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# A Monge–Kantorovich mass transport problem for a discrete distance

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To the memory of Fuensanta Andreu, our friend and colleague

**Abstract**

This paper is concerned with a Monge–Kantorovich mass transport problem in which in the transport cost we replace the Euclidean distance with a discrete distance. We fix the length of a step and the distance that measures the cost of the transport depends of the number of steps that is needed to transport the involved mass from its origin to its destination. For this problem we construct special Kantorovich potentials, and optimal transport plans via a nonlocal version of the PDE formulation given by Evans and Gangbo for the classical case with the Euclidean distance. We also study how these problems, when rescaling the step distance, approximate the classical problem. In particular we obtain, taking limits in the rescaled nonlocal formulation, the PDE formulation given by Evans–Gangbo for the classical problem.


*Keywords:* Mass transport; Nonlocal problems; Monge–Kantorovich problems

## 1. Introduction and preliminaries

The Monge mass transport problem, as proposed by Monge in 1781, deals with the optimal way of moving points from one mass distribution to another so that the total work done is min-

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imized. In general, the total work is proportional to some cost function. In the classical Monge problem the cost function is the Euclidean distance, and this problem has been intensively studied and generalized in different directions that correspond to different classes of cost functions. We refer to the surveys and books [1,3,10,17,19,20] for further discussion of Monge's problem, its history, and applications.

However, even being the case of discontinuous cost functions very interesting for concrete situations and applications, it seems not to be well covered in the literature, maybe for the lack of convexity of the associated cost functions, which, nevertheless, enhance the interest of the problem. For instance, assume that you want to transport an amount of sand located somewhere to a hole at other place, then you count the number of steps that you have to move each part of sand to its final destination in the hole and try to move the total amount of sand making as less as possible steps. This amounts to the classical Monge–Kantorovich problem for the discrete distance:

$$d_1(x,y)=\begin{cases}0 & \text{if } x=y,\\ 1 & \text{if } 0<|x-y|\leqslant 1,\\ 2 & \text{if } 1<|x-y|\leqslant 2,\\ \vdots & \end{cases}$$

that count the number of steps. This transport problem also appears naturally when one considers, in a simplified way, a transport problem between cities in which the cost is measured by the toll in the road (that is a discrete function of the number of kilometers). We want to mention that our first motivation for the study of this problem comes from an interpretation of a nonlocal model for sandpiles studied in [5] (which is a nonlocal version of the sandpile model of Aronsson–Evans–Wu [6], see also [14]); in this model the height $u$ of a sandpile evolves following the equation:

$$\begin{cases} f(t,\cdot)-u_t(t,\cdot)\in\partial\mathbb{I}_{K_{d_1}(\mathbb{R}^N)}\big(u(t,\cdot)\big) & \text{a.e. } t\in(0,T),\\ u(x,0)=u_0(x), & \end{cases}$$

where $K_{d_1}(\mathbb{R}^N)$ is the set of 1-Lipschitz $L^2$-functions w.r.t. $d_1$ and $f$ is a source. The interpretation reads as follows (it is similar to the one given in [10] for the sandpile model of Aronsson–Evans–Wu with the Euclidean distance): at each moment of time, the height function $u(t,\cdot)$ of the sandpile is deemed also to be the potential generating the Monge–Kantorovich reallocation of $\mu^+=f(t,\cdot)\,dx$ to $\mu^-=u_t(t,\cdot)\,dy$ when the cost distance considered is $d_1$. In other words, the mass $\mu^+$ is instantly and optimally transported downhill by the potential $u(t,\cdot)$ into the mass $\mu^-$.

The aim of this paper is a detailed study of the mass transport problem for the discrete cost function $d_1$. It is clear that our problem falls into the scope of lower semi-continuous metric cost functions, so that standard results, like the existence of a solution for the relaxed problem, the so called Monge–Kantorovich problem, or the Kantorovich duality, stated in terms of the Kantorovich potentials, remain true for $d_1$. Nevertheless the above standard results rely on a general theory and our interest resides in giving concrete characterizations: since $d_1$ is discrete, the characterization of the potentials, the Evans–Gangbo approach [11], as well as concrete computations of optimal transport plans and/or maps are not covered in the literature; in particular, the potentials cannot be characterized in a standard way, i.e., by using standard differentiation. It is also worth to mention that, adapting an example of [16], it is easy to see that the Monge infimum and the Monge–Kantorovich minimum does not coincide in general.

