Journal of Algebra 325 (2011) 305–320

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Pointed Hopf algebras over the sporadic simple groups ☆

N. Andruskiewitsch [a], F. Fantino [a], M. Graña [b], L. Vendramin [b,c,*]

[a] Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, CIEM – CONICET, Medina Allende s/n (5000), Ciudad Universitaria, Córdoba, Argentina

[b] Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pab. I, Ciudad Universitaria (1428), Buenos Aires, Argentina

[c] Instituto de Ciencias, Universidad de Gral. Sarmiento, J.M. Gutierrez 1150, Los Polvorines (1653), Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 7 January 2010
Available online 5 November 2010
Communicated by Gerhard Hiss

MSC:
16W30
17B37

Keywords:
Pointed Hopf algebras
Nichols algebras
Racks

ABSTRACT

We show that every finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to a sporadic group is a group algebra, except for the Fischer group Fi_{22}, the Baby Monster and the Monster. For these three groups, we give a short list of irreducible Yetter–Drinfeld modules whose Nichols algebra is not known to be finite-dimensional.

Introduction

This paper contributes to the classification of finite-dimensional pointed Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0. There are different possible approaches to this problem; one of them is to fix a finite group G and to address the classification of finite-dimensional pointed Hopf algebras H such that $G(H) \simeq G$. Due to the intrinsic difficulty of this problem, it is natural to consider separately different classes of finite groups. A considerable progress in the case when G is abelian was achieved in [AS2]. There are reasons to consider next the class of finite simple, or close to

☆ This work was partially supported by ANPCyT-Foncyt, CONICET, Ministerio de Ciencia y Tecnología (Córdoba), Secyt-UNC and Secyt-UBA.

* Corresponding author at: Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pab. I, Ciudad Universitaria (1428), Buenos Aires, Argentina.

E-mail addresses: andrus@famaf.unc.edu.ar (N. Andruskiewitsch), fantino@famaf.unc.edu.ar (F. Fantino), matiasg@dm.uba.ar (M. Graña), lvendramin@dm.uba.ar (L. Vendramin).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.10.019

Table 1
Nichols algebras still open.

G	Class	Obs.	Centralizer	Rep.
Fi_{22}	22A, 22B	quasi-real $j=3$, $g^9 \neq g$	$\mathbb{Z}/22$	χ_{-1}
B	16C, 16D	real	order 2048	
	32A, 32B	real	order 128	
	32C, 32D	quasi-real $j=3$, $g^9 \neq g$	order 128	
	34A	real	$\mathbb{Z}/34 \times \mathbb{Z}/2$	$\chi_{-1} \otimes \epsilon$, $\chi_{-1} \otimes \operatorname{sgn}$
	46A, 46B	quasi-real $j=3$, $g^9 \neq g$	$\mathbb{Z}/46$	χ_{-1}
M	32A, 32B	real	order 128	
	46A, 46B	quasi-real $j=3$, $g^9 \neq g$	$\mathbb{Z}/23 \times \mathbb{D}_4$	$\epsilon \otimes \rho_2$
	92A, 92B	quasi-real $j=3$, $g^9 \neq g$	$\mathbb{Z}/92$	χ_{-1}
	94A, 94B	quasi-real $j=3$, $g^9 \neq g$	$\mathbb{Z}/94$	χ_{-1}

simple, groups. See [AFGV1] and references therein for G symmetric or alternating. Nichols algebras over Mathieu groups are partially studied in [F]. In this paper we deal with sporadic simple groups.

Definition 1. We shall say that a finite group G *collapses* if for any finite-dimensional pointed Hopf algebra H, with $G(H) \simeq G$, then $H \simeq \Bbbk G$.

Theorem I. *If G is a sporadic simple group, then it collapses, except for the groups $G = Fi_{22}$, B, M. For these groups, the list of irreducible Yetter–Drinfeld modules $M(\mathcal{O}, \rho)$ whose Nichols algebra is not known to be finite-dimensional appears in Table* 1.

Notice that the Nichols algebras of *reducible* Yetter–Drinfeld modules over the groups in Table 1 are infinite-dimensional [HS, 8.3]. The conjugacy classes of sporadic groups are labeled as in the ATLAS; the notation for the representations in Table 1 is discussed in Section 3.1.

Another approach to the mentioned classification problem is through Nichols algebras associated to racks. Precisely, one has to attack the following question:

For every finite indecomposable rack X, for every $n \in \mathbb{N}$, and for every 2*-cocycle $q \in Z^2(X, \mathbf{GL}(n, \Bbbk^\times))$, determine if* $\dim \mathfrak{B}(X, q) < \infty$.

Here $\mathfrak{B}(X, q)$ denotes the Nichols algebra associated to the pair (X, q), see Section 1.1. We refer to [AG], [AFGV1, Introduction] for the relation between this question and the classification problem. Again, it is natural to consider separately different classes of finite racks; again, it is natural to start by the class of finite simple racks. These were classified in [AG, Theorem 3.9, Theorem 3.12], see also [J]; among them, there are the racks arising as non-trivial conjugacy classes of finite simple groups. A substantial part of the proof of Theorem I follows from a much more general result for a large family of conjugacy classes of sporadic groups. It is convenient to introduce the following terminology before stating our next main theorem.

Definition 2. We shall say that a finite simple rack X *collapses* if $\dim \mathfrak{B}(X, q) = \infty$ for any 2-cocycle $q \in Z^2(X, \mathbf{GL}(n, \Bbbk^\times))$, for any $n \in \mathbb{N}$.

See Definition 2.1 for the notion of rack of type D; such rack collapses by [AFGV1, Theorem 3.6], recalled in Theorem 2.2. This result is a translation to the language of racks of [HS, Theorem 8.6], whose proof relies on results from [AHS].

Theorem II. *If G is a sporadic simple group and $\mathcal{O}$ is a non-trivial conjugacy class of G NOT listed in Table* 2*, then $\mathcal{O}$ is of type D, hence it collapses.*

Table 2
Conjugacy classes not known of type D; those which are NOT of type D appear in bold.

G	Classes	G	Classes
M_{11}	**8A**, **8B**, **11A**, **11B**	Co_1	3A, 23A, 23B
M_{12}	**11A**, **11B**	J_1	**15A**, **15B**, **19A**, **19B**, **19C**
M_{22}	**11A**, **11B**	$O'N$	31A, 31B
M_{23}	**23A**, **23B**	J_3	5A, 5B, 19A, 19B
M_{24}	**23A**, **23B**	Ru	29A, 29B
J_2	**2A**, **3A**	He	all collapse
Suz	3A	Fi_{22}	**2A**, 22A, 22B
HS	11A, 11B	Fi_{23}	**2A**, 23A, 23B
McL	11A, 11B	HN	all collapse
Co_3	23A, 23B	Th	all collapse
Co_2	**2A**, 23A, 23B	T	2A

G	Classes
Ly	33A, 33B, 37A, 37B, 67A, 67B, 67C
J_4	29A, 37A, 37B, 37C, 43A, 43B, 43C
Fi'_{24}	23A, 23B, 27B, 27C, 29A, 29B, 33A, 33B, 39C, 39D
B	2A, 16C, 16D, 32A, 32B, 32C, 32D, 34A, 46A, 46B, 47A, 47B
M	32A, 32B, 41A, 46A, 46B, 47A, 47B, 59A, 59B, 69A, 69B, 71A, 71B, 87A, 87B, 92A, 92B, 94A, 94B

Theorem II is not merely an auxiliary step towards Theorem I; its consequences, illustrated by Theorems 2.4 and 2.6, show how crucial is the use of racks in the classification of pointed Hopf algebras.

