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De Rham and Infinitesimal Cohomology in Kapranov's Model for Noncommutative Algebraic Geometry

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Abstract. The title refers to the nilcommutative or *NC*-schemes introduced by M. Kapranov in 'Noncommutative Geometry Based on Commutator Expansions', *J. Reine Angew. Math* **505** (1998) 73–118. The latter are noncommutative nilpotent thickenings of commutative schemes. We also consider the parallel theory of nil-Poisson or *NP*-schemes, which are nilpotent thickenings of commutative schemes in the category of Poisson schemes. We study several variants of de Rham cohomology for *NC*- and *NP*-schemes. The variants include nilcommutative and nil-Poisson versions of the de Rham complex as well as of the cohomology of the infinitesimal site introduced by Grothendieck in Crystals and the de Rham Cohomology of Schemes, *Dix exposés sur la cohomologie des schémas*, Masson, Paris (1968), pp. 306–358. It turns out that each of these noncommutative variants admits a kind of Hodge decomposition which allows one to express the cohomology groups of a noncommutative scheme Y as a sum of copies of the usual (de Rham, infinitesimal) cohomology groups of the underlying commutative scheme X (Theorems 6.1, 6.4, 6.7). As a byproduct we obtain new proofs for classical results of Grothendieck (Corollary 6.2) and of Feigin and Tsygan (Corollary 6.8) on the relation between de Rham and infinitesimal cohomology and between the latter and periodic cyclic homology.

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1. Introduction

In this paper we study the de Rham theory of the nilcommutative or *NC*-schemes introduced by Kapranov in [17]. To start, let us recall the definitions of *NC*-algebras and schemes and introduce differential forms for such objects. We consider algebras and schemes over a fixed field k of characteristic zero. Recall an associative algebra R is *nilcommutative of order* $\leqslant l$ or an NC_l-algebra if for the *commutator filtration*

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$$F_0R = R, \qquad F_{n+1}R := \sum_{p=1}^{n} F_pR \cdot F_{n+1-p}R + \sum_{p=0}^{n} \langle [F_pR, F_{n-p}R] \rangle \tag{1}$$

we have $F_{l+1}R = 0$. For every NC_l-algebra R there is defined a noncommutative locally ringed space $\operatorname{Spec} R$. Its underlying topological space is the prime spectrum of the commutative algebra $A = R/F_1R$; the stalk at $\mathfrak{p} \in \operatorname{Spec} A$ is the Øre localization of R at the inverse image of $\mathfrak{p}$ under the projection $R \twoheadrightarrow A$. Prime spectra of NC_l-algebras are called *affine NC_l-schemes*. In general, an NC_l-scheme is a locally ringed space that can be covered by affine NC_l-schemes. If $Y = (Y, \mathcal{O}_Y)$ is an NC_l-scheme then $Y^{[0]} = (Y, \mathcal{O}_Y/F_1\mathcal{O}_Y)$ is a commutative (i.e. usual) scheme. In particular an NC_0-scheme is just a commutative or Comm-scheme. There is a natural notion of differential forms for NC_l-schemes, as follows. For $R \in NC_l$ we define its $NC_l - DGA$ of differential forms as the quotient

$$\Omega_{NC_l}R = \frac{\Omega R}{F_{l+1}\Omega R} \tag{2}$$

of the usual DGA of noncommutative forms ([9]) by the $l + l$th term of its commutator filtration (taken in the DG sense). For example if R is commutative and $l = 0$, then this is the usual commutative DGA of Kähler forms. One checks that Ω_{NC_l} localizes (in the Øre sense), and thus defines a sheaf of $NC_l - DGA$'s on $\operatorname{Spec} R$ which is quasi-coherent in the sense that each term $\Omega^p_{NC_l}$ comes from an R-bimodule and its (Øre) localizations (see Sections 2 and 4 below). Thus for every NC_l-scheme Y there is defined a quasi-coherent sheaf Ω_{NC_l} of $NC_l - DGA$'s. We compute its cohomology in the formally smooth case. Recall from [17] that an NC_l-algebra R is formally NC_l-smooth if $\hom_{NC_l}(R, \cdot)$ carries surjections with nilpotent kernel into surjections. We call an NC_l-scheme formally NC_l-smooth if it can be covered by spectra of formally NC_l-smooth algebras. We remark that Y formally NC_l-smooth $\Rightarrow Y^{[0]}$ formally Comm-smooth. We show (Corollary 6.2) that if Y is formally NC_l-smooth, then for $X = Y^{[0]}$,

$$\mathbb{H}^*(Y_{\mathrm{zar}}, \Omega_{NC_l}) := \mathbb{H}^*(X_{\mathrm{zar}}, \Omega_{NC_l}\mathcal{O}_Y) = \mathbb{H}^*(X_{\mathrm{zar}}, \Omega_{\mathrm{Comm}}) =: H^*_{\mathrm{d}R}X \tag{3}$$

is just the usual de Rham cohomology of the underlying commutative scheme. In the affine case, because of the quasi-coherence of the Zariski sheaf Ω_{NC_l} its hypercohomology is just the cohomology of its global sections (cf. (27)) and we have

$$H^*(\Omega_{NC_l}R) = \mathbb{H}^*(\operatorname{Spec} R, \Omega_{NC_l}) = \mathbb{H}^*(\operatorname{Spec} A, \Omega_{\mathrm{Comm}}) = H^*(\Omega_{\mathrm{Comm}}A) =: H^*_{\mathrm{d}R}A.$$

This contrasts with the fact that for every $R \in \mathrm{Ass}$ the usual DGA of noncommutative forms is acyclic, that is

$$H^n(\Omega R) = \begin{cases} k & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Recall, however, that if we divide ΩR by its commutator subspace we get the image of the periodicity map in cyclic homology ([19, 2.6.7])

$$H^n\left(\frac{\Omega R}{[\Omega R, \Omega R]}\right) = s(HC_{n+2}R) \subset HC_n R,$$

which is nontrivial in general. For example if A is smooth commutative then ([19, 5.1.12])

$$H^n\left(\frac{\Omega A}{[\Omega A, \Omega A]}\right) = \bigoplus_{2m \leqslant n} H^{n-2m}_{\mathrm{d}R} A$$

We show (see 8.3.1 below) that if R is formally NC_l-smooth, then for $A = R/F_1 R$,

$$H^n\left(\frac{\Omega_{NC_l} R}{[\Omega_{NC_l} R, \Omega_{NC_l} R]}\right) = \bigoplus_{m=0}^{l} H^{n+2m}_{\mathrm{d}R} A.$$

Note that $H^{n+2m}_{\mathrm{d}R} A$ is the nth cohomology of the complex

$$\tau_{2m}\Omega_{\mathrm{Comm}} A\colon \frac{\Omega^{2m}_{\mathrm{Comm}} A}{d\Omega^{2m-1}_{\mathrm{Comm}} A} \to \Omega^{2m+1}_{\mathrm{Comm}} A \to \cdots \tag{4}$$

which has $\Omega^{2m}_{\mathrm{Comm}} A / \mathrm{d}\Omega^{2m-1}_{\mathrm{Comm}} A$ in degree zero. We show (Corollary 6.5) that if Y is a formally NC_l-smooth scheme then for $X = Y^{[0]}$

$$\mathbb{H}^n\left(Y_{\mathrm{Zar}}, \frac{\Omega_{NC_l}}{[\Omega_{NC_l}, \Omega_{NC_l}]}\right) = \bigoplus_{m=0}^{l} \mathbb{H}^{n+2m}(X_{\mathrm{Zar}}, \tau_{2m}\Omega_{\mathrm{Comm}}). \tag{5}$$

Here $X = Y^{[0]}$, and $\Omega_{NC_l}/[\Omega_{NC_l}, \Omega_{NC_l}]$ and $\tau_{2m}\Omega_{\mathrm{Comm}}$ are the sheafified complexes. In particular the 0th term of $\tau_{2m}\Omega_{\mathrm{Comm}}$ is the sheaf cokernel of $d\colon \Omega^{2m-1}_{\mathrm{Comm}} \to \Omega^{2m}_{\mathrm{Comm}}$; it is not a quasi-coherent sheaf. There is always a map $H^{n+2m}_{dR}(X) \to \mathbb{H}^n(X, \tau_{2m}\Omega_{\mathrm{Comm}})$ induced by the projection $\Omega_{\mathrm{Comm}}[2m] \twoheadrightarrow \tau_{2m}\Omega_{\mathrm{Comm}}$ but it is not an isomorphism in general, not even if X is affine. Hence, in general

$$\mathbb{H}^*\left(\operatorname{Spec} R, \frac{\Omega_{NC_l}}{[\Omega_{NC_l}, \Omega_{NC_l}]}\right) \neq H^*\left(\frac{\Omega_{NC_l} R}{[\Omega_{NC_l} R, \Omega_{NC_l} R]}\right). \tag{6}$$

Next we consider a third type of de Rham complex; the periodic $\mathfrak{X}$-complex of [10]. Recall that if R is any algebra then $\mathfrak{X}R$ is the 2-periodic complex with

$$\mathfrak{X}^{\mathrm{even}} R = \Omega^0 R = R, \qquad \mathfrak{X}^{\mathrm{odd}} R = \Omega^1 R_\natural := \frac{\Omega^1 R}{[R, \Omega^1 R]}$$

and with the de Rham differential as coboundary from even to odd degree and the Hochschild boundary from odd to even degree. We compute the cohomology of $\mathfrak{X}$ for formally NC_∞-smooth schemes. Such a gadget consists of a commutative scheme X together with an inverse system of Zariski sheaves

$$\mathcal{O}_{Y_\infty}\colon \cdots \twoheadrightarrow \mathcal{O}_{Y_l} \twoheadrightarrow \mathcal{O}_{Y_{l-1}} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{O}_{Y_1} \twoheadrightarrow \mathcal{O}_{Y_0} = \mathcal{O}_X$$

such that each $Y_l = (X, \mathcal{O}_{Y_l})$ is a formally NC_l-smooth scheme, that $\mathcal{O}_{Y_l}/F_l\mathcal{O}_{Y_l} = \mathcal{O}_{Y_{l-1}}$ and that the map $\mathcal{O}_{Y_l} \twoheadrightarrow \mathcal{O}_{Y_{l-1}}$ is the natural projection. We compute the hypercohomology of the procomplex $\mathfrak{X}\mathcal{O}_{Y_\infty} := \{\mathfrak{X}\mathcal{O}_{Y_l}\}_l$; we show (Corollary 6.9) that

$$\mathbb{H}^n(X_{\mathrm{Pro-Zar}}, \mathfrak{X}(\mathcal{O}_{Y_\infty})) = \prod_{2j \geqslant n} H_{dR}^{2j-n} X. \tag{7}$$

Moreover, the decomposition above is induced by the commutator filtration. On the other hand we prove that for periodic cyclic homology

$$HC_*^{\mathrm{per}}(X) = \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathfrak{X}\mathcal{O}_{Y_\infty}). \tag{8}$$

Putting (7) and (8) together we get the well-known formula ([19, 5.1.12], [23, 3.3])

$$HC_n^{\mathrm{per}}(X) = \prod_{2j \geqslant n} H_{dR}^{2j-n} X. \tag{9}$$

Our proof gets rid of the usual finiteness hypothesis and shows that the so-called Hodge decomposition (9) comes from the commutator filtration.

Each of the results for formally smooth schemes mentioned up to here is deduced from a general theorem which holds without smoothness hypothesis. In the absence of formal smoothness we need to replace de Rham by infinitesimal cohomology. Recall that if X is a commutative scheme then its infinitesimal cohomology is the cohomology of the structure sheaf on the infinitesimal site, which consists of all nilpotent thickenings $U \hookrightarrow T$ of open subschemes $U \subset X$. One can also consider the NC_l-infinitesimal site of any NC_l-scheme Y, consisting of all nilpotent thickenings $U \hookrightarrow T$ of open subsets U of Y with T an NC_l-scheme. We prove that for Y formally NC_l-smooth

$$H^*(Y_{NC_l-\mathrm{inf}}, \mathcal{O}) = \mathbb{H}^*(Y_{\mathrm{Zar}}, \Omega_{NC_l}), \tag{10}$$

$$H^*\left(Y_{NC_l-\mathrm{inf}}, \frac{\mathcal{O}}{[\mathcal{O}, \mathcal{O}]}\right) = \mathbb{H}^*\left(Y_{\mathrm{Zar}}, \frac{\Omega_{NC_l}}{[\Omega_{NC_l}, \Omega_{NC_l}]}\right). \tag{11}$$

The above generalizes the theorem of Grothendieck ([14]) which establishes the case $l = 0$ of (10) (compare also [6, Th. 3.0]). For commutative but not necessarily Comm-formally smooth X, we have (as part of theorems 6.1 and 6.4) the following generalizations of (3) and (5)

$$H^*(X_{NC_l-\mathrm{inf}}, \mathcal{O}) = H^*(X_{\mathrm{Comm-inf}}, \mathcal{O})$$
$$H^*\left(X_{NC_l-\mathrm{inf}}, \frac{\mathcal{O}}{[\mathcal{O}, \mathcal{O}]}\right) = \bigoplus_{m=0}^{l} H^{*+2m}\left(X_{\mathrm{Comm-inf}}, \frac{\Omega^m_{\mathrm{Comm}}}{d\Omega^{m-1}_{\mathrm{Comm}}}\right)$$

In place of (7) and (8) we obtain (as part of Theorem 6.7)

$$HC_n^{\mathrm{per}}(X) = \mathbb{H}^n(X_{NC_\infty-\mathrm{inf}}, \mathfrak{X}) = \prod_{2m \geqslant n} H^{2m-n}(X_{\mathrm{Comm-inf}}, \mathcal{O})$$

Here n is any integer and $NC_\infty - \inf$ denotes the site of all nilpotent thickenings $U \hookrightarrow T$ with T an NC-scheme of arbitrary order. In particular, we recover Feigin and Tsygan's formula ([13, Th. 5], [23, Th. 3.4]) without finiteness hypothesis and by noncommutative methods, showing that also this instance of the Hodge decomposition comes from the commutator filtration. In the commutative case there is an equivalent definition of infinitesimal cohomology which also generalizes to NC-schemes and is as follows. Assume first that X admits a closed embedding $\iota\colon X \hookrightarrow Y$ into a formally smooth scheme Y, with ideal of definition $I \subset \mathcal{O}_Y$. Then it is known that the infinitesimal cohomology of X is the same thing as the hypercohomology of the I-adic completion

$$\mathbb{H}^*(X_{\mathrm{zar}}, \iota^{-1}\widehat{\Omega}_{\mathrm{Comm}}\mathcal{O}_Y) = H^*(X_{\mathrm{Comm-inf}}, \mathcal{O}) \tag{12}$$

The same is true in the case of NC_l-schemes, with Ω_{NC_l} and a formally NC_l-smooth scheme Y_l substituted for Ω_{Comm} and Y (cf. Theorem 6.1). A similar statement holds for $\mathcal{O}/[\mathcal{O},\mathcal{O}]$ and $\Omega_{NC_l}/[\Omega_{NC_l},\Omega_{NC_l}]$, but we have to take pro-complex cohomology rather than just complete (cf. Theorem 6.4). Back to the commutative theory, when X cannot be embedded in a formally smooth scheme, one can still take an open covering $\mathcal{U}$ of X consisting of embeddable schemes (affine schemes are embeddable). Then one can combine the completed de Rham complexes of each of the local embeddings and of their intersections into a kind of Čech complex as done in [15, p. 28]; the analogue of (12) holds for this complex. The same is true in the NC-case, and (12) as well as its version for $\mathcal{O}/[\mathcal{O},\mathcal{O}]$ hold for systems of local NC-embeddings (defined in 3.2 below) substituted for single NC-embeddings (Theorems 6.1 and 6.4).

We also consider the Poisson analogue of the commutator filtration, obtained by substituting Poisson for Lie brackets in (1). This leads one naturally to the notion of NP_l-schemes, their differential forms, their nilpotent thickenings and through the latter to their infinitesimal topologies. We show that if X is a commutative scheme, then

$$\begin{aligned}
&H^*(X_{NP_l-\mathrm{inf}}, \mathcal{O}) = H^*(X_{NC_l-\mathrm{inf}}, \mathcal{O}),\\
&H^*\left(X_{NP_l-\mathrm{inf}}, \frac{\mathcal{O}}{\{\mathcal{O},\mathcal{O}\}}\right) = H^*\left(X_{NC_l-\mathrm{inf}}, \frac{\mathcal{O}}{[\mathcal{O},\mathcal{O}]}\right),\\
&H^*(X_{NP_l-\mathrm{inf}}, \mathfrak{Y}) = H^*(X_{NC_l-\mathrm{inf}}, \mathfrak{X}).
\end{aligned} \tag{13}$$

Here we write $\mathcal{O}$ for the structure sheaf of both the NC_l- and NP_l-infinitesimal sites, and $\mathfrak{Y}$ is a Poisson adaptation of the $\mathfrak{X}$-complex, similar to the adaptation of the usual cyclic complex given in [4] and [18] (see 6.6-8 below).

