The present study is based on a model that consists of an orbiting body in the shape of a triaxial ellipsoid parametrized by the axes of inertia, $C \geq B \geq A$, spinning only about its largest axis of inertia, which is assumed to be oriented perpendicular to the orbital plane (Fig. 1). This is a reduction of the full six-dimensional system for rotational motion about all three principal axes. In the spirit of Wisdom, Peale, Mignard (1984) and Klavetter (1989b) we use this dynamical system as a numerical laboratory. (We have chosen to study the reduced system for clarity in displaying the results—the orbits we extract can be easily visualized in two phase-space dimensions as opposed to six. In practice, our methods can be applied to systems with arbitrary numbers of variables.) The above reduction yields the dynamical equation for the rotational angular variable θ,

$$\theta'' + \frac{\omega^2}{2r^3} \sin 2(\theta - f) = 0 , \tag{1}$$

where f is the true anomaly of the satellite, the prime indicates the time derivative, and ω is a measure of the deviation of the body from sphericity, i.e.,

$$\omega^2 = \frac{3(B - A)}{C} . \tag{2}$$

The units in equation (1) are such that one orbital period is 2π, and the semimajor axis of the orbit is 1. The time dependence of r, present for all eccentricities greater than zero, essentially provides a periodic forcing to the system at each periastron passage.

Note that in the case of circular orbits r is no longer a function of time. If we let $\Psi = 2(\theta - f)$ then $\Psi' = 2(\theta' - f')$. For circular orbits $f' = 1$, so $\Psi'' = 2\theta''$, and the equation of motion can be written as

$$\Psi'' + \frac{\omega^2}{a^3} \sin \Psi = 0 , \tag{3}$$

which is an *integrable* equation, namely that describing the motion of a simple pendulum. In this integrable limit, three qualitatively different types of motion are present for the system, parameterized by the total energy. For high energy the state of motion is rotational. For low energy the system will present librations about $\Psi = 0$. The two separatrices between these qualitatively different kinds of motion are known as the homoclinic orbits that connect $(\Psi, \Psi') = (\pi, 0)$ with itself. They correspond to a clockwise (counterclockwise) rotation of the pendulum of infinite period. It should be noted that these orbits are highly degenerate structures; every point on either of them belongs to both the stable and unstable manifolds of the fixed point $(\pi, 0)$.

For orbits with eccentricity $e \neq 0$, the system (1) differs from the intergrable limit by the presence of the nonlinear driving term, which acts to complicate the dynamics considerably. Note that equation (1) contains no dissipative terms, hence the system is Hamiltonian. It is routine in the study of Hamiltonian systems to consider a Poincaré section (alternatively called a "surface of section"), obtained by plotting the phase-space variables θ and θ' at each periastron passage (i.e., $f = 0$). From the patterns described by these points, we are able to determine in which regions of phase space regular islands dominate, as well as which regions are dominated by chaotic motion—regions referred to as "chaotic seas." The Poincaré section for Hyperion is shown in Figure 2a. It consists of 22,200 points (200 points for 111 separate initial conditions).

To obtain these points we integrated the equations

$$\theta' = \phi ,$$
$$\phi' = \frac{-\omega^2}{2r^3} \sin 2(\theta - f) \quad , \qquad (4)$$
$$f' = \frac{(1 + e \cos f)^2}{(1 - e^2)^{3/2}}$$

(Goldreich 1966). We take the asphericity parameter ω to be 0.89 in keeping with the results quoted by Klavetter (1989b). The main feature of Figure 2*a* is the large chaotic sea that separates islands of regular motion. An initial condition within this region displays chaotic rotation. However, it must be noted that there exist unstable periodic orbits even in the chaotic region; in fact, regular and chaotic motions are known to coexist on all scales in the phase space of Hamiltonia systems (see Dana 1990). This characteristic causes chaotic orbits to resemble a collection of almost periodic segments, each acting like (a piece of) a regular trajectory. We rely on this characteristic of chaotic data sets in our method described below.