Our proofs of Theorems I, II and 2.4 are based on reductions to problems on conjugacy classes of finite groups; we solve these problems in the present setting with the help of [GAP]. For completeness, we review these reductions in Section 1, and discuss the elements of GAP that we are using. We discuss in Section 2 the algorithms used to justify Theorem II. We discuss Theorem I in Section 3. We treat the classes not covered by Theorems II in Section 3.1. In Section 3.2, we explain why the remaining classes in Table 1 are beyond our present knowledge.

In this paper we explain the algorithms, and their theoretical support, for the proofs of our main results. The details of how these algorithms are actually implemented are given in the companion paper [AFGV2]. We believe, and hope, that these details are enough to guide the diligent reader to repeat and corroborate our calculations.

1. Preliminaries

1.1. Nichols algebras

A *braided vector space* is a pair (V, c), where V is a vector space and $c \in \mathbf{GL}(V \otimes V)$ is a solution of the braid equation, that is

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \qquad (1.1)$$

The braid equation is equivalent to the celebrated quantum Yang–Baxter equation, that plays an important role in statistical mechanics.

There is a very interesting object associated to a braided vector space, *its Nichols algebra*, defined as follows.

- The solution c of the braid equation (1.1) induces a representation of the braid group $\mathbb{B}_n$ in the n-th tensor product $V^{\otimes n}$.

- Let $M : \mathbb{S}_n \to \mathbb{B}_n$ be the so-called Matsumoto section; it preserves the product only when the length is preserved.
- Let $\Omega_n = \sum_{\sigma\in\mathbb{S}_n} M(\sigma) \in \operatorname{End}(V^{\otimes n})$ and let $J = \bigoplus_{n\geqslant 2} \ker\Omega_n$. The Nichols algebra of the braided vector space (V, c) is $\mathfrak{B}(V, c) = T(V)/J$.

The study of Nichols algebras is unavoidable in the classification problem of pointed Hopf algebras. A notable example of a Nichols algebra is the positive part $U_q^+(\mathfrak{g})$ of a quantized enveloping algebra when q is not a root of 1 [L,R,S]. It is in general very difficult to compute explicitly the Nichols algebra of a braided vector space – that is, to determine its dimension, or even better, an efficient set of generators. We refer to [AS1] for a detailed discussion of alternative definitions, basic results and examples of Nichols algebras. In the present paper, we shall make use of the following observation:

If (W, c) is a braided vector subspace of (V, c), then $\mathfrak{B}(W, c) \hookrightarrow \mathfrak{B}(V, c)$.

In particular, if we find a braided vector subspace (W, c) of (V, c) with infinite-dimensional Nichols algebra, then $\dim\mathfrak{B}(V, c) = \infty$ too. We shall reduce the search of a suitable braided vector subspace to problems in finite group theory, and then we will solve them for sporadic groups using GAP.

1.2. Yetter–Drinfeld modules

We now describe the class of braided vector spaces whose Nichols algebras we need to show that have infinite dimension. Let G be a finite group. We denote by $Z(G)$ the center of G and by $\operatorname{Irr} G$ the set of isomorphism classes of irreducible representations of G. If $g \in G$, we denote by $C_G(g)$ the centralizer of g in G. The conjugacy class of g is denoted by $\mathcal{O}_g$ or by $\mathcal{O}_g^G$, if emphasis on the group is needed.

A *Yetter–Drinfeld module* over G is a G-graded vector space $M = \bigoplus_{g\in G} M_g$ provided with a G-module structure such that $g \cdot M_t = M_{gtg^{-1}}$ for any $g, t \in G$. The category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of Yetter–Drinfeld modules over G is semisimple and its irreducible objects are all of the following form. Let $\mathcal{O}$ be a conjugacy class of G, $g \in \mathcal{O}$ fixed, $\rho \in \operatorname{Irr} C_G(g)$. We describe the corresponding irreducible Yetter–Drinfeld module $M(\mathcal{O}, \rho)$. Let $g_1 = g, \dots, g_m$ be a numeration of $\mathcal{O}$ and let $x_i \in G$ such that $x_i \triangleright g = g_i$ for all $1 \leqslant i \leqslant m$. Then

$$M(\mathcal{O}, \rho) = \operatorname{Ind}_{C_G(g)}^G V = \bigoplus_{1\leqslant i\leqslant m} x_i \otimes V.$$

Let $x_i v := x_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \leqslant i \leqslant m$, $v \in V$. The Yetter–Drinfeld module $M(\mathcal{O}, \rho)$ is a braided vector space with braiding given by

$$c(x_i v \otimes x_j w) = g_i \cdot (x_j w) \otimes x_i v = x_h \rho(\gamma)(w) \otimes x_i v \tag{1.2}$$

for any $1 \leqslant i, j \leqslant m$, $v, w \in V$, where $g_i x_j = x_h\gamma$ for unique h, $1 \leqslant h \leqslant m$ and $\gamma \in C_G(g)$. The Nichols algebra[1] of $M(\mathcal{O}, \rho)$ is simply denoted $\mathfrak{B}(\mathcal{O}, \rho)$.

Let H be a pointed Hopf algebra with $G(H) \simeq G$. Then there are two fundamental invariants of H, a Yetter–Drinfeld module V over $\Bbbk G$ (called the infinitesimal braiding of H) and its Nichols algebra $\mathfrak{B}(V)$, see [AS1,A]. A basic problem for the classification of finite-dimensional pointed Hopf algebras over G is the determination of all Yetter–Drinfeld modules V over $\Bbbk G$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional. In particular, the following statements are equivalent:

(1) *If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.*

[1] We omit to mention the braiding c in the notation of a Nichols algebra from now on.

(2) *If* $V \neq 0$ *is a Yetter–Drinfeld module over* $\Bbbk G$*, then* $\dim \mathfrak{B}(V) = \infty$.
(3) *If* V *is an* irreducible *Yetter–Drinfeld module over* $\Bbbk G$*, then* $\dim \mathfrak{B}(V) = \infty$.

Therefore, for a fixed group G, we aim to know when $\dim \mathfrak{B}(V) = \infty$ for an irreducible $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$; that is, when $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for a pair $(\mathcal{O}, \rho)$ as above. As we said, we shall look at braided vector subspaces of V, and to describe them we find convenient the language of racks and cocycles.

1.3. Racks

A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is a function such that $\phi_i : X \to X$, $\phi_i(j) := i \triangleright j$, is a bijection for all $i \in X$, and $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. For instance, a group G, and any conjugacy class in G, is a rack with $x \triangleright y = xyx^{-1}$. In this case, $j \triangleright i = i$ whenever $i \triangleright j = j$ and $i \triangleright i = i$ for all $i \in G$. A rack X is *abelian* if for any $x, y \in X$, $x \triangleright y = y$. See [AG] for a survey on racks.

Let X be a rack. Given $n \in \mathbb{N}$, a map $q : X \times X \to \mathbf{GL}(n, \Bbbk)$ is a 2-*cocycle* if

$$q_{x, y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z},$$

for all $x, y, z \in X$. Let q be a 2-cocycle, $V = \Bbbk X \otimes \Bbbk^n$, where $\Bbbk X$ is the vector space with basis e_x, for $x \in X$. We denote $e_x v := e_x \otimes v$. Consider the linear isomorphism $c^q : V \otimes V \to V \otimes V$,

$$c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x,y}(w) \otimes e_x v,$$

$x \in X$, $y \in X$, $v \in \Bbbk^n$, $w \in \Bbbk^n$. Then c^q is a solution of the braid equation; its Nichols algebra is denoted $\mathfrak{B}(X, q)$. Pointed Hopf algebras are related to Nichols algebras over racks by the following observations, see [AG, Theorem 4.14].