The rest of this paper is organized as follows. In Section 2 the notion of a quasi-coherent sheaf of bimodules on an NC-scheme is introduced, and its elementary properties are proved. Then this is used to establish the NC-analogues of several notions from elementary algebraic geometry and their basic properties. In Section 3 the NC_l-infinitesimal site of a scheme is introduced. The connection between this site and the indiscrete infinitesimal site of an algebra considered in [5] and [6] is discussed (3.4). This section also contains a useful lemma regarding the Čech-Alexander complex for infinitesimal topology (Lemma 3.3.1). Section 4 concerns *NC*-differential forms and their elementary properties. In Section 5 the Poisson analogues of what has been done in previous sections are discussed. In Section 6 the main results of the paper are stated. These are packed into three theorems. The first (6.1) computes the *NC*- and *NP*-infinitesimal cohomologies of the structure sheaf, and the Zariski hypercohomology of the complexes of *NC*- and *NP*-forms. The second (6.4) computes the cohomology of the structure sheaves modulo Lie and Poisson brackets and compares them with the hypercohomology of the complexes of forms modulo commutators and Poisson brackets. The third (6.7) computes the infinitesimal hypercohomology of the Cuntz–Quillen complex and of its Poisson analogue. All three theorems are stated in their fullest generality; for the reader's convenience the particular case of each theorem concerning formally smooth schemes has been included as a corollary. The proofs of the main theorems are given in Section 8, after a number of lemmas and auxiliary results which are the subject of Section 7. Among these auxiliary results at least one is of independent interest (Prop. 7.9). It establishes that if X is a commutative scheme, then

$$H^p(X_{\mathrm{Comm-inf}}, \Omega^q_{\mathrm{Comm}}) = 0 \quad \text{for } p \geqslant 0, q \geqslant 1.$$

2. Basic Properties of *NC*-Schemes

The first subsection below is an introduction to sheaves of bimodules on *NC*-schemes. For sheaves of bimodules in a different context, see [20].

2.1. *NC*-BIMODULES AND ASSOCIATED SHEAVES

We extend the commutator filtration (1) to arbitrary R-bimodules M by $F_0M = M$ and

$$F_{n+1}M := \sum_{p=0}^{n} F_pM \cdot F_{n+1-p}R + F_{n+1-p}R \cdot F_pM + \langle [F_pM, F_{n-p}R] \rangle. \tag{14}$$

As in [17] we write NC_l for the category of those algebras R such that $F_{l+1}R = 0$; in addition we put $NC_m(R)$ for those $M \in R-\mathrm{Bimod}$ such that $F_{m+1}M = 0$. Let $NC_\infty = \cup_{l \geqslant 0} NC_l$, $NC_\infty(R) = \cup_{m \geqslant 0} NC_m(R)$. Note that $R \in NC_l \Rightarrow R \in NC_l(R)$ $(0 \leqslant l \leqslant \infty)$. If α: $R \to R' \in NC_\infty$ is a homomorphism and N is an R'-bimodule then we can either take its commutator filtration as an $R'-$ or an $R-$ bimodule (via α). For each $0 \leqslant n$ we have the inclusion

$$F_n^R(N) \subset F_n^{R'}(N). \tag{15}$$

It follows that

$$N \in NC_n(R') \Rightarrow N \in NC_n(R). \tag{16}$$

In other words, the functor $R' - \mathrm{Bimod} \to R - \mathrm{Bimod}$ induced by α sends $NC_n(R')$ into $NC_n(R)$ $(0 \leqslant n \leqslant \infty)$. For $n < \infty >$ the functor $NC_n(R') \to NC_n(R)$ has a left adjoint given by

$$M \mapsto \frac{R' \otimes_R M \otimes_R R'}{F_{n+1}(R' \otimes_R M \otimes_R R')}.$$

Note, however, that the functor $NC_\infty(R') \to NC_\infty(R)$ does not have a left adjoint.

Fix $R \in NC_l$ and $M \in NC_m(R)$; put $A := R/F_1 R$. The proof of [17, 2.1.5] shows that any multiplicative subset $\hat{\Gamma} \subset R$ satisfies both the right and the left Øre conditions. Similarly the proof of [17, 2.1.7] shows that localization commutes with each of the terms of the filtration (1). In particular $F_1(R[\hat{\Gamma}^{-1}]) = (F_1 R)[\hat{\Gamma}^{-1}] \subset R[\hat{\Gamma}^{-1}]$ is a nilpotent ideal, whence an element of $R[\hat{\Gamma}^{-1}]$ is invertible if it is so in $R[\hat{\Gamma}^{-1}]/(F_1 R)[\hat{\Gamma}^{-1}]$ which – by exactness of Øre localization – is the same thing as $\Gamma^{-1}A$, the commutative localization at the image $\Gamma \subset A$ of $\hat{\Gamma}$. We have just shown that $R[\hat{\Gamma}^{-1}]$ depends only on Γ. We shall therefore write $R[\Gamma^{-1}]$ to mean $R[\hat{\Gamma}^{-1}]$. The identity

$$s^{m+1}x = \left(\sum_{i=0}^{m} s^{m-i} ad(sa)^i(x)\right)s \quad (s \in R, x \in M)$$

is proved in the same manner as [17, 2.1.5.1]. It shows that right multiplication by s is surjective if left multiplication is. Similarly, by the same argument as in loc. cit., $xs = 0$ implies $s^{m+1}x = 0$, whence also injectivity of left and right multiplication are equivalent. It follows that there is a canonical isomorphism

$$M[\Gamma^{-1}] := R[\Gamma^{-1}] \otimes_R M \cong M \otimes_R R[\Gamma^{-1}].$$

The same proof as in [17, 2.1.6] applies with M substituted for R and proves that

$$F_n(M[\Gamma^{-1}]) = (F_n M)[\Gamma^{-1}] \quad (n \geqslant 0). \tag{17}$$

Here the F_n on the left is taken in the sense of $R[\Gamma^{-1}]$-bimodules, while that on the right is taken in $R - \mathrm{Bimod}$. In particular $M[\Gamma^{-1}] \in NC_m(R[\Gamma^{-1}])$. Next we show that M and its localizations define a sheaf $\tilde{M}$ on Spec A. We need some notations. If $f \in A$, $\mathfrak{p} \in \mathrm{Spec} A$ and $x \in M$ we put $D(f) = \{\mathfrak{q} \in \mathrm{Spec} A : f \notin \mathfrak{q}\}$ and write M_f and $M_\mathfrak{p}$ for the localizations at $\{f^n : n \geqslant 0\}$ and at $A \backslash \mathfrak{p}$ respectively and $x_\mathfrak{p}$ for the image of x in $M_\mathfrak{p}$. If $U \subset \mathrm{Spec}\, A$ is open, we put $\tilde{M}(U) \subset \prod_{\mathfrak{p} \in U} M_\mathfrak{p}$ for the subset of all those elements σ which satisfy

$$(\forall \mathfrak{p} \in U)(\exists U \supset D(f) \ni \mathfrak{p}, x \in M_f)(\forall \mathfrak{q} \in D(f)), \quad \sigma_\mathfrak{q} = x_\mathfrak{q}. \tag{18}$$

It turns out that

$$\tilde{M}(D(f)) = M_f. \tag{19}$$

Seeing this amounts to showing that the sheaf condition holds for affine coverings of affine open subsets, and one reduces immediately to the case when the affine open is all of $\operatorname{Spec} A$. For $M = R$, this is [17, 2.2.1(b)]. In view of (17) the same proof as in *loc. cit.* works for arbitrary M. We put $X = \operatorname{Spec} R := (\operatorname{Spec} A, \tilde{R})$ and $\mathcal{O}_X := \tilde{R}$. Note that $\operatorname{Spec} R$ is a locally ringed space (in the obvious noncommutative sense) and that $\tilde{M}$ is an $\mathcal{O}_X$-bimodule. Note further the adjoint property

$$\hom_{\mathcal{O}_X-\mathrm{Bimod}}(\tilde{M}, \mathcal{G}) = \hom_{R-\mathrm{Bimod}}(M, \mathcal{G}(X)) \tag{20}$$

for $\mathcal{G} \in \mathcal{O}_X - \mathrm{Bimod}$. The $\mathcal{O}_X$-bimodule $\tilde{M}$ is an example of the general notion of NC_m-bimodule over $\mathcal{O}_X$, which is defined as follows. If $\mathcal{G}$ is an $\mathcal{O}_X$-bimodule, we write $F_n\mathcal{G}$ for the sheafification of the presheaf $U \mapsto F_n(\mathcal{G}(U))$ $(n \geqslant 0)$, with $F_n(\mathcal{G}(U))$ taken in the sense of $\mathcal{O}_X(U)$-bimodules. Note the inclusion

$$F_n(\mathcal{G}(U)) \subset (F_n\mathcal{G})(U). \tag{21}$$

We say that $\mathcal{G}$ is an *NC_m-bimodule over $\mathcal{O}_X$* and write $\mathcal{G} \in NC_m(\mathcal{O}_X)$ if $F_{m+1}\mathcal{G} = 0$. For $\mathcal{G} = \widetilde{F_n M}$ the inclusion (21) together with the adjunction (20) give a sheaf map

$$\widetilde{F_n M} \xrightarrow{\cong} F_n\tilde{M} \tag{22}$$

which is an isomorphism by (17) and (19). In particular $\tilde{M}$ is an object of $NC_m(\mathcal{O}_X)$. We remark that the functor $NC_m(R) \to NC_m(\mathcal{O}_X)$ which sends M to $\tilde{M}$ is exact, because Øre localization is. Suppose now a homomorphism $\alpha\colon R \to R' \in NC_\infty$ is given. Then α descends to a homomorphism $A \to A' = R'/F_1R'$, which in turn induces a continuous map $\hat{\alpha}\colon \operatorname{Spec} A' \to \operatorname{Spec} A$. The map $\hat{\alpha}$ together with the induced homomorphisms

$$\tilde{R}(D(f)) = R_f \to R'_{\alpha f} = \hat{\alpha}_*(\tilde{R}')(D(f)) \quad (f \in A)$$

give rise to a map of locally ringed spaces $X' := \operatorname{Spec} R' \to X$. If $\mathcal{G} \in NC_m(\mathcal{O}_{X'})$ then $\hat{\alpha}_*\mathcal{G} \in NC_m(\mathcal{O}_X)$, by (16). If furthermore $\mathcal{G} = \tilde{N}$ for some $N \in NC_m(R')$ then $\hat{\alpha}_*\mathcal{G} = {}_\alpha\tilde{N}_\alpha$, where the subscript indicates that R acts through α. A left adjoint of the functor $\hat{\alpha}_*\colon NC_m(\mathcal{O}_{X'}) \to NC_m(\mathcal{O}_X)$ is given by

$$\hat{\alpha}^*\mathcal{G} = \frac{O_{X'} \otimes_{\hat{\alpha}^{-1}\mathcal{O}_X} \otimes \mathcal{G} \otimes_{\hat{\alpha}^{-1}\mathcal{O}_X} O_{X'}}{F_{m+1}(O_{X'} \otimes_{\hat{\alpha}^{-1}\mathcal{O}_X} \otimes \mathcal{G} \otimes_{\hat{\alpha}^{-1}\mathcal{O}_X} O_{X'})}. \tag{23}$$

2.2. QUASI-COHERENT SHEAVES

Let $X = (X, \mathcal{O}_X) \to \operatorname{Spec} k$ be a (not necessarily commutative) locally ringed space over $\operatorname{Spec} k$ and $0 \leqslant l \leqslant \infty$. We say that X is an *affine NC_l-scheme* if it is isomorphic – as a locally ringed space over $\operatorname{Spec} k$ – to the spectrum of some $R \in NC_l$, and in general that it is an *NC_l-scheme* if every point $p \in X$ has an open neighborhood U such that $(U, \mathcal{O}_{X|U})$ is an affine NC_l-scheme. We write $NC_l - Sch$ for the category of NC_l-schemes and morphisms of locally ringed spaces and put $NC_\infty - Sch = \cup_{l \geqslant 0} NC_l - Sch$. Note that $NC_0 - Sch$ is the usual category of commutative schemes over Spec k. Like in the commutative case, the global sections functor is right adjoint to Spec; we have

$$\mathrm{Hom}_{NC_l-Sch}(X, \mathrm{Spec}\ R) = \mathrm{Hom}_{NC_l}(R, \mathcal{O}_X(X)).$$

This is proved in two steps, first for X affine and then in general; the arguments of the proofs of [12, 1.7.3] and [12, 2.2.4] apply verbatim to the NC-case.

Fix $X \in NC_l$; an NC_m-bimodule over $\mathcal{O}_X$ as defined in 2.1 above is called *quasi-coherent* if every point $p \in X$ has an open affine neighborhood U such that the natural map

$$\widetilde{M(U)} \xrightarrow{\sim} M_{|U} \tag{24}$$

is an isomorphism. Put $QCoh_m(X)$ for the category of NC_m-quasi-coherent bimodules. Recall [16, Prop. 5.4] that for $l = 0$ the definition we have just given is equivalent to the condition that (24) be an isomorphism for *every* affine open subset U. The same is true for arbitrary m and l. To see this note that in 2.1 we have already proved the analogues of Prop. 5.1 and 5.2 and of Ex. 5.3 of loc. cit. One checks, using these results, together with elementary properties of Øre localization, that the proof of Lemma 5.3 in loc. cit. goes through for arbitrary m and l. The NC_m-analogue of Prop. 5.4 of loc. cit. is then immediate. As an application of all this as well as of (22) and of the exactness of the functor $\tilde{}$ we get that if $M \in QCoh(X)$ and U is affine then

$$F_n M_{|U} \cong F_n \widetilde{M(U)} \quad \text{and} \quad \frac{M}{F_n M}_{|U} = \frac{\widetilde{M(U)}}{F_n M(U)} \quad (U \text{ affine}). \tag{25}$$

In particular, for each $0 \leqslant n \leqslant l < \infty$, the locally ringed space

$$X^{[n]} := (X, \mathcal{O}_X), \qquad \mathcal{O}_{X^{[n]}} := \frac{\mathcal{O}_X}{F_{n+1}\mathcal{O}_X}$$

is an NC_n-scheme. We have a canonical identification

$$QCoh_m(X) = QCoh_m(X^{[n]}) \quad (0 \leqslant m \leqslant n).$$

If $\mathcal{S}$ is any abelian sheaf on the commutative scheme $X^{[0]}$, we put

$$H^*(X_{\mathrm{Zar}}, \mathcal{S}) := H^*(X^{[0]}_{\mathrm{Zar}}, \mathcal{S}), \tag{26}$$

where the subscript indicates that cohomology is taken with respect to the Zariski topology. If M is NC_m-quasi-coherent then the commutator filtration induces a cohomology spectral sequence

$$E_2^{p,q} = H^p\left(X^{[0]}_{\mathrm{Zar}}, \frac{F_q M}{F_{q+1} M}\right) \Rightarrow H^{p+q}(X_{\mathrm{Zar}}, M).$$

We remark that – by (25) – the sheaves $F_n M / F_{n+1} M$ are quasi-coherent ($n \geqslant 0$). Thus for example if X happens to be affine then

$$H^n(X_{\mathrm{Zar}}, M) = 0 \quad (n > 0). \tag{27}$$

As an application of (27) one obtains that the subcategory $QCoh_m(X) \subset NC_m(\mathcal{O}_X)$ is closed under extensions; indeed the proof of [16, Prop. 5.7] applies. Now let $f\colon X \to Y \in NC_\infty - Sch$ be a homomorphism. The inclusion (15) implies that the functor $f_*\colon \mathcal{O}_X - \text{Bimod} \to \mathcal{O}_Y - \text{Bimod}$ sends $NC_m(\mathcal{O}_X)$ into $NC_m(\mathcal{O}_Y)$. Formula (23) defines a left adjoint functor f_m^* of the induced functor $f_*^m\colon NC_m(\mathcal{O}_X) \to NC_m(\mathcal{O}_Y)$. It is clear from the affine case (2.1) that f_m^* always sends $QCoh_m(Y)$ into $QCoh_m(X)$. The proof of [16, 5.8 c)] shows that if $X^{[0]}$ is noetherian, then also f_*^m preserves quasi-coherence.

We say that the morphism f is a *closed* or an *open immersion* if it is so in the sense of locally ringed spaces. The argument of the proof of [12, 4.2.2-b)] shows that if f is a closed immersion and Y is an affine NC_l-scheme then also $X \in NC_l$ and is affine. One shows using this that for any closed immersion f the functor

$$f_*^m\colon NC_m(\mathcal{O}_X) \to NC_m(\mathcal{O}_Y) \tag{28}$$

preserves quasi-coherence. As an application one obtains a one-to-one correspondence between equivalence classes of closed immersions $X \hookrightarrow Y$ and quasi-coherent two sided ideals of $\mathcal{O}_Y$.