We point out that a useful Poincaré section for Hyperion cannot be generated using observational data because of the limited number of data points obtained, even in a long-baseline observing program. In addition, an observer is restricted to only one point in the space of initial conditions, (i.e., the point that corresponds to that particular trajectory being observed), whereas numerous initial conditions must be explored when generating a meaningful Poincaré section. As an example, we generate a Poincaré section from a numerically generated data set of 950 nightly observations. In Figure 2*b* we show a hypothetical attempt to display almost 2.6 yr (=950 nights) worth of "observational" data as a Poincaré map. With so few points corresponding to near-periastron passage (45 points in this case), it is impossible to make out any but the slightest resemblance to Figure 2*a*. It is for this reason that metric techniques are not appropriate when dealing with the typical astronomical data set. Fortunately, the method of close returns allows for the investigation of shorter time series, most often with more meaningful results than those provided by the metric approach. In what follows, we analyze the same 2.6 yr numerical data set with the method of close returns.

3. THE SEARCH FOR CLOSE RETURNS

The method of close returns is based on the fact that numerous unstable perodic orbits exist in regions of phase space where chaotic motion takes place. This is true in general for dynamical systems; in particular, for Hamiltonian systems the unstable periodic orbits are dense.

As the point describing the state of a system evolves through phase space, it eventually enters the neighborhood of some unstable orbit of low period. At this point, it is attracted to that orbit along the orbit's stable direction and repelled from the orbit along its unstable direction. The point eventually moves away from the unstable periodic orbit after a certain time, depending on how near to the orbit it is in the stable direction and how strong the repulsion is in the unstable direction (measured by the unstable eigenvalue of the periodic orbit, λ). If it is not too strongly repelled, and the orbit is of sufficiently low period (P), measured in multiples of the driving period, then the trajectory may stay close to the unstable periodic orbit and return to its initial neighborhood one or more times before eventually evolving to other regions of phase space. In other words, during these stretches of the evolution, the trajectory mimics the neighboring unstable periodic orbit. In this way, *pulling out the close returns is akin to extracting from the evolution those unstable periodic orbits to which the trajectory stays near.*

