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Ordinary Differential Equations in Linear Topological Spaces, I

H. O. FATTORINI[1]

Consejo Nacional de Investigaciones Cientificas y Tecnicas, Argentina, and Departamento de Matematicas, Universidad de Buenos Aires, Facultad de Ciencias Exactas y Naturales,

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INTRODUCTION

We discuss in this paper the differential equation

$$u^{(n)}(t) = Au(t) \tag{1}$$

($u^{(n)}(t) = (d/dt)^n u$), A a linear operator in the linear topological space E. The Cauchy problem for (1) is assumed to be *well posed*, i.e., solutions of (1) are assumed to exist, to be unique and to depend continuously on their initial data for each t. The above conditions are natural enough when $n = 1$, and the problem has been studied essentially in this formulation by various authors (see [V_1][2], chapter III, Section 2 for a survey of results). When $n > 1$ (say, $n = 2$) the equation (1) rarely arises in applications with the degree of generality considered here. Usually additional information on A is available that allows one to reduce the problem to a first order one by the usual device of introducing derivatives as new unknowns. For instance if E is a Hilbert space, A a nonpositive self-adjoint operator the Cauchy problem for (1) can be reduced to a first order problem in the product space $D((-A)^{\frac{1}{2}}) \times E$ [$D((-A)^{\frac{1}{2}})$ endowed with the graph norm], namely $u' = u_1$, $u_1' = Au$. In the general case, however, nothing like $(-A)^{\frac{1}{2}}$ or its graph norm is available a priori, and thus it is natural to study the Cauchy problem directly. We obtain later as a result (Theorem (6.9) that if mild additional restrictions are satisfied then a reduction to a first order problem quite similar to the one outlined for selfadjoint nonpositive A can be carried out in the general case. For $n \geqslant 3$ the situation is even simpler (see Remark 3.6).

Section 1 of this paper is of an introductory nature and deals with some facts on linear topological spaces to be used later. We give in 2 the precise

[1] Present address: University of California at Los Angeles, Department of Mathematics, Los Angeles, California.

[2] All references are referred by letters.

definition of a well posed (uniformly well posed) Cauchy problem, deduce some relations between certain operator-valued solutions of (1), the propagators (Lemma 2.1, 2.2) and give a criterion for uniform well posedness of the Cauchy problem for (1) (Theorem 2.4) for Fréchet spaces. We apply these results in 3 to show for a class of spaces E that if $n \geqslant 3$ the Cauchy problem for (1) is uniformly well posed if and only if A is continuous and satisfies an additional condition (Theorem 3.1). Next we consider the case when the solutions of (1) increase at infinity less than a given exponential and characterize the operators A for which this happens (Theorem 3.3). Paragraph 4 is devoted to the case $n = 1$; here and in following paragraphs we only consider the case in which the solutions of (1) have exponential growth at infinity. The (more or less well known) result in this case is that the Cauchy problem for (1), $n = 1$ is uniformly well posed if and only if A is the infinitesimal generator if a strongly continuous semigroup (Theorem 4.1). We consider in 5 families of continuous operators in E satisfying $S(0) = I$, $S(t + s) + S(t - s) = 2S(t)S(s)$ (the "cosine functional equation"), prove several results for them in the spirit of semigroup theory (Lemma 5.3 and following results) and apply them to the Cauchy problem for (1) (Theorem 5.9) to obtain a result similar to the one in 4. We construct in 6 square roots of certain translates of A; under an additional condition (Assumption 6.4) these square roots generate strongly continuous groups and the Cauchy problem for (1), $n = 2$ can be reduced to a first-order Cauchy problem in the product space $E \times E$ (Theorem 6.9).

In case E is a Banach space, we are able to improve somewhat our results or to obtain new ones; see for instance Remark 3.4, Theorem 4.2, Lemmas 5.2, 5.3 and 5.5.

Results in this paper have been announced in F_1.

1. Linear Topological Spaces

Throughout this paper $E = \{u, v, ...\}$ will be a (Hausdorff) complete, barreled locally convex linear topological space (LTS) over the field C of complex numbers ([B_1], II, Section 2, III, Section 1 and Section 2). We shall denote by $\mathscr{E}$ a set $\{| \cdot |, ...\}$ of semi-norms determining the topology of E, i.e. such that a generalized sequence $\{u_\alpha\}$ converges to zero if and only if $\lim_\alpha | u_\alpha | = 0$ for all $| \cdot | \in \mathscr{E}$. If $\mathscr{E}$ can be chosen countable, then E is said to be a Fréchet space. A Fréchet (in particular a Banach) space is always barreled. For these spaces we also have

1.1 Theorem (*Closed graph theorem*). *Let E, F be Fréchet spaces, and A a closed, everywhere defined linear map from E to F. Then A is continuous.*

For a proof see [B_1], I, Section 3 (local convexity is not necessary here). Let E, F be two complete, barreled locally convex LTS. We shall denote $L(E, F)$ the space of all continuous linear maps from E to F endowed with the topology of uniform convergence on bounded sets of E; $L(E, F)$ is a quasi-complete ([B_1], III, Section 3) locally convex LTS. The topology of $L(E, F)$ is determined by the family $\mathscr{L}$ of semi-norms

$$|A| = \sup\{|Au|, u \in K\}$$

where K ranges over all bounded sets of E, $|\cdot|$ over a family $\mathscr{F}$ of semi-norms determining the topology of F. We shall only consider the cases $F = E$, $F = C$ and write $L(E, E) = L(E) = \{A, B,...\}, L(E, C) = \{u^*, v^*,...\} = E^*$ (the dual space of E). If E is Banach so is $L(E)$ (resp. E^*) under the norm $|A| = \sup\{|Au|, u \in E, |u| \leqslant 1\}$ (resp. $|u^*| = \sup\{|u^*(u)|, u \in E, |u| \leqslant 1\}$), $|\cdot|$ the norm in E. We shall write

$$u^*(u) = \langle u^*, u\rangle = \langle u, u^*\rangle, \qquad u^* \in E^*, \quad u \in E.$$

Frequent use will be made of

1.2 THEOREM. *Let $\mathscr{A} = \{A,...\}$ be a set in $L(E)$ such that $\{Au; A \in \mathscr{A}\}$ is bounded in E for each $u \in E$. Then $\mathscr{A}$ is equicontinuous*

and of its corollary

1.3 THEOREM (*Banach–Steinhaus theorem*). *Let $\{A_a\}$ be a generalized sequence in $L(E)$ such that $\{A_a u\}$ converges and is bounded in E for each $u \in E$. Then $A =$ (pointwise) $\lim_a A_a$ is a continuous operator.*

For a proof see [B_1], III, Section 3. Note that the boundedness assumption in Theorem 1.3 is automatically satisfied if $\{A_a\}$ is a sequence.

Let D be a domain in the complex plane, $f(\cdot)$ an E-valued function. f is said to be *analytic* in D if $\lim_{h\to 0} h^{-1}(f(z+h) - f(z)) = f'(z)$ exists for all $z \in D$.

1.4 THEOREM. (a) *Let $T(\cdot)$ be an $L(E)$-valued function defined in D such that the scalar-valued function $\langle u^*, T(\cdot)u\rangle$ is analytic in D for all $u \in E$, $u^* \in E^*$. Then $T(\cdot)$ is analytic.* (b) *Let $f(\cdot)$ be an E-valued function defined in D such that $\langle u^*, f(\cdot)\rangle$ is analytic for all $u^* \in E^*$. Then $f(\cdot)$ is analytic.*

For a proof in the Banach space case see [H_1], 3.10.1. Since it is based in Theorem 1.1 and quasi-completeness of $L(E)$, it extends to our case without major changes.

Most properties of scalar-valued analytic functions extend to the E-valued case (See [H_1, III). If f is analytic in $|z - z_0| < a$ then it has derivatives of all orders there that can be developed in power series in the customary way. The radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{1.1}$$

is given by $1/r$, $r = \sup\{\limsup_{n\to\infty} | a_n |^{1/n}; | \cdot | \in \mathscr{E}\}$, i.e. (1.1) converges absolutely and uniformly on compacts of $| z - z_0 | < 1/r$, diverges for $| z - z_0 | > 1/r$. The same properties hold for $L(E)$-valued functions.

Finally, a word about integration in E. If $f(s)$ is continuous in $-\infty < a \leqslant s \leqslant b < \infty$ the integral $\int_a^b f(s)\, ds$ can be defined and seen to exist by means of Riemann sums in the same way as for ordinary functions. Improper integrals like

$$\int_a^{\infty} f(s)\, ds \tag{1.2}$$

and similar ones will be defined as the limit when $b \to \infty$ of the integral over (a, b). A simple criterion for existence of such integrals is given by

1.5 THEOREM. (a) *Let the E-valued function $f(\cdot)$ be defined and continuous in (a, ∞); assume that the real-valued function $|f(\cdot)|$ is integrable in (a, ∞) for all $| \cdot | \in \mathscr{E}$. Then* (1.2) *exists.* (b) *Let the $L(E)$-valued function $A(\cdot)$ be defined in (a, ∞) and strongly continuous there; assume $| A(\cdot)u |$ is integrable in (a, ∞) for all $u \in E$, $| \cdot | \in \mathscr{E}$. Then* (1.2) *exists for $A(\cdot)u$, $u \in E$ and defines a linear continuous operator in E.*

The proof of (a) is immediate; the proof of (b) is a simple application of Theorem 1.3. For Banach spaces we shall make use (in Corollary 5.3) of Bochner's integration theory; see [H_1], 3.5.

2. THE CAUCHY PROBLEM

We shall write through this and following paragraphs $R = (-\infty, \infty)$, $R_+ = (0, \infty)$, $\bar{R}_+ = (0, \infty)$; $C^{(n)}(E)$ (resp. $C_+^{(n)}(E)$, $\bar{C}_+^{(n)}(E)$) shall denote the space of all E-valued functions $u(\cdot)$ defined and n times (strongly) continuously differentiable in R (resp. R_+, $\bar{R}_+$). A will be a linear operator with domain $D(A)$ dense in E and range in E; we shall assume that $\rho(A)$, the resolvent set of A is non-void, i.e. that there exists λ such that $R(\lambda; A) = (\lambda I - A)^{-1}$ exists and is continuous. This implies that A is closed. The subindex k (unless otherwise stated) will always take the values $0, 1, \ldots, n - 1$.

The E-valued function $u(\cdot)$ will be called a *solution* of

$$u^{(n)}(t) = Au(t) \tag{2.1}$$

in R(resp. R_+, $\bar{R}_+$) if $u(\cdot) \in C^{(n)}(E)$ (resp. $C_+^{(n)}(E)$, $\bar{C}_+^{(n)}(E)$), $u(t) \in D(A)$ for all $t \in R$(resp. R_+, $\bar{R}_+$) and (2.1) is satisfied everywhere.

The Cauchy problem for (2.1) will be *well posed* (w.p.) in R_+ if

(a) There exists a dense subspace D of E such that if $u_0, \ldots, u_{n-1} \in D$ then there exists a solution $u(\cdot)$ of (2.1) in R_+ such that

$$u^{(k)}(0+) = \lim_{h \to 0+} u^{(k)}(h) = u_k .$$

(b) Let $\{u_a(\cdot)\}$ be a generalized sequence of solutions of (2.1) in R_+ such that $u_a^{(k)}(0+) \to 0$ (we *assume* $u_a^{(k)}(0+)$ to exist). Then $u_a(\cdot) \to 0$ pointwise in R_+.

The Cauchy problem for (2.1) will be *uniformly well posed* (u.w.p.) in R_+ if (b) is strenghtened to

(b′) Let $\{u_a(\cdot)\}$ satisfy the assumptions in (b). Then $\{u_a(\cdot)\}$ converges to zero uniformly on compacts of R_+.

Similar definitions for R, $\bar{R}_+$.