LEMMA 2.2.1. *Let* $X \in NC_\infty$, $M \xrightarrow{g} N \in QCoh_l(X)$, $\bar{g}\colon \bar{M} := M/F_1M \to \bar{N}$ *the induced map. Then*

(i) *g is surjective $\iff$ $\bar{g}$ is.*
(ii) *Assume $M = N$. If $\bar{g} = id_{\bar{M}}$, then $g_{|F_lM} = id_{F_lM}$.*

Proof. Part $\Rightarrow$ of (i) is trivial. To prove the converse we may assume X affine. Furthermore, by (27) it suffices to show that if $R \in NC_\infty$ and $h\colon P \to Q \in NC_l(R)$ is such that $\bar{h}$ is surjective then so is h. To prove this it suffices to show $h_n\colon G_nP := F_nP/F_{n+1}P \to G_nQ$ is surjective for all $n \geqslant 1$. Every element of G_nQ is represented by a sum of elements of the form $a \cdot x \cdot b$ where $x = ad(r_1) \circ \cdots \circ ad(r_j)(q)$. Here $q \in Q$, $r_1, \dots, r_j \in R$, $a \in F_iR$, $b \in F_kR$ and $i + j + k = n$. If $h(p) \equiv q \bmod F_1Q$, then

$$h(F_nP) \ni h(a \cdots ad(r_1) \circ \cdots \circ ad(r_j)(p) \cdot b) \equiv a \cdot x \cdot b \bmod F_{n+1}Q.$$

This proves (i). To prove (ii) assume that $P = Q$ and that $\bar{h}$ is the identity. Then for n, a, x and b as above,

$$\begin{aligned} h(a \cdot x \cdot b) &= a \cdot ad(r_1) \circ \cdots \circ ad(r_j)(h(q)) \cdot b \\ &\equiv a \cdot x \cdot b \bmod F_{n+1}P. \end{aligned}$$

For $n = l$, $F_{n+1}P = 0$, so $\equiv$ can be replaced by $=$. $\square$

COROLLARY 2.2.2. *Let $f\colon X \to Y \in NC_\infty - Sch$. Then f is a closed immersion $\iff$ $f^{[0]}$ is.*

Remark 2.2.3. For open instead of closed immersions we still have

$$f\colon X \to Y \text{ open immersion} \Rightarrow f^{[n]}\colon X^{[n]} \to Y^{[n]} \text{ open immersion},$$

for each $n \geqslant 0$. □

2.3. PRODUCTS OF NC-SCHEMES; SEPARATED SCHEMES

For $l < \infty$ the categorical product of two affine NC_l-schemes as objects of $NC_l - Sch$ is given by

$$\operatorname{Spec} R \times_l \operatorname{Spec} R' = \operatorname{Spec} \frac{R * R'}{F_{l+1}(R * R')} \tag{29}$$

where $*$ is the coproduct in the category Ass of associative algebras. The product of not necessarily affine $X, X' \in NC_l - Sch$ denoted $X \times_l X'$ is constructed by glueing together products of affine ones, just as in the commutative case. Note that products do not exist in NC_∞. We say that an NC_l scheme is *separated* over Spec k–if the diagonal map $\delta_l\colon X \to X \times_l X$ is a closed immersion.

LEMMA 2.3.1. *Let* $X, Y \in NC_l$, $\infty > l \geqslant m \geqslant 0$. *Then*

(i) $(X \times_l Y)^{[m]} = X^{[m]} \times_m Y^{[m]}$.
(ii) X *is separated* $\Leftrightarrow$ $X^{[0]}$ *is separated.*

Proof. To prove part (i). The projections $X \times_l Y \to X, Y$ induce a map $f_m\colon (X \times_l Y)^{[m]} \to X^{[m]} \times_m Y^{[m]} \in NC_m - Sch$. To show f_n is an isomorphism we may assume X, Y are affine, in which case the lemma is immediate from (29). Part (ii) is immediate from (i) and Corollary 2.2.2. □

2.4. THICKENINGS

In this paper by a *thickening* of an NC_∞-scheme X we understand a closed immersion $\tau\colon X \to T \in NC_\infty - Sch$ such that $J_\tau := \ker(\mathcal{O}_T \twoheadrightarrow \tau_*\mathcal{O}_X)$ is a nilpotent ideal. If both $X, T \in NC_l - Sch$ then we say that τ is an NC_l-thickening. For example if $X \in NC_\infty$ then for each $0 \leqslant m \leqslant l$ the inclusion

$$X^{[m]} \hookrightarrow X^{[l]} \tag{30}$$

is an NC_l-thickening. We remark that all NC-thickenings considered in [17] are either of the form (30) or colimits of such. However the definition given here is more general, as it includes for example all thickenings of commutative schemes in the commutative sense ([14, 4.1]); indeed these are precisely the NC_0-thickenings. In fact we have

LEMMA 2.4.1. $\tau\colon X \to T$ *is an* NC_l*-thickening* $\Leftrightarrow$ $\tau^{[0]}\colon X^{[0]} \to T^{[0]}$ *is an* NC_0*-thickening*.

Proof. Immediate from 2.2.2. □

LEMMA 2.4.2. *Let* $\tau\colon X \hookrightarrow T \in NC_\infty - Sch$ *be a thickening. Then X is affine* $\Leftrightarrow$ *T is.*

Proof. If T is affine then X must be affine since it is closed (cf. the discussion just before (28)). To prove the converse, we may assume $J_\tau^2 = 0$. Because τ is a closed immersion, $\tau_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O} - NC_\infty$-bimodule (cf. (28)). Hence, J_τ is quasi-coherent, and an object of $NC_\infty(\mathcal{O}_X)$ since $J_\tau^2 = 0$ and τ is a homeomorphism. Because $X = \operatorname{Spec} R$ is affine, there is an R-bimodule M such that $J_\tau = \tilde{M}$. Put $E = \mathcal{O}_T(T)$. Taking global sections in the exact sequence

$$0 \to J_\tau \to \mathcal{O}_T \to \tau_*\mathcal{O}_X \to 0$$

and using (27) we get an exact sequence of E-modules

$$0 \to M \to E \to R \to 0.$$

Applying the functor $\tilde{}$ to the latter sequence we get

$$0 \to J_\tau \to \tilde{E} \to \tau_*\mathcal{O}_X \to 0.$$

It follows that the canonical adjunction map $\tilde{E} \to \mathcal{O}_T$ is an isomorphism. $\square$

COROLLARY 2.4.3. *Let X be an* NC_∞*-scheme. Then*

(i) *X is affine* $\Leftrightarrow$ $X^{[0]}$ *is.*
(ii) *If* $U, V \subset X$ *are open affine subschemes and X is separated then* $U \cap V$ *is affine.*

2.5. PRO-SHEAVES

If C is any category, we write $\text{Pro} - C$ for the category of countably indexed pro-objects in C (cf. [2, 11]). Recall that $\text{Pro} - C$ is Abelian if C is, and by [11, Prop. 1.1] has sufficiently many injectives if C does. In particular if X is an NC-scheme, then the category $\text{Pro} - ShAb(X_{\text{Zar}})$ of pro-sheaves of Abelian groups has sufficiently many injectives, and thus the right derived functors of the total global section functor

$$\hat{H}^0\colon \text{Pro} - ShAb(X_{\text{Zar}}) \ni \mathcal{S} = \{\mathcal{S}_i\}_{i\in I} \mapsto \lim_{i\in I} H^0(X, \mathcal{S}_i) \in Ab$$

are defined. We write

$$H^*(X_{\text{Pro-Zar}}, \mathcal{S}) := R^*(\hat{H}^0)\mathcal{S}$$

There is a cohomology spectral sequence

$$E_2^{p,q} = H^p(X_{\text{Zar}}, \lim{}^q \mathcal{S}) \Rightarrow H^{p+q}(X_{\text{Pro-Zar}}, \mathcal{S}). \tag{31}$$

For example if $\mathcal{M}$ is an inverse system of quasi-coherent sheaves with surjective maps

$$\cdots \twoheadrightarrow \mathcal{M}_2 \twoheadrightarrow \mathcal{M}_1 \quad (\mathcal{M}_n \in QCoh_\infty(X)),$$

then by (27) and [15 Ch. 1 §4], the derived functors of lim vanish and we get

$$H^*(X_{\text{Pro-Zar}}, \mathcal{M}) \cong H^*(X_{\text{Zar}}, \lim \mathcal{M}). \tag{32}$$

An important application of pro-sheaves is to fix the problem that usual sheaf cohomology does not commute with infinite products of Abelian sheaves; that is

$$H^*\left(X_{\mathrm{Zar}}, \prod_{n=1}^{\infty} \mathcal{S}_n\right) \neq \prod_{n=1}^{\infty} H^*(X_{\mathrm{Zar}}, \mathcal{S}_n).$$

However, for the pro-sheaf

$$`\textstyle\prod\text{'}\mathcal{S} : \cdots \twoheadrightarrow \bigoplus_{n=1}^{3} \mathcal{S}_n \twoheadrightarrow \bigoplus_{n=1}^{2} \mathcal{S}_n \twoheadrightarrow \mathcal{S}_1,$$

we have

$$H^*\left(X_{\mathrm{Pro-Zar}}, `\textstyle\prod\text{'}\mathcal{S}\right) = \prod_{n=0}^{\infty} H^*(X_{\mathrm{Zar}}, \mathcal{S}_n). \tag{33}$$

Hypercohomology of pro-sheaves in the Cartan–Eilenberg sense [22, App] is defined in the obvious way, and the obvious generalizations of (32) and (33) hold for hypercohomology. These observations will be used in the proofs of the main theorems (Section 8) to obtain Hodge-type decompositions for various variants of de Rham cohomology.

3. Infinitesimal Topologies

3.1. THE INFINITESIMAL TOPOLOGIES OF AN NC-SCHEME

Let $0 \leqslant l \leqslant \infty$, X an NC_l-scheme. The *NC_l-infinitesimal site* on X is the Grothendieck topology $X_{NC_l-\mathrm{inf}}$ defined as follows. The underlying category $\mathrm{Cat}(X_{NC_l-\mathrm{inf}})$ has as objects the NC_l-thickenings $U \hookrightarrow T$. We write (U, T) or even T to mean $U \hookrightarrow T$. A map $(U, T) \to (U', T')$ in $\mathrm{Cat}(X_{NC_l-\mathrm{inf}})$ exists only if $U \subset U'$ in which case it is a morphism of NC_l-schemes $T \to T'$ such that the obvious diagram commutes. A covering of an object T is a family $\{T_i \to T\}$ of morphisms such that each $T_i \to T$ is an open immersion and $\cup T_i = T$. A sheaf $\mathcal{S}$ on $X_{NC_l-\mathrm{inf}}$ is the same thing as a compatible collection of Zariski sheaves $\{\mathcal{S}_T \in Sh(T_{\mathrm{Zar}}) : T \in X_{NC_l-\mathrm{inf}}\}$ (cf. [3 §5], [14, 4.1]). For example the *infinitesimal structure sheaf* $\mathcal{O}$ is defined by the collection $\{\mathcal{O}_T\}_T$ of the structure sheaves of $T \in X_{NC_l-\mathrm{inf}}$. We remark that a sequence of NC_l-infinitesimal sheaves $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ is exact $\iff$ the sequence of Zariski sheaves

$$0 \to \mathcal{S}'_T \to \mathcal{S}_T \to \mathcal{S}''_T \to 0 \tag{34}$$

is exact for all $T \in X_{NC_l-\mathrm{inf}}$. An important feature of infinitesimal cohomology is that it depends only on the underlying commutative scheme. Precisely, if $X \in NC_l - Sch$ then the inclusion $\iota\colon X^{[0]} \hookrightarrow X$ is an object of $X^{[0]}_{NC_l-\mathrm{inf}}$; thus by composition we obtain a morphism of topologies $F\colon X_{NC_l-\mathrm{inf}} \to X^{[0]}_{NC_l-\mathrm{inf}}$. With the notations of [1], we put

$$\iota_* := F^s\colon ShAb(X^{[0]}_{NC_l-\mathrm{inf}}) \to ShAb(X_{NC_l-\mathrm{inf}}).$$

LEMMA 3.1.1. $H^*(X_{NC_l-\mathrm{inf}}, \imath_*\mathcal{S}) = H^*(X^{[0]}_{NC_l-\mathrm{inf}}, \mathcal{S})$.

Proof. One checks that the left adjoint $\imath^*$ of $\imath_*$ is exact. On the other hand $\imath_*$ is exact by (34). The lemma follows from the Leray spectral sequence associated to the morphism of topoi $\imath = (\imath_*, \imath^*)$: $Sh(X^{[0]}_{NC_l-\mathrm{inf}}) \to Sh(X_{NC_l-\mathrm{inf}})$. □

If X is a commutative scheme and $m < l \leqslant \infty$, then the natural inclusion $\hat{j}$: $X_{NC_m-\mathrm{inf}} \hookrightarrow X_{NC_l-\mathrm{inf}}$ is a morphism of topologies. An argument similar to that of the proof of 3.1.1 shows that for $j_* := \hat{j}^s$ and $\mathcal{S} \in ShAb(X_{NC_m-\mathrm{inf}})$ we have

LEMMA 3.1.2. $H^*(X_{NC_l-\mathrm{inf}}, j_*\mathcal{S}) = H^*(X_{NC_m-\mathrm{inf}}, \mathcal{S})$

3.2. FORMAL NC_l-SMOOTHNESS; SYSTEMS OF EMBEDDINGS

Let $0 \leqslant l < \infty$. An NC_l-scheme X is *formally NC_l-smooth* ($l < \infty$) if it can be covered by open affine schemes of the form Spec R with R formally l-smooth in the sense of [17] and [8]. Equivalently, X is formally NC_l-smooth if the representable sheaf $\tilde{X}$ covers the final object $*$ of the NC_l-infinitesimal topos, i.e. the map $\tilde{X} \twoheadrightarrow *$ is an epimorphism (cf. [3, 5.28]). An *NC_l-embedding* of X is a closed immersion $\tau: X \hookrightarrow Y$ with Y formally l-smooth. If $\mathfrak{I} = \ker(\mathcal{O}_Y \twoheadrightarrow \tau_*\mathcal{O}_X)$, we consider the *$n$-th formal neighborhood* of X along $X \hookrightarrow Y$

$$(Y)_n := \left(X, \tau^{-1}\frac{\mathcal{O}_Y}{\mathfrak{I}^n}\right) \in X_{NC_l-\mathrm{inf}}.$$

An NC_∞-embedding is a direct system $\mathcal{Y} = \{X \hookrightarrow Y_l \hookrightarrow Y_{l+1} \hookrightarrow \cdots\}$ where $X \hookrightarrow Y_l$ is an NC_l-embedding and for $m \geqslant l+1$ each $Y_{m-1} \hookrightarrow Y_m$ is an NC_m embedding. A *system of (local) NC_l-embeddings* of X is a family $\mathcal{Y} = \{\tau_i : U_i \hookrightarrow Y_i : i \in I\}$ indexed by a *well ordered set I* such that $\mathcal{U} = \{U_i : i \in I\}$ is an open covering of X and each τ_i is an NC_l-embedding. The utility of the order on I will be clear in 3.3 below. The definition of a system of NC_∞-embeddings is analogous.

3.3. ČECH–ALEXANDER COMPLEX

Let $l < \infty$, X be a separated NC_l-scheme, $\mathcal{Y} := \{U_i \hookrightarrow Y_i : i \in I\}$ a system of NC_l-embeddings, and $\mathcal{S}$ a sheaf of Abelian groups on $X_{NC_l-\mathrm{inf}}$. For $i_0 < \cdots < i_p$ ($i_j \in I$) we consider the following object of $X_{NC_l-\mathrm{inf}}$

$$(Y_{i_0,\ldots,i_p})_n := (U_{i_0} \cap \cdots \cap U_{i_p} \hookrightarrow (Y_{i_0} \times_l \cdots \times_l Y_{i_p})_n).$$

The *Čech–Alexander (pro–)complex* of X relative to $\mathcal{Y}$ is the double pro-complex of Zariski sheaves

$$\mathcal{C}^{p,q}_{\mathcal{Y}}(\mathcal{S})_n = \prod_{i_0 < \cdots < i_p} \mathcal{S}_{\left(Y^{\times_l q+1}_{i_0,\ldots,i_p}\right)_n} \tag{35}$$

with the horizontal coboundary being the alternating sum of the cofaces induced by the natural inclusions and the natural projections

$$U_{i_0,\dots,i_p} \subset U_{i_0,\dots,\underset{i_j}{\vee},\dots,i_p}, \qquad Y_{i_0,\dots,i_p} \twoheadrightarrow Y_{i_0,\dots,\underset{i_j}{\vee},\dots,i_p}$$

and with the vertical coboundary being the alternating sum of the cofaces induced by the $q+1$ distinct projections $Y_{i_0,\dots,i_p}^{\times_l q+1} \twoheadrightarrow Y_{i_0,\dots,i_p}^{\times_l q}$. In other words $\mathcal{C}_{\mathcal{Y}}^{*,*}(\mathcal{S})$ is a semi-cosimplicial-cosimplicial pro-sheaf, regarded as a double cochain pro-complex in the usual fashion. Next assume $\mathcal{Y}$ extends to a system of NC_∞-embeddings $\mathcal{Z}$ and let $\mathcal{G}$ be an abelian sheaf on $X_{NC_\infty-\mathrm{inf}}$. By definition $\mathcal{Z}$ is a sequence $\mathcal{Y}_l \hookrightarrow \mathcal{Y}_{l+1} \hookrightarrow \cdots$ of systems of embeddings and compatible maps. Hence it gives rise to the following double complex in $ShAb(X)^{\mathbb{N}\times\mathbb{N}}$

$$\mathcal{C}_{\mathcal{Z}}^{*,*}(\mathcal{G})_{m,n} := \mathcal{C}_{\mathcal{Y}_m}^{*,*}(\mathcal{G}_{|X_{NC_l-\mathrm{inf}}})_n \qquad (m \geqslant l, n \geqslant 0). \tag{36}$$

As a pro-object, (36) is isomorphic to the inverse system

$$\mathcal{C}_{\mathcal{Z}}^{*,*}(\mathcal{G})_m := \mathcal{C}_{\mathcal{Z}}^{*,*}(\mathcal{G})_{m+l,m} \qquad (m \geqslant 0).$$

LEMMA 3.3.1. *Let $0 \leqslant l \leqslant \infty$, X a separated NC_l-scheme, $\mathcal{Y} = \{U_i \hookrightarrow Y_i : i \in I\}$ a system of formally NC_l-smooth embeddings and $\mathcal{S}$ an abelian sheaf on $X_{NC_l-\mathrm{inf}}$. Assume* either *of the following hypothesis holds*

(i) $\mathcal{U} = \{U_i : i \in I\}$ *is locally finite.*
(ii) $(\forall\ T \in X_{NC_l-\mathrm{inf}})$ *the Zariski sheaf $\mathcal{S}_T$ is quasi-coherent in the sense of* 2.2 *above.*

Then with the notations of 2.5 *and* 3.3*,*

$$H^*(X_{NC_l-\mathrm{inf}}, \mathcal{S}) = \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathcal{C}_{\mathcal{Y}}(\mathcal{S})).$$