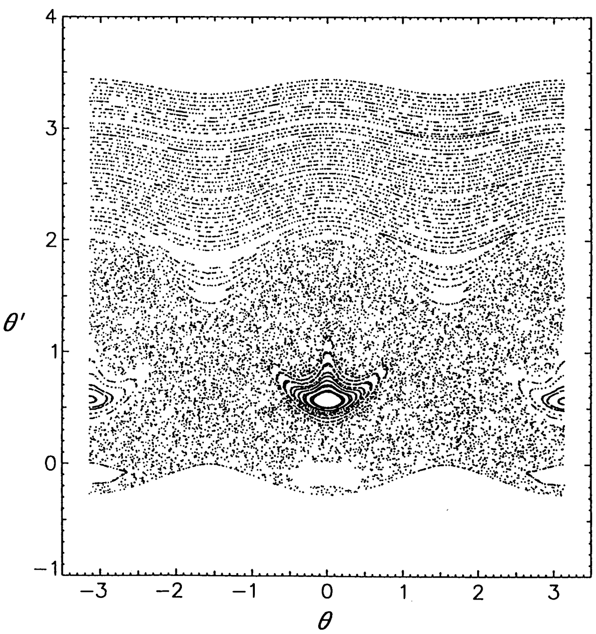

FIG. 2*a*

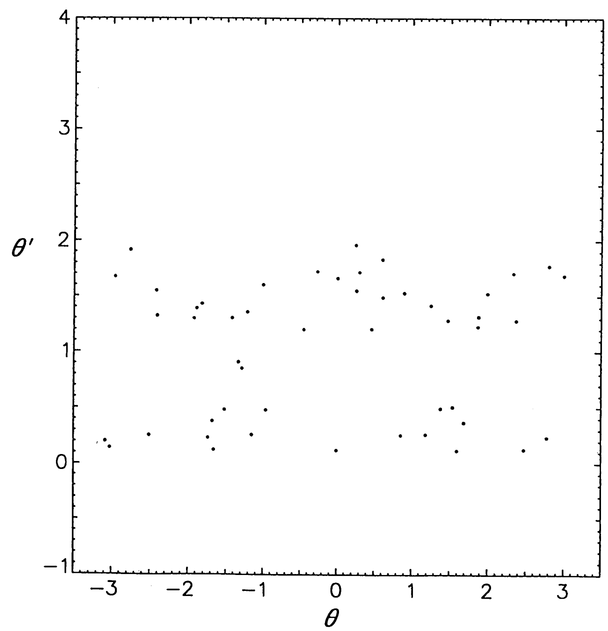

FIG. 2*b*

FIG. 2.—Poincaré section for the two-dimensional model of Hyperion. (*a*) This section was generated by collecting at each periapsis θ and θ' at 200 passages for each of 111 initial conditions. Note the large chaotic zone for a body such as Hyperion with large asphericity. (*b*) This section is the result of plotting 40 hypothetical observations, occurring near periapsis, over the course of 2.6 yr. Only the slightest resemblance to (*a*) can be detected from this plot.

Such close returns are more likely for shorter periods and smaller unstable eigenvalues. The product λP is a useful indicator that the system will evolve in the neighborhood of an unstable periodic orbit for a long time; the smaller the product, the more likely a close return near that orbit will be seen in the data. From the value of λ we can determine how far a trajectory is from the unstable periodic orbit at some time, given by $d(t)$, as a function of its distance at some initial time, given by $d(0)$, i.e., $d(t) = d(0) \exp(\lambda t)$. Here, $d(0)$ is assumed small. For close returns of period P, the difference $d(t) - d(0)$ must be on the order of a few times $d(0)$. In other words,

$$\frac{d(P) - d(0)}{d(0)} = 1 + \eta = \exp(\lambda P) - 1 , \qquad (5)$$

where η is small. Since $2 + \eta = e^{\lambda P}$, this implies that λP is on the order of 1, ($\lambda P \sim 1$). Equivalently, since $P = nT$, where T is the driving period of the system, then for close returns of period P to be present in the data,

$$\lambda_n \approx \frac{1}{nT} . \qquad (6)$$

Equation (6) is just an estimate. In general, successful analysis with the method of close returns involves the extraction of orbits with periods in the range of 1 to 5 times the driving period ($1 \leq P/T \leq 5$), with unstable eigenvalues in the range of $1 < \lambda < 3$.

Operationally, the observer is in the position of having a data set only, usually with no a priori knowledge of the values of λ for different orbits. Based on the above, then, if in a particular data set close returns are often found of period P_1, but rarely of period P_2, we can infer the relative values of λ_1 and λ_2.

The search for close returns can be carried out easily in a discretely sampled scalar or vector data set, $\boldsymbol{x}(i) = [x_1(i), x_2(i), \ldots, x_n(i)]$, $i = 1, 2, \ldots, t$. Here n is the dimension of the vector $\boldsymbol{x}$, and t is the number of data points in the time series. This is most conveniently done by computing the phase-space distance between $\boldsymbol{x}(i)$ and $\boldsymbol{x}(i + p)$, given by $d[\boldsymbol{x}(i) - \boldsymbol{x}(i + p)]$, where i locates the vector in the time series, p identifies the lag, or separation between two points being compared, in terms of the time sample rate, and d is any reasonable metric. This gives a two-dimensional (i vs. p) grid of d values—the *close returns image*. This information is then visually implemented by color-coding the (i, p)th pixel black if $d[\boldsymbol{x}(i) - \boldsymbol{x}(i + p)] \leq \epsilon$, and white otherwise. The threshold ϵ is typically taken to be in the range 10 ± 1% of the maximum $d[\boldsymbol{x}(i) - \boldsymbol{x}(i + p)]$. In this visual representation, close returns appear as nearly horizontal line segments in the close returns image, which are easily seen by inspection (see Fig. 3).