We find a special class of Kantorovich potentials and perform a detailed study of the one-dimensional case with concrete examples that illustrate the obstructions to the existence of optimal transport maps; we show that the Monge problem is, in fact, ill-posed. In any dimension, we give an equation for the Kantorovich potentials, in the way of Evans–Gangbo, obtained as a limit of nonlocal $p$-Laplacian problems, and, what is quite important, we use it to construct optimal transport plans. We want to remark that all these developments can be done in the same way for the discrete distance with steps of size $\varepsilon$,

$$d_\varepsilon(x,y)=\begin{cases}0 & \text{if } x=y,\\ \varepsilon & \text{if } 0<|x-y|\leqslant\varepsilon,\\ 2\varepsilon & \text{if } \varepsilon<|x-y|\leqslant 2\varepsilon,\\ \vdots & \end{cases}$$

Then, finally, we give the connection between the Monge–Kantorovich problem with the discrete distance $d_\varepsilon$ and the classical Monge–Kantorovich problem with the Euclidean distance, proving that, when the length of the step tends to zero, these discrete/nonlocal problems give an approximation to the classical one; in particular, we recover the PDE formulation given by Evans–Gangbo in [11].

Whenever $T$ is a map from a measure space $(X,\mu)$ to an arbitrary space $Y$, we denote by $T\#\mu$ the pushforward measure of $\mu$ by $T$. Explicitly, $(T\#\mu)[B]=\mu[T^{-1}(B)]$. When we write $T\#f=g$, where $f$ and $g$ are non-negative functions, this means that the measure having density $f$ is pushed-forward to the measure having density $g$.

The general framework in which we will move is in a bounded convex domain $\Omega$ in $\mathbb{R}^N$.

**The Monge problem for the cost function $d_1$.** Take two non-negative Borel function $f^+, f^- \in L^1(\Omega)$ satisfying the mass balance condition

$$\int_\Omega f^+(x)\,dx=\int_\Omega f^-(y)\,dy. \tag{1.1}$$

Let $\mathcal{A}(f^+,f^-)$ be the set of transport maps pushing $f^+$ to $f^-$, that is, the set of Borel maps $T:\Omega\to\Omega$ such that $T\#f^+=f^-$. The Monge problem consists in finding a map $T^*\in\mathcal{A}(f^+,f^-)$ which minimizes the cost functional

$$\mathcal{F}_{d_1}(T):=\int_\Omega d_1\big(x,T(x)\big)f^+(x)\,dx$$

in the set $\mathcal{A}(f^+,f^-)$. $T^*$ is called an optimal transport map pushing $f^+$ to $f^-$.

The original problem studied by Monge corresponds to the cost function $d_{|\cdot|}(x,y):=|x-y|$ the Euclidean distance. In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, L.V. Kantorovich (1942) [15] proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle.

We will use the usual convention of denoting by $\pi_i : \mathbb{R}^N \times \mathbb{R}^N$ the projections, $\pi_1(x, y) := x$, $\pi_2(x, y) := y$. Given a Radon measure $\mu$ in $\Omega \times \Omega$, its marginals are defined by $\mathrm{proj}_x(\mu) := \pi_1 \# \mu$, $\mathrm{proj}_y(\mu) := \pi_2 \# \mu$.

**The Monge–Kantorovich relaxed problem for $d_1$.** Fix $f^+$ and $f^-$ satisfying (1.1). Let $\pi(f^+, f^-)$ the set of transport plans between $f^+$ and $f^-$, that is the set of non-negative Radon measures $\mu$ in $\Omega \times \Omega$ such that $\mathrm{proj}_x(\mu) = f^+(x)\,dx$ and $\mathrm{proj}_y(\mu) = f^-(y)\,dy$. The Monge–Kantorovich problem is to find a measure $\mu^* \in \pi(f^+, f^-)$ which minimizes the cost functional

$$\mathcal{K}_{d_1}(\mu) := \int\limits_{\Omega \times \Omega} d_1(x, y)\, d\mu(x, y),$$

in the set $\pi(f^+, f^-)$. A minimizer $\mu^*$ is called an *optimal transport plan* between $f^+$ and $f^-$. Remark that we say plans between $f^+$ and $f^-$ since this problem is reversible, which is not true in general for the Monge problem.

As a consequence of [1, Propostion 2.1], we have

$$\inf\{\mathcal{K}_{d_1}(\mu)\colon\ \mu \in \pi\left(f^+, f^-\right)\} \leqslant \inf\{\mathcal{F}_{d_1}(T)\colon\ T \in \mathcal{A}\left(f^+, f^-\right)\}.$$

On the other hand, since $d_1$ is a lower semi-continuous cost function, it is well known the existence of an optimal transport plan (see [1,16] and the references therein). Therefore we have the following result.