- If X is a rack, $n \in \mathbb{N}$, and $q : X \times X \to \mathbf{GL}(n, \Bbbk)$ is a 2-cocycle, then there exists a group G such that $V = \Bbbk X \otimes \Bbbk^n$ is a Yetter–Drinfeld module over G and the braiding of V as an object in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ coincides with c^q. If the image of q is contained in a finite subgroup $\Gamma \subset \mathbf{GL}(n, \Bbbk)$, then G can be chosen to be finite.
- Conversely, if G is a finite group and $V = M(\mathcal{O}, \rho) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is irreducible, then there exists a finite subgroup Γ of $\mathbf{GL}(n, \Bbbk)$, $n = \dim \rho$, and a 2-cocycle $q : X \times X \to \Gamma$ such that V is given as above and the braiding $c \in \operatorname{Aut}(V \otimes V)$ in the category ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ coincides with c^q.

1.4. Abelian techniques

Let G be a finite group, $\mathcal{O}$ a conjugacy class of G, $g \in \mathcal{O}$ and $(\rho, V) \in \operatorname{Irr} C_G(g)$. Our next goal is to describe techniques to conclude that $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ from the analysis of abelian subracks; these will be applied in the proof of Theorem I. Abelian subracks give rise to braided subspaces of $M(\mathcal{O}, \rho)$ of diagonal type; then the classification of braided vector spaces of diagonal type with finite-dimensional Nichols algebra [H] may be invoked. We shall freely use the notations and results from [H], in particular the notion of *generalized Dynkin diagram*. We start by subracks with one element. By the Schur lemma, $\rho(g)$ is a scalar. It is well known that

$$\rho(g) = 1 \quad \Rightarrow \quad \dim \mathfrak{B}(\mathcal{O}, \rho) = \infty. \tag{1.3}$$

We consider next subracks with two or three elements.

Lemma 1.1. *(See [AZ, 2.2].) Assume that* $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$*. If* $g \in G$ *is* real*, that is* $g^{-1} \in \mathcal{O}$*, then* $\rho(g) = -1$*. In particular, the order of* g *is even.*

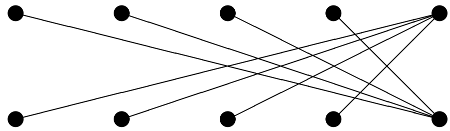

Fig. 1.

If g is not real, but it is conjugated to $g^j \neq g$ for some $j \in \mathbb{Z}$, then we shall say that $g \in G$ and $\mathcal{O}$ are *quasi-real*.

Lemma 1.2. *(See [AF, 1.8 and 1.9], [FGV, 2.2].) Assume that* $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$ *and that there exists j such that $g \neq g^j \in \mathcal{O}$.*

(i) *If* $\deg \rho > 1$, *then* $\rho(g) = -1$ *and g has even order.*
(ii) *If* $\deg \rho = 1$, *then* $\rho(g) = -1$ *and g has even order or* $\rho(g) \in \mathcal{R}_3$.
(iii) *If* $g^{j^2} \neq g$, *then* $\rho(g) = -1$.

Involutions, that is elements of order 2, are real but Lemma 1.1 does not give useful information. Our next criterium deals with classes of involutions and representations of degree greater than one. Another useful criterium for classes of involutions is Proposition 1.8 below.

Lemma 1.3. *Let G be a finite group, $\mathcal{O}$ the conjugacy class of an involution $g \in G$ and* $(\rho, V) \in \operatorname{Irr} C_G(g)$. *Assume that there exists an involution x such that $h = xgx$ and $gh = hg$. If $\rho(h)$ has a eigenvalues* 1, *b eigenvalues* -1, *where either* $a > 0$ *and* $b > 3$ *or* $a > 3$ *and* $b > 0$, *then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. By (1.3) we can assume $\rho(g) = -1$. Let $x_1 = e$ and $x_2 = x$. Since $gh = hg$, there exists a linear basis $\{v_1, \dots, v_n\}$ of V such that $\rho(g)$ and $\rho(h)$ are simultaneously diagonalizable in this basis. Define W as the subspace generated by $x_\ell v_i = x_\ell \otimes v_i$, where $\ell = 1, 2$ and $1 \leqslant i \leqslant n$. Then W is a braided vector space of diagonal type with braiding given by $c(x_\ell v_i \otimes x_\ell v_j) = -x_\ell v_j \otimes x_\ell v_i$ and

$$c(x_1 v_i \otimes x_2 v_j) = x_2 \rho(h) v_j \otimes x_1 v_i,$$

$$c(x_2 v_i \otimes x_1 v_j) = x_1 \rho(h) v_j \otimes x_2 v_i,$$

for $\ell = 1, 2$, and $1 \leqslant i, j \leqslant n$. Therefore, the results follow from [H] because the generalized Dynkin diagram has at least one vertex with valency > 3. For instance assume that $a = 4$ and $b = 1$. Then the generalized Dynkin diagram is Fig. 1.

This completes the proof. $\square$

The computation of the multiplicities a and b in the statement of the lemma can be performed using the following remark.

Remark 1.4. Let G be a finite group, $g \in G$ and (ρ, V) a representation of G. The multiplicities of the eigenvalues of $\rho(g)$ are the scalar products of the restriction of the representation to the cyclic group generated by g with the irreducible characters of this cyclic group. The GAP function EigenvaluesChar can be used for this computation.

1.5. The technique of the subgroup

Let G be a finite group, $\sigma \in G$, $\mathcal{O}_\sigma^G = \mathcal{O}^G$ its conjugacy class, $C_G(\sigma)$ its centralizer and $\rho \in \operatorname{Irr} C_G(\sigma)$. If H is a subgroup of G and $\sigma \in H$, then $\mathcal{O}_\sigma^H = \mathcal{O}^H$ denotes the conjugacy class of σ in H. Let $\rho|_{C_H(\sigma)} = \tau_1 \oplus \cdots \oplus \tau_s$ where $\tau_j \in \operatorname{Irr} C_H(\sigma)$, $1 \leqslant j \leqslant s$.

Lemma 1.5. *(See [AFGV1, 3.2].)*

(i) *If* $\dim \mathfrak{B}(\mathcal{O}^H, \lambda) = \infty$ *for all* $\lambda \in \operatorname{Irr} C_H(\sigma)$*, then* $\dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty$ *for all* $\rho \in \operatorname{Irr} C_G(\sigma)$.

(ii) *Let* $\sigma_1, \sigma_2 \in \mathcal{O}^G \cap H$*. Let* $\mathcal{O}_i = \mathcal{O}^H_{\sigma_i}$ *and assume that* $\mathcal{O}_1 \neq \mathcal{O}_2$*. If* $\dim \mathfrak{B}(M(\mathcal{O}_1, \lambda_1) \oplus M(\mathcal{O}_2, \lambda_2)) = \infty$ *for all pairs* $\lambda_1 \in \operatorname{Irr} C_H(\sigma_1)$, $\lambda_2 \in \operatorname{Irr} C_H(\sigma_2)$*, then* $\dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty$ *for all* $\rho \in \operatorname{Irr} C_G(\sigma)$.

1.6. The group $\mathbb{A}_4$ and conjugacy classes of involutions

In this section we give another criterium for Nichols algebras over the conjugacy class of an involution.

Lemma 1.6. *Let* $g, h \in G$ *with* g *an involution,* h *and* gh *of order* 3; *let* $\mathcal{O}$ *be the conjugacy class of* g*. Then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. The alternating group in four letters $\mathbb{A}_4$ can be presented by generators g and h with relations $g^2 = h^3 = (gh)^3 = e$ [D]. Thus, the subgroup H of G generated by g and h is isomorphic to $\mathbb{A}_4$. Indeed, the hypothesis implies that the elements $g, h, h^2, gh, (gh)^2$ of G are all distinct, but a proper subgroup of $\mathbb{A}_4$ has at most 4 elements. From [AF], the conjugacy class of involutions in $\mathbb{A}_4$ gives infinite-dimensional Nichols algebras for every representation. Hence Lemma 1.5 applies. □

Now we need an efficient way of checking if $\mathbb{A}_4$ is a subgroup. To this purpose, first recall a very useful result from the theory of groups.