The appearance of the close returns image is often enlightening in itself. A close returns image for a random evolution has the appearance of a scatter diagram, with very short, if any, horizontal segments, i.e., no close returns. On the other hand periodic evolution manifests itself by producing a close returns image with continuous horizontal segments that stretch the entire i axis and are separated from each other in the p-direction by a lag corresponding to the periodicity of the motion.

For the present Hyperion study, the vector $\boldsymbol{x}(i) = [\theta(i), \theta(i)']$, where $i = 1, 2, \ldots, 950$. To obtain $[\theta(i), \theta(i)']$, the equations of motion (4) were integrated with a 4th order Runge-Kutta procedure. The step size was reduced until convergence was achieved. The sampling rate corresponded to one-night intervals. One orbital period corresponded to approximately 21 points. We recorded 950 points ($\sim$45 orbital periods) for each of the numerical experiments carried out, four of which are discussed in the following section. Our data sets simulate observations of $\sim$2.6 yr. While this may seem like an enormous data set to observers accustomed to relatively short observing runs, it is staggeringly small to workers in the field of nonlinear dynamics who employ metric techniques.

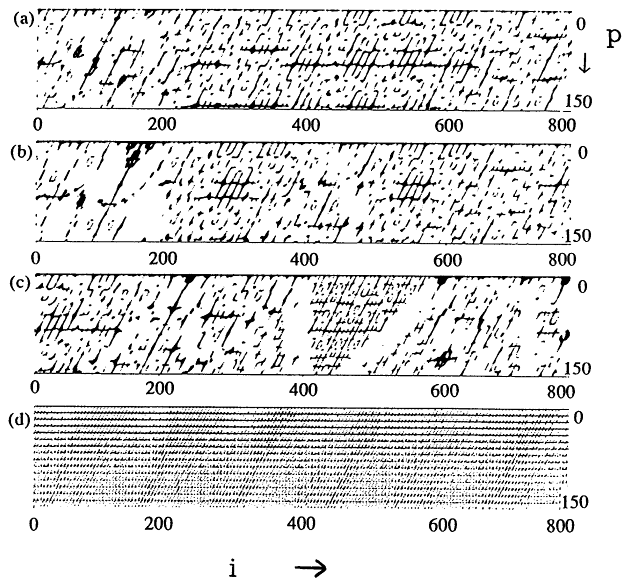

FIG. 3.—Close return images for four separate initial conditions of the Hyperion system. The body parameters are fixed, and the initial value of f is taken as zero in each experiment. The initial conditions are (*a*) $\theta = 1.0$, $\theta' = 0.5$; (*b*) $\theta = 0.7$, $\theta' = 1.0$; (*c*) $\theta = 0.5$, $\theta' = 0.5$; and (*d*) $\theta = 2.0$, $\theta' = 2.0$. The horizontal axis is the position i of a point in the data file that is compared with another point in the data file separated by p nights, with p plotted vertically downward. If the phase-space distance between the two points is less than 10% of the maximum, we plot a black point at (i, p).

In general, our experience indicates that the method of close returns works well when a data set contains at least 15 readings per cycle (driving period) and 20 to 40 cycles of the evolution of a dynamical system. In this case, it is generally possible to extract four or five unstable periodic orbits. The reliability of the results improve with the sampling rate and length of the data set.

Once close returns have been extracted from a data set, the next step is to study the topological properties of these orbits. The idea behind this involves considering the orientation of the low-period obits with respect to one another in phase space. A comprehensive discussion of this method and its application to a dissipative dynamical system can be found in Mindlin & Gilmore (1992). We do not carry out further topological analysis on the orbits extracted from our numerical data set in this paper.

4. DISCUSSION

Chaotic solutions behave like a series of different (parts of) regular episodes associated with periodic orbits. These episodes give rise to close returns in a chaotic time series, associated with the lower period unstable periodic orbits. This means that, during some times, the chaotic solution behaves nearly like a periodic orbit.