**Proposition 1.1.** *Let $f^+, f^- \in L^1(\Omega)$ be two non-negative Borel functions satisfying the mass balance condition* (1.1). *Then, there exists an optimal transport plan $\mu^* \in \pi(f^+, f^-)$ solving the Monge–Kantorovich problem $\mathcal{K}_{d_1}(\mu^*) = \min\{\mathcal{K}_{d_1}(\mu)\colon\ \mu \in \pi(f^+, f^-)\}$.*

**The Kantorovich dual problem for $d_1$.** Since the cost function $d_1$ is a lower semi-continuous metric, we have the following result (see for instance [19, Theorem 1.14]).

**Theorem 1.2** *(Kantorovich–Rubinstein Theorem). Let $f^+, f^- \in L^1(\Omega)$ be two non-negative Borel functions satisfying the mass balance condition* (1.1). *Then,*

$$\min\{\mathcal{K}_{d_1}(\mu)\colon\ \mu \in \pi\left(f^+, f^-\right)\} = \sup\{\mathcal{P}_{f^+, f^-}(u)\colon\ u \in K_{d_1}(\Omega)\}, \tag{1.2}$$

*where*

$$\mathcal{P}_{f^+, f^-}(u) := \int\limits_{\Omega} u(x)\left(f^+(x) - f^-(x)\right) dx,$$

*and $K_{d_1}(\Omega)$ is the set of* 1*-Lipschitz functions w.r.t. $d_1$,*

$$K_{d_1}(\Omega) := \left\{u \in L^2(\Omega)\colon\ \left|u(x) - u(y)\right| \leqslant d_1(x, y) \text{ for all } x, y \in \Omega\right\}.$$

*The maximizers $u^*$ of the right-hand side of* (1.2) *are called* Kantorovich (transport) potentials.

The Kantorovich dual problem consists in finding this Kantorovich potentials. Although it can be studied for masses being Borel measures, we will restrict ourselves to Lebesgue integrable functions in order to avoid more technicalities.

If we denote by $\mathbb{I}_{K_{d_1}(\Omega)}$ to the indicator function of $K_{d_1}(\Omega)$,

$$\mathbb{I}_{K_{d_1}(\Omega)}(u) := \begin{cases} 0 & \text{if } u \in K_{d_1}(\Omega), \\ +\infty & \text{if } u \notin K_{d_1}(\Omega), \end{cases}$$

we have that the Euler–Lagrange equation associated with the variational problem

$$\sup\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_{d_1}(\Omega)\}$$

is the equation

$$f^+ - f^- \in \partial \mathbb{I}_{K_{d_1}(\Omega)}(u). \tag{1.3}$$

That is, the Kantorovich potentials of (1.2) are solutions of (1.3).

In the particular case of the Euclidean distance $d_{|\cdot|}(x,y)$ and for adequate masses $f^+$ and $f^-$, Evans and Gangbo in [11] find a solution of the related equation (1.3) as a limit, as $p \to \infty$, of solutions to the local $p$-Laplace equation with Dirichlet boundary conditions in a sufficiently large ball $B_R(0)$:

$$\begin{cases} -\Delta_p u_p = f^+ - f^-, & B_R(0), \\ u_p = 0, & \partial B_R(0). \end{cases}$$

Moreover, they characterize the solutions to the limit equation (1.3) by means of a PDE.

**Theorem 1.3** *(Evans–Gangbo Theorem). Let $f^+, f^- \in L^1(\Omega)$ be two non-negative Borel functions satisfying the mass balance condition* (1.1)*. Assume additionally that $f^+$ and $f^-$ are Lipschitz continuous functions with compact support such that* $\operatorname{supp}(f^+) \cap \operatorname{supp}(f^-) = \emptyset$*. Then, there exists $u^* \in \operatorname{Lip}_1(\Omega, d_{|\cdot|})$ such that*

$$\int_\Omega u^*(x)\big(f^+(x) - f^-(x)\big)\,dx = \max\left\{\int_\Omega u(x)\big(f^+(x) - f^-(x)\big)\,dx\colon u \in \operatorname{Lip}_1(\Omega, d_{|\cdot|})\right\};$$

*and there exists $0 \leqslant a \in L^\infty(\Omega)$ (the transport density) such that*

$$f^+ - f^- = -\operatorname{div}\big(a \nabla u^*\big) \quad \text{in } \mathcal{D}'(\Omega). \tag{1.4}$$

*Furthermore $|\nabla u^*| = 1$ a.e. on the set $\{a > 0\}$.*

The function $a$ that appear in the previous result is the Lagrange multiplier corresponding to the constraint $|\nabla u^*| \leqslant 1$, and it is called the *transport density*. Moreover, what is very important from the point of view of mass transport, Evans and Gangbo use this PDE to find a proof of the existence of an optimal transport map for the classical Monge problem, different to the first one given by Sudakov in 1979 by means of probability methods ([18], see also [1] and [3]).