Proposition 1.7. *(See [Go, Theorem 4.2.12].) Let* G *be a finite group and let* $\mathcal{O}_i$, $\mathcal{O}_j$, $\mathcal{O}_k$ *be some conjugacy classes. If* $S(\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k)$ *is the number of times that a given element of* $\mathcal{O}_k$ *can be expressed as an ordered product of an element of* $\mathcal{O}_i$ *with an element of* $\mathcal{O}_j$*, then*

$$S(\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k) = \frac{|\mathcal{O}_i||\mathcal{O}_j|}{|G|} \sum_{\chi} \frac{\chi(\mathcal{O}_i)\chi(\mathcal{O}_j)\overline{\chi(\mathcal{O}_k)}}{\chi(1)}, \tag{1.4}$$

where χ *runs over all irreducible characters of* G.

The last proposition will be used in connection with the following criterium.

Proposition 1.8. *Let* $\mathcal{O}$ *be the conjugacy class of an involution, and* $\mathcal{K}_1, \mathcal{K}_2$ *conjugacy classes of elements of order* 3*. If* $S(\mathcal{O}, \mathcal{K}_1, \mathcal{K}_2) \geqslant 1$ *then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. By hypothesis, there exist $a \in \mathcal{O}$ and $b_i \in \mathcal{K}_i$, $i = 1, 2$, such that $ab_1 = b_2$. Then Lemma 1.6 applies. □

1.7. A technique with simple affine racks

We now apply [HS, Theorem 8.2] to racks arising as union of two affine racks. This is used for the conjugacy classes labeled 8A, 8B of the Mathieu group M_{11}, that cannot be treated otherwise.

If G is a finite group and $g, h \in G$, then the conjugacy classes $\mathcal{O}_g$ and $\mathcal{O}_h$ *commute* if $st = ts$ for all $s \in \mathcal{O}_g$, and for all $t \in \mathcal{O}_h$. Let

$$\mathcal{F}(G) = \{\mathcal{O} \text{ conjugacy class of } G\colon \dim \mathfrak{B}(\mathcal{O}, \rho) < \infty \text{ for some } \rho\}.$$

Theorem 1.9. *(See [HS, Theorem 8.2].) Let* G *be a finite group such that any two conjugacy classes in* $\mathcal{F}(G)$ *do not commute. Let* $0 \neq U \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ *such that* $\dim \mathfrak{B}(U) < \infty$*. Then* U *is irreducible.*

If A is an abelian group and $T \in \operatorname{Aut}(A)$, then A becomes a rack with $x \triangleright y = (1-T)x + Ty$. It will be denoted by (A, T) and called an *affine rack*. We realize it as a conjugacy class in the semi-direct product $G = A \rtimes \langle T \rangle$. The conjugation in G gives

$$\big(v, T^h\big) \triangleright \big(w, T^j\big) = \big(T^h(w) + \big(\mathrm{id} - T^j\big)(v), T^j\big). \tag{1.5}$$

We denote $\mathcal{Q}^j_{A,T} := \{(w, T^j)\colon\ w \in A\}$, $j \in \mathbb{Z}/d$, a subrack of G isomorphic to the affine rack (A, T^j).

We assume that (A, T) is a *simple* affine rack; that is, $A = \mathbb{F}_p^t$, p a prime, and $T \in \mathbf{GL}(t, \mathbb{F}_p) - \{\mathrm{id}\}$ of order d, acting irreducibly.

Lemma 1.10. *Suppose that $p > 2$ when d is even. Let $G = A \rtimes \langle T \rangle$. If $0 \neq U \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ satisfies $\dim \mathfrak{B}(U) < \infty$, then U is irreducible.*

Proof. By Theorem 1.9, we have to show that any two conjugacy classes in $\mathcal{F}(G)$ do not commute. We claim that

(a) The conjugacy classes of G are either $\mathcal{Q}^j_{A,T}$ with $j \neq 0$, or else the orbits of T in A.
(b) $\mathcal{F}(G) \subset \{\mathcal{Q}^j_{A,T}\colon\ j \neq 0\}$; hence any two conjugacy classes in $\mathcal{F}(G)$ do not commute, cf. (1.5).

Part (a) is elementary, but we sketch the argument. It is evident from (1.5) that the conjugacy class of (w, id) is the orbit of w under T. If $0 < j < d$, then $\mathrm{id} - T^j$ is bijective, since its kernel is a T-invariant subspace, but we are assuming that T is irreducible. Hence the class of $(0, T^j)$ is $\mathcal{Q}^j_{A,T}$.

We prove (b). Let $v \in A - 0$; the centralizer of v is A. Set $\sigma_k = T^k(v)$ and $x_k = (0, T^k)$; thus $x_k \triangleright \sigma_0 = \sigma_k$ and $\sigma_\ell x_k = x_k \sigma_{\ell-k}$, $0 \leqslant \ell, k \leqslant d-1$. Let $\chi \in \operatorname{Irr} A$; then the braiding in $M(\mathcal{O}_v, \chi)$ is given by $c(x_k \otimes x_\ell) = \chi(\sigma_{k-\ell}) x_\ell \otimes x_k$. In other words, this is of diagonal type with matrix $q_{k\ell} = \chi(T^{k-\ell}(v))$. Let Δ be the generalized Dynkin diagram associated to $(q_{k\ell})$. Now we can identify $A = \mathbb{F}_q$, with $q = p^t$, in such a way that $T(v) = \xi v$, where $\xi \in \mathbb{F}_q^\times$ has order d. Hence $q_{k\ell} q_{\ell k} = \chi((\xi^{\ell-k} + \xi^{k-\ell})v)$. Notice that the order of $q_{k\ell} q_{\ell k}$ divides p. Also,

$$1 + \xi + \xi^2 + \cdots + \xi^{d-1} = 0. \tag{1.6}$$

We can assume $\chi(v) \neq 1$. Now, we consider different cases.

Suppose that d is odd.

(I) Assume that $3 \nmid d$. If there exists ℓ, with $1 \leqslant \ell \leqslant d-1$, such that $q_{0\ell} q_{\ell 0} \neq 1$, then $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$. Indeed, Δ contains a cycle of length greater than 3, namely the set of vertices $0, \ell, 2\ell, \dots, (M-1)\ell$, where M is the order of ℓ. Then the result follows from [H]. If there is not such ℓ, then $v = -\sum_{1 \leqslant \ell \leqslant (d-1)/2} (\xi^\ell + \xi^{d-\ell}) v \in \ker \chi$ by (1.6), a contradiction.

(II) Suppose that $3 \mid d$.
  (i) If there exists ℓ, with $1 \leqslant \ell \leqslant d-1$ and $\ell \neq \frac{d}{3}, \frac{2d}{3}$, such that $q_{0\ell} q_{\ell 0} \neq 1$, then $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$ because Δ contains a cycle as in (I).
  (ii) Assume that $q_{0\ell} q_{\ell 0} = 1$ for all ℓ, with $1 \leqslant \ell \leqslant d-1$ and $\ell \neq \frac{d}{3}, \frac{2d}{3}$. If $q_{0\frac{d}{3}} q_{\frac{d}{3}0} = q_{0\frac{2d}{3}} q_{\frac{2d}{3}0} = 1$, then $\chi(v) = 1$ by (1.6), a contradiction. On the other hand, if $\omega := q_{0\frac{d}{3}} q_{\frac{d}{3}0} = q_{0\frac{2d}{3}} q_{\frac{2d}{3}0} \neq 1$, then Δ contains a triangle like in Fig. 2, but no triangle of this sort appears in [H, Table 2].