Assume the Cauchy problem for (2.1) is well posed, let $0 \leqslant j \leqslant n-1$ and let $u(\cdot)$ be a solution of (2.1) with $u^{(k)}(0+) = \delta_{jk}u$, δ_{jk} the Kronecker delta. The linear operators

$$S_j(t)u = u(t)$$

are, by virtue of (a) and (b) densely defined and continuous. We can then extend $S_j(t)$ to all of E by continuity; we shall denote the extensions with the same symbols. The operators $S_j(t)$ will be called the *propagators* or *solution operators* associated with (2.1). If $u(\cdot)$ is any solution of (2.1) such that $u^{(k)}(0+)$ exist, we have

$$u(t) = \sum_{k=0}^{n-1} S_k(t)u^{(k)}(0+) \tag{2.2}$$

This is clear if the initial values $u^{(k)}(0+)$ of $u(\cdot)$ belong to D; if not we may approximate them by elements of D and use the continuity of the S_k.

If the Cauchy problem for (2.1) is u.w.p., then for any $u \in E$, $S_0(\cdot)u, \ldots, S_{n-1}(t)u$ are continuous functions of t. In fact, $S_k(\cdot)$ is the limit, uniform on compacts of $\{S_k(\cdot)u_a\}$, $\{u_a\}$ any generalized sequence in D with $u_a \to u$.

Assume the Cauchy problem for (2.1) is w.p. Then we have

2.1 LEMMA. (a) *Let* $1 \leqslant k \leqslant n-1$, $u \in D$. *Then*

$$S_k'(t)u = S_{k-1}(t)u \tag{2.3}$$

(b)

$$S_0'(t)u = AS_{n-1}(t)u$$

Proof. We shall carry it out for R_+; the proofs for R, $\bar{R}_+$ are similar. Let $u \in D$,

$$u(t) = \int_0^t S_{k-1}(s)u\,ds \qquad t > 0 \tag{2.4}$$

Plainly $u(\cdot) \in C_+^{(n+1)}(E)$, $u^{(j)}(s) = S_{k-1}^{(j-1)}(s)u$, which implies

$$u^{(j)}(0+) = \delta_{jk}u, \qquad j = 0, 1, \ldots, n-1 \tag{2.5}$$

Since $S_{k-1}(s)u \in D(A)$ for $s > 0$ and $AS_{k-1}(s)u = S_{k-1}^{(n)}(s)u$ is continuous for $s > 0$, $\int_r^t S_{k-1}(s)u\,ds \in D(A)$ for $r > 0$, $A\int_r^t S_{k-1}(s)u\,ds = \int_r^t AS_{k-1}(s)u\,ds = \int_r^t S_{k-1}^{(n)}(s)u\,ds = S_{k-1}^{(n-1)}(t)u - S_{k-1}^{(n-1)}(r)u$. Thus $u(t) \in D(A)$ and $Au(t) = S_{k-1}^{(n-1)}(t)u - S_{k-1}^{(n-1)}(0+)u = S_{k-1}^{(n-1)}(t)u = u^{(n)}(t)$, which shows that $u(\cdot)$ is a solution of (2.1). In view of (2.5) $u(t) = S_k(t)u$; differentiating we get (2.3). Applying repeatedly (2.3) to $S_{n-1}(t)u$ we get $S_{n-1}^{(n-1)}(t)u = S_0(t)u$; differentiating once more, $S_0'(t)u = S_{n-1}^{(n)}(t)u = AS_{n-1}(t)u$.

2.2 Lemma. *The operators A, $S_0(t_0), \ldots, S_{n-1}(t_{n-1})$ all commute for any $t_0, \ldots, t_{n-1}$.*

Proof. As in Lemma 2.1 we only give the proof for R_+. Let $\lambda \in \rho(A)$ and let $u(t) = R(\lambda; A)S_k(t)u$, $u \in D$. It is easy to see that $u(\cdot)$ is a solution of (2.1); since $u^{(k)}(0+) = \delta_{jk}R(\lambda; A)u$ we have

$$R(\lambda; A)S_k(t)u = S_k(t)R(\lambda; A)u \tag{2.6}$$

Since u is dense in E, (2.6) holds for any $u \in E$. Applying (2.6) to $u = (\lambda I - A)v$, $v \in D(A)$ and applying $\lambda I - A$ to both sides we see that A commutes with $S_k(t)$. The rest of the proof is similar.

2.3 Lemma. (a) *Let $0 \leqslant k \leqslant n-2$. Then*

$$S_k(s+t) = \sum_{j=0}^{k} S_j(s)S_{k-j}(t) + A\sum_{j=k+1}^{n-1} S_j(s)S_{n-j+k}(t) \tag{2.7}$$

(b)

$$S_{n-1}(s+t) = \sum_{j=0}^{n-1} S_j(s)S_{n-1-j}(t) \tag{2.8}$$

Proof. Let $u \in D$. The function $u(s) = S_k(t+s)u$ (t fixed) is a solution of (2.1). Then we have, by (2.2)

$$S_k(s+t)u = \sum_{j=0}^{n-1} S_j(s)S_k^{(j)}(t)u$$

Applying now Lemma 2.1 to compute $S_k^{(j)}(t)u$ we get (2.7), (2.8) for $u \in D$. Since D is dense in E, Lemma 2.3 follows.

2.4 *Remark.* If E is a Fréchet space we can drop the continuous dependence hypothesis (b') at the cost of assuming existence and uniqueness of solutions for certain choices of initial data. In fact, we have

2.5 THEOREM. *Let E be a Fréchet space. Assume that for each $u \in D(A)$ there exists a unique solution of* (2.1) *in R_+ with $u^{(k)}(0+) = \delta_{0k}u$. Then the Cauchy problem for* (2.1) *is uniformly well posed in R_+. Same conclusions for $\bar{R}_+$, R.*

Proof. Let $\mathscr{E}$ be a (countable) set of semi-norms determining the topology of E and let $\lambda \in \rho(A)$. It is easy to see that $D(A)$ becomes a Fréchet space if topologized by the family of semi-norms

$$\| u \| = |(\lambda I - A)u|, \qquad |\cdot| \in \mathscr{E}$$

Let now M be the subspace of $C_+^{(n)}(E)$ consisting of those functions $u(\cdot)$ for which $u^{(k)}(0+)$ exist. M becomes a Fréchet space if topologized by the family of semi-norms

$$\| u(\cdot)\|_m = \sum_{k=0}^{n-1} \sup_{0 \leqslant t \leqslant m} | u^{(k)}(t)|$$

$$+ \sup_{1/m \leqslant t \leqslant m} | u^{(n)}(t)|, \qquad |\cdot| \in \mathscr{E}, \qquad m = 1, 2, ...$$

(here we have written $u^{(k)}(0) = u^{(k)}(0+)$). Consider the linear operator from $D(A)$ to M

$$u \to Ku = Ku(\cdot) \tag{2.9}$$

$Ku(\cdot)$ the solution of (2.1) with initial data $u^{(k)}(0+) = \delta_{0k}u$. It is easy to see that K is closed; by the closed graph theorem it is as well continuous.

Let $u_0, ..., u_{n-1} \in D(A)$. The function

$$u(t) = Ku_0(t) + \sum_{j=1}^{n-1} \frac{1}{(j-1)!} \int_0^t (t-s)^{j-1} Ku_j(s)\, ds \tag{2.10}$$

is easily seen to be a solution of (2.1) with $u^{(k)}(0+) = u_k$; thus by uniqueness it represents *any* solution of (2.1) with $u^{(k)}(0+) \in D(A)$. Let $\{u_m(\cdot)\}$ be a sequence of solutions of (2.1) such that $u_m^{(k)}(0+) \to 0$. Since $v_m(\cdot) = R(\lambda; A)u_m(\cdot)$ is a solution of (2.1) with $v_m^{(k)}(0+) = R(\lambda; A)u_m^{(k)}(0+) \in D(A)$ we get from (2.10) and the continuity of K that $\{v_m(\cdot)\}$ tends to zero in the topology of M; in particular, $\lambda R(\lambda;\, A)u_m(\cdot) - R(\lambda;\, A)u_m^{(n)}(\cdot) = u_m(\cdot) \to 0$ uniformly on compacts of R_+. Existence of solutions of (2.1) with $u^{(k)}(0+) = u_k$ for any $u_0, \ldots, u_{n-1} \in D(A)$ is shown by (2.10).

2.6 *Remark.* The idea of the proof of Theorem 2.5 is essentially the same used in H_1 to prove Theorem 23.8.3.

3. The Case $n \geqslant 3$

3.1 Theorem. *Assume the Cauchy problem for the equation*

$$u^{(n)}(t) = Au(t), \qquad n \geqslant 3 \tag{3.1}$$

is u.w.p. in R_+. Then $D(A) = E$, A is continuous and the series

$$\sum_{j=0}^{\infty} t^j A^j/(nj)! \tag{3.2}$$

converges in $L(E)$ for all $t > 0$. Conversely, assume $D(A) = E$, A continuous and (3.2) *convergent for all $t > 0$. Then the Cauchy problem for* (3.1) *is* u.w.p. *in R. The propagators S_k can be extended to $L(E)$-valued entire functions with McLaurin series*

$$S_k(z) = \sum_{j=0}^{\infty} z^{nj+k} A^j/(nj+k)! \tag{3.3}$$

Proof. We shall carry it out by first extending S_{n-1} to the complex plane and then obtaining A by differentiation. Let $w = \exp(2\pi i/n)$. Divide the complex plane in sectors

$$W_k = \{z;\, 2\pi k/n \leqslant \arg z \leqslant 2\pi(k+1)/n\}$$

Extend S_j to the rays sw^k, $s \geqslant 0$ by setting $S_j(sw^k) = w^{kj}S_j(t)$ and then extend S_{n-1} to the interior of each sector on the basis of equality (2.8) extended to the complex plane, i.e.

$$S_{n-1}(z) = S_{n-1}(sw^k + tw^{k+1}) = \sum_{j=0}^{n-1} S_j(sw^k) S_{n-1-j}(tw^{k+1})$$

$$= w^{-(k+1)} \sum_{j=0}^{n-1} w^{-j} S_j(s) S_{n-1-j}(t) \tag{3.4}$$

It is easy to see that if $u \in D$, $S_{n-1}u$ has first partial derivatives in the interior $W_k{}^0$ of each W_k given by

$$
\begin{aligned}
w^{k+1}\partial S_{n-1}u/\partial s &= w^k \partial S_{n-1}u/\partial t \\
&= \sum_{j=1}^{n-1} w^{-j} S_{j-1}(s) S_{n-1-j}(t)u + A S_{n-1}(s) S_{n-1}(t)u \\
&= \sum_{j=1}^{n-1} (w^{-j} - 1) S_{j-1}(s) S_{n-1-j}(t)u + S_{n-2}(s+t)u \qquad (3.5)
\end{aligned}
$$

(in the last step we have made use of equality (2.7) for the case $k = n - 2$). Since $S_k(t)u$ is continuous in R_+ for each $u \in E$ (see 2) it follows from Theorem 1.3 that $S_k(\,\cdot\,)$ is equicontinuous on compacts of R_+; this, and the fact that $S_k(t)u$ is continuous in $\bar{R}_+$ for each $u \in D$ show $S_{n-1}u$ and its first partials to be continuous in the whole plane—except perhaps at the origin. Next, observe that the relation (3.5) between the partials of $S_{n-1}u$ are the Cauchy–Riemann equations with respect to the directions given by w^k, w^{k+1}. Then we obtain

(i) If $u \in D$ $S_{n-1}(z)u$ is holomorphic in the whole plane, except perhaps at the origin.

Let now $u \in E$, $\{u_a\}$ a generalized sequence in D such that $u_a \to u$. It follows again from equicontinuity of S_{n-1} on compacts of R_+ that $S_{n-1}(z)u_a \to S_{n-1}(z)u$ uniformly on compacts of each $W_k{}^0$. Then $S_{n-1}(\,\cdot\,)$ is holomorphic in each $W_k{}^0$; by Theorem 1.4,

(ii) $S_{n-1}(\,\cdot\,)$ is holomorphic (as an $L(E)$-valued function) in each $W_k{}^0$.

