Faster recognition of clique-Helly and hereditary clique-Helly graphs

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Abstract

A family of subsets of a set is Helly when every subfamily of it, which is formed by pairwise intersecting subsets contains a common element. A graph $G$ is clique-Helly when the family of its (maximal) cliques is Helly, while $G$ is hereditary clique-Helly when every induced subgraph of it is clique-Helly. The best algorithms currently known to recognize clique-Helly and hereditary clique-Helly graphs have complexities $O(nm^2)$ and $O(n^2m)$, respectively, for a graph with $n$ vertices and $m$ edges. In this Note, we describe algorithms which recognize both classes in $O(m^2)$ time. These algorithms also reduce the complexity of recognizing some other classes, as disk-Helly graphs.

Keywords: Algorithms; Clique-Helly graphs; Disk-Helly graphs; Helly property; Hereditary clique-Helly graphs; Hereditary disk-Helly graphs

1. Introduction

The Helly property has been studied in many contexts, as in combinatorics and geometry. Within graph theory, the Helly property has been applied to some different families of sets. Its application to the (maximal) cliques of a graph is one of the most common. It has lead to the classes of clique-Helly and hereditary clique-Helly graphs. Clique-Helly graphs have been considered in many papers, [4,15,10,11,9], among others. Hereditary clique-Helly graphs have been studied in, e.g., [13,5]. Clique-Helly graphs have been characterized in [8,16], while [13,17] contain characterizations for hereditary clique-Helly graphs. These characterizations lead to recognition algorithms whose complexities are $O(nm^2)$ for clique-Helly and $O(n^2m)$ for hereditary clique-Helly graphs, where $n$ and $m$ are the number of vertices and edges of the graph (see [6]). In this Note, we describe algorithms which reduce the complexities to $O(m^2)$, in both cases. We remark that all mentioned algorithms are based on Berge’s basic test for Helly hypergraphs [3]. Finally, we mention that hereditary Helly hypergraphs can be also recognized in polynomial time [7].

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Let \( G \) be an undirected graph with vertex set \( V(G) \) and edge set \( E(G) \). Write \( ab \) to denote the edge of \( G \), formed by vertices \( a, b \in V(G) \). Represent by \( N(a) \) the subset of vertices adjacent to \( a \), and let \( N[a] = N(a) \cup \{a\} \). Write \( d_a = |N(a)| \). For an edge \( ab \in E(G) \), define \( N(ab) = N(a) \cap N(b) \) and \( N[ab] = N[a] \cap N[b] \). Clearly, \( N[ab] = N(ab) \cup \{a, b\} \). A vertex \( a \in V(G) \) satisfying \( N[a] = V(G) \) is a universal vertex of \( G \). For \( V' \subseteq V(G) \), denote by \( G[V'] \) the subgraph of \( G \) induced by the vertices of \( V' \). Denote by \( U(ab) \) the set of universal vertices of \( G[N(ab)] \). Similarly, \( U(ab) \) is the set of universal vertices of \( G[N[ab]] \). Again, \( U(ab) = U(ab) \cup \{a, b\} \).

Say that a family of subsets of some set is Helly when every subfamily of it, which is formed by pairwise intersecting subsets, contains a common element. A clique of a graph \( G \) is a maximal subset of pairwise adjacent vertices. Say that \( G \) is clique-Helly when the family of cliques of \( G \) form a Helly family, while \( G \) is hereditary clique-Helly when every induced subgraph of \( G \) is clique-Helly.

Let \( T \subseteq V(G) \) be a subset of three vertices, forming a triangle. Denote by \( T^* \subseteq V(G) \) the subset formed by the vertices of \( G \) which are adjacent to at least two vertices of \( T \). Clique-Helly graphs have been characterized in terms of \( T^* \), as follows.

**Proposition 1.** (See [8,16].) A graph \( G \) is clique-Helly if and only if every triangle \( T \) of \( G \) is such that \( T^* \) has a universal vertex.

Hereditary clique-Helly graphs have been characterized by forbidden subgraphs, as below.

**Proposition 2.** (See [13,17].) A graph is hereditary clique-Helly if and only if it does not contain any of the graphs of Fig. 1, as induced subgraphs.

The recognition algorithms for these classes are based on the above characterizations.

![Fig. 1. Forbidden induced subgraphs of hereditary clique Helly graphs.](image)

2. The algorithms

In this section, we describe the new implementations for recognizing clique-Helly and hereditary clique-Helly graphs. The former class is considered first. The method is based on the following proposition.

**Proposition 3.** A graph \( G \) is clique-Helly if and only if \( U[ab] \cap U[bc] \cap U[ac] \neq \emptyset \), for any triple of vertices \( a, b, c \in V(G) \) forming a triangle.

The proof follows directly from Proposition 1. The recognition algorithm basically checks the above conditions for every triangle of the graph.

**Algorithm 1 (Clique-Helly graphs).** Let \( G \) be the input graph. In the initial step, compute the sets \( N(ab), N[ab] \) and \( U(ab) \), for every edge \( ab \in E(G) \). In the general step, perform the following operations, for each edge \( ab \in E(G) \).

For each \( c \in N(ab) \), compute \( S_c := U[bc] \cup U[ac] \). If every \( w \in S_c \) satisfies \( w \notin U(ab) \) then report "\( G \) is not clique-Helly" and **stop**.

Report "\( G \) is clique-Helly" and **stop**.