Assume that d is even. Then $q_{0\frac{d}{2}} q_{\frac{d}{2}0} = \chi(v)^{-2} \neq 1$ because $p > 2$; hence, 0 is connected to $\frac{d}{2}$ and the sub-diagram spanned by 0 and $\frac{d}{2}$ is of Cartan type $A_1^{(1)}$. By [H], $\dim \mathfrak{B}(\mathcal{O}_v, \chi) = \infty$. In conclusion, $\mathcal{O}_v \notin \mathcal{F}(G)$. This proves (b). $\square$

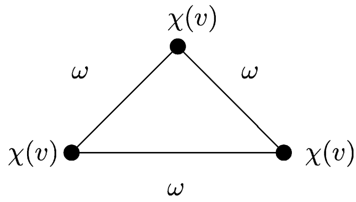

Fig. 2.

Proposition 1.11. *Let G be a finite group; (A, T) an affine simple rack with $|T| = d$ and $p > 2$ as above; and $\psi : A \rtimes \langle T\rangle \to G$ a monomorphism of groups. Assume that the conjugacy class $\mathcal{O}$ of $\sigma = \psi(0, T)$ is quasi-real of type j, $1 < j < d$. If $\rho \in \operatorname{Irr} C_G(\sigma)$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. This follows from Lemmata 1.5(ii) and 1.10 applied to $H =$ the image of ψ. Indeed, $\mathcal{O}^H_\sigma$ and $\mathcal{O}^H_{\sigma^j}$ are both contained in $\mathcal{O}$, but $\mathcal{O}^H_\sigma \cap \mathcal{O}^H_{\sigma^j} = \emptyset$. □

Example 1.12. Let $T \in \mathbf{GL}(2, \mathbb{F}_3)$ of order 8 and $G = \mathbb{F}_3^2 \rtimes \langle T\rangle \simeq (\mathbb{Z}/3 \oplus \mathbb{Z}/3) \rtimes \mathbb{Z}/8$. Then there is a monomorphism of groups ψ from G to the Mathieu group M_{11} such that $\psi(T)$ belongs to the class $\mathcal{O}$ labeled 8A (resp. 8B) in ATLAS, which is quasi-real of type 3. Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any ρ.

1.8. The dihedral group and conjugacy class of involutions

Let n be an even number. Recall that the dihedral group of $2n$ elements is given by

$$\mathbb{D}_{2n} = \langle r, s \mid r^n = s^2 = 1,\ srs = r^{-1}\rangle.$$

The involutions of $\mathbb{D}_{2n}$ split in three conjugacy classes $S = \{r^{2i}s \mid 0 \leqslant i \leqslant \frac{n}{2} - 1\}$, $R = \{r^{2i+1}s \mid 0 \leqslant i \leqslant \frac{n}{2} - 1\}$ and $\{r^{n/2}\}$.

Lemma 1.13. *Let $n > 4$ be an even number. Then $Y = R \sqcup S$ is of type D.*

Proof. Let $\sigma_1 = s \in S$ and $\sigma_2 = rs \in R$; clearly, $(\sigma_1\sigma_2)^2 \neq (\sigma_2\sigma_1)^2$. □

Lemma 1.14. *Let $\mathcal{A}$ be a conjugacy class of involutions in a finite group G and let $\mathcal{B}$ be a conjugacy class with representative of order n, with $n > 2$ even. If $S(\mathcal{A}, \mathcal{A}, \mathcal{B}) > 0$, then the conjugacy class $\mathcal{A}$ is of type D.*

Proof. Since $S(\mathcal{A}, \mathcal{A}, \mathcal{B}) > 0$, there exist $s, t \in \mathcal{A}$ and $r \in \mathcal{B}$ such that $ts = r$, Proposition 1.7. Hence, $\langle r, s\rangle \simeq \mathbb{D}_{2n}$ and $\mathcal{A}$ contains the subrack $Y = R \sqcup S$ which is of type D by Lemma 1.13. □

Example 1.15. The conjugacy classes of involutions of the Conway group Co_1 are of type D. In fact, $S(2A, 2A, 6E) = 6$, $S(2B, 2B, 6A) = 2592$ and $S(2C, 2C, 6A) = 25\,920$.

Example 1.16. The conjugacy class 2C of the Baby Monster group B is of type D, since $S(2C, 2C, 6C) = 82\,752$.

1.9. The ATLAS

The ATLAS of Finite Groups, often simply known as the ATLAS, is a group theory book by John Conway, Robert Curtis, Simon Norton, Richard Parker and Robert Wilson (with computational assistance from J.G. Thackray), published in 1985 – see [CC+]. It lists basic information about finite simple groups such as presentations, conjugacy classes of maximal subgroups, character tables and power maps on the conjugacy classes.

The ATLAS is being continued in the form of an electronic database – see [WWT+]. It currently contains information (including 5215 representations) on about 716 groups. In order to access to the information contained in the ATLAS, we use the `AtlasRep` package for GAP – see [WPN+].

We recall some notations from the ATLAS for the reader not used to it.

The notation for families of simple groups can be found in pp. x–xiii of [CC+]:

- $L_n(q) = \mathbf{PSL}(n, \mathbb{F}_q)$ is the projective special linear group.
- $U_n(q) = \mathbf{PSU}(n, \mathbb{F}_q)$ is the projective special unitary group.
- $S_{2n}(q) = \mathbf{PSp}(2n, \mathbb{F}_q)$ is the projective symplectic group.
- $O_n(q) < \mathbf{PSO}(n, \mathbb{F}_q)$, for n odd, and $O^{\epsilon}_{2k}(q) < \mathbf{PSO}(2k, \mathbb{F}_q)$, for $n = 2k$ even, is the usual simple subgroup of the projective special orthogonal group, where $\epsilon = \pm$ means the plus/minus type of the corresponding quadratic form.
- $G_2(q)$, $E_6(q)$ are exceptional groups in the family of Chevalley groups.

There are various ways to combine groups or abbreviate some groups structures – see p. xx of [CC+]. Assume that K and G are groups. Then:

- $K.G$ means a group L fitting into an extension $1 \to K \to L \to G \to 1$; which extension should be clear from the context. Besides, $K:G$ means that the extension is split (i.e. when $K.G$ is a semi-direct product) and $K \cdot G$ means that the extension is non-split.
- $K \times G$ is the direct product of K and G.
- K^m denotes the direct product of m groups isomorphic to K.
- p^m, for p prime, denotes the elementary abelian group of order p^m.
- $[m]$, for $m \in \mathbb{N}$, denotes an arbitrary group of order m.
- m denotes the cyclic group of m elements.
- p^{n+m} indicates a case of $p^n.p^m$.
- p^{1+2n} or p_+^{1+2n} or p_-^{1+2n} is used for the particular case of an extraspecial group.

Product of three or more groups are left-associated. That is, $A.B.C$ means $(A.B).C$, and implies the existence of a normal subgroup isomorphic to A.

We extracted from the ATLAS:

- The relevant information about maximal subgroups of a given sporadic simple group G and, if possible, the information about the fusion of conjugacy classes from maximal subgroups of G into G.
- The notation for the conjugacy classes. The conjugacy classes that contain elements of order n are named nA, nB, nC, $\dots$, and notice that the alphabet used here is potentially infinite. The conjugacy classes computed with GAP of a group given by a particular representation (taken from the ATLAS or not) are not necessarily named following the ATLAS notation. To avoid problems, in all these cases, we print sizes of the centralizers to recognize the classes we are working with.

1.10. Notations and results from GAP

We checked with GAP the following information:

- Real and quasi-real conjugacy classes. To recognize real conjugacy classes we use the GAP function `RealClasses`. To recognize quasi-real conjugacy classes we developed the GAP function `QuasiRealClasses`. This function uses the GAP function `PowerMaps`. See [AFGV2] for more details.
- If H is a subgroup of a group G, we use the function `PossibleClassFusions` or `FusionConjugacyClasses` for computing, or recovering, the fusion of conjugacy classes from H to G. See [AFGV2] for more details.