In our numerical experiments to simulate observational data of the rotation state of Hyperion, the physical parameters are kept constant (see Table 1); the initial conditions (θ, θ') are chosen to be in different regions of phase space. Each data set has 950 points corresponding to nightly observations. Our close returns analyses compare each of the first 800 points with the 150 points following them. The corresponding 800×150 images which contain the close return information are shown in Figures 3*a*–3*d*. Figures 3*a*, 3*b*, and 3*c* were generated from initial conditions lying within the large chaotic region, while Figure 3*d* was generated from an initial condition belonging to a regular region of phase space (see Fig. 2*a*).

According to the definition of a close return given in the previous section, a location in the i versus p close returns image is black if, at that particular value of i, the trajectory was at a phase-space location that is very closely matched p time steps later. A horizontal segment of black then means that the trajectory very nearly repeated itself for that duration of time. Thus, a relatively longer horizontal segment in a close returns image corresponds to a trajectory that repeats its earlier behavior for a longer time. For Figures 3*a*–3*d*, a horizontal segment of length l starting at location (i, p) corresponds to a close return in the data set of period p, beginning at location i and lasting l days.

In each of the four numerical experiments, close returns of low periodicity (i.e., 3, 4, and 7 times the orbital period) are evident as dark horizontal line segments that endure for at least one orbit. These results are summarized in the Tables 2, 3, and 4, where we have listed the locations and duration of the most prominent close returns for the images of Figures 3*a*, 3*b*, and 3*c*. Since the experiment corresponding to Figure 3*d* was run using initial conditions within a regular region, (θ, θ') = 2.0, 2.0, close returns are regularly spaced and endure for the entire data set. The return after $\sim$10.5 days clearly indicates a 2:1 spin-orbit resonance. This is qualitatively different from the results obtained in any of the chaotic experiments, where several different regular orbits are visited, for short amounts of time.

TABLE 2

PERIODS OF CLOSE RETURNS IN FIGURE 3*a*

PARAMETER	P: 3	7	4	3	4	2
Duration	37	154	66	66	254	57
Starting i	125	215	221	309	367	737
Comments	...	~5 months	...	...	~8.5 months	...

NOTES.—P is given in units of the orbital period. The duration is given in days.

TABLE 3

PERIODS OF CLOSE RETURNS IN FIGURE 3*b*

PARAMETER	P: 4	4	3	2
Duration	40	94	56	55
Starting i..........	1	235	264	680
Comments	...	~3 months	...	...

TABLE 4

PERIOD OF CLOSE RETURNS IN FIGURE 3*c*

PARAMETER	P: 4	3	4	2	6	3
Duration	111	49	66	99	40	25
Starting i	14	247	406	449	587	627
Comments	~3.5 months	...	...	~3 months	...	...

The close return information for Figure 3*a* is displayed in Table 2. The initial conditions for this run are $\theta = 1.0$ radians, $\theta' = 0.5$ rotations per orbital period, and $f = 0.0$. The periods of the most obvious close returns are listed along with their duration and their starting (i) position in the image, i.e., the first close return is of period 3 ($\sim$63 days) and begins 125 units from the left edge of Figure 3*a* (corresponding to 125 days after the first observation). Also included in the table is a "comments" row, in which we list the duration in months of the longer lived close return episodes. The very long period 4 close return in Figure 3*a* lasts approximately 8.5 months; hence, an observer confined to this span of time might erroneously conclude a periodic rotation state from an inherently chaotic evolution.

In Figures 4*a* we plot two separate period 4 close return orbits occurring during this 8.5 month interval. In these figures, the pentagons correspond to nightly "observations" and the dotted lines to the full numerical trajectories from which we extracted the nightly points. The similarity of the two period 4 trajectories is obvious, as well as the fact that the orbits do not exactly close on themselves in four orbital periods, but come very close to doing so (hence the term "close return"). Figures 3*b* and 3*c* are similar to 3*a*; their quantitative aspects can be found in the two remaining tables. In all of the images, the close return features become weaker and of a shorter duration as the time increases (i.e., as we go down the p-axis). This is due to the fact that the close return behavior we are studying occurs only for short times. Thus, the study of close returns in a chaotic data set is ideally suited to observational data files of lengths previously mentioned; in the particular case of Hyperion, for which the period of the forcing is on the order of 21 days, a data file corresponding to 2.6 yr ($\sim$45 orbital periods) was adequate to observe at least four close returns for each of the three chaotic initial conditions.