Let $u \in D$, $H(s, z)u = \sum_{j=0}^{u-1} S_j(s) S_{n-1}^{(j)}(z)u$, $s \in R_+$. Using Lemmas 2.1 and 2.3 we see that $H(s, z)u = S_{n-1}(s + z)u$ for $z \in R_+$; by analytic continuation this holds as well for any complex $z \neq 0$. Another analytic continuation argument and (ii) show that

(iii) $\sum_{j=0}^{n-1} S_{n-1}^{(j)}(z) S_{n-1}^{(n-1-j)}(\zeta)u = S_{n-1}(z + \zeta)u$ for $z \in W^0 = W_0{}^0 \cup \cdots \cup W_{k-1}^0$ ($S_{n-1}^{(j)}(z)$ is bounded there), $\zeta \neq 0$. For z, $\zeta \in W^0$ all operators in (iii) are continuous, thus (iii) holds as well for any $u \in E$. But this plainly implies that $S_{n-1}(\,\cdot\,)$ is holomorphic in $W_0{}^0 + \cdots + W_{n-1}^0$ which is the entire complex plane, i.e. S_{n-1} is an $L(E)$-valued entire function. By Lemma 2.1, so are $S_0, \ldots, S_{n-2}$—they can be obtained from S_{n-1} by differentiation. If $u \in D$, by Lemma 2.1 $S_{n-1}^{(n-1)}(t)u = S_0(t)u$, $t \in R_+$; then $S_{n-1}^{(2n-1)}(t)u = S_0^{(n)}(t)u = AS_0(t)u$. Letting $t \to 0$ we get $S_{n-1}^{(2n-1)}(0)u = AS_0(0)u = Au$; since A is closed $D(A) = E$ and $A = S_{n-1}^{(2n-1)}(0) \in L(E)$. It follows easily from the definitions of the S_k that their McLaurin series are given by (3.3). Since (3.2) equals $S_0(t^{1/n})$ its convergence follows from that of (3.3) for the case $k = 0$.

Assume now A is continuous and the series (3.2) converges for any $t > 0$. It is easy to see that this implies that all the series (3.3) converge uniformly on compacts of the plane and define entire functions $S_0, ..., S_{n-1}$, which are the required solutions of (3.1). As for uniqueness, let $u(\cdot)$ be a solution of (3.1) with $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$. Integrating (3.1) repeatedly we get (H_1, p. 625)

$$u(t) = \frac{1}{(nj-1)!} \int_0^t (t-s)^{nj-1} A^j u(s)\, ds, \qquad j = 1, 2, ... \tag{3.6}$$

It follows from (3.2), from the definition of the topology of $L(E)$ and from the fact that $u(s), 0 \leqslant s \leqslant t$ is bounded in E that $\lim_{j\to\infty}((nj-1)!)^{-1} (t-s)^{nj-1} A^j u(s) \to 0$ uniformly with respect to s, $0 \leqslant s \leqslant t$ and then (3.6) implies $u(t) = 0$ for all t, which ends the proof of Theorem 3.1.

3.2 *Remark.* Convergence of the series (3.2) is equivalent to convergence of $\sum t^j A^j u/(nj)!$ in E for all $u \in E$; this, in turn is equivalent to the relation $\lim_{j\to\infty}((nj)!)^{-1} | A^j u |^{1/j} = 0$ for all semi-norms $|\cdot|$ in $\mathscr{E}$. In fact, if the series $\sum t^j A^j u/(nj)!$ converges for each $u \in E$ then the limit $\lim_{n\to\infty} \sum_{j=1}^{n} z^j A^j u/(nj)!$ exists for each complex z and each $u \in E$; by Theorem 1.3 it defines a continuous operator $F(z)$. But $F(\cdot)u$ is entire for each $u \in E$, then $F(\cdot)$ is entire as an $L(E)$-valued function (Theorem 1.5) which implies convergence of (3.2) in $L(E)$.

3.3 Theorem. *Assume E is a Banach space, and let the Cauchy problem for* (3.1) *be* w.p. *in R_+. Then A is bounded.*

Proof. Consider the real-valued finite function $m(t) = \max\{| S_k(t)|, k = 0, 1, ..., n-1\}$ defined in $\bar{R}_+$. Since $\{t; | S_k(t)| > b\} = \bigcup_{u\in E} \{t; | S_k(t)u | > b| u |\}$, each of the sets on the union being open, $m(\cdot)$ is lower semicontinuous. Let $a > 0, j = 1, 2, ...$ and define

$$e_{j,a} = \{t; 0 \leqslant t \leqslant a, m(t) \leqslant j\}$$

Since each $e_{j,a}$ is closed and $e_{1,a} \cup e_{2,a} \cup \cdots = [0, a]$ we get from the Baire category theorem that some $e_{j,a}$ contains an interval $(\alpha(a), \beta(a))$, $\alpha(a) < \beta(a) \leqslant a$.

Apply now $R(\lambda; A)$ to both sides of (2.7). We get from this and from (2.8)

$$| R(\lambda; A) S_k(s+t)| \leqslant cm(s)m(t), \qquad 0 \leqslant k \leqslant n-2$$
$$| S_{n-1}(s+t)| \leqslant cm(s)m(t)$$

$s, t \in R_+$, c a constant independent of s, t. This and the preceding considerations show that $R(\lambda; A)S_k(t)$ is bounded in some interval around any point in R_+, thus is bounded on compacts of R_+. Extend $R(\lambda; A)^2 S_{n-1}$ to the complex plane by means of the formula

$$R(\lambda; A)^2 S_{n-1}(sw^k + tw^{k+1})$$
$$= w^{-(k+1)} \sum_{j=0}^{n-1} w^{-j} R(\lambda; A) S_j(s) R(\lambda; A) S_{n-1-j}(t). \tag{3.7}$$

Proceeding as in Theorem 3.1 ((iii) is now replaced by

$$\text{(iii}') \quad \sum_{j=1}^{n-1} R(\lambda; A)^2 S_{n-1}^{(j)}(z) R(\lambda; A)^2 S_{n-1}^{(n-1-j)}(\zeta) = R(\lambda; A)^4 S_{n-1}(z + \zeta))$$

we can show that $R(\lambda; A)^4 S_{n-1}$ can be extended to an $L(E)$-valued entire function.

Now let $u \in R(\lambda; A)^3 D$. It is easy to see that $A^m S_k(\cdot)u \in C_+^{(n)}(E)$, $0 \leqslant m \leqslant 3$ and that

$$S_{n-1}^{(mn)}(t)u = A^m S_{n-1}(t)u, \qquad 1 \leqslant m \leqslant 4, \qquad t \in R_+ \tag{3.8}$$

Consider the $L(E)$-valued function

$$T(\cdot) = \sum_{m=0}^{4} \binom{4}{m} \lambda^{4-m} (-R(\lambda; A)^4 S_{n-1}(\cdot))^{(mn)} \tag{3.9}$$

For $u \in R(\lambda; A)^3 D$, $t \in R_+$

$$T(t)u = R(\lambda; A)^4 \sum_{m=0}^{4} \binom{4}{m} \lambda^{4-m} (-S_{n-1}^{(mn)}(t)u)$$
$$= R(\lambda; A)^4 (\lambda I - A)^4 S_{n-1}(t)u = S_{n-1}(t)u \tag{3.10}$$

Since $R(\lambda; A)^3 D$ is dense in E, (3.10) shows that $S_{n-1}(\cdot)$ can be extended to an entire function and the proof ends now like that of Theorem 3.1.

An $L(E)$-valued function $S(\cdot)$ defined in R will be said to be *of type* $\leqslant \omega$ *in* R (ω a real number) if

$$\{e^{-\omega|t|} S(t)u, \mid t \mid \geqslant 1\}$$

is a bounded set in E for each $u \in E$. Similar definitions for R_+, $\bar{R}_+$. The Cauchy problem for (3.1) (for $n \geqslant 1$) will be of type $\leqslant \omega$ if the propagators $S_0, \ldots, S_{n-1}$ are of type $\leqslant \omega$. For the case $n \geqslant 3$ we have

3.3 THEOREM. *The Cauchy problem for* (3.1) *is* u.w.p. *and of type* $\leqslant \omega$

in R_+ for some $\omega < \infty$ if and only if $R(\lambda; A)$ exists for $|\lambda|$ large enough and is analytic at ∞.

Proof. Assume $R(\lambda; A)$ exists for $|\lambda| > a$ and is analytic at ∞. Let $R(\lambda; A) = \sum_{j=1}^{\infty} \lambda^{-j} A_j$ for $|\lambda| > a$. Then if $\rho > a$,

$$A_j = \frac{1}{2\pi i} \int_{|\lambda|=\rho} \lambda^{j-1} R(\lambda; A)\, d\lambda \tag{3.11}$$

Since $AR(\lambda; A) = -I + \lambda R(\lambda; A)$ is an $L(E)$-valued analytic function, AA_j can be computed by introducing A under the integral defining A_j; we obtain in this way

$$AA_0 = -I + A_1, \qquad AA_j = A_{j+1}, \qquad j = 1, 2,.... \tag{3.12}$$

Since $A_0 = \lim_{|\lambda|\to\infty} R(\lambda; A)$, taking $u \in D(A)$, writing $\lambda^{-1}R(\lambda; A)Au = -\lambda^{-1}u + R(\lambda; A)u$ and letting $|\lambda| \to \infty$ we see that $A_0 u = 0$ for $u \in D(A)$ and thus $A_0 = 0$. This and (3.12) show that A is continuous ($= A_2$) and that $A_j = A^{j-1}, j = 1, 2,....$ Consequently we can write

$$R(\lambda; A) = \sum_{j=0}^{\infty} \lambda^{-(j+1)} A^j, \qquad |\lambda| > a$$

Then (see 1) if $|\cdot|$ is a semi-norm in $\mathscr{L}$

$$\limsup_{j\to\infty} |A^j|^{1/j} \leqslant a$$

thus if $\omega > a$ there exists a constant K such that

$$|A^j| \leqslant K\omega^j$$

This implies that the series (3.2) converges; moreover, the series (3.3) can be estimated as follows:

$$|S_k(z)| \leqslant \sum_{j=0}^{\infty} |z|^{nj+k} |A^j|/(nj+k)!$$

$$\leqslant K \sum_{j=0}^{\infty} \omega^j |z|^j/j! = Ke^{\omega|z|}$$

This shows that the Cauchy problem for (3.1) is of type $\leqslant\omega$. Conversely, assume the Cauchy problem for (3.1) is u.w.p. and of type $\leqslant\omega$. Then it follows from Theorem 1.5 that the Laplace transform

$$R(\lambda)u = \int_0^{\infty} e^{-\lambda t} S_{n-1}(t)u\, dt \tag{3.13}$$

exists for $\operatorname{Re}\lambda > \omega$ and defines a continuous linear operator in E. It is easy to see that it is permissible to integrate by parts n times the right-hand side of (3.13) when $u \in D$; in doing this and applying Lemma 2.1 we get

$$R(\lambda) = \lambda^{-n}(I + AR(\lambda))$$

in D and thus in all of E. This implies that $R(\lambda) = R(\lambda^n; A)$. Since $\{\lambda^n; \operatorname{Re}\lambda > w\}$ is a neighborhood of ∞ if $n \geqslant 3$, we see that $R(\lambda; A)$ is holomorphic in a deleted neighborhood of ∞. But (3.13) implies that $R(\lambda)$ remains bounded—in fact, tends to zero— when $\operatorname{Re}\lambda \to \infty$, then $R(\lambda; A)$ is analytic at ∞.

3.4 *Remark.* If E is a Banach space, the hypothesis of Theorem 3.3 is always satisfied. Then the Cauchy problem for (3.1), if w.p. is always of type $\leqslant\omega$ for some $\omega < \infty$. Theorem 3.3 is closely related to Theorem 23.9.6 in [H_1]; there A is assumed to have the form V^n (V a closed operator with $\rho(V) \neq \varnothing$) and a more stringent definition of solution is used. However, solutions are assumed to exist only for certain particular choices of initial data and no continuous dependence on them is required.

3.6 *Remark.* If the Cauchy problem for (3.1) in u.w.p., then it always can be reduced to an u.w.p. first-order problem in the product space $E^n = E \times E \times \cdots \times E$ (E^n endowed with pointwise operations and the product topology). In fact, we only have to set $u_k(t) = u^{(k)}(t)$; the equation in the product space is $u_k'(t) = u_{k+1}(t)$, $0 \leqslant k \leqslant n-1$, $u_{n-1}'(t) = Au_1(t)$. See Theorem 6.9 and Remark 6.10 for a similar problem in the case $n = 2$.