- We computed with GAP sums $S(\mathcal{O}, \mathcal{K}_1, \mathcal{K}_2)$, where $\mathcal{O}$ is the conjugacy class of an involution and $\mathcal{K}_1, \mathcal{K}_2$ are conjugacy classes of elements of order 3, in order to apply Proposition 1.8. See (1.4).

2. On Theorem II

2.1. Type D

Let X be a rack, $n \in \mathbb{N}$, $\Gamma \subset \mathbf{GL}(n, \Bbbk)$ a subgroup, $q : X \times X \to \Gamma$ a 2-cocycle. Let $V = \Bbbk X \otimes \Bbbk^n$ as in Section 1.3 and $g : X \to \mathbf{GL}(V)$ be the morphism of racks given by

$$g_x(e_y w) = e_{x \triangleright y} q_{x,y}(w), \quad x, y \in X, \ w \in V. \tag{2.1}$$

Definition 2.1. Let $(X, \triangleright)$ be a rack and q a 2-cocycle.

- We say that X is *faithful* if $\phi : X \to \mathbb{S}_X$ is injective.
- We say that (X, q) is *faithful* if $q : X \to \mathbf{GL}(V)$ is injective; if X is clear from the context, we shall also say that q is faithful. If X is faithful, then (X, q) is faithful for any q.
- We say that X *collapses* if for any faithful cocycle q with values in a finite group $\Gamma \subset \mathbf{GL}(n, \Bbbk)$, for any $n \in \mathbb{N}$, $\dim \mathfrak{B}(X, q) = \infty$.
- A rack X is *decomposable* iff there exist disjoint subracks $X_1, X_2 \subset X$ such that $X = X_1 \sqcup X_2$. Otherwise, X is *indecomposable*.
- We say that X is *of type D* if there exists a decomposable subrack $Y = R \sqcup S$ of X such that

$$r \triangleright \big(s \triangleright (r \triangleright s)\big) \neq s, \quad \text{for some } r \in R, \ s \in S. \tag{2.2}$$

Our main tool is the following rack-theoretical version of [HS, Theorem 8.6], whose proof uses the main result of [AHS].

Theorem 2.2. *(See [AFGV1, 3.6].) If X is a finite rack of type D, then X collapses.*

Let X be a simple rack that is not a permutation rack. Then X is faithful, hence any cocycle is faithful, and the notion of collapsing in Definitions 2.1 and 2 in the Introduction coincide. Therefore, for the purposes of this paper, we first need to check when a conjugacy class in a sporadic simple group is of type D. We collect some useful consequences of Theorem 2.2.

Lemma 2.3.

(i) *Let X be a rack of type D and let Z be a finite rack that admits a rack epimorphism $f : Z \to X$, X of type D. Then Z is of type D.*
(ii) *Let $f : G \to H$ be a group epimorphism and let $g \in G$ such that $f(g) = h$. If the conjugacy class of h is of type D, then the conjugacy class of g is of type D.*

Proof. For (i), $\pi^{-1}(Y) = \pi^{-1}(R) \sqcup \pi^{-1}(S)$ is a decomposable subrack of Z satisfying (2.2). Clearly, (ii) follows from (i). $\square$

Notice the inference of Theorem 2.4 from Theorem II by Lemma 2.3(1).

Theorem 2.4. *Let G be any finite group, $\mathcal{Q}$ a conjugacy class of G and $g \in \mathcal{Q}$. Assume that there is a rack epimorphism $\mathcal{Q} \to \mathcal{O}$, where $\mathcal{O}$ is a non-trivial conjugacy class of a sporadic group NOT listed in Table 2. Then $\dim \mathfrak{B}(\mathcal{Q}, \rho) = \infty$ for every $\rho \in \operatorname{Irr} C_G(g)$.*

Let us say that a finite group is *of type D* if all its non-trivial conjugacy classes are of type D. By Theorem 2.2, a finite group of type D collapses.

Proposition 2.5. *Let $0 \to K \xrightarrow{i} G \xrightarrow{p} H \to 0$ be a short exact sequence of finite groups such that K and H are of type D. Then G is of type D.*

Proof. Let $g \in G$. If $p(g) = 1$, then $g \in i(K)$ and the conjugacy class of g in G is of type D (since the conjugacy class in K of $k \in K$, where $i(k) = g$, is of type D). If $1 \neq p(g)$, then the conjugacy class of $h = p(g)$ in H is of type D and then, by Lemma 2.3, the conjugacy class of g in G is of type D. $\square$

We get immediately the following result.

Theorem 2.6. *Let G be any finite group. Assume that the simple factors of G in the Jordan–Hölder decomposition of G are of type D. Then G is of type D, hence it collapses.*

By Theorem II, the groups *Th*, *He* and *HN* are of type D; we also know that the groups $G_2(3)$, $G_2(5)$ are of type D.

2.2. Algorithms

We now explain our algorithms to implement the technique of the previous subsection.

Algorithm I. Let Γ be a finite group and let $\mathcal{O}$ be a conjugacy class. Fix $r \in \mathcal{O}$.

(1) For any $s \in \mathcal{O}$, check if $(rs)^2 \neq (sr)^2$; this is equivalent to (2.2).
(2) If such s is found, consider the subgroup H generated by r, s. If $\mathcal{O}_r^H \cap \mathcal{O}_s^H = \emptyset$, then $Y = \mathcal{O}_r^H \sqcup \mathcal{O}_s^H$ is the decomposable subrack we are looking for and $\mathcal{O}$ is of type D.

We have found that a useful variant, instead of going over all the elements, is to choose randomly a certain number of $s \in \mathcal{O}$ and check the conditions above. This turns out to be very often a much quicker way to see if $\mathcal{O}$ is of type D.

In the practice, for large groups, it is more economical to implement the algorithm in a recursive way.

Algorithm II. Let G be a finite group.

(1) List all maximal subgroups of G up to conjugacy, say $\mathcal{M}_1, \dots, \mathcal{M}_k$, with $|\mathcal{M}_1| \leqslant |\mathcal{M}_2| \leqslant \dots \leqslant |\mathcal{M}_k|$.
(2) Perform Algorithm I for every conjugacy class of $\Gamma = \mathcal{M}_1$. Let $\mathcal{D}_1$ be the set of conjugacy classes of G that contain a conjugacy class of $\mathcal{M}_1$ of type D.
(3) Perform Algorithm I for every conjugacy class of $\Gamma = \mathcal{M}_2$ that is not contained in any $\mathcal{O} \in \mathcal{D}_1$. Let $\mathcal{D}_2$ be the set of conjugacy classes of G that either contain a conjugacy class of $\mathcal{M}_2$ of type D, or else are in $\mathcal{D}_1$.
(4) Continue in this way, performing Algorithm I for every conjugacy class of the various maximal subgroups $\mathcal{M}_j$ and producing at each step a set of discarded classes $\mathcal{D}_j$.
(5) Perform Algorithm I for every conjugacy class $\mathcal{O}$ of G not in $\mathcal{D}_k$. Let $\mathcal{D}$ be the set of conjugacy classes of G that either are of type D by this argument, or else are in $\mathcal{D}_k$. This set $\mathcal{D}$ is the output of the algorithm.

The treatment of a maximal subgroup $\mathcal{M}$ can be simplified if it fits into a short exact sequence of groups $1 \to K \xrightarrow{i} \mathcal{M} \xrightarrow{p} H \to 1$, where we know the conjugacy classes in H of type D. To apply Lemma 2.3 to a conjugacy class in $\mathcal{M}$, we just need to know that some specific conjugacy classes in H are of type D.