To date, the most ambitious observational study of Hyperion is that performed by Klavetter, who collected 63 points covering approximately 3.5 orbital periods of the satellite. His attempts to fit these data to periodic rotation ranging between 2 hr and 30 days all exhibited large deviations (Klavetter 1989b). Klavetter further analyzed the light curve by finding the best numerical fit to the initial conditions of the full six-dimensional nonlinear system. This involved a tremendous amount of computer time resulting in a well-determined point in the six-dimensional phase space. It is certainly worthwhile pinpointing the region of phase space in which the dynamics of Hyperion lives. With a longer data set of the same sampling rate, the close return analysis can be performed yielding, as shown above, the unstable periodic orbits underlying the chaotic rotation.

We note that our analysis is meant to exemplify the application of the method of close returns to a particular observational data set. This method is in no way restricted to the study of the rotation state of celestial bodies. In fact, numerous astronomical data sets exist for which a better understanding of the time evolution is desired. These include climate records, variations in sunspot number, cataclysmic variable light

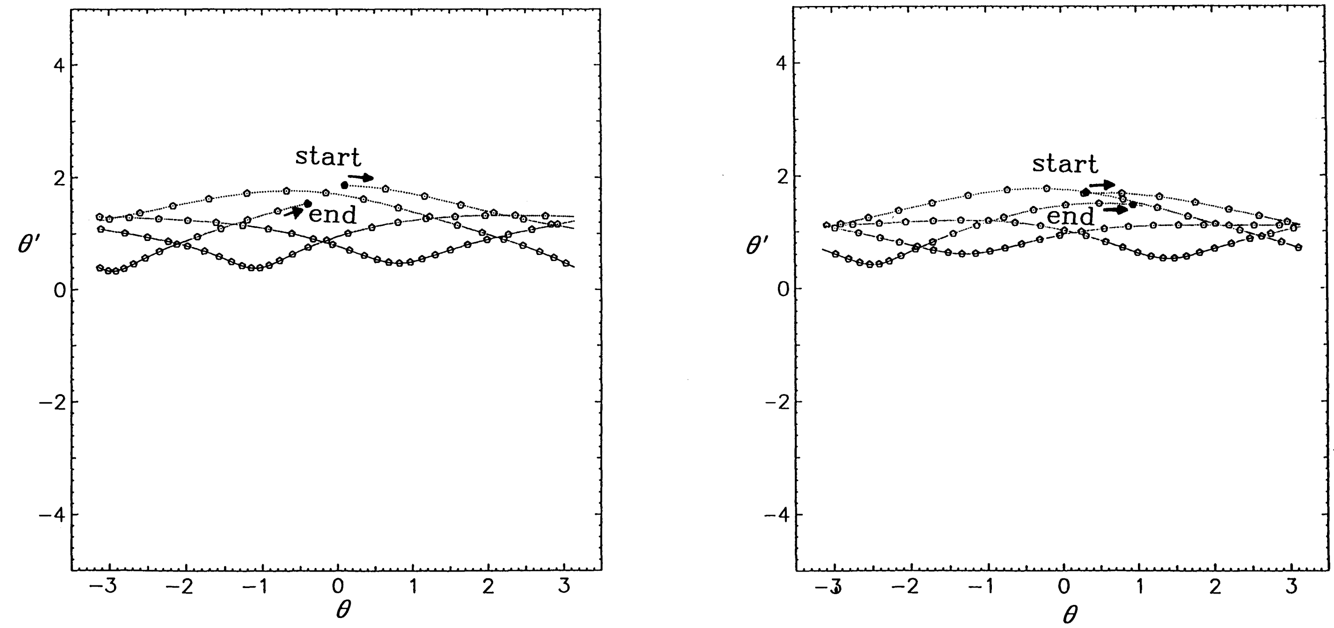

FIG. 4.—Two period 4 orbit segments corresponding to the long-lived close return in Fig. 3*a*. The pentagons denote data points corresponding to nightly observations. The dotted line shows the full integration. These segments do not quite close up.

curves, as well as observations of perhaps "multiply periodic" pulsating variables.