Proof. Assume $l < \infty$. Consider the following objects of the infinitesimal NC_l-topos

$$\widetilde{(Y_i)}_\infty := \operatorname*{colim}_n \widetilde{(Y_i)}_n, \qquad \tilde{\mathcal{Y}} = \coprod_{i\in I} \widetilde{(Y_i)}_\infty,$$

where $\widetilde{(Y_i)}_n$ is the representable sheaf. Then

$$\begin{aligned}\mathbb{H}^0(X_{\mathrm{Pro-inf}}, \mathcal{C}_{\mathcal{Y}}(\mathcal{S})) &= h^0(\hat{H}^0 \mathcal{C}_{\mathcal{Y}}(\mathcal{S})) \\ &= \ker\left(\mathrm{Hom}(\tilde{\mathcal{Y}}, \mathcal{S}) \to \mathrm{Hom}\left(\coprod_{i<j} \widetilde{(Y_i)}_\infty \times \widetilde{(Y_j)}_\infty, \mathcal{S}\right)\right)\end{aligned} \tag{37}$$

where the map is the difference of those induced by the two projections $\widetilde{(Y_i)}_\infty \times \widetilde{(Y_j)}_\infty) \to \tilde{\mathcal{Y}}$. Because $\tilde{\mathcal{Y}} \twoheadrightarrow *$ is an effective epimorphism, (37) equals $H^0(X_{NC_l-\mathrm{inf}}, \mathcal{S})$ It remains to show that

$$\mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathcal{C}_{\mathcal{Y}}(\cdot)) : \mathcal{S} \mapsto \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathcal{C}_{\mathcal{Y}}(\mathcal{S}))$$

is a unversal δ-functor. Under either of the hypothesis (i), (ii) of the lemma, the products appearing in (35) are exact. Indeed in the case of (i) this is clear, and for (ii) it follows from (27) and [15, Ch.1§4]. Thus we may assume $\mathcal{Y}$ consists of a single embedding $X \hookrightarrow Y$. It is clear from (34) that $\mathcal{C}^q_{\mathcal{Y}}(\cdot)$ is an exact functor for each $q \geqslant 0$, whence $\mathbb{H}^*(X_{\text{Pro}-\text{Zar}}, \mathcal{C}_{\mathcal{Y}}(\cdot))$ is a δ-functor. It remains to show that the functor $\mathcal{C}^q_{\mathcal{Y}}(\cdot)$ preserves injectives; this will follow once we show it has an exact left adjoint. For the product embedding $\mathcal{Y}^{(q)} := \{X \hookrightarrow Y^{\times q+1}\}$ we have $\mathcal{C}^q_{\mathcal{Y}}(\cdot) = \mathcal{C}^0_{\mathcal{Y}^{(q)}}(\cdot)$. Thus we may assume $q = 0$. For each $T = (\tau\colon U \hookrightarrow T) \in X_{NC_l-\text{inf}}$ let

$$n_T = \min\{m \geqslant 1\colon \ker(\mathcal{O}_T \to \tau_*\mathcal{O}_U)^m = 0\}.$$

One checks that the exact functor

$$\text{Pro} - ShAb(X_{\text{Zar}}) \ni \{\mathcal{S}_n\}_n \mapsto \left\{ \bigoplus_{\text{Hom}(T, Y_{n_T})} \tau_*(\mathcal{S}_{n_T|U}) \right\}_T \in ShAb(X_{NC_l-\text{inf}})$$

is left adjoint to $\mathcal{C}^0_{\mathcal{Y}}(\cdot)$. This finishes the proof for $l < \infty$; the case $l = \infty$ is proven similarly. □

3.4. THE INDISCRETE INFINITESIMAL TOPOLOGIES OF AN *NC*-ALGEBRA

Let $0 \leqslant l \leqslant \infty$, $A \in NC_l$. We write $\inf(NC_l/A)$ for the category of all surjective homomorphisms with nilpotent kernel $B \twoheadrightarrow A$. We equip the opposite category $\inf(NC_l/A)^{op}$ with the *indiscrete topology*; this means that if $B = (B \twoheadrightarrow A)$ then $Cov(B)$ is the set of all isomorphisms $B \xrightarrow{\cong} B' \in \inf(NC_l/A)$. A sheaf of abelian groups on $\inf(NC_l/A)^{op}$ with this topology is the same thing as a presheaf, which in turn is just a covariant functor $\mathcal{G}\colon \inf(NC_l/A) \to Ab$. Let $X = \operatorname{Spec} A$. With the notations of [1] the functor

$$F\colon \inf(NC_l/A)^{op} \to X_{NC_l-\text{inf}}, \qquad B \mapsto \operatorname{Spec} B$$

is a morphism of topologies, and induces a functor between the categories of sheaves of sets

$$\begin{aligned} f_* := F^s : Sh(X_{NC_l-\text{inf}}) &\to Sh(\inf(NC_l/A), \\ \mathcal{S} &\mapsto (B \mapsto H^0(\operatorname{Spec} B, \mathcal{S})). \end{aligned}$$

The left adjoint of f_* is the functor

$$f^* := F_s : Sh(\inf(NC_l/A)) \to Sh(X_{NC_l-\text{inf}}), \quad \mathcal{G} \mapsto \widetilde{\mathcal{G}}$$

where for each $T \in X_{NC_l-\text{inf}}$, $\widetilde{\mathcal{G}}_T$ is the Zariski sheaf defined by (18). Because f^* is exact (i.e. (f_*, f^*) is a morphism of topoi) we have a Leray spectral sequence

$$E_2^{p,q} = H^p(\inf(NC_l/A), (R^q f_*)(\mathcal{S})) \Rightarrow H^{p+q}(X_{NC_l-\text{inf}}, \mathcal{S}). \tag{38}$$

Here

$$(R^q f_*)(\mathcal{S})(B) = H^q(\operatorname{Spec} B_{\text{Zar}}, \mathcal{S}_{\operatorname{Spec} B}) \quad (B \in \inf(NC_l/A)) \tag{39}$$

For example, if M is a sheaf of NC-bimodules on $\inf(NC_l/A)$ then the Zariski sheaves $\mathcal{M}_T$ ($T \in X_{NC_l-\text{inf}}$) are all quasi-coherent, whence (39) vanishes for $q > 0$, and

$$H^*(\inf(NC_l/A), M) = H^*(X_{NC_l-\inf}, \tilde{M}). \tag{40}$$

If $\mathcal{G} : \inf(NC_l/A) \to Ab$ is arbitrary, we still have a natural map

$$H^*(\inf(NC_l/A), \mathcal{G}) \to H^*(X_{NC_l-\inf}, \tilde{\mathcal{G}}),$$

but this is not an isomorphism in general.The Čech–Alexander pro-complex for the indiscrete topology is constructed as follows. Assume first that $l < \infty$. Given a sheaf $\mathcal{G}$: $\inf(NC_l/A) \to Ab$ and a presentation $0 \to J \to R \twoheadrightarrow A \to 0$ of A as a quotient of an algebra $R \in$ Ass, we put

$$C^p_{NC_l}(R, J, \mathcal{G})_m = \mathcal{G}\left(\frac{Cyl^p(R, J)_m}{F_l Cyl^p(R, J)_m}\right) \quad (m \geqslant 0),$$

where $Cyl^p(R, J)$ is the pro-algebra of [6]. In case $R_l := R/F_l R$ is formally l-smooth, the procomplex $C_{NC_l}(R, J, \mathcal{G})$ computes sheaf cohomology (cf. [5, 5.1])

$$\begin{aligned} H^*(\mathrm{holim}_n C_{NC_l}(R, J, \mathcal{G})_n) &:= \mathbb{H}^*(\mathrm{Spec}\, k_{\mathrm{Pro-Zar}}, C_{NC_l}(R, J, \mathcal{G})) \\ &= H^*(\inf(NC_l/A), \mathcal{G}). \end{aligned} \tag{41}$$

We also put

$$C^p_{NC_\infty}(R, J, \mathcal{G})_m = C^p_{NC_m}(R, J, \mathcal{G})_m \quad (m \geqslant 0)$$

If R_l is formally l-smooth for all l (e.g. if R is quasi-free in the sense of [9]) then (41) holds for $l = \infty$ as well. As a particular case of (31) (or rather of its hypercohomology version) we obtain a spectral sequence

$$E_2^{p,q} = H^p(\lim{}^q C_{NC_l}(R, J, \mathcal{G})) \Rightarrow H^{p+q}(\inf(NC_l/A), \mathcal{G}) \quad (0 \leqslant l \leqslant \infty)$$

This spectral sequence degenerates for example when $\mathcal{G}$ maps surjections with nilpotent kernel to surjections, as $\lim^q C_{NC_l}(R, J, \mathcal{G})) = 0$ for $q > 0$. Hence if M is as in (40) and in addition maps surjections with nilpotent kernel to surjections, then

$$H^*(X_{NC_l-\inf}, \tilde{M}) = H^*(\lim C_{NC_l}(R, J, M)).$$

Remark 3.4.1. One can also consider the indiscrete topology on the category $\inf(\mathrm{Ass}/A)$ of all nilpotent extensions $B \twoheadrightarrow A$, where B runs in the category Ass of associative algebras. It was proved in [6] that for $A \in$ Ass

$$H^*\left(\inf(\mathrm{Ass}/A), \frac{\mathcal{O}}{[\mathcal{O}, \mathcal{O}]}\right) = HC^{\mathrm{per}}_* A$$

We remark that the indiscrete infinitesimal cohomology of $A \in NC_l$ as an associative algebra does not agree with its cohomology as an NC_l-algebra. For example if A is formally Comm-smooth then $HC^{\mathrm{per}}_* A$ is as in (9) while the indiscrete NC_l-infinitesimal cohomology $H^*(\inf(NC_l/A), \mathcal{O}/[\mathcal{O}, \mathcal{O}])$ is as calculated in 8.3.1 below. □

4. *NC*-Differential Forms

4.1. *NC*-FORMS FOR *NC*-ALGEBRAS AND SCHEMES

We make some remarks regarding the definition of NC_l-forms given in the introduction (2). We observe that the bimodule filtration (14) is included in the DG-commutator filtration; we have $F_m\Omega^p R \subset (F_m\Omega R)^p$ $(m \geqslant 0)$. In particular, $\Omega^p_{NC_l} R \in NC_l(R)$. Moreover, one checks that Ω_{NC_l} localizes; if $\Gamma \subset R$ is a multiplicative system, then

$$\Omega_{NC_l}(R[\Gamma^{-1}]) \cong (\Omega_{NC_l}R)[\Gamma^{-1}].$$

Thus the following NC-$\mathcal{O}_{\mathrm{Spec}R}$-bimodules are isomorphic

$$\widetilde{\Omega_{NC_l}R} = \Omega_{NC_l}\mathcal{O}_{\mathrm{Spec}R}.$$

It follows that, in general, if $X \in NC_l - Sch$ then the sheaf $\Omega_{NC_l} := \Omega_{NC_l}\mathcal{O}_X$ is a quasi-coherent sheaf of DG- algebras over $\mathcal{O}_X$. All this generalizes to the case $l = \infty$ as follows. With the notations above, put

$$\Omega_{NC_\infty}R = \{\Omega_{NC_l}R\}_l \in \mathrm{Pro} - R - \mathrm{Bimod}.$$

If X is a scheme, then the Pro-Zariski sheaf Ω_{NC_∞} is defined in the obvious way.

Remark 4.1.1. With the definitions above, the functor Ω_{NC_l}: $NC_l \to DGNC_l$ is left adjoint to $DGNC_l \to NC_l$, $\Lambda \mapsto \Lambda^0$. Indeed for $R \in NC_l$ and $\Lambda \in DGNC_l$

$$\begin{aligned}\mathrm{Hom}_{DGNC_l}(\Omega_{NC_l}R, \Lambda) &= \mathrm{Hom}_{DG\mathrm{Ass}}(\Omega R, \Lambda)\\ &= \mathrm{Hom}_{\mathrm{Ass}}(R, \Lambda^0) = \mathrm{Hom}_{NC_l}(R, \Lambda^0).\end{aligned}$$

It follows from this that $\Omega_{NC_l}R$ is formally NC_l-smooth in the obvious DG-sense $\Longleftrightarrow$ R is formally NC_l-smooth. On the other hand

$$\frac{\Omega_{NC_l}R}{F_1\Omega_{NC_l}R} = \Omega_{\mathrm{Comm}}\left(\frac{R}{F_1R}\right)$$

for every $R \in NC_l$. Thus if A is smooth commutative and if R_l is an NC_l-smooth thickening of A in the sense of [17, 1.6.1] then $\Omega_{NC_l}R_l$ is a DG-NC_l-smooth thickening of the smooth DG Comm-algebra $\Omega_{\mathrm{Comm}}A$. □

4.2. FORMS AND EMBEDDINGS

If $R \in \mathrm{Ass}$ and $J \lhd R$ is an ideal we put

$$\Omega_{NC_l}(R, J) := \Omega_{NC_l}\left(\frac{R}{F_lR + J^\infty}\right) \in \mathrm{Pro} - Ab$$

for $0 \leqslant l \leqslant \infty$. Here as in [11], J^∞ is the pro-ideal of the powers of J. Similarly for $l < \infty$ if $\mathcal{Y} = \{\tau\colon X \hookrightarrow Y\}$ is an NC_l-embedding with ideal of definition J, we put

$$\Omega^{\mathcal{Y}}_{NC_l} = \Omega_{NC_l}\left(\tau^{-1}\left(\frac{\mathcal{O}_Y}{J^\infty}\right)\right) \in \mathrm{Pro} - ShAb(X_{\mathrm{Zar}}). \tag{42}$$

In general, if $\mathcal{Y} = \{U_i \hookrightarrow Y_i : i \in I\}$ is a system of NC_l-embeddings, $\Omega^{\mathcal{Y}}_{NC_l}$ is the total complex of the double pro-complex whose pth column is $(\Omega^{\mathcal{Y}}_{NC_l})^{p,*} = \prod_{i_0<\cdots<i_p} \iota_*\Omega^{\mathcal{Y}^{i_0,\ldots,i_p}}_{NC_l}$. Here $\mathcal{Y}^{i_0,\ldots,i_p} := \{U_{i_0,\ldots,i_p} \hookrightarrow Y_{i_0,\ldots,i_p}\}$ and $\iota : U_{i_0,\ldots,i_p} \hookrightarrow X$ is the open immersion. If $\mathcal{Z} = \mathcal{Y}_l \hookrightarrow \mathcal{Y}_{l+1} \hookrightarrow \cdots$ is a system of NC_∞-embeddings we put $(\Omega^{\mathcal{Z}}_{NC_\infty})_{l,n} = (\Omega^{\mathcal{Y}_l}_{NC_l})_n$.

5. *NP*-Algebras and Schemes

The forgetful functors going from the category of Poisson algebras to vectorspaces and to commutative algebras have each a left adjoint, which we write respectively Poiss and P. We have an isomorphism of Poisson algebras

$$PSV = SLV = \mathrm{Poiss}\,V, \tag{43}$$

where V is a vectorspace, $L : \mathrm{Lie} - \mathrm{Alg} \to \mathrm{Vect}$ is left adjoint to the forgetful functor and S is the symmetric algebra. The Poisson bracket on SLV is induced by the Lie bracket of LV. Recall from [8] that if A is any commutative algebra then PA carries a natural grading

$$PA = \bigoplus_{l=0}^{\infty} P_l A \tag{44}$$

such that

$$\{P_l A, P_m A\} \subset P_{l+m+1} A, \qquad P_l A \cdot P_m A \subset P_{l+m} A.$$

In the case $A = SV$ this grading is the same as the grading

$$SLV = \bigoplus_{m=0}^{\infty} S_m LV, \tag{45}$$

induced by $L_0 V = V$, $L_{m+1} V = [L_0 V, L_m V]$. Note that this is different from the usual grading $S = \bigoplus_{m=0}^{\infty} S^m$ of the symmetric algebra. The analogue of the commutator filtration for Poisson algebras is the *Poisson filtration* defined as follows. Let P be a Poisson algebra. Put $F_0 P = P$ and inductively

$$F_{m+1} P := \sum_{i=1}^{m} F_i P \cdot F_{m+1-i} P + \sum_{i=0}^{m} \langle \{F_i P, F_{m-i} P\} \rangle. \tag{46}$$

For example, $F_lPA = \bigoplus_{m \geqslant l} P_mA$. The analogues of NC_l-algebras and schemes in the Poisson setting are called NP_l-algebras and schemes. All what has been done for NC-algebras and schemes translates immediately to the NP-setting. We shall not go into the details of this translation here but shall make a few remarks about it. First of all we note that the coproduct of two Poisson algebras as such is not the same as their coproduct as commutative algebras or tensor product. If KP and KQ are the kernels of the canonical maps Poiss $P \to P$ and Poiss $Q \to Q$ then

$$P \coprod Q = \frac{\mathrm{Poiss}(P \oplus Q)}{\langle\langle KP, KQ \rangle\rangle}, \tag{47}$$

where $\langle\langle X \rangle\rangle$ denotes the smallest Poisson ideal containing X (cf. [8]). Note that as Poiss has a right adjoint, it must preserve coproducts, and that (47) simply expresses this fact. Coproducts in NP_l and products in $NP_l - Sch$ are defined accordingly. Second of all the right definition for the DGP of differential forms of $\mathcal{A} \in$ Poiss is not $\Omega_{\mathrm{Comm}}\mathcal{A}$, but is defined by the adjointness property $\mathrm{Hom}_{DGP}(\Omega_{\mathrm{Poiss}}\mathcal{A}, Q) = \mathrm{Hom}_{\mathrm{Poiss}}(\mathcal{A}, Q^0)$. For example,

$$\Omega_{\mathrm{Poiss}}SLV = S(L(V \oplus \mathrm{d}V))S(LV \oplus \mathrm{d}LV) = \Omega_{\mathrm{Comm}}SLV,$$

where $\mathrm{d}V$ and $\mathrm{d}LV$ are intended to be meaningful notations for the graded vectorspaces $V[-1]$ and $(LV)[-1]$. In general if $A \in$ Comm, then

$$\Omega_{\mathrm{Poiss}}PA = P\Omega_{\mathrm{Comm}}A, \tag{48}$$

where the P on the right hand side is the left adjoint of the forgetful functor $DGP \to DG$ Comm. With this definition, the same considerations as to formal smoothness remarked for NC-algebras (4.1.1) hold in the NP-case (see [8, 3.3]).