Clearly, for recognizing \( G \) as a clique-Helly graph, the algorithm checks whether \( U[ab] \cap U[bc] \cap U[ac] \neq \emptyset \), for every triangle \( a, b, c \) of \( G \). Consequently, the algorithm is correct.

Next, we evaluate the complexity of the algorithm. The sets \( N(ab), N[ab] \) and \( U(ab) \) can be computed in overall \( O(m^2) \) time, for all edges of \( G \), with no difficulty. The operations performed within the general step of the algorithm depend on the size of the set \( S_c = U[bc] \cap U[ac] \). Clearly, \( |U[bc]| \leq |N[c]| = d_c + 1 \), and \( |U[ac]| \leq |N[c]| = d_c + 1 \). Consequently, we need no more than \( O(m) \) operations, in order to compute all sets \( S_c \), for each edge \( ab \in E(G) \). In addition, the sum of the sizes of the sets \( S_c \) is also \( O(m) \), for each considered edge \( ab \). To verify whether \( w \notin U(ab) \) can be done in constant time, by employing boolean vectors. Consequently, we require \( O(m) \) steps for each edge \( ab \). That is, the general step of the algorithm also requires \( O(m^2) \) time, which is the overall complexity.

In the sequel, we consider hereditary clique-Helly graphs. The proposed algorithm is based on the following proposition.

**Proposition 4.** A graph \( G \) is hereditary clique-Helly if and only if \( c \in U(ab) \), or \( b \in U(ac) \), or \( a \in U(bc) \), for each triangle with vertices \( a, b, c \in V(G) \).
**Proof.** Clearly, \( c \in U(ab) \) if and only if \( N(ab) \subseteq N(c) \). Similar relations hold for \( b \in U(ac) \) and \( a \in U(bc) \). Consequently, \( c \in U(ab) \), or \( b \in U(ac) \), or \( a \in U(bc) \) holds if and only if \( N(ab) \subseteq N(c) \), or \( N(bc) \subseteq N(a) \), or \( N(ac) \subseteq N(b) \).

Suppose \( G \) is hereditary clique-Helly. By Proposition 2, \( G \) does not contain any of the graphs of Fig. 1 as induced subgraphs. Consequently, for any triangle with vertices \( a, b, c \), there are no vertices \( x, y, z \in V(G) \) which simultaneously satisfy \( x \in N(ab) \setminus N(c) \), \( y \in N(bc) \setminus N(a) \) and \( z \in N(ac) \setminus N(b) \). Consequently, \( N(ab) \subseteq N(c) \), or \( N(bc) \subseteq N(a) \), or \( N(ac) \subseteq N(b) \), meaning that the proposition is true. The proof of the converse is similar.  

The algorithm for recognizing hereditary clique-Helly graphs is a direct implementation of Proposition 4.

**Algorithm 2** (Hereditary clique-Helly graphs). Let \( G \) be the input (connected) graph. In the initial step, compute \( U(ab) \), for each edge \( ab \in E(G) \). In the general step, for each triangle \( a, b, c \) of \( G \), perform the following operations.

If \( c \notin U(ab) \) and \( b \notin U(ac) \) and \( a \notin U(bc) \) then report “\( G \) is not hereditary clique-Helly” and stop.

Report “\( G \) is hereditary clique-Helly” and stop.

As for the complexity, the initial step requires \( O(m^2) \) time. In the general step, to verify if \( c \notin U(ab) \) can be done in constant time, again employing boolean vectors. Similarly, for the checks \( b \notin U(ac) \) and \( a \notin U(bc) \). Consequently, the total complexity of the general step is that of the number of triangles of \( G \), that is, \( O(nnm) \). Therefore the overall complexity of the algorithm is \( O(m^2) \).

### 3. Conclusions

We have described new implementations for performing the required checks on the triangles of a graph \( G \), which leads to recognizing whether \( G \) is clique-Helly or hereditary clique-Helly. The complexity of the proposed methods is \( O(m^2) \), for any of these two classes. Currently known algorithms recognize clique-Helly graphs in \( O(nm^2) \) time, and hereditary clique-Helly in \( O(n^2m) \) time.

Besides improving the complexity of recognizing these classes, the proposed algorithms also represent improvements in the recognition of some other classes. Disk-Helly graphs and hereditary disk-Helly graphs are examples of such classes.

For disk-Helly graphs, the currently best recognition algorithm has complexity \( O(n^2m) \) [1]. On the other hand, in [2] it has been proved that a graph is disk-Helly if and only if it is dismantlable and clique-Helly. Dismantlable graphs can be recognized in \( O(nm) \) time [12, 14]. Consequently, by applying Algorithm 1, we can reduce to \( O(m^2) \) the complexity of recognizing disk-Helly graphs.

For hereditary disk-Helly graphs, the recognition is based on the characterization [8] (cf. [6]), which states that a graph \( G \) is hereditary disk-Helly if and only if \( G \) is chordal and does not contain the graph \( G_1 \) as an induced subgraph. Since chordal graphs can be recognized in linear time, the complexity is dominated by the operations of verifying if \( G \) contains \( G_1 \) as an induced subgraph, which would require \( O(n^2m) \) time. However, since the graphs \( G_2, G_3 \) and \( G_4 \) are not chordal, we can apply Algorithm 2 after the chordality test, and therefore reduce the total complexity to \( O(m^2) \) time.

We leave the question whether clique-Helly graphs or hereditary clique-Helly graphs can be recognized in time proportional to the number of triangles of the graph.

### References