5. CONCLUSIONS

We have presented a method for analyzing fairly small data sets which may be exhibiting chaotic motion. The method of close returns is ideally suited to data sets which contain about 15 readings per cycle (or driving period) and 20 to 40 cycles of the evolution of a dynamical system. The technique introduces two additional observables in an observational data set, namely the periods of the major close returns and the duration of these features. The appearance of the resulting *close returns image* can yield information about whether the system under study is random, periodic, or chaotic in nature and, if chaotic, which unstable periodic orbits are more likely to be revisited during the dynamical evolution of the object. The power of this method is that with a relatively small number of data points, it yields information about the unstable periodic orbits embedded within a chaotic time series. These orbits are not only the tracers of the topology of the phase space in which the chaotic solution evolves, they also allow us to test different theoretical models against the observational evidence by studying the spectrum of orbits present in the chaotic data sets (Mindlin & Gilmore 1992). The study of the unstable periodic orbits as a tool for the modeling of dynamical systems has been shown to be successful when applied to dissipative systems (Mindlin et al. 1991). In these systems, the number of twists present in various attractors embedded in phase space was determined by studying the lower period close returns. Our results indicate the applicability of these techniques to small data sets arising in Hamiltonian systems as well.

We have shown four numerical examples of well-sampled observations which all show the ubiquitous nature of close return episodes for the $2 + 1$ dimensional reduced model of Hyperion. In each case, the close returns analysis yielded at least four, and up to six, episodes of nearly repeated behavior which occur when the dynamical system is near in phase space to unstable periodic orbits. From the close returns episodes, we were able pull out pieces of the trajectories that most closely mimicked the true unstable periodic orbits embedded within the phase space.

It should be mentioned that close returns occurring in the full six-dimensional Hyperion system would also be present in any lower dimensional subset of the phase space, since the six-dimensional vector describing the phase space "distance" of two points is minimized only when all of its components are likewise minimized. In other words, if it were possible to directly observe only one or two of the six-phase-space variables, a close returns analysis on such a restricted set would still produce close returns that related to the full six-dimensional dynamics. We also note that, for a system of arbitrary dimension, a two-dimensional close return image such as those shown in Figure 3 will always result from this analysis, since the image is the comparison of a scalar distance between two points in an n-dimensional phase space.

We foresee two independent directions for the continuation of this work. The first is the complete topological analysis of the full six-dimensional problem with the techniques outlined in this work. As mentioned previously, the method of close returns for finding unstable periodic orbits will yield insight into the nearly periodic states of the six-dimensional motion that take place during the chaotic evolution. It is expected that the topological organization of the extracted orbits, such as their relative rotation rates and linking numbers (Tufillaro et al. 1990) will provide insight into the topological organization of phase space in Hamiltonian systems.

Another useful direction is a topological analysis of the spin-orbit coupling problem for a generic nonspherical satellite well approximated by an ellipsoid, which is a $2 + 1$ dimensional Hamiltonian system. In this case, the unstable orbits are closed curves embedded in a three-dimensional phase space and will therefore be knotted in a particular way. The topological description of these knots gives insight into the mechanism underlying the transition to chaos in the dynamical system under study. To date, this method has never been implemented in the study of Hamiltonian systems, although it has proven fruitful in the study of dissipative systems (Mindlin et al. 1991).

The close returns analysis has the further advantage that it clearly distinguishes between chaotic and stochastic processes, the former giving rise to horizontal segments, the latter to a nearly uniform distribution of short accidental close returns. It may be fruitful to apply these techniques to astrophysical data sets for which the periodicity as well as the underlying dynamics are not as yet determined.

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