4. The Case $n = 1$

Equality (2.8) reduces to

$$S_0(s + t) = S_0(s)S_0(t)$$

i.e. the propagator $S_0 = S$ is a *semigroup* of continuous operators in E (a *group* in the case of R). We shall only consider the case in which the Cauchy problem for

$$u'(t) = Au(t) \tag{4.1}$$

is of type $\leqslant\omega$ for some $\omega < \infty$ (for a study of semigroups not necessarily satisfying this condition see [M_1]). Also, we shall confine ourselves to the cases R, R_+; the case R_+ is treated (for Banach spaces) in [F_2], also in [P_1] with additional conditions.

4.1 THEOREM. *The Cauchy problem for* (4.1) *is* u.w.p. *and of type* $\leqslant\omega$ *in* $\bar{R}_+$ *(in R) if and only if A is the infinitesimal generator of a strongly continuous semigroup (group) T of type $\leqslant\omega$. If S_0 is the propagator associated with* (4.1), $S_0 = T$; *for each* $u \in D(A)$ *there exists a solution of* (4.1) *with u as initial value.*

4.2 THEOREM. *Let E be a Banach space. Assume the Cauchy problem for* (4.1) *is* w.p. *in R. Then it is* u.w.p. (*This result is false if we replace R by* $\bar{R}_+$).

Proof of Theorem 4.1. Assume A is the infinitesimal generator of a strongly continuous semigroup $T(\cdot)$ of type $\leqslant\omega$. Then $u(\cdot) = T(\cdot)u$ is a solution of (4.1) for any $u \in D(A)$. It is the only such solution. For, let $u(\cdot)$ be a solution of (4.1) with $u(0) = 0$. For $t > 0$ set

$$h(s) = T(t-s)u(s), \qquad 0 \leqslant s \leqslant t$$

It is easy to see that h is continuous in $[0, t]$, continuously differentiable in $(0, t)$ and that $h'(s) = T(t-s)Au(s) - AT(t-s)u(s) = 0$. But then $0 = u(0) = h(0) = h(t) = u(t)$, which shows that $u(\cdot) = 0$. Continuous dependence of the solutions on the initial data is evident. Conversely, assume the Cauchy problem for (4.1) is u.w.p. and of type $\leqslant\omega$ in $\bar{R}_+$, and let $S = S_0$ be the propagator associated with (4.1). Plainly S is a strongly continuous semigroup of type $\leqslant\omega$ in $\bar{R}_+$. Let B be its infinitesimal generator, $u \in D(B)$, $\{u_a\}$ a generalized sequence in D such that $u_a \to u$, $t > 0$. We have

$$\int_0^t S(s)u_a\,ds \to \int_0^t S(s)u\,ds$$

$$\begin{aligned} A\int_0^t S(s)u_a\,ds &= \int_0^t AS(s)u_a\,ds = \int_0^t S'(s)u_a\,ds \\ &= S(t)u_a - u_a = S(t)u - u \end{aligned}$$

thus $u_t = \int_0^t S(s)u\,ds \in D(A)$ and $Au_t = S(t)u - u$. But, since $t^{-1}u_t \to u$, $A(t^{-1}u_t) \to Bu$, $u \in D(A)$ and $Au = Bu$, i.e. $B \subseteq A$. Let us show in fact that $B = A$. Assume this is false; let us $u \in D(A)$, $u \notin D(B)$, $\lambda \in \rho(B)$. Since $(\lambda I - B)D(B) = E$, there exists $v \in D(B)$ such that $(\lambda I - A)v = (\lambda I - B)v = (\lambda I - A)u$, thus, if $w = u - v$, $w \neq 0$ and

$$Aw = \lambda w$$

But then $w(t) = e^{\lambda t}w$ is a solution of (4.1) with $w(0) = w \in D(B)$, which is absurd. The proof is similar for the case of a group.

Proof of Theorem 4.2. It is clear that it will follow from

4.4 LEMMA. *Let $S(\cdot)$ be a group in the Banach space E such that $t \to S(t)u$ is strongly measurable for any $u \in E$. Then $S(\cdot)$ is strongly continuous.*

For a proof of Lemma 4.4 see [D_1] or [H_1].

Operators A which are infinitesimal generators of strongly continuous semigroups in Banach space are caracterized by the Hille–Yosida–Feller–Phillips theorem (see [D_1], VIII.1.13]. This characterization has been extended by Miyadera [M_2] for semigroups of type $\leqslant\omega < \infty$ in Fréchet space. The essential feature of this proofs being the equicontinuity criterion furnished by Theorem 1.2, they extend without changes to the case of a barreled, complete locally convex LTS. We have ([M_2], Theorems 5.1 and 5.2).

4.5 Theorem. *The operator A is the infinitesimal generator of a strongly continuous semigroup (group) $S(\cdot)$ of type $\leqslant\omega$ in $R_+(R)$ if and only if* (a) *for each λ, $\lambda > \omega(|\lambda| > \omega)$, $\lambda \in \rho(A)$,* (b) *for each $u \in E$ the set*

$$\{(|\lambda| - w)^n R(\lambda; A)^n u;\ \lambda > \omega(|\lambda| > \omega),\ n = 1, 2,...\}$$

is bounded in E.

The resolvent $R(\lambda; A)$ can be obtained from $S(\cdot)$ by means of the formula

$$R(\lambda; A)u = \int_0^\infty e^{-\lambda t} S(t)u\, dt \tag{4.2}$$

valid for $\operatorname{Re}\lambda > \omega$. We shall make use later of

4.6 Lemma. *Let $S(\cdot)$ be a strongly continuous family of continuous operators of type $\leqslant\omega$ in $\bar{R}_+$, let A be an operator in E such that $\lambda \in \rho(A)$ for $\lambda > \omega$ and* (4.2) *holds. Assume $S(0) = I$. Then $S(\cdot)$ is a semigroup and A is its infinitesimal generator.*

The proof is similar to the one for the Banach space case (see [D_1], VIII).

4.7 *Remark.* In the Banach space case every semigroup is of type $\leqslant\omega$ for some $\omega < \infty$; thus any w.p. Cauchy problem is of type $\leqslant\omega$ for some ω.

5. The Case $n = 2$

Assume the Cauchy problem for

$$u''(t) = Au(t) \tag{5.1}$$

is w.p. Equalities (2.7) and (2.8) take for $n = 2$ the form

$$S_0(s + t) = S_0(s)S_0(t) + AS_1(s)S_1(t) \tag{5.2}$$

$$S_1(s + t) = S_0(s)S_1(t) + S_1(s)S_0(t) \tag{5.3}$$

5.1 LEMMA. *Let the Cauchy problem for* (5.1) *be* w.p. *in* $\bar{R}_+$ (u.w.p. *in* R_+). *Then it is* w.p. *in* R (u.w.p. *in* R).

Proof. Let the Cauchy problem for (5.1) be w.p. in $\bar{R}_+$. If $u_+(\cdot)$ is a solution of (5.1) in $\bar{R}_+$ with $u_+'(0) = 0$ it is easy to see that $u(t) = u_+(|t|)$ is a solution of (5.1) in R. On the other hand, if $v_+(\cdot)$ is a solution of (5.1) with $v_+(0) = 0$ can be extended to a solution in R by setting $v(t) = \operatorname{sgn} tv_+(|t|)$. Thus any solution $u(t) = S_0(t)u(0) + S_1(t)u'(0)$ can be extended to R and is clear that uniqueness and continuous dependence on initial data hold as well in R.

Assume now the Cauchy problem for (5.1) is u.w.p. in R_+. Let $s, t \geqslant 0$, $s + t > 0$,

$$H(r; s, t) = 2S_0(s + r)S_0(t + r) - S_0(s + t + 2r) \tag{5.4}$$

On account of the fact that $S_k(\cdot)$ is equicontinuous on compacts of R_+, $S_k(\cdot)$ is continuous in R_+ for $u \in D$ one easily sees that $H(\cdot; s, t)u$, $u \in D$ is continuous for $r \geqslant 0$, continuously differentiable for $r > 0$. We have

$$\begin{aligned}(d/dr)H(r; s, t)u &= 2AS_1(s + r)S_0(t + r)u \\ &\quad + 2AS_0(s + r)S_1(t + r)u - 2AS_1(s + t + 2r)u = 0\end{aligned}$$

(we have made use of (5.3) in the last step). This, the fact that H is symmetric with respect to s and t and the equality

$$H(r; s + h, t + h) = H(r + h; s, t)$$

imply the existence of a $L(E)$-valued function $K(r)$, $r \geqslant 0$ such that $H = K(|s - t|)$. Setting $t = 0$ we see that $K(|s|) = S_0(s)$; extending S_0 to the entire real axis by means of $S_0(t) = S_0(|t|)$, we obtain

$$S_0(s + t) + S_0(s - t) = 2S_0(s)S_0(t), \qquad s, t \in R \tag{5.5}$$

(Strictly speaking, we obtain (5.5) only for $s, t \geqslant 0$, $s + t > 0$; however, it can be readily extended for all values of s, t making use of the symmetry of S_0 and of the fact that $S_0(0) = I$.)

It is easy to show in the same way that if we extend S_1 to R by setting $S_1(t) = \operatorname{sgn} tS_1(|t|)$,

$$S_1(s + t) + S_1(s - t) = 2S_0(s)S_1(t), \qquad s, t \in R \tag{5.6}$$

Let now $u \in D$. The functions $S_0(\cdot)u$, $S_1(\cdot)u$ are solutions of (5.1) if $t \neq 0$; however, (5.5) and (5.6) allow us to express $S_0(t)u$, $S_1(t)u$ for t in the vicinity of 0 by means of their values away from the origin, and thus they are solutions in R. This takes care of existence of solutions in R, uniqueness being clear.

As for continuous dependence on initial data, let $\{u_a(t)\} = \{S_0(t)u_a(0) + S_1(t)u_a'(0)\}$ be a generalized sequence of solutions of (5.1) with $u_a(0), u_a'(0) \to 0$. $S_0(\cdot)u_a(0)$, $S_1(\cdot)u_a'(0)$ converge to zero uniformly on compacts of $(-\infty, 0) \cup (0, \infty)$; making use again of (5.5), (5.6) we see that this is also true for compacts of R.

We shall study in the sequel $L(E)$-valued functions $S(\cdot)$ satisfying (5.5) in the style of semigroup theory. Here we use some results and methods in [K_1], [K_2] where such functions are considered although with somewhat different continuity and measurability assumptions.

Through the rest of this paragraph $S(t)$, $t \in R$ will be an $L(E)$-valued function satisfying (5.5) and such that $S(0) = I$.

5.2 Lemma. *Let E be a Banach space. Assume $t \to S(t)u$ is a strongly measurable function of t for each $u \in E$. Then* (a) *$S(\cdot)$ is bounded on compacts of R.* (b) *$t \to S(t)u$ is continuous for each $u \in E$.* (c) *$t \to |S(t)|$ is measurable.*

Proof. Assume $S(\cdot)$ is not bounded in some compact of R. Proceeding like in ([D_1], VIII.1.3) we can construct a null set e_0 and a separable subspace F of E such that

(i) $S(t)F \subseteq F$ for $t \notin e_0$.

(ii) There exists a bounded sequence $t_1, t_2, \ldots$ of real numbers and a sequence $u_1, u_2, \ldots$ in F, $|u_n| = 1$ such that $|S(t_n)u_n| > n$, $n = 1, 2, \ldots$.

Define now $m(t) = \sup\{|S(t)u|, u \in F, |u| \leqslant 1\}$.

Since F is separable, the sup can be taken over a countable subset of the unit sphere of F, and then $m(\cdot)$ is measurable. It is easy to see that if s or $t \notin e_0$

$$m(s + t) \leqslant 2m(s)m(t) + m(s - t) \tag{5.7}$$

Thus (a) will follow from the auxiliary result.

(iii) Let $m(\cdot)$, $0 \leqslant m(t) < \infty$ be a measurable function in R such that $m(-t) = m(t)$ and (5.7) holds when s or t do not belong to a fixed null set e_0. Then m is bounded on compacts of R.

The proof of (iii) is an immediate generalization of 2 in K_1 (there $e_0 = \varnothing$).