6. Statement of the Main Theorems

Before stating the first theorem we need some more notations. Recall that if $\mathfrak{g}$ is a Lie algebra and $U\mathfrak{g}$ its universal enveloping algebra then there is an isomorphism of vectorspaces

$$e : S\mathfrak{g} \xrightarrow{\cong} U\mathfrak{g}, \qquad g_1 \dots g_n \mapsto \frac{1}{n!}\sum_{\sigma} sg(\sigma) g_{\sigma_1} \dots g_{\sigma n} \tag{49}$$

Here σ runs among all permutations of n elements. The map e is called the *symmetrization map*. In Theorem 6.1 below we use the particular case when $\mathfrak{g} = LV$ is as in (43), so that $U\mathfrak{g} = TV$, the tensor algebra. We use a ˆ to indicate completion of a prosheaf; thus for example if $\mathcal{Y}$ is a system of NC_l-embeddings of a scheme X then $\hat{\Omega}^{\mathcal{Y}}_{NC_l} = \lim_n (\Omega^{\mathcal{Y}}_{NC_l})_n$. In the statement of 6.1 below we use the fact that, as follows from Lemma 3.1.2, if X is a commutative scheme and $0 \leqslant l \leqslant \infty$ then

$$H^*(X_{NC_l-\mathrm{inf}}, \mathcal{O}/F_1\mathcal{O}) = H^*(X_{\mathrm{Comm-inf}}, \mathcal{O}), \tag{50}$$

where the $\mathcal{O}$ on the left-hand side is the structure sheaf of $X_{NC_l-\mathrm{inf}}$ while the one on the right-hand side is that of $X_{\mathrm{Comm-inf}}$. The same is true with $X_{NP_l-\mathrm{inf}}$ substituted for $X_{NC_l-\mathrm{inf}}$.

THEOREM 6.1. *Let X be a separated commutative scheme, $0 \leqslant l \leqslant \infty$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ systems of local NC_l-, NP_l- and* Comm *embeddings of X. Write $\mathcal{O}$ for the structure sheaf of each of the infinitesimal sites on X. Then there is a commutative square of natural isomorphisms*

$$\begin{array}{ccc} H^*(X_{NP_l-\mathrm{inf}}, \mathcal{O}) & & \mathbb{H}^*(X_{\mathrm{Zar}}, \hat{\Omega}^{\mathcal{Z}}_{NP_l}) \\ \big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle e'} \\ H^*(X_{NC_l-\mathrm{inf}}, \mathcal{O}) & \xrightarrow{\alpha_l} & \mathbb{H}^*(X_{\mathrm{Zar}}, \hat{\Omega}^{\mathcal{Y}}_{NC_l}) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ H^*(X_{\mathrm{Comm-inf}}, \mathcal{O}) & \xrightarrow{\alpha_0} & \mathbb{H}^*(X_{\mathrm{Zar}}, \hat{\Omega}^{\mathcal{W}}_{\mathrm{Comm}}) \end{array}$$

Here e and e′ are induced by the symmetrization map (49) *and each of π, π', $\pi \circ e$ and $\pi' \circ e'$ by the natural projection $\mathcal{O} \twoheadrightarrow \mathcal{O}/F_1\mathcal{O}$ and the isomorphism* (50)*. Moreover, if we equip each of the four vertices of the top of the diagram with the filtration induced by the corresponding commutator or Poisson filtration then all three edges are filtered isomorphisms.*

COROLLARY 6.2. *Assume X is formally smooth. Then*

$$\begin{aligned} H^*(X_{\mathrm{Comm-inf}}, \mathcal{O}) &= \mathbb{H}^*(X_{\mathrm{Zar}}, \Omega_{\mathrm{Comm}}) \\ &= \mathbb{H}^*(X_{\mathrm{Zar}}, P_{\leqslant l}\Omega_{\mathrm{Comm}}), \end{aligned} \tag{51}$$

for each $0 \leqslant l < \infty$. If moreover X admits a formally NC_l-smooth thickening $X \hookrightarrow Y_l$, then (51) *equals*

$$\mathbb{H}^*((Y_l)_{\mathrm{Zar}}, \Omega_{NC_l}) = H^*((Y_l)_{NC_l-\mathrm{inf}}, \mathcal{O}),$$

for each $l < \infty$.

Proof. Immediate from 6.1, (26), (48) and 3.1.1. □

Notation for sheaf cokernels 6.3. In Theorem 6.4 and further below, the expression $\mathcal{O}/[\mathcal{O}, \mathcal{O}]$ denotes the quotient of the sheaf of rings $\mathcal{O}$ by the subsheaf generated by the sheafification of the presheaf $[\mathcal{O}, \mathcal{O}](U) = [\mathcal{O}(U), \mathcal{O}(U)]$. In other words, the sheafification symbol is omitted. This is done to avoid further decorating already involved symbols. The same abuse is committed with the subpresheaf generated by brackets in a sheaf of Poisson algebras, and the subpresheaf generated by exact differentials in the sheaves of differential forms.

THEOREM 6.4. *Let X, l, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{W}$ and $\mathcal{O}$ be as in Theorem* 6.1. *Assume the underlying open coverings of $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ are locally finite. Then with the convention of 6.3 there is a commutative square of natural isomorphisms*

$$
\begin{array}{ccc}
H^*\left(X_{NP_l-\mathrm{inf}}, \frac{\mathcal{O}}{\{\mathcal{O},\mathcal{O}\}}\right) & & \mathbb{H}^*\left(X_{\mathrm{Pro-Zar}}, \frac{\Omega^{\mathcal{Z}}_{NP_l}}{\{\Omega^{\mathcal{Z}}_{NP_l},\Omega^{\mathcal{Z}}_{NP_l}\}}\right) \\
\downarrow \bar{e} & & \downarrow \bar{e}' \\
H^*\left(X_{NC_l-\mathrm{inf}}, \frac{\mathcal{O}}{[\mathcal{O},\mathcal{O}]}\right) & \xrightarrow{\bar{\alpha}_l} & \mathbb{H}^*\left(X_{\mathrm{Pro-Zar}}, \frac{\Omega^{\mathcal{Y}}_{NC_l}}{[\Omega^{\mathcal{Y}}_{NC_l},\Omega^{\mathcal{Y}}_{NC_l}]}\right) \\
\downarrow \gamma & & \downarrow \gamma' \\
\prod_{m=0}^{l} H^*\left(X_{\mathrm{Comm-inf}}, \frac{\Omega^m}{\mathrm{d}\Omega^{m-1}}\right) & \xrightarrow{\beta} & \prod_{m=0}^{l} \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \tau_m\Omega^{\mathcal{W}}_{\mathrm{Comm}})
\end{array}
\tag{52}
$$

Here $\tau_m\Omega_{\mathrm{Comm}}$ *is the complex of sheaves*

$$\tau_m\Omega_{\mathrm{Comm}}\colon \quad \frac{\Omega^m_{\mathrm{Comm}}}{\mathrm{d}\Omega^{m-1}_{\mathrm{Comm}}} \xrightarrow{d} \Omega^{m+1}_{\mathrm{Comm}} \xrightarrow{d} \Omega^{m+2}_{\mathrm{Comm}} \xrightarrow{d} \cdots,$$

where $\Omega^m_{\mathrm{Comm}}/\mathrm{d}\Omega^{m-1}_{\mathrm{Comm}}$ *is in degree* 0. *Each of* $\bar{\alpha}_l$, $\bar{e}$, *and* $\bar{e}'$ *in* (52) *is induced by the unbarred map with the same name in Theorem* 6.1*; it is filtered for the respective commutator and Poisson filtrations. Each of* γ, γ' *is a filtered isomorphism for the commutator filtration of its source and the filtration* $F_r = \prod_{m \leqslant r}(\cdot)$ *of its target. The map* β *is a product of isomorphisms*

$$\beta_m : H^*\left(X_{\mathrm{Comm-inf}}, \frac{\Omega^m_{\mathrm{Comm}}}{\mathrm{d}\Omega^{m-1}_{\mathrm{Comm}}}\right) \xrightarrow{\sim} \mathbb{H}^*\left(X_{\mathrm{Pro-Zar}}, \tau_m\Omega^{\mathcal{W}}_{\mathrm{Comm}}\right)$$

of which $\beta_0 = \alpha_0$ *is the map of Theorem* 6.1.

COROLLARY 6.5. *Assume* X *is formally* Comm-*smooth. Then*

$$H^*\left(X_{\mathrm{Comm-inf}}, \frac{\Omega^m}{\mathrm{d}\Omega^{m-1}}\right) = \mathbb{H}^*(X_{\mathrm{Zar}}, \tau_m\Omega_{\mathrm{Comm}}), \tag{53}$$

$$\mathbb{H}^*\left(X_{\mathrm{Zar}}, \frac{P_{\leqslant l}\Omega_{\mathrm{Comm}}}{\{P_{\leqslant l}\Omega_{\mathrm{Comm}}, P_{\leqslant l}\Omega_{\mathrm{Comm}}\}}\right) = \prod_{m=0}^{l} \mathbb{H}^*(X_{\mathrm{Zar}}, \tau_m\Omega_{\mathrm{Comm}}). \tag{54}$$

If, moreover, X *admits a formally* NC_l*-smooth thickening* $X \hookrightarrow Y_l$ *then* (54) *equals*

$$= \mathbb{H}\left((Y_l)_{\mathrm{Zar}}, \frac{\Omega_{NC_l}}{[\Omega_{NC_l}, \Omega_{NC_l}]}\right) = \mathbb{H}\left((Y_l)_{NC_l-\mathrm{inf}}, \frac{\mathcal{O}}{[\mathcal{O}, \mathcal{O}]}\right).$$

Notations for cyclic homology 6.6. Let X be an NC_∞-scheme. The *periodic cyclic homology* of X is

$$HC^{\mathrm{per}}_*(X) := \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathcal{CC}^{\mathrm{per}}),$$

where $\mathcal{CC}^{\text{per}}$ is the sheafification at each level of the 2-periodic pro-complex

$$\mathcal{CC}^{\text{per}} = \left\{ \left(\bigoplus_{m=0}^{n-1} \Omega^m \right) \oplus \Omega^n_\natural \right\}_n$$

called $\theta\Omega$ in [10]. In particular for the Hochschild boundary b, $\Omega^n_\natural := \Omega^n / b\Omega^{n+1}$. We remark that as $\mathcal{CC}^{\text{per}}$ is 2-periodic, the Cartan-Eilenberg resolution can also be taken 2-periodic. Indeed the procedure for the construction of CE-resolutions described in [21, Proof of 5.7.2] yields periodic resolutions for periodic complexes. At level $n = 1$, CC^{per} is the periodic de Rham complex

$$\mathfrak{X}\colon\ \Omega^0 \underset{b}{\overset{d}{\rightleftarrows}} \Omega^1_\natural$$

called X in [10]. We also consider the analogue of the latter complex for Poisson algebras, which is defined as follows. Recall from [4] that if $\mathcal{A}$ is a Poisson algebra then there is a boundary map

$$\begin{aligned}
&\delta : \Omega^*_{\text{Comm}}\mathcal{A} \longrightarrow \Omega^{*-1}_{\text{Comm}}\mathcal{A},\\
&\delta(p_0 \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n)\\
&\quad = \sum_{i=0}^{n} (-1)^{i+1} \{p_0, p_i\} \mathrm{d}p_1 \wedge \cdots \wedge \widehat{\mathrm{d}p_i} \wedge \cdots \wedge \mathrm{d}p_n +\\
&\qquad + \sum_{i<j} (-1)^{i+j} p_0 \mathrm{d}\{p_i, p_j\} \wedge \mathrm{d}p_1 \wedge \cdots \wedge \widehat{\mathrm{d}p_i} \wedge \cdots \wedge \widehat{\mathrm{d}p_j} \wedge \cdots \wedge \mathrm{d}p_n.
\end{aligned}$$

We put

$$\mathfrak{Y} : \mathcal{A} \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^1_\delta \mathcal{A},$$

where $\Omega^n_\delta = \Omega^n_{\text{Comm}} / \delta\Omega^{n+1}_{\text{Comm}}$. We shall abuse notation and write $\mathfrak{X}$ and $\mathfrak{Y}$ for the sheafification of $\mathfrak{X}$ and $\mathfrak{Y}$ on the various topologies for schemes considered in this paper. If $\mathcal{Y}$ and $\mathcal{Z}$ are systems of NC_l- and NP_l-embeddings then one can form the procomplexes of sheaves $\mathfrak{X}^{\mathcal{Y}}$ and $\mathfrak{Y}^{\mathcal{Z}}$ in the same way as was done with the complex $\Omega^{\mathcal{Y}}$ in (42).

THEOREM 6.7. *Let $n \in \mathbb{Z}$, X , $\mathcal{Y}$, $\mathcal{Z}$ as in the case $l = \infty$ of theorem* 6.4 *above, and $\mathfrak{X}$, $\mathfrak{Y}$ as in* 6.6. *Then there is a commutative diagram of natural isomorphisms*

$$\begin{tikzcd}
\prod_{2j \geqslant n} H^{2j-n}(X_{\text{Comm}-\text{inf}}\, \mathcal{O}) \arrow[d, "f_1"] & \\
\mathbb{H}^n(X_{NP_\infty-\text{inf}}, \mathfrak{Y}) \arrow[r, "f_2"] \arrow[d, "f_3"] & \mathbb{H}^n(X_{\text{Pro}-\text{Zar}}, \mathfrak{Y}^{\mathcal{Z}}) \arrow[d, "f_{3'}"] \\
\mathbb{H}^n(X_{NC_\infty-\text{inf}}, \mathfrak{X}) \arrow[r, "f_{2'}"] & \mathbb{H}^n(X_{\text{Pro}-\text{Zar}}, \mathfrak{X}^{\mathcal{Y}}) \arrow[d, "f_4"] \\
& HC^{\text{per}}_n(X)
\end{tikzcd}$$

The map f_1 *sends the filtration* $F'_r = \prod_{m \leqslant r}(\cdot)$ *isomorphically onto the filtration induced by the Poisson filtration* (46)*; both* f_2 *and* $f_{2'}$ *are filtered isomorphisms. The isomorphisms* f_3 *and* $f_{3'}$ *are induced by the symmetrization map* (49)*; they map the filtration induced by* (46) *isomorphically onto that induced by* (1).

COROLLARY 6.8 (Compare [13, Th. 5], [23, Th. 3.4]). *There is a natural isomorphism* $HC_n^{\mathrm{per}}(X) \cong \prod_{2j \geqslant n} H^{2j-n}(X_{\mathrm{Comm-inf}}, \mathcal{O})$ □

COROLLARY 6.9 *Assume* X *is formally smooth. Then with the notations of* 2.5 *above*

$$HC_*^{\mathrm{per}}(X) = \mathbb{H}^*\left(X_{\mathrm{Pro-Zar}}, \mathfrak{Y}\left(\left\{\frac{P\mathcal{O}}{P_{\geqslant n}\mathcal{O}}\right\}_n\right)\right) \tag{55}$$

If in addition X *admits a formally* NC_∞*-thickening* $X \hookrightarrow Y_\infty$ *then the group* (55) *is also isomorphic to* $\mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathfrak{X}\mathcal{O}_{Y_\infty})$. □

7. Auxiliary Results

PROPOSITION 7.1. *Let* $R \in \mathrm{Ass}$ *be a quasi-free algebra,* TR *the tensor algebra,* $JR := \ker(TR \twoheadrightarrow R)$*,* $s \in \mathrm{Hom}_{\mathrm{Ass}}(R, TR/JR^2)$ *a section of the canonical projection,* $J \triangleleft R$ *an ideal and* $l \geqslant 0$*. Then there are maps of pro-complexes*

$$C_{NC_l}(R, J, \mathcal{O}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \Omega_{NC_l}(R, J)$$

and homotopies $\delta\colon \alpha\beta \to 1$ *and* $\gamma\colon \beta\alpha \to 1$ *all of which are natural with respect to* R*,* s*,* J *and* l*, and interchange commutators and graded commutators. In particular* $C_{NC_l}(R, J, \mathcal{O}/[\mathcal{O}, \mathcal{O}])$ *is naturally homotopic to* $\Omega_{NC_l}(R, J)/[\Omega_{NC_l}(R, J), \Omega_{NC_l}(R, J)]$.