Lemma 2.7. *Let $g \in \mathcal{M}$ of order m. Assume that every conjugacy class in H with representative of order k is of type D, for every k such that $k \mid m$ and $\frac{m}{k} \mid |K|$. Then the conjugacy class of g in $\mathcal{M}$ is also of type D.*

Proof. If $h = p(g)$ has order k, then $k \mid m$. Also, since $g^k \in \ker(p) = i(K)$ and i is a monomorphism, we have that $\frac{m}{k} = \frac{m}{(m,k)} \mid |K|$. Thus, it suffices to have these conjugacy classes of type D to conclude from Lemma 2.3. □

Here is another useful remark.

Lemma 2.8. *Let $G = H \times K$ be a direct product of finite groups. Let $\mathcal{O}$ be a conjugacy class of H with representative h, and $k \in K$. If $\mathcal{O}$ is of type D, then the conjugacy class of $h \times k$ is of type D.*

The proof of Theorem II follows by applying Algorithm II either to the sporadic groups or their maximal subgroups. The details of the computations are in [AFGV2], except for the involutions treated in Examples 1.15 and 1.16.

3. On Theorem I

3.1. Proof of Theorem I

As explained in Section 1.2, Theorem I is equivalent to the following statement.

Theorem 3.1.

(1) *If G is a sporadic group but not Fi_{22}, B, M, then the Nichols algebra of any irreducible Yetter–Drinfeld module has infinite dimension.*
(2) *If G is Fi_{22}, B or M and the pair $(\mathcal{O}, \rho)$ is not listed in Table* 1*, then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. By Theorems II and 2.2, it remains to consider the conjugacy classes listed in Table 2; we summarize the methods for each class in Table 3. The corresponding Nichols algebras have infinite dimension by the following reasons:

- If the class is real, then Lemma 1.1 applies.
- If the class is quasi-real, then Lemma 1.2 applies.
- If the class contains an involution, then often Proposition 1.8 applies, since some sum (shown in the table) is not zero.
- The conjugacy classes labeled 2A of the groups Co_2, Fi_{22}, Fi_{23} or B do not collapse, but $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any irreducible representation ρ of the corresponding centralizer. Indeed, it is enough to consider the representations ρ such that $\rho(g) = -1$. By Lemma 1.3, we are reduced to find an involution x such that $gh = hg$, for $h = xgx^{-1}$, and to compute the multiplicities of the eigenvalues of $\rho(h)$. For this last task, we use Remark 1.4. In the case of the group Fi_{23}, we first apply Lemma 1.3 to $S_8(2)$, that is isomorphic to a subgroup of Fi_{23}, and then apply Lemma 1.5. Finally, the class labeled 2A of Fi_{23} embeds as a subrack of the class labeled 2A of B, hence the technique of the subgroup applies. See [AFGV2] for details.

We finally explain the representations appearing in Table 1.

Let $\mathcal{O}$ be one of the conjugacy classes 22A, 22B of Fi_{22}, 46A, 46B of B, 92A, 92B, 94A, 94B of M. Then the centralizer of $g \in \mathcal{O}$ is the cyclic group $\langle g\rangle$. Thus $\operatorname{Irr} C_G(g)$ is also cyclic; say χ_ω is the representation such that $\chi_\omega(g) = \omega$, ω a root of 1 whose order divides the order of g. On the other hand these classes are quasi-real, as stated in Table 1. By Lemmata 1.1 or 1.2, only χ_{-1} survives.

Table 3
Proof of Theorem I.

G	Classes	Relevant information
M_{11}	8A, 8B	Example 1.12
	11A, 11B	quasi-real $j=3$, $g^9 \neq g$
M_{12}	11A, 11B	quasi-real $j=3$, $g^9 \neq g$
M_{22}	11A, 11B	quasi-real $j=3$, $g^9 \neq g$
M_{23}	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
M_{24}	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
J_2	2A	$S(2A, 3B, 3B) = 18$
	3A	real
Suz	3A	real
HS	11A, 11B	quasi-real $j=3$, $g^9 \neq g$
McL	11A, 11B	quasi-real $j=3$, $g^9 \neq g$
Co_3	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
Co_2	2A	abelian subrack
	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
Co_1	3A	real
	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
J_1	15A, 15B, 19A, 19B, 19C	real
$O'N$	31A, 31B	quasi-real $j=2$, $g^4 \neq g$
J_3	5A, 5B	real
	19A, 19B	quasi-real $j=4$, $g^{16} \neq g$
Ly	33A, 33B	quasi-real $j=4$, $g^{16} \neq g$
	37A, 37B, 67A, 67B, 67C	real
Ru	29A, 29B	real
J_4	29A, 37A, 37B, 37C, 43A, 43B, 43C	real
Fi_{22}	2A	abelian subrack
Fi_{23}	2A	abelian subrack
	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
Fi'_{24}	27B, 27C, 29A, 29B	real
	33A, 33B, 39C, 39D	real
	23A, 23B	quasi-real $j=2$, $g^4 \neq g$
B	2A	abelian subrack
	47A, 47B	quasi-real $j=2$, $g^4 \neq g$
M	41A	real
	47A, 47B, 69A, 69B, 71A, 71B, 87A, 87B	quasi-real $j=2$, $g^4 \neq g$
	59A, 59B	quasi-real $j=3$, $g^9 \neq g$
T	2A	$S(2A, 3A, 3A) = 108$

Let $\mathcal{O}$ be the conjugacy class 34A of B and $g \in \mathcal{O}$. This class is real. By Lemma 1.1, we discard all irreducible representations ρ of the centralizer $\mathbb{Z}/34 \times \mathbb{Z}/2$ except those satisfying $\rho(g) = -1$, i.e. $\rho = \chi_{-1} \otimes \epsilon$, $\chi_{-1} \otimes \mathrm{sgn}$, where ϵ and sgn mean the trivial and the sign representation of $\mathbb{Z}/2$.

Let $\mathcal{O}$ be one of the conjugacy classes 46A, 46B of M and $g \in \mathcal{O}$. The class $\mathcal{O}$ is quasi-real, as stated in Table 1. By Lemma 1.2, we discard all irreducible representations ρ of the centralizer $\mathbb{Z}/23 \times \mathbb{D}_4$

except the one satisfying $\rho(g) = -1$, i.e. $\rho = \epsilon \otimes \rho_2$, where ϵ is the trivial representation of $\mathbb{Z}/2$ and ρ_2 is the unique irreducible representation of $\mathbb{D}_4$ of degree 2. $\square$

In the proof of Theorem 3.1, we needed the structure of the centralizers of some specific elements in some sporadic groups; this was kindly communicated to us by Thomas Breuer, when not available in the literature.

3.2. *Remarks on the remaining irreducible Yetter–Drinfeld modules in Table 1*

Let G be a sporadic group and $\mathcal{O}$ a conjugacy class as in Table 1. Assume that r and s are two elements in $\mathcal{O}$ such that (2.2) holds and let H be the subgroup of G generated by r and s. If the conjugacy classes in H of r and s are disjoint, then $\mathcal{O}$ would be of type D. Notice that H should be contained in a maximal subgroup $\mathcal{M}$ of G and, clearly, it would be enough to perform the necessary computations in $\mathcal{M}$. Also, if G is not the Monster group, the list of all maximal subgroups is known. So, we proceed to investigate these maximal subgroups by the fusion of conjugacy classes, see Section 1.10.

3.2.1. *The classes $\mathcal{O}$ labeled* 22A, 22B *in Fi_{22}*

We know that H should be contained in $\mathcal{M}_1 \simeq 2.U_6(2)$. It is not possible with our computational resources to determine if these classes are of type D in $\mathcal{M}_1$, i.e. due to the size of these conjugacy classes it is not possible to complete all the possible elections of r and s. Actually, in all the computed cases H yields to be the centralizer of the representative of the class or $\mathcal{M}_1$ itself.

The same occurs if we want to determine if the conjugacy classes 11A, 11B in $U_6(2)$ are of type D. So we cannot decide that the corresponding Nichols algebra is infinite-dimensional by lifting racks of type D from the classes 11A, 11B of $U_6(2)$.