Let $t_0, t, r \in R$. After some manipulations with (5.5) we get

$$\begin{aligned} S(t_0 + r) - S(t_0) = {} & 2S(t)(S(t_0 + r - t) - S(t_0 - t)) \\ & - (S(t_0 + r - 2t) - S(t_0 - 2t)) \end{aligned} \tag{5.8}$$

Let now $\alpha < \beta$. Applying both sides of (5.8) to an element $u \in E$, integrating between α and β and using the fact that $S(\cdot)$ si bounded on compacts we obtain, after simple changes of variable

$$\begin{aligned}(\beta - \alpha)|(S(t_0 + r) - S(t_0))u| &\leqslant \text{const.} \int_{t_0-\beta}^{t_0-\alpha} |(S(t + r) - S(t))u| \, dt \\ &+ (1/2) \int_{t_0-2\beta}^{t_0-2\alpha} |(S(t + r) - S(t))u| \, dt \end{aligned} \tag{5.9}$$

The right-hand side of (5.9) tends to zero with r, thus proving (b). As for (c), observe that

$$\{t; |S(t)| > a\} = \bigcup_{u \in E} \{t; |S(t)u| > a|u|\}$$

5.3 COROLLARY. *Let E be a Banach space. Assume the Cauchy problem for* (5.1) *is* w.p. *in* R_+. *Then it is* u.w.p. *in* R.

Proof. In view of Lemma 5.2, Corollary 5.3 will follow if we can show that the propagator S_0 satisfies (5.5) (it is easy to see that S_0 is strongly measurable). Proceeding like in the proof of Theorem 3.3, we see that $|R(\lambda; A)S_k(\cdot)|$ is bounded on compacts of R_+. Using now $R(\lambda; A)^2H(r)$ instead of $H(r)$ in the proof of Lemma 5.1 we obtain (5.5) multiplied by $R(\lambda; A)^2$. Since $R(\lambda; A)^2$ is one-to-one, (5.5) holds.

Return now to the case of a general E. Assume $S(\cdot)$ to be strongly continuous. The *infinitesimal generator* of $S(\cdot)$ is the linear operator defined as follows:

$$D(A) = \{u \in E;\ S(\cdot)u \in C^{(2)}(E)\}, \qquad Au = S''(0)u$$

5.4 LEMMA. (a) *$D(A)$ is dense in E and A is closed.* (b) *If $u \in D(A)$, $S(t)u \in D(A)$ and $S''(t)u = AS(t)u = S(t)Au$.*

Proof. Let $u \in E$, $b > 0$, $v_b = \int_0^b S(s)u \, ds$. It is easy to see using (5.5) that $S(\cdot)v_b \in C^{(1)}(E)$; in fact $S'(t)v_b = \frac{1}{2}(S(t + b) - S(t - b))u$. Consequently, if $u_b = \int_0^b S(s)v_b \, ds$, $u_b \in D(A)$. Since $b^{-2}u_b \to u$ as $b \to 0$, $D(A)$ is dense in E.

We obtain easily from (5.5) that

$$\begin{aligned}&\frac{1}{2h^2}(S(t + h) - 2S(t) + S(t - h))u \\ &\qquad = S(t)\frac{1}{h^2}(S(h) - I)u = \frac{1}{h^2}(S(h) - I)S(t)u\end{aligned}$$

Letting $h \to 0$ we get (b). Let $\{u_a\}$ be a generalized sequence in $D(A)$ such that $u_a \to u$, $Au_a \to v \in E$. We have

$$S''(t)u_a = S(t)Au_a, \qquad S'(t)u_a = \int_0^t S(s)Au_a\, ds$$

thus $\{S(\cdot)u_a\}$ converges uniformly on compacts of R together with its first two derivatives. This clearly shows that $S(\cdot)u = \lim S(\cdot)u_a \in C^{(2)}(E)$, $Au = S''(0)u = \lim S(0)Au_a = v$.

As in the case $n = 1$, $S(\cdot)$ need not be of type $\in \omega$ for any $\omega < \infty$. However,

5.5 Lemma. *Let E be a Banach space. Then $S(\cdot)$ is of type $\leqslant \omega$ for some $\omega < \infty$.*

Proof. Choose K, ω such that

$$S(t)| \leqslant Ke^{\omega|t|} \tag{5.10}$$

$0 \leqslant t \leqslant 1$, and

$$2|S(1)|e^{-\omega} + e^{-2\omega} \leqslant 1 \tag{5.11}$$

Assume now (5.10) holds for $t \leqslant n$. Then, making use of (5.11)

$$|S(t+1)| \leqslant 2|S(1)||S(t)| + |S(t-1)| \leqslant Ke^{\omega(t+1)}$$

thus (5.10) holds as well for $t \leqslant n + 1$. By induction, it holds for all $t \geqslant 0$; since $S(t) = S(-t)$, for all $t \in R$.

5.6 Lemma. *Assume $S(\cdot)$ has type $\leqslant \omega$, let A be its infinitesimal generator and let* $\operatorname{Re}\lambda > \omega$. *Then $\lambda^2 \in \rho(A)$ and*

$$\lambda R(\lambda^2; A)u = \int_0^\infty e^{-\lambda t}S(t)u\, dt, \tag{5.12}$$

$R(\lambda^2; A)$ is analytic in $\operatorname{Re}\lambda > \omega$.

Proof. Call $R(\lambda)$ the right-hand side of (5.12). It is easy to see that if $u \in D$, we can integrate it by parts twice; after so doing we get

$$(\lambda^2 - A)R(\lambda)u = \lambda u. \tag{5.13}$$

Since A is closed, this implies that for any $u \in E$, $R(\lambda)u \in D(A)$ and (5.13) holds. A commutes with $S(\cdot)$, thus also commutes with $R(\lambda)$. The fact that $R(\lambda^2; A)$ is analytic in $\operatorname{Re}\lambda > \omega$ can be seen by applying functionals to both sides of (5.12) and making use of Theorem 1.4.

5.7 *Remark.* Lemma 5.6 shows that $\sigma(A) \subseteq \{\lambda^2; \operatorname{Re} \lambda \leqslant \omega\} = \{\lambda; \operatorname{Re} \lambda \leqslant \omega^2 - (\operatorname{Im} \lambda)^2/4\omega^2\}$, the region to the left of a parabola passing through the points ω^2, $\pm 2i\omega^2$. In particular, if $\omega = 0$, $\sigma(A)$ is contained in the negative real axis.

5.8 Lemma. *Let A be a linear operator with domain $D(A)$ such that $\lambda \in \rho(A)$ if $\lambda \geqslant \omega$, $S(\cdot)$ a $L(E)$-valued strongly continuous function of type $\leqslant \omega$, $S(0) = I$. Assume* (5.12) *holds for $\lambda > \omega$. Then $S(\cdot)$ satisfies* (5.5) *and A is its infinitesimal generator.*

Proof. Let $u \in E$, $\lambda, \mu > \omega$. Integrating the expression

$$e^{-(\lambda s + \mu t)}(S(s + t) + S(s - t) - 2S(s)S(t))u, \tag{5.14}$$

with respect to s and t in $s, t \geqslant 0$ we get, after some changes of variable and making use of (5.12),

$$\begin{aligned} &- (\lambda - \mu)^{-1}(\lambda R(\lambda^2; A) - \mu R(\mu^2; A))u \\ &+ (\lambda + \mu)^{-1}(\lambda R(\lambda^2; A) + \mu R(\mu^2; A)u - 2\lambda\mu R(\lambda^2; A)R(\lambda^2; A)u. \end{aligned} \tag{5.15}$$

It is not difficult to see with the help of the resolvent equation that (5.15) vanishes identically. Then, by uniqueness of double Laplace transforms so does (5.14) if $s, t \geqslant 0$, a fortiori for all $s, t \in R$, i.e. $S(\cdot)$ satisfies (5.5). If B is the infinitesimal generator of $S(\cdot)$ and $\lambda > \omega$, $R(\lambda^2; A) = R(\lambda^2; B)$, thus $A = B$.

5.9. Theorem. *The Cauchy problem for* (5.1) *is* u.w.p. *in R and of type $\leqslant \omega$ if and only if A is the infinitesimal generator of an $L(E)$-valued strongly continuous function $S(\cdot)$ of type $\leqslant \omega$ satisfying* (5.5), $S(0) = I$. *If S_0, S_1 are the propagators associated with* (5.1), $S_0(t) = S(t)$, $S_1(t) = T(t)$, *where*

$$T(t)u = \int_0^t S(s)u\, ds, \tag{5.16}$$

(*if $\omega = 0$ we have to require $T(\cdot)$ itself to be of type $\leqslant \omega$*).

Proof. Assume A generates a $L(E)$-valued, strongly continuous function, $S(0) = I$, satisfying (5.5). Then (Lemma 5.4) if $u, v \in D(A)$, $u(t) = S(t)u + T(t)v$ is a solution of (5.1) in R with $u(0) = u$, $u'(0) = v$.

This takes care of existence. As for uniqueness, let $u(\cdot)$ be a solution of (5.1) with $u(0) = u'(0) = 0$. Let $t > 0$, $\lambda \in \rho(A)$ and consider the E-valued function $h(s) = R(\lambda; A)S(t - s)u'(s) + R(\lambda; A)AT(t - s)u(s)$, $0 \leqslant s \leqslant t$. It is easy to see that $h(\cdot)$ has zero derivative in $(0, t)$, is strongly continuous in $[0, t]$. Then

$$0 = h(0) = h(t) = u'(t) = 0$$

This shows that u is constant; since $u(0) = 0$ it vanishes identically. Same reasoning for $t < 0$.

Assume now the Cauchy problem for (5.1) is u.w.p. and to type $\leqslant \omega$. Then $S(\cdot) = S_0(\cdot)$ is of type $\leqslant \omega$, satisfies (5.5) and $S(0) = I$. Let

$$D_0 = \{u \in E;\ S(\cdot)u \text{ is a solution of } (5.1)\},$$

B the infinitesimal generator of $S(\cdot)$, $u \in D(B)$, $\{u_\alpha\}$ a generalized sequence in D such that $u_\alpha \to u$, $t > 0$. We have

$$T(t)^2 u_\alpha \to T(t)^2 u$$

$$AT(t)^2 u_\alpha = \tfrac{1}{2}(S(2t) - 2S(t) + I)u_\alpha + (S(t) - I)u_\alpha$$

Thus $u_t = T(t)^2 u \in D(A)$ and $Au_t = \frac{1}{2}(S(2t) - 2S(t) + I)u + (S(t) - I)u$. But $t^{-2}u_t \to u$, $A(t^{-2}u_t) \to Bu$, which shows that $u \in D(A)$, $Au = Bu$, i.e. $B \subseteq A$, $D_0 = D(B)$.

Assume now $D_0 \neq D(A)$; let $u \in D(A)$, $u \notin D_0$, $\lambda \in \rho(B)$. Since $(\lambda - B)D(B) = (\lambda - B)D_0 = E$, there exists $v \in D_0$ such that $(\lambda - B)v = (\lambda - A)v = (\lambda - A)u$, i.e. there exists $w(= u - v)$ such that $(\lambda - A)w = 0$, $w \notin D_0$. Let $w(t) = \cosh(\lambda^{\frac{1}{2}}t)w$. Plainly $w(\cdot)$ is a solution of (5.1) with $u'(0) = 0$ whose initial value does not belong D_0, absurd.

5.10 *Remark.* We obtain as a by product of the proof of Theorem 5.9 that $D_0 = D(A)$; similarly, if we define $D_1 = \{u \in E;\ T(\cdot)u$ is a solution of (5.1)$\}$, $D_1 \supseteq D(A)$. We give later more precise information about D_1.