Proof. It suffices to check that the map $\alpha := 1 \otimes p : \bar{C}(R, 0, \mathcal{O}) \mapsto \Omega R$ of the proof of [6, 2.4] which by the proof of [6, 3.1] preserves both the J-adic filtration and the commutator subspace, preserves also the commutator filtration, and to construct a natural homotopy inverse for it with the same properties. The proof that α preserves the commutator filtration is similar to the proof that it preserves the commutator subspace; one just considers the action of the full symmetric group Σ_m on $\bar{T}^m$ rather than only that of the cyclic group. The map s of the proof of [6, 3.1] extends to a Σ_m-equivariant contracting homotopy θ of the augmented resolution $\bar{T}^m \twoheadrightarrow k$. Using θ and the perturbation lemma [6, 2.5] one obtains a contracting homotopy of the mapping cone of α, $h\colon M^n = \bar{C}^n \oplus \Omega^n R \to M^{n-1}$ with the matricial form

$$h = \begin{pmatrix} \gamma & \beta \\ 0 & 0 \end{pmatrix} : M^p = \overline{\mathrm{Cyl}}^p R \oplus \Omega^{p-1} \to M^{p-1}$$

It follows that β is a cochain map with $\alpha\beta = 1$ and that γ is a homotopy $1 \to \beta\alpha$. One checks, using the equivariance of θ and s and the formulas of [6, 2.5] that both γ and β preserve both the commutator subspace and the commutator filtration and are continuous for the J-adic filtration. □

LEMMA 7.2. *Let* $A \in \mathrm{Comm}$, $R \in \mathrm{Ass}$, $\pi : R \twoheadrightarrow A$ *a surjective homomorphism,* $\pi_{l,n}$: $R_{l,n} := R/F_{l+1}R + (\ker\pi)^{n+1} \twoheadrightarrow A$ *the induced map,* $\Gamma \subset B := R/F_1 R$ *a multiplicative system and* $\hat{\Gamma} = \pi^{-1}(\Gamma)$. *Then there is a commutative diagram with horizontal isomorphism*

$$\begin{tikzcd} R_{l,n}[\Gamma^{-1}] \arrow[r, "\phi", "\sim"'] \arrow[dr, "\pi_{l,n}[\Gamma^{-1}]"'] & R[\hat{\Gamma}^{-1}]_{l,n} \arrow[d, "\pi[\hat{\Gamma}^{-1}]_{l,n}"] \\ & A[\Gamma^{-1}] \end{tikzcd}$$

Proof. Both $R \mapsto R_l[\Gamma^{-1}]$ and $R \mapsto (R[\hat{\Gamma}^{-1}])_l$ are universal (initial) among all those algebra homomorphisms going from R to an NC_l-algebra which invert Γ. Therefore they are isomorphic R-algebras. By naturality we get a commutative diagram

$$\begin{tikzcd} R_l[\Gamma^{-1}] \arrow[r, "\psi", "\sim"'] \arrow[dr, "\pi_l[\Gamma^{-1}]"'] & (R[\hat{\Gamma}^{-1}])_l \arrow[d, "(\pi[\hat{\Gamma}^{-1}])_l"] \\ & \Gamma^{-1}A \end{tikzcd}$$

where ψ is the natural isomorphism of R-algebras just defined. Thus ψ maps $K := \ker(\pi_l[\Gamma^{-1}]) = (\ker\pi_l)[\Gamma^{-1}]$ isomorphically to $K' := (\ker\pi[\hat{\Gamma}^{-1}])_l$. One checks, using [17, 2.1.5.1] that $K^n = (\ker\pi_l)^n[\Gamma^{-1}]$. The map ϕ of the lemma is that induced by ψ upon passage to the quotient. □

LEMMA 7.3. *Let* $G = G_0 \oplus G_1 \oplus \cdots \oplus G_l$ *be a graded commutative algebra. Assume* G *is additionally equipped with an associative but not necessarily commutative product* $\Phi = \sum_{p=0}^{l} \Phi_p$: $G \otimes G \to G$ *such that* Φ_0 *is the original commutative product, and* Φ_p *is homogeneous of degree p and a bidifferential operator. Consider the associative algebra* $R = (G, \Phi)$. *If* $\Gamma \subset G_0$ *is a multiplicative system, then* $\hat{\Gamma} = \{s + g_+ | s \in \Gamma, g_+ \in \oplus_{n \geqslant 1} G_n\} \subset R$ *is a multiplicative system and* $R[\hat{\Gamma}^{-1}] \cong (\Gamma^{-1}G, \Gamma^{-1}\Phi)$.

Proof. Note first that the product $\Gamma^{-1}\Phi$ is associative because the associator localizes

$$\begin{aligned} \mathcal{A}(\Gamma^{-1}\Phi) &= \Gamma^{-1}\Phi(\Gamma^{-1}\Phi(\,,\,),\,) - \Gamma^{-1}\Phi(\,,\Gamma^{-1}\Phi(\,,\,)) \\ &= \Gamma^{-1}\mathcal{A}(\Phi) = 0. \end{aligned}$$

On the other hand, that $\hat{\Gamma}$ is multiplicatively closed is clear from the fact that $\Phi(G_{\geqslant n}\otimes G_{\geqslant m})\subset G_{\geqslant n+m}$. One checks that, upon localization of G_0-modules, the projection $G\twoheadrightarrow G_0$ becomes a surjective algebra homomorphism $\Gamma^{-1}\pi\colon R'=(\Gamma^{-1}G,\Gamma^{-1}\Phi)\twoheadrightarrow\Gamma^{-1}G_0$. Thus an element $x\in R'$ is invertible if and only if $\Gamma^{-1}\pi(x)$ is invertible. It follows that the obvious homomorphism $R\to R'$ maps each element of $\hat{\Gamma}$ to an invertible element, whence we have a natural map $\phi\colon R[\hat{\Gamma}^{-1}]\to R'$. To prove that ϕ is an isomorphism proceed as follows. Consider the filtration $R=R_0\supset R_1\supset\cdots\supset R_l$, $R_n=(G_{\geqslant n},\Phi_{|G_{\geqslant n}\otimes G_{\geqslant n}})$. Each R_n is an ideal of R, and by exactness of Øre localization, the associated graded ring is

$$\bigoplus_{n=0}^{l}\frac{R_n[\hat{\Gamma}^{-1}]}{R_{n+1}[\hat{\Gamma}^{-1}]}=\bigoplus_{n=0}^{l}\Gamma^{-1}G_n=\Gamma^{-1}G.$$

Thus ϕ is an isomorphism because it is so at the graded level. $\square$

LEMMA 7.4. *Let V be a vectorspace, $T=TV$ the tensor algebra, $T_{\leqslant l}=T/F_{l+1}T$. Also let $P=\mathrm{Poiss}V$ be the free Poisson algebra, and for the Poisson analogue of the commutator filtration, $P_{\leqslant l}=P/F_{l+1}P$. Assume a multiplicative system $\Gamma\subset P_{\leqslant 0}=T_{\leqslant 0}=S:=SV$ is given, and let $\hat{\Gamma}\subset T_{\leqslant l}$ be the inverse image of Γ under the projection $T_{\leqslant l}\twoheadrightarrow S$. Then the symmetrization map* (49) *induces an isomorphism*

$$\frac{\Gamma^{-1}P_{\leqslant l}}{\{\Gamma^{-1}P_{\leqslant l},\Gamma^{-1}P_{\leqslant l}\}}\cong\frac{T_{\leqslant l}[\hat{\Gamma}^{-1}]}{[T_{\leqslant l}[\hat{\Gamma}^{-1}],T_{\leqslant l}[\hat{\Gamma}^{-1}]]}.$$

Proof. Apply Lemma 7.2 with $R=T$, $A=B=S$ to obtain an isomorphism $T_{\leqslant l}[\Gamma^{-1}]\cong T[\hat{\Gamma}^{-1}]/F_{l+1}T[\hat{\Gamma}^{-1}]$. Thus

$$\frac{T_{\leqslant l}[\Gamma^{-1}]}{[T_{\leqslant l}[\Gamma^{-1}],T_{\leqslant l}[\Gamma^{-1}]]}=\frac{T[\hat{\Gamma}^{-1}]}{F_{l+1}T[\hat{\Gamma}^{-1}]+[T[\hat{\Gamma}^{-1}],T[\hat{\Gamma}^{-1}]]}.$$

Now $[T[\hat{\Gamma}^{-1}],T[\hat{\Gamma}^{-1}]]$ is the image of the Hochschild boundary

$$b\colon\Omega^1T[\hat{\Gamma}^{-1}]\to T[\hat{\Gamma}^{-1}],\ xdy\mapsto[x,y].$$

But $\Omega^1T[\hat{\Gamma}^{-1}]=T[\hat{\Gamma}^{-1}]dVT[\hat{\Gamma}^{-1}]\cong T[\hat{\Gamma}^{-1}]\otimes V\otimes T[\hat{\Gamma}^{-1}]$ as $T[\hat{\Gamma}^{-1}]$-bimodules. Hence,

$$\frac{\Omega^1T[\hat{\Gamma}^{-1}]}{b\Omega^2T[\hat{\Gamma}^{-1}]}=\frac{\Omega^1T[\hat{\Gamma}^{-1}]}{[T[\hat{\Gamma}^{-1}],\Omega^1T[\hat{\Gamma}^{-1}]]}\cong T[\hat{\Gamma}^{-1}]\otimes V$$

and, therefore,

$$[T[\hat{\Gamma}^{-1}],T[\hat{\Gamma}^{-1}]]=[T[\hat{\Gamma}^{-1}],V].\tag{56}$$

On the other hand, by [8, 2.1 (1)] the map e induces a vectorspace isomorphism $P_{\leqslant l} \cong T_{\leqslant l}$, whence $T_{\leqslant l}$ is identified with the algebra with underlying vectorspace $P_{\leqslant l}$ and multiplication

$$\Phi(x, y) := \mathrm{e}^{-1}(\mathrm{e}x\mathrm{e}y). \tag{57}$$

By [8, 2.1 (2)] and [7, 2.2], Lemma 7.3 applies to $G = P_{\leqslant l}$ whence $T_{\leqslant l}[\hat{\Gamma}^{-1}] \cong (\Gamma^{-1}P_{\leq l}, \Gamma^{-1}\Phi)$. Thus, modulo $F_{l+1}T[\hat{\Gamma}^{-1}]$, (56) gets identified with the subspace generated by the elements of the form

$$\sum_{p=1}^{l}(\Gamma^{-1}\Phi)_p(s^{-1}x, v) - (\Gamma^{-1}\Phi)_p(v, s^{-1}x) \quad (x \in P \leqslant l, s \in \Gamma, v \in V). \tag{58}$$

By [7, 1.1] the homogeneous part of degree one of (58) is $\{s^{-1}x, v\} \in \{\Gamma^{-1}P_{\leqslant l}, V\}$. I claim that each homogeneous part of degree $p \geqslant 2$ of (58) is zero. For $s = 1$, the claim is just the fact that e commutes with the adjoint action of the Lie subalgebra $L \subset T$ generated by V –[19, 3.3.5]– and in particular with its restriction to V. Recall both $\Phi_p(v,)$ and $\Phi_p(, v)$ are differential operators of order $\leqslant p$ ([7, 2.2]). Thus the identity

$$F(s^{-1}a) = \sum_{j=0}^{p}\left(\sum_{i=j}^{p}(-1)^i\binom{i}{j}\right)s^{-(j+1)}F(s^{-1}a) \quad (s \in \Gamma, a \in P_{\leqslant l})$$

holds for both $F = \Phi_p(v,), \Phi_p(, v)$. The general case of the claim follows from this observation and the case $s = 1$. We have shown that under our identifications $[T_{\leqslant l}[\Gamma^{-1}], T_{\leqslant l}[\Gamma^{-1}]]$ gets identified with $\{\Gamma^{-1}P_{\leqslant l}, V\}$. It is clear that the latter coincides with $\{\Gamma^{-1}P_{\leqslant l}, \Gamma^{-1}P_{\leqslant l}\}$. □

LEMMA 7.5. *Let V be a vectorspace, $S = \Omega_{\mathrm{Comm}}SV = S(V \oplus \mathrm{d}V)$ the commutative DGA. Let $P_+S = \oplus_{n \geqslant 1}P_nS$ be the part of positive degree in the DG-Poisson envelope* (44). *Then there is a contracting homotopy h: $P_+S \to P_+S$ which is right S-linear, homogeneous of degree zero for the Poisson gradation and maps $S(LV \oplus \mathrm{d}LV) \cap P_+S$ to itself.*

Proof. Define a k-linear map ∂: $W := V \oplus \mathrm{d}V \to W$, $\partial\mathrm{d}v = v$, $\partial v = 0$. Extend ∂ first to $\mathfrak{g} = LW$ as a derivation for the Lie bracket and then to all of $S\mathfrak{g} = PS$ as a derivation for the (skew-) commutative product. Put $\Delta = [\partial, d]$. Write || for homogenous degree with respect to (45). Consider the grading ω of $S\mathfrak{g}$ determined by $\omega(g) = |g| + 1$. If x is homogeneous with respect to ω, then $\Delta x = \omega(x)x$. Rescale the restriction of Δ to $S_+\mathfrak{g}_+$ (notation as in [8, 1.0]) to obtain a k-linear map κ: $S_+\mathfrak{g}_+ \to S_+\mathfrak{g}_+$ with $\kappa d + d\kappa = 1$.

$$h\colon P_+S = S\mathfrak{g}_0 \otimes S_+\mathfrak{g}_+ \to P_+S, \qquad h(x \otimes y) = (-1)^{\deg x}x \otimes \kappa(y).$$

One checks that h is right $S\mathfrak{g}_0$-linear and that $dh + hd = 1$. □

LEMMA 7.6. *Let V be a vectorspace, $P = \mathrm{Poiss}\, V$, $i \geqslant 0$. Write ${}_i\Omega_{\mathrm{Comm}}P$ for the homogeneous part of degree i with respect to* (45). *Then there is a k-linear map*

$$\nabla\colon \Omega^r_{\mathrm{Comm}}P \to \Omega^{r+1}_{\mathrm{Comm}}P, \quad r \geqslant 1$$

such that

(i) $(\nabla\delta + \delta\nabla)\omega = \omega$ *if* $\omega \in \Omega^r_{\mathrm{Comm}}P$, $r \geqslant 2$.
(ii) $\nabla({}_i\Omega^*_{\mathrm{Comm}}P) \subset_{i-1} \Omega^{*+1}_{\mathrm{Comm}}P$.
(iii) *The restriction of ∇ to ${}_i\Omega^r_{\mathrm{Comm}}P$ is a differential operator of SV-modules.*

Proof. Put $L = LV$,

$$C^\alpha_{r,i} := P_i \otimes (\Lambda^r L)_{\alpha-(r+i)}, \qquad C^\alpha_r :=_{\alpha-r} \Omega^r_{\mathrm{Comm}}P = \bigoplus_{i=0}^{\alpha-r} C^\alpha_{r,i}. \tag{59}$$

We have $\delta C^\alpha_{r,i} \subset \bigoplus_{j \geqslant i} C^\alpha_{r-1,j}$, whence C^α is a subcomplex of the complex $C = (\Omega_{\mathrm{Comm}}P, \delta)$, and $C = \bigoplus_{\alpha \geqslant 0} C^\alpha$. The homogeneous component of degree 0 of $\delta\colon C^\alpha_r \to C^\alpha_{r-1}$ is the restriction of $1 \otimes \delta'$, where $\delta'\colon \Lambda^r L \to \Lambda^{r-1}L$ is the Chevalley-Eilenberg boundary

$$\delta'(g_1 \wedge \cdots \wedge g_r) = \sum_{i<j} (-1)^{i+j}[g_i, g_j] \wedge g_1 \wedge \cdots \overset{i}{\vee} \cdots \overset{j}{\vee} \cdots \wedge g_r$$

Because L is free, there is a k-linear map $\nabla'\colon \Lambda^n L \to \Lambda^{n+1}L$ $(n \geqslant 1)$ such that $\nabla'\delta' + \delta'\nabla' = 1$ on $\Lambda^{\geqslant 2}L$. Because δ' is homogeneous of degree $+1$ for the chain complex decomposition induced by $L = \bigoplus_{n \geqslant 0} L_n$, we may assume ∇' homogeneous of degree -1. Put

$$\nabla^{\alpha,0} := 1 \otimes \nabla' : C^\alpha_r \to C^\alpha_{r+1} \quad (r \geqslant 1)$$

Then $\nabla^{\alpha,0}$ is homogeneous of degree 0 for the decomposition (59). By the perturbation Lemma ([6, 2.5]) there exists, for each $n \geqslant 1$, a k-linear map $\nabla^{\alpha,n} : C^\alpha_r \to C^\alpha_{r+1}$ homogeneous of degree n (with respect to (59)) such that $\nabla^\alpha := \sum_{n=0}^\infty \nabla^{\alpha,n}$ verifies $\nabla^\alpha\delta + \delta\nabla^\alpha = 1$ on C^α_r $(r \geqslant 2)$. Moreover, from the formulas of [6, 2.5] and the fact that each of the components of δ is a differential operator -because $\{,\}$ is bidifferential- it follows that the same is true of each $\nabla^{\alpha,n}$. Therefore the map $\nabla = \bigoplus_\alpha \nabla^\alpha$ satisfies the conditions of the lemma. □

LEMMA 7.7. *Let V be a vectorspace, $S = SV$, $P = \mathrm{Poiss}\, V$. Consider the complex*

$$\mathfrak{N}^n = \begin{cases} P_i, & \text{if } n = 2i \\ \dfrac{{}_i\Omega^1_{\mathrm{Comm}}P}{\delta_{i-1}\Omega^2_{\mathrm{Comm}}P}, & \text{if } n = 2i+1 \end{cases} \qquad (n \geqslant 0),$$

with coboundary maps $\mathrm{d}\colon \mathfrak{N}^{2i} \to \mathfrak{N}^{2i+1}$ *and* $\delta\colon \mathfrak{N}^{2i+1} \to \mathfrak{N}^{2i+2}$ *($i \geqslant 0$). Then $\mathfrak{N}$ is naturally homotopy equivalent to $\Omega_{\mathrm{Comm}}S$ in such a way that each of the natural homotopy* *equivalences and homotopies involved is continuous for the adic topology induced by any ideal $I \lhd S$.*