3.2.2. *The classes $\mathcal{O}$ labeled* 16C, 16D, 32A, 32B, 32C, 32D, 34A *in B*

The classes 16C, 16D meet the following maximal subgroups:

$$\begin{aligned} &\mathcal{M}_1 \simeq 2.\,^2E_6(2):2, && \mathcal{M}_2 \simeq 2^{1+22}.Co_2, \\ &\mathcal{M}_4 \simeq 2^{9+16}.S_8(2), && \mathcal{M}_7 \simeq 2^{2+10+20}.(M_{22}:2 \times \mathbb{S}_3), \\ &\mathcal{M}_8 \simeq \big[2^{30}\big].L_5(2), && \mathcal{M}_{10} \simeq \big[2^{35}\big].\big(\mathbb{S}_5 \times L_3(2)\big); \end{aligned}$$

the classes 32A, 32B meet $\mathcal{M}_2$, $\mathcal{M}_4$, $\mathcal{M}_7$ or $\mathcal{M}_{10}$; the classes 32C, 32D meet $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$, $\mathcal{M}_7$ or $\mathcal{M}_{10}$; the class 34A meets $\mathcal{M}_1$. The known matrix representations of these subgroups do not allow us to perform the necessary computations. Notice that the sixth maximal subgroup of B is excluded of our analysis, because we do not know the fusion of conjugacy classes $\mathcal{M}_6 \to B$.

3.2.3. *The classes $\mathcal{O}$ labeled* 46A, 46B *in B*

We know that H should be contained in the second maximal subgroup $\mathcal{M}_2$. As already said, the computations for this case are out of our resources. Note also that Co_2 has no elements of order 46; thus, if $g \in \mathcal{M}_2$ has order 46, then its projection in Co_2 has order 23, but the classes of elements of order 23 in Co_2 are not known to be of type D.

3.2.4. *The classes $\mathcal{O}$ labeled* 32A, 32B, 46A, 46B, 92A, 92B, 94A, 94B *in M*

Among the known maximal subgroups of the Monster, [B] provides the fusion of only 4 of them: $\mathcal{M}_1 \simeq 2.B$, $\mathcal{M}_2 \simeq 2^{1+24}.Co_1$, $\mathcal{M}_3 \simeq 3.Fi_{24}$, and $\mathcal{M}_9 \simeq \mathbb{S}_3 \times Th$. The classes 32A, 32B meet $\mathcal{M}_1$, $\mathcal{M}_2$; the classes 46A, 46B meet $\mathcal{M}_1$, $\mathcal{M}_2$; the classes 92A, 92B meet $\mathcal{M}_2$; the classes 94A, 94B meet $\mathcal{M}_1$. As before, the necessary computations are out of our resources.

Acknowledgments

Some of the results of the present paper were announced in [AFGV3] and at several conferences like: IV Encuentro Nacional de Álgebra, August 2008, La Falda, Argentina; Groupes quantiques dynamiques et catégories de fusion, April 2008, CIRM, Luminy, France; First De Brún Workshop on Computational Algebra, August 2008, Galway, Ireland; Hopf algebras and related topics (A conference in honor of Professor Susan Montgomery), February 2009, University of Southern California; XVIII Latin American Algebra Colloquium, August 2009, São Pedro, Brazil; Coloquio de Álgebras de Hopf, Grupos Cuánticos y Categorías Tensoriales, August, 2009, La Falda, Argentina; XIX Encuentro Rioplatense de Álgebra y Geometría Algebraica, November 2009, Montevideo, Uruguay; VII workshop in Lie Theory and its applications, November 2009, Córdoba, Argentina. We are grateful to the organizers for the kind invitations.

We thank Sebastián Freyre for interesting conversations. We are very grateful to Alexander Hulpke, John Bray, Robert Wilson and very specially to Thomas Breuer for answering our endless questions on GAP. We also thank Enrique Tobis and the people from shiva and ganesh who allowed us to use their computers.

References

[A] N. Andruskiewitsch, Some remarks on Nichols algebras, in: Bergen, Catoiu, Chin (Eds.), Hopf Algebras, M. Dekker, 2004, pp. 25–45.
[AF] N. Andruskiewitsch, F. Fantino, On pointed Hopf algebras associated with alternating and dihedral groups, Rev. Un. Mat. Argentina 48 (3) (2007) 57–71.
[AFGV1] N. Andruskiewitsch, F. Fantino, M. Graña, L. Vendramin, Finite-dimensional pointed Hopf algebras with alternating groups are trivial, Ann. Mat. Pura Appl., doi:10.1007/s10231-010-0147-0.
[AFGV2] N. Andruskiewitsch, F. Fantino, M. Graña, L. Vendramin, The *logbook* of pointed Hopf algebras over the sporadic simple groups, J. Algebra 325 (1) (2011) 282–304.
[AFGV3] N. Andruskiewitsch, F. Fantino, M. Graña, L. Vendramin, Pointed Hopf algebras over some sporadic simple groups, C. R. Math. Acad. Sci. Paris 348 (2010) 605–608.
[AG] N. Andruskiewitsch, M. Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2003) 177–243.
[AHS] N. Andruskiewitsch, I. Heckenberger, H.-J. Schneider, The Nichols algebra of a semisimple Yetter–Drinfeld module, arXiv:0803.2430.
[AS1] N. Andruskiewitsch, H.-J. Schneider, Pointed Hopf algebras, in: New Directions in Hopf Algebras, in: Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, 2002, pp. 1–68.
[AS2] N. Andruskiewitsch, H.-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Ann. Math. 171 (1) (2010) 375–417.
[AZ] N. Andruskiewitsch, S. Zhang, On pointed Hopf algebras associated to some conjugacy classes in $\mathbb{S}_n$, Proc. Amer. Math. Soc. 135 (2007) 2723–2731.
[B] T. Breuer, The GAP Character Table Library, Version 1.2 (unpublished), http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib/.
[CC+] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Oxford University Press, 1985.
[D] L. Dickson, The alternating group on eight letters and the quaternary linear congruence group modulo two, Math. Ann. 54 (1901) 564–569.
[F] F. Fantino, On pointed Hopf algebras associated with Mathieu groups, J. Algebra Appl. 8 (5) (2009) 633–672.
[FGV] S. Freyre, M. Graña, L. Vendramin, On Nichols algebras over $\mathbf{GL}(2, \mathbb{F}_q)$ and $\mathbf{SL}(2, \mathbb{F}_q)$, J. Math. Phys. 48 (2007) 123513-1–123513-11.
[GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12, 2008, available at http://www.gap-system.org.
[Go] D. Gorenstein, Finite Groups, Harper & Row Publ., New York, 1968, xv+527.
[H] I. Heckenberger, Classification of arithmetic root systems, Adv. Math. 220 (2009) 59–124.
[HS] I. Heckenberger, H.-J. Schneider, Root systems and Weyl groupoids for semisimple Nichols algebras, Proc. London Math. Soc., doi:10.1112/plms/pdq001.
[J] D. Joyce, Simple quandles, J. Algebra 79 (2) (1982) 307–318.
[L] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, 1993.
[R] M. Rosso, Quantum groups and quantum shuffles, Invent. Math. 133 (1998) 399–416.
[S] P. Schauenburg, A characterization of the Borel-like subalgebras of quantum enveloping algebras, Comm. Algebra 24 (1996) 2811–2823.
[WWT+] R.A. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, R. Abbott, A world-wide-web Atlas of finite group representations, http://brauer.maths.qmul.ac.uk/Atlas/v3/.
[WPN+] R.A. Wilson, R.A. Parker, S. Nickerson, J.N. Bray, T. Breuer, AtlasRep, A GAP Interface to the ATLAS of Group Representations, Version 1.4, refereed GAP package, http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep, 2007.