5.11 *Remark.* Assume the Cauchy problem for (5.1) is u.w.p. and of type $\leqslant \omega < \infty$. Then the Cauchy problem for $u'(t) = Au(t)$ is as well u.w.p. in $\bar{R}_+$ and of type $\leqslant \omega^2$; more precisely, A is the infinitesimal generator of a strongly continuous semigroup $U(t)$, $t \geqslant 0$ of type $\leqslant \omega^2$ that can be analytically extended to the right half-plane. In fact, define

$$U(t)u = (\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-s^2/4t} S(s)u\, ds \tag{5.17}$$

for $u \in E$, $t > 0$. On the basis of Theorem 1.5 it is easy to see that (5.17) defines a continuous operator in E for each $t > 0$. If $u \in E$, $|\cdot| \in \mathscr{E}$ then there exists a constant K such that $|S(s)u| \leqslant Ke^{\omega s}$; using this estimate in (5.17) we get, after some manipulations

$$|U(t)u| \leqslant 2K\pi^{-\frac{1}{2}} e^{\omega^2 t} \int_{-\omega t^{\frac{1}{2}}}^\infty e^{-s^2}\, ds \leqslant 2Ke^{\omega^2 t}$$

which shows that $U(\cdot)$ is of type $\leqslant w^2$ in $\bar{R}_+$. It is plain that $U(\cdot)$ is strongly continuous for $t > 0$; as for the origin, write

$$U(t)u - u = (\pi t)^{-\frac{1}{2}} \int_0^\infty e^{-s^2/4t}(S(s) - I)u \, ds \tag{5.18}$$

Divide now the right-hand side of (5.12) in two parts, I_1 the integral from 0 to η, I_2 from η to ∞. If $|\cdot|$, K are as before, then

$$| I_2 | \leqslant 2K\pi^{-\frac{1}{2}}e^{\omega^2 t} \int_{\eta(4t)^{-\frac{1}{2}}-\omega t^{\frac{1}{2}}}^\infty e^{-s^2} \, ds$$

as for I_1 it can be made small by taking η small enough and exploiting the continuity of $S(\cdot)$ at the origin. Thus we see that (strong) $\lim_{t\to 0+} U(t) = I$.

Let now $\lambda > \omega$, $u \in E$. Making use of (5.17) and interchanging orders of integration we obtain

$$\int_0^\infty e^{-\lambda^2 t} U(t)u \, dt = \frac{1}{\lambda} \int_0^\infty e^{-\lambda s} S(s)u \, ds = R(\lambda^2; A)$$

(in the last step we have made use of Lemma 5.6). Applying now Lemma 4.6 we see that $U(\cdot)$ is a semigroup, A its infinitesimal generator. The fact that $U(\cdot)$ can be analytically extended to the right half-plane follows from the fact that the integral in the right-hand side of (5.17) converges for t in the right half-plane and differentiation under the integral sign is possible.

6. The Case $n = 2$ (continuation)

Through this paragraph, as in 5, $S(\cdot)$ (sometimes with a subindex) shall be a $L(E)$-valued, strongly continuous function satisfying (5.5), $S(0) = I$ of type $\leqslant\omega$ for some $\omega < \infty$, $T(\cdot)$ (with the same subindex) will be defined from S by means of (5.16).

6.1 Lemma. *Let A be the infinitesimal generator of S. Then*

$$A_b = A - b^2 I$$

(b any complex number) is the infinitesimal generator of a strongly continuous $L(E)$-valued function $S_b(\cdot)$ satisfying (5.5), *$S_b(0) = I$. If S has type $\leqslant\omega$, S_b has type $\leqslant\omega + | b |$.*

Proof. Define, inductively

$$S_0(t) = S(t), \qquad S_n(t)u = \int_0^t T(t - s)S_{n-1}(s)u \, ds \tag{6.1}$$

$n = 1, 2, \ldots$. Plainly, $S_0, S_1, \ldots$ are strongly continuous functions of t. Let $W^{(n)}$ be the set of all n-tuples $(e_1, e_2, \ldots, e_n)$, $e_k = \pm 1$; $W^{(n)}$ has 2^n elements. It is easy to see by induction, starting from (5.5) that if $t_0, s_1, \ldots, s_n \in R$

$$\begin{aligned} S(s_n)S(s_{n-1}) \cdots S(s_2)S(s_1)S(t_0) \\ = \frac{1}{2^n} \sum S(t_0 + e_1 s_1 + \cdots + e_n s_n) \end{aligned} \tag{6.2}$$

$(e_1, e_2, \cdots, e_n) \in W^{(n)}$. Let $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = t > 0$, $0 \leqslant s_k \leqslant t_k - t_{k-1}$. Since $S(\cdot)$ is of type $\leqslant \omega$ we have

$$| S(t)u | \leqslant K e^{\omega |t|}, \qquad t \in R$$

for any $u \in E$ and any semi-norm $| \cdot |$ in $\mathscr{E}$. Since $| t_0 + e_1 s_1 + \cdots + e_n s_n | \leqslant t$ we can estimate (6.2) as follows:

$$| S(s_n)S(s_{n-1}) \cdots S(s_2)S(s_1)S(t_0)u | \leqslant K e^{\omega t} \tag{6.3}$$

Integrating now (6.2) with respect to $s_1, \ldots, s_n$ in the n-dimensional parallelopiped $0 \leqslant s_k \leqslant t_k - t_{k-1}$, $k = 1, \ldots, n$ $(t_n = t)$ and making use of (6.3) we get

$$\begin{aligned} | T(t - t_{n-1})T(t_{n-1} - t_{n-2}) \cdots T(t_1 - t_0)S(t_0)u | \\ \leqslant K(t_n - t_{n-1})(t_{n-1} - t_{n-2}) \cdots (t_1 - t_0) e^{\omega t} \end{aligned}$$

Now, since

$$S_n(t)u = \int T(t - t_{n-1}) \cdots T(t_1 - t_0)S(t_0)u \, dt_0 \cdots dt_{n-1}$$

the integral taken on the region $0 \leqslant t_0 \leqslant t_1 \cdots \leqslant t_{n-1} \leqslant t$, it follows that

$$| S_n(t)u | \leqslant K e^{\omega t} \frac{t^{2n}}{(2n)!}$$

Consequently, the series

$$S_b(t)u = \sum_{n=0}^{\infty} (-b^2)^n S_n(t)u$$

converges uniformly on compacts of R for each $u \in E$. This plainly implies that $S_b(\cdot)$ is an $L(E)$-valued function, strongly continuous, and that $S_b(0) = I$. If $| \cdot | \in \mathscr{E}$,

$$| S_b(t)u | \leqslant K e^{\omega t} \sum_{n=0}^{\infty} \frac{| bt |^{2n}}{(2n)!} \leqslant K e^{(\omega + |b|)t}$$

which shows that S_b is of type $\leqslant \omega + | b |$.

It is easy to see by induction, using (5.12) that

$$\int_0^\infty e^{-\lambda t} S_n(t) u \, dt = \lambda R(\lambda^2; A)^{n+1} u, \qquad n = 1, 2, \ldots$$

and thus, finally

$$\int_0^\infty e^{-\lambda t} S_b(t) u \, dt = \lambda \sum_{n=0}^{\infty} (-b^2)^n R(\lambda^2; A)^{n+1} u \tag{6.4}$$

The $L(E)$-valued function $R(\,\cdot\,; A)$ is analytic in $\operatorname{Re} \lambda^{\frac{1}{2}} > w$ (Lemma 5.6). Expanding it as a power series (the derivatives of $R(\,\cdot\,; A)$ are easily computed by means of the resolvent equation) we see that the right-hand side of (6.4) equals $\lambda R(\lambda^2 + b^2; A) = \lambda R(\lambda^2; A_b)$. Applying now Lemma 5.8 we get the desired result.

Let A be the infinitesimal generator of a $S(\,\cdot\,)$ of type $\leqslant \omega$. If $|\cdot|$ is any semi-norm in $\mathscr{E}$ we easily obtain, taking into account (5.12) and the fact that $|S(t)u| \leqslant K e^{\omega t}$ for some constant K

$$\begin{aligned} |R(\lambda^2; A_b)u| &= |R(\lambda^2 + b^2; A)u| \\ &\leqslant \frac{K}{\lambda((\lambda^2 + b^2)^{1/2} - \omega)} \leqslant \frac{K'}{\lambda^2} \end{aligned} \tag{6.5}$$

for $b \geqslant \omega$, which in the Banach space case is condition (H_0) in [B_3], p. 420 for A_b. Following [B_3] we define, for $u \in D(A) = D(A_b)$, $0 < \alpha < 1$

$$J_b^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} R(\lambda; A_b)(-A_b) u \, d\lambda \tag{6.6}$$

It is plain that (6.6) converges at ∞; for λ near zero we use the fact that $R(\lambda; A_b)A_b u = \lambda R(\lambda; A_b)u - u$. The following facts about the operators J_b^α are proved just like in the Banach space case (see [B_3]) replacing the norm of the space by the family $\mathscr{E}$ of semi-norms.

(i) J_b^α is an analytic function of α for $u \in D(A)$.

(ii) Let $u \in D(A^2)$, $0 < \alpha, \beta < 1$. Then $J_b^\alpha u \in D(A)$ and $J_b^\alpha J_b^\beta = J_b^{\alpha+\beta} u$.

(iii) $\lim_{\alpha \to 1-} J_b^\alpha u = -A_b u$ for $u \in D(A)$.

(iv) Each J_b^α is closable.

6.2 Lemma. *Let $A_b^{\frac{1}{2}}$ = closure of $iJ_b^{\frac{1}{2}}$. Then $(A_b^{\frac{1}{2}})^2 = A_b$; if $\lambda^2 \in \rho(A_b)$, $\lambda \in \rho(A_b^{\frac{1}{2}})$ and $R(\lambda; A_b^{\frac{1}{2}}) = (\lambda I + A_b^{\frac{1}{2}}) R(\lambda^2; A_b)$.*

Proof. Let $u \in D(A^2)$. Then, in view of (ii) and (iii) $J_b^{\frac{1}{2}} J_b^{\frac{1}{2}} u = \lim_{\alpha \to \frac{1}{2}-} J_b^{\alpha} J_b^{\frac{1}{2}} u = \lim_{\alpha \to \frac{1}{2}-} J_b^{\alpha+\frac{1}{2}} u = -A_b u$. Let $\lambda^2 \in \rho(A_b)$. The operator

$$R(\lambda) = (\lambda I + iJ_b^{\frac{1}{2}}) R(\lambda^2; A_b) = (\lambda I + A_b^{\frac{1}{2}}) R(\lambda^2; A_b)$$

is easily seen to be continuous. ($J_b^{\frac{1}{2}} R(\lambda^2; A_b)$ can be seen to be continuous by introducing $R(\lambda^2; A_b)$ under the integral defining $J_b^{\frac{1}{2}}$). This and the fact that $(A_b^{\frac{1}{2}})^2 u = A_b u$ for $u \in D(A^2)$ show that $(\lambda I - A_b^{\frac{1}{2}}) R(\lambda) u = u = R(\lambda)(\lambda I - A_b^{\frac{1}{2}}) u$ for $u \in D(A^2)$. Using now the fact that $A_b^{\frac{1}{2}}$ is closed we see that the left-hand side of the preceding equality holds for all $u \in E$, the right hand side for all $u \in D(A)$. Thus $R(\lambda) = R(\lambda; A_b^{\frac{1}{2}})$. The fact that $A_b^{\frac{1}{2}} D(A_b) \subseteq D(A_b^{\frac{1}{2}})$ and that $A_b^{\frac{1}{2}} A_b^{\frac{1}{2}} u = A_b u$ for all $u \in D(A_b)$ follow from the easily verifiable identities

$$A_b^{\frac{1}{2}} R(\lambda^2; A_b) = R(\lambda; A_b^{\frac{1}{2}}) - \lambda R(\lambda^2; A_b),$$

$$A_b^{\frac{1}{2}} A_b^{\frac{1}{2}} R(\lambda^2; A_b) = (-I + \lambda^2 R(\lambda^2; A_b)) = A_b R(\lambda^2; A_b)$$

6.3 Lemma. *Let $b, b' \geqslant w$, $0 < \alpha < 1$. Call $\bar{J}_b^{\alpha} =$ closure of J_b^{α}. Then $D(\bar{J}_b^{\alpha}) = D(\bar{J}_{b'}^{\alpha})$ and $\bar{J}_b^{\alpha} - \bar{J}_{b'}^{\alpha}$, is a continuous operator in E.*

Proof. We can suppose that $b' > \omega$ so that $R(\lambda; A_b)$ is continuous at $\lambda = 0$. Let $u \in D(A)$. Divide the integral (6.6) defining J_b^{α} in two parts, I_b^{α} taken from 0 to 1, K_b^{α} taken from 1 to ∞. Same notation for $J_{b'}^{\alpha}$. Write

$$J_b^{\alpha} u - J_{b'}^{\alpha} u = (I_b^{\alpha} - I_{b'}^{\alpha}) u + K_b^{\alpha} u - K_{b'}^{\alpha} u$$

It follows from (6.5) and from Theorem 1.5 that K_b^{α}, $K_{b'}^{\alpha}$ define continuous operators. As for $I_b^{\alpha} - I_{b'}^{\alpha}$ it can be expressed (save for a constant factor) as an integral in (0, 1) with integrand

$$\begin{aligned} &\lambda^{\alpha-1}(R(\lambda; A_b) - R(\lambda; A_{b'}))(-A_b) \\ &\quad = \lambda^{\alpha-1}(R(\lambda + b^2; A) - R(\lambda + b'^2; A))(-A_b) \\ &\quad = \lambda^{\alpha-1}(b'^2 - b^2) R(\lambda; A_{b'})(I - \lambda R(\lambda; A_b)) \end{aligned}$$

thus it also defines a bounded operator.