Proof. Put

$$\mathfrak{R}\Omega_i^n = \begin{cases} \mathfrak{R}^n, & \text{if } n \leqslant 2i+1 \\ \frac{{}_i\Omega^r_{\mathrm{Comm}} P}{\delta_{i-1}\Omega^{r+1}_{\mathrm{Comm}} P}, & \text{if } n = 2i+r \qquad (r \geqslant 2). \end{cases}$$

Make $\mathfrak{R}\Omega_i^n$ into a cochain complex with boundary map $\partial^n \colon \mathfrak{R}\Omega_i^n \to \mathfrak{R}\Omega_i^{n+1}$ given by

$$\partial^n = \begin{cases} \delta, & n \text{ odd and } \leqslant 2i-1, \\ d, & n \text{ even or } \geqslant 2i. \end{cases}$$

Let h and ∇ be as in Lemmas 7.5 and 7.6. Define maps

$$\mathfrak{R}\Omega_i^n \underset{\beta_n}{\overset{\alpha_n}{\rightleftarrows}} \mathfrak{R}\Omega_{i+1}^n, \ \mathfrak{R}\Omega_i^n \xrightarrow{\gamma_n} \mathfrak{R}\Omega_i^{n-1}, \text{ and } \mathfrak{R}\Omega_{i+1}^n \xrightarrow{\epsilon_n} \mathfrak{R}\Omega_{i+1}^{n-1}$$

as follows

$$\alpha_n = \begin{cases} 1, & n \leqslant 2i+1, \\ -h\delta, & n \geqslant 2i+2, \end{cases} \qquad \beta_n = \begin{cases} 1, & n \leqslant 2i+1, \\ -\nabla d, & n = 2i+2, \\ -(\nabla d + d\nabla), & n \geqslant 2i+3, \end{cases}$$

$$\gamma_n = \begin{cases} 0 & n \leqslant 2i+2, \\ \nabla h\delta & n \geqslant 2i+3, \end{cases} \qquad \epsilon_n = \begin{cases} 0 & n \leqslant 2i+2, \\ h(\delta\nabla - 1) & n \geqslant 2i+3, \end{cases}$$

One checks that α and β are cochain maps as well as that the following identities hold

$$\alpha\beta - 1 = \epsilon\partial + \partial\epsilon, \qquad \beta\alpha - 1 = \gamma\partial + \partial\gamma.$$

Thus $(\Omega_{\mathrm{Comm}}S, \mathrm{d}) = (\mathfrak{R}\Omega_0, \partial)$ is naturally and adically continuously homotopy equivalent to $\mathfrak{R} = \mathrm{colim}(\mathfrak{R}\Omega_0 \xrightarrow{\beta} \mathfrak{R}\Omega_1 \xrightarrow{\beta} \mathfrak{R}\Omega_2 \xrightarrow{\beta} \cdots)$. $\square$

LEMMA 7.8. *Let U and V be vectorspaces, $\alpha, \beta \geqslant 0$, $\gamma = \alpha + \beta$. Let $\Sigma_{\alpha,\beta} := \Sigma_\alpha \times \Sigma_\beta$ act on $T^\gamma U \otimes T^\gamma V$ as follows:*

$$\begin{aligned}&(\sigma, \tau)(u_1 \ldots u_\gamma \otimes v_1 \ldots v_\gamma) \\ &\quad = (\mathrm{sg}\tau) u_{\sigma 1} \ldots u_{\sigma\alpha} u_{\alpha+\tau 1} \ldots u_{\alpha+\tau\beta} \otimes v_{\sigma 1} \ldots v_{\sigma\alpha} v_{\alpha+\tau 1} \ldots v_{\alpha+\tau\beta}.\end{aligned}$$

Then

$$S^\alpha(U \otimes V) \otimes \Lambda^\beta(U \otimes V) \cong (T^\gamma U \otimes T^\gamma V)_{\Sigma_{\alpha,\beta}}.$$

Proof. Straightforward. $\square$

PROPOSITION 7.9. *Let X be a separated commutative scheme.*

Then

$$H^n(X_{\mathrm{Comm-inf}}, \Omega^p_{\mathrm{Comm}}) = 0 \quad (p \geqslant 1, n \geqslant 0).$$

Proof. By (38), (39) and (40) it suffices to show that if U is a vectorspace and $I \subset S = SU$ is an ideal, then the normalized pro-complex $\bar{C}(S, I, \Omega^p)$ is contractible. We shall show this for the case $I = 0$; a routine verification shows that all the cochain maps and homotopies we shall define are continuous for the adic topology of any ideal I, proving the general case. Let V^* and W^* be the cosimplicial vectorspaces of [6, 1.2]. We have

$$S^{\otimes m+1} = S(U \oplus U \otimes V^m) = S(W^m). \tag{60}$$

Hence by the lemma above

$$\Omega^p_{\mathrm{Comm}} S^{\otimes m+1} = \bigoplus_{\alpha \geqslant 0} \bigoplus_{q+\beta=p} \Omega^q_{\mathrm{Comm}} SU \otimes (T^{\alpha+\beta} U \otimes T^{\alpha+\beta} V^m)_{\Sigma_{\alpha,\beta}}.$$

Pro-completion with respect to the ideal $\langle U \otimes V^m \rangle \subset S^{\otimes m+1}$ gives the pro-space

$$C^m_n := C^m(S, 0, \Omega^p)_n = \bigoplus_{0 \leqslant \alpha \leqslant n} \bigoplus_{q+\beta=p} \Omega^q_{\mathrm{Comm}} SU \otimes (T^{\alpha+\beta} U \otimes T^{\alpha+\beta} V^m)_{\Sigma_{\alpha,\beta}},$$

where ($m \geqslant 1$). Recall from the proof of [6, 2.4] that for the normalized complex $\overline{T^r V^*}$ we have

$$\overline{T^r V^m} = 0, \quad \text{for } r > m \quad \text{and} \quad \overline{T^m V^m} = k[\Sigma_m].$$

Thus $\bar{C}^m$ is the constant pro-vectorspace

$$\bar{C}^m = \bigoplus_{0 \leqslant r \leqslant m} \bigoplus_{p-r \leqslant q \leqslant p} \Omega^q_{\mathrm{Comm}} SU \otimes (T^r U \otimes \overline{T^r V}^m)_{\Sigma_{r-p+q,p-q}}.$$

Recall from the proof of Proposition 7.1 that there is a Σ_m-equivariant homotopy equivalence $p\colon \overline{T^m V} \to k[-m]$. It follows that $1 \otimes p$ passes to the quotient modulo the action of the symmetric group, giving a homotopy equivalence between $\overline{C}$ and a complex having

$$D^m := \bigoplus_{p-m \leqslant q \leqslant p} \Omega^q_{\mathrm{Comm}} SU \otimes \Lambda^{m-p+q} U \otimes S^{p-q} U$$

in degree m. This vectorspace can be interpreted as a piece of the DG-module of m-differential forms of the DGA $\Omega_{\mathrm{Comm}} SU$. Namely $D^m =_p \Omega^m_{DG-\mathrm{Comm}}(\Omega_{\mathrm{Comm}} SU)$. Here the subindex p denotes weight with respect to the grading of $\Omega_{\mathrm{Comm}} SU$ determined by $\deg(u) = 0$, $\deg(du) = 1$. One checks further that the coboundary map is the restriction of d', the de Rham differential for forms on Ω_{SU}. We have

$$\begin{aligned}
\Omega_{DG-\mathrm{Comm}} \Omega_{\mathrm{Comm}} SU &= S(U \oplus \mathrm{d}U \oplus \mathrm{d}'(U \oplus dU)) \\
&= S(U \oplus \mathrm{d}'U) \otimes S(\mathrm{d}U \oplus \mathrm{d}'dU) \\
&\cong \Omega_{DG-\mathrm{Comm}} SU \otimes \Omega_{DG-\mathrm{Comm}} S(\mathrm{d}U).
\end{aligned}$$

It is clear that $\Omega_{DG-\mathrm{Comm}}S(\mathrm{d}U)$ is contractible by means of a weight preserving contracting homotopy h. Thus $1\otimes h$ is a contracting homotopy for D. This concludes the proof. $\square$

LEMMA 7.10. *Let $n, m \geqslant 0$. Then*

(i) *Let $\Omega^1_\natural$ be the $NC_\infty - \inf$ sheaf* $\operatorname{coker}(b\colon \Omega^2 \to \Omega^1)$. *Then $H^n(X_{NC_\infty-\inf}, \Omega^1_\natural) = 0$.*
(ii) *Let $({}_m\Omega^1_{\mathrm{Comm}}P)_\delta$ be the* $\mathrm{Comm} - \inf$ *sheaf* $\operatorname{coker}(\delta\colon {}_{m-1}\Omega^2_{\mathrm{Comm}}P \to {}_m\Omega^1_{\mathrm{Comm}}P)$,

where the subscript on the left hand corner indicates degree with respect to the grading (44). *Then $H^n(X_{\mathrm{Comm}-\inf}, ({}_m\Omega^1_{\mathrm{Comm}}P)_\delta) = 0$.*

Proof. (i) It suffices to show that if $\mathrm{Comm} \ni A = R/I$ with R quasi-free then for the presheaf cokernel $\bar\Omega^* = \Omega^*/b\Omega^{*+1}$, the complex $C(R, I, \bar\Omega^1)$ is naturally contractible. A similar argument as that given in the proof of 7.1 above shows that the homotopy equivalence of the proof of [6, Lemma 5.6]

$$1\otimes p\colon C(R, I, \bar\Omega^1) \to \frac{\Omega^{*+1}R \oplus \Omega^*R}{N^* + \mathcal{G}^{1,\infty}} = \frac{\bar\Omega^{*+1}R \oplus \bar\Omega^*R}{\mathcal{G}^{1,\infty}}$$

preserves the commutator filtration.

(ii) By the proof of Lemma 7.4, for $R = TV$, $S = SV$, and the presheaf cokernel $\bar\Omega^*_{\mathrm{Comm}} := \Omega^*_{\mathrm{Comm}}/\delta\Omega^{*+1}_{\mathrm{Comm}}$ the symmetrization map induces an isomorphism of pro-complexes $\prod_{m=0}^\infty C(S, I, {}_m\bar\Omega^1_{\mathrm{Comm}}P) \cong C(R, I, \bar\Omega^1)$. $\square$

LEMMA 7.11. *Let V be a vectorspace, $S = SV$, $T = TV$, $L = LV$, $P = \mathrm{Poiss}\, V$, $I \subset S$ an ideal, $A = S/I$ and $J \subset T$ the inverse image of I under the projection $T \twoheadrightarrow T/F_1T = S$. Then the map*

$$\eta\colon \Omega_{\mathrm{Comm}}P \to \Omega T,$$

$$\eta(a \otimes \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n) = e(a) \otimes \sum_{\sigma\in\Sigma_n} \mathrm{sg}(\sigma)\mathrm{d}g_{\sigma 1}\ldots\mathrm{d}g_{\sigma n} \quad (g_i \in L)$$

induces a homotopy equivalence of pro-complexes

$$\left(\Omega_{\mathrm{Comm}}\left(\frac{P}{P_{\geqslant\infty} + I^\infty P}\right), \delta\right) \xrightarrow{\sim} \left(\Omega\left(\frac{T}{F_\infty T + J^\infty}\right), b\right).$$

Proof. Consider the associative product $x \star y = \mathrm{e}^{-1}(\mathrm{e}x\mathrm{e}y)$ for $x, y \in SL = P$; put $Q := (P, \star)$. The map e induces a chain isomorphism between $\Omega Q = (\Omega P, b^Q)$ and

$(\Omega T, b)$. Moreover, it follows from [7, 2.2], [8, 2.1] and [8, 2.6] that the map e induces a chain pro-isomorphism

$$\left(\Omega\left(\frac{P}{P_\infty + I^\infty P}\right), b^Q\right) \cong \left(\Omega\left(\frac{T}{F_\infty T + J^\infty}\right), b\right).$$

Let $\alpha\colon S\mathrm{d}L = \Lambda L \to \mathrm{d}\Omega P$,

$$\alpha(\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n) = \sum_{\sigma\in\Sigma_n} \mathrm{sg}(\sigma)\mathrm{d}g_{\sigma 1} \ldots \mathrm{d}g_{\sigma n}.$$

Note $\eta = e \circ (1 \otimes \alpha)$. By [18, Th. 3-a)], $1 \otimes \alpha$ is a chain map. It induces a chain pro-map

$$\Omega_{\mathrm{Comm}}\left(\frac{P}{P_{\geqslant\infty} + I^\infty P}\right) \to \Omega \frac{P}{P_{\geqslant\infty} + I^\infty P}. \tag{61}$$

We must show (61) is a homotopy equivalence. For this we shall construct a homotopy inverse $\beta\colon \Omega Q \to \Omega_{\mathrm{Comm}} P$ of $1 \otimes \alpha$ and homotopies $\gamma\colon \beta(1 \otimes \alpha) \to 1$ and $\kappa\colon (1 \otimes \alpha)\beta \to 1$ each of which will be continuous for the linear topologies of the filtrations

$$\begin{aligned}&\{\ker(\Omega_{\mathrm{Comm}} P \twoheadrightarrow \Omega_{\mathrm{Comm}} P/P_{\geqslant n} + I^n P)\}_n \\ &\qquad \text{and} \quad \{\ker(\Omega P \twoheadrightarrow \Omega P/P_{\geqslant n} + I^n P)\}_n.\end{aligned}$$

We point out that the first of these topologies coincides with that of the filtration $\{I^n \Omega_{\mathrm{Comm}} P + \bigoplus_{l\geqslant n} {}_l\Omega_{\mathrm{Comm}} P\}_n$. Write $K\colon (\Omega_{\mathrm{Comm}} \otimes P, \delta') \overset{\epsilon}{\to} P$ for the augmented $Q \otimes Q^{op}$-resolution denoted (L', b') in [18, Prop. 3] and $R\colon (\Omega P \otimes P, b') \overset{\mu}{\to} P$ for the augmented Hochschild resolution. By [18, Lemma 9] the continuous map $1 \otimes \alpha \otimes 1\colon K \to R$ a chain $Q \otimes Q^{op}$-module homomorphism. It suffices to construct continuous $Q \otimes Q^{op}$-homomorphisms $\beta'\colon R \to K$, $\gamma'\colon R \to R[1]$ and $\kappa'\colon K \to K[1]$ such that $\beta' b' = \delta' \beta$, and such that γ' and κ' be homotopies $\beta'(1 \otimes \alpha \otimes 1) \to 1$ and $1 \to (1 \otimes \alpha \otimes 1)\beta'$. In turn for this it suffices to show that both R and K have continuous k-linear contracting homotopies. For then the standard procedure for lifting the identity in dimension zero to a chain map β' using a contracting homotopy for K yields a continous β', and similarly for the standard procedure for constructing the homotopies γ' and κ'. The map $a \mapsto 1 \otimes a$, $\omega \otimes x \mapsto \mathrm{d}w \otimes x$ defines a continuous contracting homotopy for the augmented resolution R. To obtain a continuous contracting homotopy for K proceed as follows. Put ${}_mK_* := \bigoplus_{i+j=m} {}_i\Omega_{\mathrm{Comm}} P \otimes P_j$ We have

$$K = \bigoplus_{m=0}^{\infty} {}_mK, \;\; \delta'({}_mK) \subset \bigoplus_{p\geqslant m} {}_pK.$$

Let δ'_n be the homogeneous component of degree $n \geqslant 0$. By [7, 2.2] each δ'_n is a continuous map. Hence, if h_0 is a continuous contracting homotopy for (K, δ'_0) then the map $h = \sum_{m=0}^{\infty} h_m$ of the perturbation Lemma [6, 2.5] is a continous contracting homotopy for (K, ∂'). We remark that (K, ∂'_0) is the standard Koszul resolution of P as a module over $P \otimes P$ with its commutative structure. Thus, essentially the same argument as in the proof of Lemma 7.5 gives a continuous homotopy h_0 as wanted. $\square$