We shall henceforth assume that $S(\cdot)$, A satisfy

6.4 Assumption. Let $b \geqslant \omega$. Then $T(t)E \subseteq D(A_b^{\frac{1}{2}})$ and $A_b^{\frac{1}{2}} T(t) u$ is a strongly continuous function of t.

We begin by showing that the operator A in Assumption 6.4 can be conveniently translated. In fact,

6.5 Lemma. *Let $S_b(\cdot)$, $T_b(\cdot)$ be as in Lemma 6.1. Then $T_b(t)E \subseteq D(A_b^{\frac{1}{2}})$ and $A_b^{\frac{1}{2}} T_b(t) u$ is a strongly continuous function of t.*

Proof. Consider the series

$$\sum_{n=0}^{\infty} (-b^2)^n T_n(\,\cdot\,)u \tag{6.7}$$

$$\sum_{n=0}^{\infty} (-b^2)^n P_n(\,\cdot\,)u \tag{6.8}$$

$$T_0(t) = T(t),\ T_n(t)u = \int_0^t T(t-s)T_{n-1}(s)u\,ds$$

$$P_0(t) = A_b^{\frac{1}{2}}\,T(t),\ P_n(t)u = \int_0^t T(t-s)P_{n-1}(s)u\,ds$$

$n = 1, 2, \ldots$. Proceeding much like in the proof of Lemma 6.1 we can show that (6.7) and (6.8) converge in the topology of E uniformly with respect to t on compacts of R. Since $T_n(t)u = \int_0^t S_n(s)u\,ds$, S_n given by (6.1) it is plain that (6.7) equals $T_b(t)$. Since $A_b^{\frac{1}{2}}$ commutes with $T(\,\cdot\,)$, $P_n(t) = A_b^{\frac{1}{2}}T_n(t)$. But $A_b^{\frac{1}{2}}$ is closed, thus $T_b(t)E \subseteq D(A_b^{\frac{1}{2}})$ and $A_b^{\frac{1}{2}}T_b(t)$ equals (6.8) which is a continuous function of t.

6.6 Theorem. *For each $b \geqslant \omega$, $A_b^{\frac{1}{2}}$ generates a strongly continuous group*

$$U_b(t) = S_b(t) + A_b^{\frac{1}{2}}T_b(t), \qquad t \in R$$

Iif $S(\,\cdot\,)$ is of type $\leqslant \omega$, $U_b(\,\cdot\,)$ is of type $\leqslant \omega + |\,b\,| + \eta$ for all $\eta > 0$.

Proof. The fact that U_b is a group follows easily from (5.2) and (5.3). We estimate now its type. It follows from (5.2) and (5.5) that $A_b T_b(t)^2 = S_b(2t) - S_b(t)^2 = \frac{1}{2}(S_b(2t) - I)$, thus

$$(A_b^{\frac{1}{2}}T_b(t))^{2n} = \frac{1}{2^n}\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} S_b(2t)^k \tag{6.9}$$

Using now (6.2) for $k - 1$, $t_0 = s_0 = \cdots = s_{n-1} = 2t$, we get

$$S_b(2t)^k = \frac{1}{2^{k-1}}\sum S_b(2t(1 + e_1 + \cdots + e_{k-1})) \tag{6.10}$$

$(e_1, \ldots, e_{k-1}) \in W^{(k-1)}$. Since $|\,1 + e_1 + \cdots + e_{k-1}\,| \leqslant k$, if S_b is of type $\leqslant \omega'$, $u \in E$, $|\cdot| \in \mathscr{E}$ we get easily from (6.10) and (6.9)

$$|(A_b^{\frac{1}{2}}T_b(t))^{2n}u\,| \leqslant \frac{K}{2^n}\sum_{k=0}^{n}\binom{n}{k}(e^{2\omega' t})^k \leqslant Ke^{2n\omega' t} \tag{6.11}$$

This implies that there exists K' such that

$$|(A_b^{\frac{1}{2}} T_b(t))^n u| \leqslant K' e^{n\omega' t} \tag{6.12}$$

In fact, (6.12) reduces to (6.11) for n even; if $n = 2m + 1$ we only have to write $(A_b^{\frac{1}{2}} T_b(t))^{2m+1} = (A_b^{\frac{1}{2}} T_b(t))(A_b^{\frac{1}{2}} T_b(t))^{2m}$ and use the fact that $A_b^{\frac{1}{2}} T_b(t)$, being a continuous operator, maps bounded sets into bounded sets.

Let us now estimate U_b. We have, from its definition

$$U_b(nt) = U_b(t)^n = \sum_{k=0}^{n} \binom{n}{k} S_b(t)^k (A_b^{\frac{1}{2}} T_b(t))^{n-k} \tag{6.13}$$

An application of (6.12), of the fact that S_b is of type $\leqslant \omega'$ and of Theorem 1.2 shows the existence of a constant K'' such that

$$| S_b(t)^k (A_b^{\frac{1}{2}} T_b(t))^{n-k} u | \leqslant K'' e^{k\omega' t} e^{(n-k)\omega' t} = K'' e^{n\omega' t},$$

$n = 1, 2,...$; thus, in view of (6.13)

$$| U_b(nt)u | \leqslant K''(2e^{\omega' t})^n, \qquad n = 1, 2,...$$

the constant K'' depending on u, $| \cdot |$, t. Let now $\eta > 0$. Choose t_0 so large that $2 \leqslant e^{\eta t_0}$. Write $t = nt_0 + r$, $0 \leqslant r < t_0$ for any $t \geqslant 0$. By Theorem 1.2 $\{U(r); 0 \leqslant r \leqslant t_0\}$ is an equicontinuous family in $L(E)$; then there exists K''' such that $| U_b(t)u | = | U_b(r) U_b(st_0)u | \leqslant K''' e^{(\omega' + \eta)t_0}$. Repeating the preceding argument for any $u \in E$, $| \cdot | \in \mathscr{E}$ as well as for $t < 0$ we see that U_b is of type $\leqslant \omega' + \eta$ for all $\eta > 0$.

For all $u \in E$, $\lambda > \omega'$

$$\int_0^\infty e^{-\lambda t} U_b(t)u \, dt = (\lambda I + A_b^{\frac{1}{2}}) R(\lambda^2; A_b)u = R(\lambda; A_b^{\frac{1}{2}})u,$$

which shows (Lemma 4.6) that $A_b^{\frac{1}{2}}$ generates U_b.

6.7 *Remark.* Lemma 6.3 shows that if Assumption 6.4 holds for some $b \geqslant \omega$, then it holds for *all* $b \geqslant \omega$. We do not know at present whether Assumption 6.4 is true in every case; this is the case, for instance when A is a self adjoint operator in Hilbert space. Also, if we replace $A_b^{\frac{1}{2}}$ by $J_b{}^\alpha$, $0 < \alpha < \frac{1}{2}$ we obtain a true statement, independent of b by virtue of Lemma 6.3.

Assumption 6.4 is equivalent to the following Assumption 6.8, which is cast in terms of the Cauchy problem for (5.1).

6.8 Assumption. Let $u(\cdot)$ be a solution of (5.1) with $u'(0) \in D(A_b^{\frac{1}{2}})$, $b \geqslant \omega$. Then $u'(t) \in D(A_b^{\frac{1}{2}})$ for all $t \in R$ and $A_b^{\frac{1}{2}} u'(t)$ is continuous.

In fact, assume this is true. Take $u(t) = S(t)u$, $u \in D(A)$. Since $u'(t) = T(t)Au$ it follows that $T(t)Au \in D(A_b^{\frac{1}{2}})$, $A_b^{\frac{1}{2}}T(t)Au$ is continuous for all $u \in D(A)$. If now $\lambda \in \rho(A)$ it is easy to see that the same happens replacing A by $\lambda I - A$, and since $(\lambda I - A)D(A) = E$ the result follows. On the other hand, let Assumption 6.4 hold, and let $u(t) = S(t)u(0) + T(t)u'(0)$ be a solution of (5.1) with $u(0) \in D(A)$, $u'(0) \in D(A_b^{\frac{1}{2}})$. Since $u'(t) = T(t)Au(0) + S(t)u'(0)$ and $S(t)D(A_b^{\frac{1}{2}}) \subseteq D(A_b^{\frac{1}{2}})$ ($S(t)$ commutes with $A_b^{\frac{1}{2}}$) it follows that $u'(t) \in D(A_b^{\frac{1}{2}})$ and that $A_b^{\frac{1}{2}}u'(t)$ is continuous.

When Assumption 6.4 holds we can reduce the Cauchy problem for (5.1) to a first-order Cauchy problem, u.w.p. in R in the product space $\mathfrak{E} = E \times E$. Recall that $\mathfrak{E}$, endowed with pointwise operations and the product topology is a barreled, complete locally convex LTS. We shall use in Theorem 6.9 a matrix notation for operators in $\mathfrak{E}$ (and a vector notation for elements of $\mathfrak{E}$) whose meaning is clear.

6.9 THEOREM. *Assume the Cauchy problem for* (5.1) *is* u.w.p. *and of type* $\leqslant\omega$ *and that Assumption* 6.4 *holds. Then the Cauchy problem* (*in* $\mathfrak{E}$)

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & A_b^{\frac{1}{2}} + ibI \\ A_b^{\frac{1}{2}} - ibI & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6.14}$$

($b \geqslant \omega$) *is* u.w.p. *and of type* $\leqslant \omega + 2|b| + \eta$ *for all* $\eta > 0$. *There is a* $1 - 1$ *correspondence between the solutions of* (5.1) *and those of* (6.14) *with* $u_1(0) \in D(A)$ *given by*

$$u(\cdot) \leftrightarrow \begin{bmatrix} (A_b^{\frac{1}{2}} + ibI)u(\cdot) \\ u'(\cdot) \end{bmatrix} \tag{6.15}$$

Proof. Call $\mathfrak{A}_b$ the operator in the right-hand side of (6.14); write $\mathfrak{A}_b = \mathfrak{B}_b + b\mathfrak{P}$,

$$\mathfrak{B}_b = \begin{bmatrix} 0 & A_b^{\frac{1}{2}} \\ A_b^{\frac{1}{2}} & 0 \end{bmatrix}$$

It is easy to see that if $\lambda^2 \in \rho(A_b)$

$$R(\lambda; \mathfrak{B}_b) = \begin{bmatrix} \lambda R(\lambda^2; A_b) & A_b^{\frac{1}{2}}R(\lambda^2; A_b) \\ A_b^{\frac{1}{2}}R(\lambda^2; A_b) & \lambda R(\lambda^2; A_b) \end{bmatrix} \tag{6.16}$$

Now let

$$\mathfrak{U}_b(t) = \begin{bmatrix} S_b(t) & A_b^{\frac{1}{2}}T_b(t) \\ A_b^{\frac{1}{2}}T_b(t) & S_b(t) \end{bmatrix}, \qquad t \in R$$