8. Proofs of the Main Theorems

8.1. PROOF OF THEOREM 6.1.

We first do the case $l < \infty$. To start, we prove the existence of the isomorphism α_l. Assume first $X = \mathrm{Spec}\, A$. Choose a presentation

$$0 \to J \to R \overset{\pi}{\to} A \to 0 \tag{62}$$

of A as quotient of a quasi-free $R \in \mathrm{Ass}$. Put $R_l = R/F_{l+1}R$ and assume $\mathcal{Y}$ is the following system of embeddings

$$\mathcal{Y} := \{X \hookrightarrow Y := \mathrm{Spec}\, R_l\} \tag{63}$$

By Lemma 7.2, for $f \in A$ and $R \supset \Gamma = \pi^{-1}\{f^n : n \geqslant 0\}$ we have canonical isomorphisms

$$\mathcal{C}_{\mathcal{Y}}(\mathcal{O})^p(D(f)) = \{\mathcal{O}_{Y_n^p}(D(f))\} = C_{NC_l}(R[\Gamma^{-1}], J_f, \mathcal{O}), \tag{64}$$

where $J_f = \ker(R[\Gamma^{-1}] \twoheadrightarrow A_f)$. Note $R[\Gamma^{-1}]$ is quasi-free, whence by Proposition 7.1 we have a natural homotopy equivalence between the pro-complex (64) and

$$\Omega_{NC_l}(R[\Gamma^{-1}], J_f). \tag{65}$$

Now because the $D(f)$ form a basis for the Zariski topology, (65) determines a unique pro-complex of sheaves; by Lemma 7.2 this pro-complex must be $\Omega^{\mathcal{Y}}$. Next, if $0 \to J' \to R'_l \to A \to 0$ is any presentation of A as quotient of a formally l-smooth $R'_l \in NC_l$, then by [5, 3.3] a choice of a map $R_l \to R'_l$ covering the identity of A induces a homotopy equivalence $\Omega_{NC_l}(R, J) \tilde{\to} \Omega_{NC_l}(R', J')$ which, in turn, gives a homotopy equivalence $\Omega^{\mathcal{Y}}_{NC_l} \tilde{\to} \Omega^{\mathcal{Y}'}_{NC_l}$ for $\mathcal{Y}' = \{X \hookrightarrow Y'\}$. Now no longer assume X is affine. If $\mathcal{Y}$ consists entirely of affine embeddings, say $\mathcal{Y} = \{\mathrm{Spec}\, A_i \hookrightarrow \mathrm{Spec}\, R_l^i\}$ for some affine open covering $\{\mathrm{Spec}\, A_i\}$ of $\mathrm{Spec}\, A$, then each n-fold intersection

$$\mathrm{Spec}\, A_{i_0} \cap \cdots \cap \mathrm{Spec}\, A_{i_n} \hookrightarrow \mathrm{Spec}\, R_l^{i_j} \times_l \cdots \times_l \mathrm{Spec}\, R_l^{i_n} = \mathrm{Spec}(R_l^{i_0} * \cdots * R_l^{i_n})_l \tag{66}$$

is of the form (63) (by 2.4.3), so from the affine case we obtain a homotopy equivalence $\mathcal{C}_{\mathcal{Y}}(\mathcal{O}) \overset{\alpha}{\to} \Omega^{\mathcal{Y}}_{NC_l}$. If $\mathcal{Y}$ is arbitrary then there is a finer system $\mathcal{Y}'$ which consists entirely of affine embeddings; the argument of [15, Remark on page 28] shows the refinement map $\Omega^{\mathcal{Y}}_{NC_l} \overset{\alpha}{\to} \Omega^{\mathcal{Y}'}_{NC_l}$ is a quasi-isomorphism. Next we construct the map e of the theorem. Assume first $X = \mathrm{Spec}\, A$, choose a presentation (62) with $R = TV$, a tensor algebra and let $\mathcal{Y}$ be as in (63). Further, consider the system of NP_l-embeddings $\mathcal{Z}$ consisting of the single embedding $\mathrm{Spec}\, A \hookrightarrow \mathrm{Spec}\, \mathrm{Poiss}_{\leqslant l} V$ induced by the composite

$$\mathrm{Poiss}_{\leqslant l} V = P_{\leqslant l} SV \twoheadrightarrow SV \cong \frac{R}{F_1 R} \twoheadrightarrow A.$$

Then by Lemmas 7.2 and 7.3 and by [8, 2.1 and 2.6], the map (49) induces an isomorphism of pro-complexes of Abelian sheaves $\mathcal{C}_{\mathcal{Z}}(\mathcal{O}) \underset{\sim}{\overset{e}{\to}} \mathcal{C}_{\mathcal{Y}}(\mathcal{O})$. This proves the isomorphism e in the affine case. For general X one chooses $\mathcal{Y}$ to consist entirely of affine embeddings as above, and $\mathcal{Z}$ as the associated NC_l- system; then use the affine case and the argument of (66), taking into account that the free product of tensor algebras is again a tensor algebra. To construct the map e' in the affine case one uses a DG-version of the same argument as for the construction of e. As per the arguments above, the affine case generalizes to the case when $\mathcal{Y}$ consists of affine NC_l-embeddings, and $\mathcal{Z}$ is the associated NP_l-system. It has already been shown that the hypercohomology of $\Omega^{\mathcal{Y}}_{NC_l}$ is independent of the choice of $\mathcal{Y}$; the same argument shows that of $\Omega^{\mathcal{Z}}_{NP_l}$ is independent of the choice of $\mathcal{Z}$. To finish the proof it suffices to show that there is a choice of $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ for which $\pi' \circ e'$ is a cohomology isomorphism. Choose an affine open covering $\mathcal{U}$ of X. For each $\mathcal{U} \ni U = \operatorname{Spec} A_U$ choose a presentation $A_U = SV_U/I_U$ and let

$$\mathcal{W} = \{\operatorname{Spec} A_U \hookrightarrow \operatorname{Spec} SV_U : U \in \mathcal{U}\},$$
$$\mathcal{Z} = \{\operatorname{Spec} A_U \hookrightarrow \operatorname{Spec} \mathrm{Poiss}_{\leqslant l} V_U : U \in \mathcal{U}\},$$
$$\mathcal{Y} = \{\operatorname{Spec} A_U \hookrightarrow \operatorname{Spec} T_{\leqslant l} V_U : U \in \mathcal{U}\}.$$

Then, by the argument of (66), we are reduced to showing that if $A = SV/I$ then the projection

$$\begin{aligned}\theta : P_{\leqslant l}\Omega_{\mathrm{Comm}}(SV/I^{\infty}) &\cong \frac{\mathrm{Poiss}_{\leqslant l}(V \oplus \mathrm{d}V)}{I^{\infty}\mathrm{Poiss}_{\leqslant l}(V \oplus \mathrm{d}V)} \\ &\twoheadrightarrow \frac{S(V \oplus \mathrm{d}V)}{I^{\infty}S(V \oplus \mathrm{d}V)} \cong \Omega_{\mathrm{Comm}}(SV/I^{\infty})\end{aligned}$$

is a homotopy equivalence. This follows from Lemma 7.5, and the fact that

$$\ker\theta = \bigoplus_{m=1}^{l} \frac{\mathrm{Poiss}_m(V \oplus \mathrm{d}V)}{I^{\infty}\mathrm{Poiss}_m(V \oplus \mathrm{d}V)}.$$

This concludes the proof of the case $l < \infty$ of the theorem. Because the cohomology isomorphisms we found come from natural cochain equivalences which are compatible with the inclusions $NC_l \subset NC_{l+1}$ and $NP_l \subset NP_{l+1}$, the case $l = \infty$ follows.

Remark 8.2. A similar argument as that of the last part of the proof above shows that

$$H^n(X_{\mathrm{comm-inf}}, P_m) = 0 \qquad (m \geqslant 1, n \geqslant 0).$$

Indeed because P_m is quasi-coherent it suffices by (40) to show that for A a commutative algebra,

$$H^n(\inf(\mathrm{Comm}/A), P_m) = 0 \tag{67}$$

With the notations of the proof, we see using Proposition 7.1 that (67) equals

$$= H^n(\lim_r P_m\Omega_{\mathrm{Comm}}(SV/I^r))$$

which is zero by Lemma 7.5.

8.3. PROOF OF THEOREM 6.4.

We shall assume X affine and $l < \infty$. The same argument as in the proof of Theorem 6.1 applies to deduce the theorem from this particular case. Let $X = \operatorname{Spec} A$; choose R, π and J as in (62) and let $\mathcal{Y}$ be as in (63). Proceed as in the proof of Theorem 6.1 to obtain a natural homotopy equivalence of pro-complexes of vectorspaces

$$\frac{\mathcal{C}_{\mathcal{Y}}(\mathcal{O})(D(f))}{[\mathcal{C}_{\mathcal{Y}}(\mathcal{O})(D(f)), \mathcal{C}_{\mathcal{Y}}(\mathcal{O})(D(f))]} \underset{\sim}{\overset{\bar{\alpha}_l}{\to}} \frac{\Omega_{NC_l}(R[\Gamma^{-1}], J_f)}{[\Omega_{NC_l}(R[\Gamma^{-1}], J_f), \Omega_{NC_l}(R[\Gamma^{-1}], J_f)]} \tag{68}$$

for each $f \in A$. Because $\bar{\alpha}_l$ is natural and because sheafification depends only on the value of the presheaf on a basis of the topology, (68) induces a homotopy equivalence of pro-complexes of sheaves

$$\mathcal{C}_{\mathcal{Y}}\left(\frac{\mathcal{O}}{[\mathcal{O}, \mathcal{O}]}\right) = \frac{\mathcal{C}_{\mathcal{Y}}(\mathcal{O})}{[\mathcal{C}_{\mathcal{Y}}(\mathcal{O}), \mathcal{C}_{\mathcal{Y}}(\mathcal{O})]} \underset{\sim}{\overset{\bar{\alpha}_l}{\to}} \frac{\Omega^{\mathcal{Y}}_{NC_l}}{[\Omega^{\mathcal{Y}}_{NC_l}, \Omega^{\mathcal{Y}}_{NC_l}]}. \tag{69}$$

This cochain map gives the cohomology isomorphism of the theorem. The same argument as in the proof of Theorem 6.1 shows that the homotopy type of the complexes (69) is the same as that of those obtained from a different choice of NC_l-embedding $\mathcal{Y}' = \{X \hookrightarrow \operatorname{Spec} R'_l\}$. Assume now $R = TV$, a tensor algebra, and choose $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ as in the proof of Theorem 6.1. Then by Lemmas 7.2, 7.3, and 7.4, [8, 2.1 and 2.6] and sheafification, we have isomorphisms of pro-complexes of Zariski sheaves

$$\bigoplus_{m=0}^{l} \mathcal{C}_{\mathcal{W}}((P_m)_\delta) \underset{\cong}{\overset{\gamma}{\to}} \mathcal{C}_{\mathcal{Z}}\left(\frac{\mathcal{O}}{\{\mathcal{O}, \mathcal{O}\}}\right) \underset{\cong}{\overset{\bar{e}}{\to}} \mathcal{C}_{\mathcal{Y}}\left(\frac{\mathcal{O}}{[\mathcal{O}, \mathcal{O}]}\right),$$

where P_m is as in Remark 8.2 above and the subscript indicates the sheaf cokernel of the restriction of the coboundary map δ of 6.6 to ${}_{m-1}\Omega_{\mathrm{Comm}}P$, i.e. the sheafification of

$$D(f) \mapsto \frac{P_m A_f}{\sum_{i+j=m-1}\{P_i A_f, P_j A_f\}}.$$

A DG-version of the same argument gives isomorphisms

$$\bigoplus_{m=0}^{l} (P_m \Omega^{\mathcal{W}}_{\mathrm{Comm}})_\delta \underset{\cong}{\overset{\gamma'}{\leftarrow}} \frac{\Omega^{\mathcal{Z}}_{NP_l}}{\{\Omega^{\mathcal{Z}}_{NP_l}, \Omega^{\mathcal{Z}}_{NP_l}\}} \underset{\cong}{\overset{\bar{e}'}{\to}} \frac{\Omega^{\mathcal{Y}}_{NC_l}}{[\Omega^{\mathcal{Y}}_{NC_l}, \Omega^{\mathcal{Y}}_{NC_l}]}.$$

By naturality, we get a homotopy equivalence $\mathcal{C}_{\mathcal{W}}((P_m)_\delta) \overset{\sim}{\to} (P_m \Omega^{\mathcal{W}}_{\mathrm{Comm}})_\delta$ Next, consider the truncation $\tau_{2m}\mathfrak{R}$ of the complex of Lemma 7.7. By 7.9, 8.2 and Proposition 7.9, we have a commutative diagram of homotopy equivalences

$$\begin{array}{ccc}
\mathrm{Tot}\mathcal{C}_{\mathcal{W}}(\tau_m\mathfrak{R}) & \overset{\sim}{\longrightarrow} & \mathcal{C}_{\mathcal{W}}((P_m)_\delta) \\
\downarrow & & \downarrow \\
\mathrm{Tot}\mathcal{C}_{\mathcal{W}}(\tau_m\Omega) & \overset{\sim}{\longrightarrow} & \mathcal{C}_{\mathcal{W}}\left(\frac{\Omega^m_{\mathrm{Comm}}}{d\Omega^{m-1}_{\mathrm{Comm}}}\right)
\end{array}$$

with as rows the natural projections and as columns the maps induced by that of Lemma 7.7. To finish the proof we must show that the natural projection

$$\mathrm{Tot}\mathcal{C}_{\mathcal{W}}(\tau_m\Omega_{\mathrm{Comm}}) \overset{\sim}{\to} \mathcal{C}^0_{\mathcal{W}}(\tau_m\Omega_{\mathrm{Comm}}) = \tau_m({\Omega_{\mathrm{Comm}}}^{\mathcal{W}}) \tag{70}$$

is a homotopy equivalence. Recall $\mathcal{W} = \{\mathrm{Spec}\ A \hookrightarrow \mathrm{Spec}\ S\}$, where $S = SU$ is the symmetric algebra of some vector space U and $A = S/I$. Thus by (60) we have an isomorphism (where Cyl is as in [6])

$$Cyl^p(SU) = \frac{S(U \oplus U \otimes V^p)}{\langle U \otimes V^p\rangle^\infty}.$$

Grade $S(U \oplus U \otimes V^p)$ by $|(u,0)| = 0$, $|(0, u \otimes v)| = 1$. Then there is an inclusion

$$\tau_m\Omega_{\mathrm{Comm}}SU \overset{\imath}{\hookrightarrow} \tau_m\Omega_{\mathrm{Comm}}Cyl^*SU = C^*(SU, 0, \tau_m\Omega_{\mathrm{Comm}})$$

of the constant co-simplicial cochain pro-complex as the part of degree zero of the Čech–Alexander pro-complex. The map $\imath$ is a right inverse for the natural projection $\mu: C^*(SU, 0, \tau_m\Omega_{\mathrm{Comm}}) \twoheadrightarrow \tau_m\Omega_{\mathrm{Comm}}SU$. The Cartan homotopy associated to the degree derivation $D(x) = |x|x$ gives a homotopy $\imath\mu \to 1$ which is compatible with the cosimplicial structure, localization and the I-adic topology. Thus upon sheafification we get that (70) is a homotopy equivalence. □

Remark 8.3.1. It follows from 7.1 and the proof above that, if R is formally NC_l-smooth ($l < \infty$) and $A = R/F_1R$, then

$$H^n\left(\inf(NC_l/A), \frac{\mathcal{O}}{[\mathcal{O},\mathcal{O}]}\right) = H^n\left(\frac{\Omega_{NC_l}R}{[\Omega_{NC_l}R, \Omega_{NC_l}R]}\right) = \bigoplus_{m=0}^{n} H^{n+2m}_{dR}A.$$

8.4. PROOF OF THEOREM 6.7.

Let $A \in \mathrm{Comm}$, $X = \mathrm{Spec}\ A$, R, J and π as in (62). For $l \geqslant 0$ put $R_l = R/F_{l+1}R \supset J_l = J + F_{l+1}R/F_{l+1}R$, $Y_l = \mathrm{Spec}\ R_l$, $\mathcal{Y} = \{X \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots\}$ the associated formally NC_∞-smooth embedding. By Goodwillie's theorem ([10, Th. 10.1]), we have a natural isomorphism

$$HC^{\mathrm{per}}_*(X) := \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, \mathcal{CC}^{\mathrm{per}}) \cong \mathbb{H}^*(X_{\mathrm{Pro-Zar}}, (\mathcal{CC}^{\mathrm{per}})^{\mathcal{Y}}).$$

To prove the isomorphism f_4 it suffices to show that the natural pro-complex projection

$$\mathcal{CC}^{\mathrm{per}}(R_\infty/J^\infty_\infty) \twoheadrightarrow \mathfrak{X}(R_\infty/J^\infty_\infty) \tag{71}$$

is a quasi-isomorphism. Because (71) comes from a map of mixed complexes, it will suffice to show that the map between the corresponding Hochschild complexes

$$(\Omega(R_\infty/J^\infty_\infty), b) \twoheadrightarrow (\mathfrak{X}(R_\infty/J^\infty_\infty), b)$$

is a quism. By [5, 3.3] it suffices to prove this in the case when $R = TV$, a tensor algebra. With the notation of Lemma 7.11, we have a commutative diagram

$$\begin{array}{ccc}
(\Omega_{\mathrm{Comm}}(PSV/I^{\infty}PSV + P_{\infty}SV), \delta) & \underset{\sim}{\xrightarrow{\eta}} & (\Omega(R_{\infty}/J^{\infty}_{\infty}), b) \\
\downarrow & & \downarrow \\
((PSV/I^{\infty}PSV + P_{\infty}SV), \delta) & \underset{\cong}{\xrightarrow{\eta}} & (\mathfrak{X}(R_{\infty}/J^{\infty}_{\infty}), b)
\end{array}$$

where the top and bottom rows are, respectively, a homotopy equivalence and an isomorphism by Lemma 7.11 and its proof. By Lemma 7.6, the first vertical arrow is a quism, whence so is the second. This gives isomorphisms f_4 and $f_{3'}$. It follows that all the coface maps of the Čech–Alexander co-simplicial pro-complex $C_{\mathcal{Y}}(\mathfrak{X})$ are homotopy equivalences, which gives isomorphism f_2. A similar argument produces an isomorphism $f_{2'}$ for $\mathcal{Z} = \{X \hookrightarrow \mathrm{Spec}\ P_{\leqslant 1}SV \hookrightarrow \mathrm{Spec}\ P_{\leqslant 2} \hookrightarrow \cdots\}$; the passage from this to the case when $P_{\leqslant\infty}SV$ is replaced by an arbitrary formally NP_{∞}-smooth pro-algebra is done as in the NC_{∞}-case. The Poisson grading (44) induces a pro-cochain complex decomposition

$$\mathfrak{Y}\left(\left\{\frac{PSV}{P_{\geqslant n}SV}\right\}_n\right) \cong \frac{\mathfrak{Y}(PSV)}{F_{\infty}\mathfrak{Y}(PSV)} = \left\{\prod_{l=0}^{n} \mathfrak{R}[-2l]\right\} n. \tag{72}$$

From (72), Lemma 7.7 and (33) we obtain the isomorphism f_1. This finishes the proof of the theorem in the affine case; the general case follows from this by the same argument as in Theorems 6.1 and 6.4. □

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