A simple computation based in (5.2) and (5.3) shows that $\mathfrak{U}_b(\cdot)$ is a strongly continuous group; as a consequence of Theorem 6.6 it has type $\leqslant \omega + |b| + \eta$ for all $\eta > 0$. Taking the Laplace transform of $\mathfrak{U}_b(\cdot)$ for $\lambda > \omega + |b|$, applying Lemma 5.6, Lemma 4.6 and the representation (6.16) for $R(\lambda; \mathfrak{B}_b)$, we see that $\mathfrak{U}_b$ is generated by $\mathfrak{B}_b$. By Theorem 4.5, for any $\eta > 0$ the set

$$\{(|\lambda| - \omega')^n R(\lambda; \mathfrak{B}_b)^n \mathfrak{u};\ |\lambda| > \omega',\ n = 1, 2,...\} \tag{6.17}$$

$(\omega' = \omega + |b| + \eta)$ is bounded in $\mathfrak{E}$ for each $\mathfrak{u} \in \mathfrak{E}$. Consider the series

$$\sum R(\lambda; \mathfrak{B}_b)(R(\lambda; \mathfrak{B}_b)b\mathfrak{P})^{k_1} \cdots R(\lambda; \mathfrak{B}_b)(R(\lambda; \mathfrak{B}_b)b\mathfrak{P})^{k_n}\mathfrak{u} \tag{6.18}$$

$k_1, k_2, ..., k_n = 0, 1, 2, ..., |\lambda| > \omega'$. It is easy to see that

$$R(\lambda; \mathfrak{B}_b)\mathfrak{P} = -\mathfrak{P}R(-\lambda; \mathfrak{B}_b)$$

thus each term of (6.18) can be written

$$(b\mathfrak{P})^k R(-\lambda; \mathfrak{B}_b)^p R(\lambda; \mathfrak{B}_b)^q, \tag{6.19}$$

$k = \sum k_j$, $p + q = k + n$ (incidentally, $p = \sum[(k_j + 1)/2]$ where $[r]$ = integer part of r). Observe now that the three sets $\{\mathfrak{P}^k\}$, $\{(|\lambda| - \omega')^p R(-\lambda; \mathfrak{B}_b)^p\}$, $\{(|\lambda| - \omega')^q R(\lambda; \mathfrak{B}_b)^q\}$, $k, p, q = 1, 2, ..., |\lambda| > \omega'$ are equicontinuous in $L(\mathfrak{E})$ (the last two by virtue of (6.17) and Theorem 1.2). Then so is the set $\{(|\lambda| - w')^{p+q}\mathfrak{P}^k R(-\lambda; \mathfrak{B}_b)^p R(\lambda; \mathfrak{B}_b)^q\}$, all the parameters as before. This implies that if $|\cdot|$ is a semi-norm in $\mathfrak{E}$ the series (6.18) is dominated (with respect to $|\cdot|$) by a constant times the numerical series

$$\frac{1}{(|\lambda| - \omega')^n} \sum \left[\frac{b}{|\lambda| - \omega'}\right]^{k_1 + k_2 + \cdots + k_n}$$
$$= \left[\frac{1}{(|\lambda| - \omega')} \sum \left[\frac{b}{|\lambda| - \omega'}\right]^k\right]^n = \left[\frac{1}{|\lambda| - (\omega' + b)}\right]^n$$

Thus (6.18) converges in $\mathfrak{E}$ for each $\mathfrak{u} \in \mathfrak{E}$. Now, it is easy to see by direct computation that (6.18) equals $R(\lambda; \mathfrak{B}_b + b\mathfrak{P})\mathfrak{u} = R(\lambda; \mathfrak{A}_b)\mathfrak{u}$ when $n = 1$, thus equals $R(\lambda; \mathfrak{A}_b)^n\mathfrak{u}$ when $n \geqslant 1$. Collecting our results we see that

$$\{(|\lambda| - (\omega' + b))^n R(\lambda; \mathfrak{A}_b)^n \mathfrak{u};\ |\lambda| > \omega' + |b|,\ n = 1, 2,...\}$$

is bounded in $\mathfrak{E}$ for each $\mathfrak{u} \in \mathfrak{E}$ which shows, via Theorem 4.5 that $\mathfrak{A}_b$ generates a strongly continuous group of type $\omega' + |b| = \omega + 2|b| + \eta$, in particular that the Cauchy problem for (6.14) is u.w.p. in R.

If $u(\cdot)$ is a solution of (5.1), it is clear that the image of $u(\cdot)$ by the map (6.15) is a solution of (6.14). The correspondence (6.15) will be shown to be $1 - 1$ as soon as we demonstrate the existence of a solution of (5.1) for every $u(0) \in D(A)$, $u'(0) \in D(A_b^{\frac{1}{2}})$; this in turn amounts to show that $T(\cdot)u$ is a solution of (5.1) for every $u \in D(A_b^{\frac{1}{2}})$. Clearly $T(t) \in D(A)$; in fact

$$A_b T(t) = A_b^{\frac{1}{2}} A_b^{\frac{1}{2}} T(t) u = A_b^{\frac{1}{2}} T(t) A_b^{\frac{1}{2}} u$$

Let now $v \in D$; by Lemma 2.1 we have

$$A \int_0^t T(s) v \, ds = \int_0^t A T(s) v \, ds = S(t) v \tag{6.20}$$

Using the facts that A is closed and D dense we easily see that (6.20) holds as well for u; then $T(\cdot)u \in C^{(2)}(E)$. $(T'(\cdot)u = S(\cdot)u)$, $(T(t)u)'' = AT(t)u$, which shows that $T(\cdot)u$ is a solution of (5.1). This ends the proof of Theorem 6.9.

6.10 *Remark.* It should be noted that the Cauchy problem for (5.1) can always be reduced to a first-order problem in E by means of the procedure outlined in Remark 3.6. However, this first-order problem may not be well posed.

6.11 *Remark* (See Remark 5.10). We have shown in the course of the proof of Theorem 6.9 that

$$D_1 \supseteq D(A_b^{\frac{1}{2}}), \qquad b \geqslant \omega$$

6.12 *Remark.* Theorem 6.6 shows that if A generates a $S(\cdot)$ satisfying (5.5) and Assumption 6.4 holds then

$$A = B^2 + b^2 I \tag{6.21}$$

$(B = A_b^{\frac{1}{2}})$, B the infinitesimal generator of a group $U_b(\cdot)$. Conversely, assume A admits the representation (6.21). If $U(\cdot)$ is the semigroup generated by B, then $S(t) = \frac{1}{2}(U(t) + U(-t))$ satisfies (5.5), $S(0) = I$ and it is easy to see (taking its Laplace transform) that B^2 is its infinitesimal generator. Applying Lemma 6.1 we see that $B^2 + b^2 I$ generates a $S(\cdot)$ satisfying (5.5). But it is not clear whether Assumption 6.4 holds for it.

6.13 *Remark.* It is possible to cast Assumption 6.4 in a form that does not involve explicitly the operator $A_b^{\frac{1}{2}}$. In fact, let

$$V(t) = \int_0^1 \log s(T(s+t) - T(s-t)) \, ds$$

Then Assumption 6.4 is equivalent to

6.14 Assumption. $V(t)E \in D(A)$ and $AV(t)$ is a strongly continuous function of t.

To prove the equivalence we shall express the operator $A_b^{\frac{1}{2}}$ directly on terms of $S(\cdot)$ for $b > \omega$. If we replace $R(\lambda; A)$ in the definition (6.6) of $J_b^{\frac{1}{2}}$ we get (after an easily justifiable interchange in the order of integration) that

$$J_b^{\frac{1}{2}}u = -A_b F_b u \tag{6.22}$$

for $u \in D(A)$, where

$$F_b = \int_0^\infty f(b; t) S(t)\, dt, \tag{6.23}$$

$$f(b; t) = \frac{1}{\pi} \int_0^\infty (\lambda(\lambda + b^2))^{-\frac{1}{2}} \exp(-t(\lambda + b^2)^{\frac{1}{2}})\, d\lambda = \frac{2}{\pi} K_0(bt)$$

([R_1], p. 356). Here K_0 is the McDonald or modified Bessel function of order zero; see ([R_1], p. 975) for its expression in terms of Bessel and elementary functions. We only need the following properties of K_0; there exist two entire functions g_1, g_2 such that

$$K_0(t) = g_1(t) \log t + g_2(t) \tag{6.24}$$

and K_0 and its derivatives of any order satisfy an estimate of the type

$$| K_0^{(k)}(t)| = 0(e^{-t}) \qquad \text{as} \qquad t \to \infty \tag{6.25}$$

This shows, via Theorem 1.5 that the operator F_b defined by (6.22) is continuous; since $A_b^{\frac{1}{2}} = iJ_b^{\frac{1}{2}}$, we also have

$$A_b^{\frac{1}{2}} = -iA_b F_b .$$

Using now equation (5.6) we can write

$$A_b^{\frac{1}{2}} T(t) = -\frac{2i}{\pi} A_b \int_0^\infty K_0(bt)(T(s + t) - T(s - t))\, ds,$$

and then, using the expression (6.24) for K_0,

$$\begin{aligned} -\frac{\pi}{2i} A_b^{\frac{1}{2}} T(t) &= g_1(0) A_b V(t) \\ &\quad + A_b \int_0^1 h(s)(T(s + t) - T(s - t))\, ds \\ &\quad + A_b \int_1^\infty K_0(bs)(T(s + t) - T(s - t))\, ds \\ &= A_b(g_1(0)V(t) + W_1(t) + W_2(t)), \end{aligned} \tag{6.26}$$

$$h(s) = (g_1(s) - g_1(0)) \log s + g_1(s) \log b + g_2(s)$$

Equality (6.26) makes clear that the equivalence between Assumptions 6.4 and 6.14 will be established as soon as we have shown that $W_i(t) \in D(A)$

and that $AW_i(t)$, $i = 1, 2$ are strongly continuous functions of t. In view of the representation (6.24) and the estimate (6.25) this is taken care of by

6.15 LEMMA. *Let $\eta(\cdot)$ be a (scalar-valued) function defined in (α, β), $0 \leqslant \alpha \leqslant \beta \leqslant \infty$. Assume η is continuously differentiable in (α, β) and that $|\eta(s)|e^{\omega s}$, $|\eta'(s)|e^{\omega s}$ are summable there. Then if $u \in E$,*

$$u(t) = \int_{\alpha}^{\beta} \eta(s)T(s + t)u \, ds \tag{6.27}$$

$u(t) \in D(A)$ for all t and $Au(t)$ is a continuous function of t.

Proof. Let $u \in D(A)$. Then the operator A can be introduced under the integral sign in (6.27); integrating by parts we obtain

$$Au(t) = \eta(\beta)S(\beta + t) - \eta(\alpha)S(\alpha + t) - \int_{\alpha}^{\beta} \eta'(s)S(s + t)u \, ds \tag{6.28}$$

which makes the required property evident. If $u \in E$, take $\{u_a\} \subseteq D(A)$ such that $u_a \to u$, and let $u_a(t)$ be the function that (6.27) attaches to each u_a. Since the right hand side of (6.28) depends continuously on u uniformly for t on compacts, $Au_a(\cdot) \to$ some continuous function $v(\cdot)$ uniformly on compacts. But A is closed, then $Au(t) = v(t)$. This ends the proof.

6.16 *Remark.* We have been able to prove that Assumption 6.4 holds whenever E is an L^p space, $1 < p < \infty$. This will be the subject of a forthcoming paper.

Note added in proof: The author has become aware of a paper by M. Sova (Cosine operator functions, Rozprawy Matematyczne XLIX, 1–46 (1966)). Here operator-valued functions $S(\cdot)$ satisfying $S(0) = I$ and (5.5) are considered in Banach space. There is some overlapping of results with 5 and 6 of the present paper. See also G. Da Prato-E. Giusti, Una caratterizzazione dei generatori di funzioni coseno astratte, *Bulletino Unione Matematica Italiana* **22**, 357–362 (1967).

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[2] We have not considered in this paper relations between properties of individual solutions of (1) and properties of A, as well as existence or uniqueness of isolated solutions; see for instance [A_1], [K_2], [L_3], [P_2]. See also [Y_1], where the Cauchy problem for a particular second-order equation is considered. Numerous sufficient conditions for the Cauchy problem for (1) or for more general time-dependent or nonlinear equations to be well-posed in various senses are known; see for instance [L_1], [S_2]. The Cauchy problem for certain equations in linear topological (distribution) spaces has also been considered in [S_1].

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