One of our main aims will be to perform such program for the discrete distance. Before starting with it, we want to remark that, as it is known (see [16]), the equality between Monge's infimum and Kantorovich's minimum is not true in general if the cost function is not continuous. The example given by Pratelli in [16] can be adapted to get a counterexample also for the case of the cost function given by the metric $d_1$.

**Example 1.4.** Consider $R$, $S$ and $T$ the parallel segments in $\mathbb{R}^2$ given by $R := \{(-1, y)\colon\ y \in [-1,1]\}$, $S := \{(0, y)\colon\ y \in [-1,1]\}$ and $Q := \{(1, y)\colon\ y \in [-1,1]\}$. Let $f^+ := 2\mathcal{H}^1 \llcorner S$ and $f^- := \mathcal{H}^1 \llcorner R + \mathcal{H}^1 \llcorner Q$. It is not difficult to see that $\min\{\mathcal{K}_{d_1}(\mu)\colon\ \mu \in \pi(f^+, f^-)\} = 2$ and the minimum is achieved by the transport plan splitting the central segment $S$ in two parts and translating them on the left and on the right. On the other hand, we claim that

$$\inf\{\mathcal{F}_{d_1}(T)\colon\ T \in \mathcal{A}(f^+, f^-)\} \geqslant 4. \tag{1.5}$$

To prove (1.5), fix $T \in \mathcal{A}(f^+, f^-)$ and consider $I(T) := \{x \in S\colon\ d_1(x, T(x)) = 1\}$. If we see that

$$f^+\big(I(T)\big) = 0, \tag{1.6}$$

then

$$\mathcal{F}_{d_1}(T) = \int_S d_1\big(x, T(x)\big)\, df^+(x) \geqslant 2 \int_{S\setminus I(T)} d\mathcal{H}^1(x) = 4,$$

and (1.5) follows. Finally, let us see that (1.6) holds. If we define

$$I(T)_R := \big\{x \in I(T)\colon\ T(x) \in R\big\} \quad \text{and} \quad I(T)_Q := \big\{x \in I(T)\colon\ T(x) \in Q\big\},$$

we have $I(T) = I(T)_R \cup I(T)_Q$ and $I(T)_R \cap I(T)_Q = \emptyset$, and by the definition of $I(T)$, if $E = T(I(T))$, it is easy to see that

$$\mathcal{H}^1(E) = \mathcal{H}^1(E \cap R) + \mathcal{H}^1(E \cap Q) = \mathcal{H}^1\big(I(T)_R\big) + \mathcal{H}^1\big(I(T)_R\big) = \mathcal{H}^1\big(I(T)\big).$$

Therefore, $f^+(I(T)) = 2f^-(E)$. But since $T \in \mathcal{A}(f^+, f^-)$ one has $f^-(E) = f^+(T^{-1}(E)) \geqslant f^+(I(T)) = 2f^-(E)$, that implies $f^+(I(T)) = 0$ and (1.6) is proved.

## 2. Kantorovich potentials

The aim of this section is the study of the Kantorovich potentials that maximize

$$\sup\big\{\mathcal{P}_{f^+,f^-}(u)\colon\ u \in K_1\big\},$$

where $K_1 := K_{d_1}(\Omega)$ for shortness.

Following ideas from [11], we first show that it is possible to construct Kantorovich potentials for the cost function $d_1$ taking limit, as $p$ goes to $\infty$, in some $p$-Laplacian problems but of nonlocal nature. Afterwards, we prove the existence of Kantorovich potentials with a finite number of jumps of size one (a specially interesting result for searching/constructing optimal transport maps and plans).

Let

$$\begin{cases} J:\mathbb{R}^N\to\mathbb{R} \text{ be a non-negative continuous radial function with} \\ \operatorname{supp}(J)=\overline{B_1(0)},\ J(0)>0 \text{ and } \displaystyle\int_{\mathbb{R}^N} J(x)\,dx=1. \end{cases} \tag{2.1}$$

We will use the following Poincaré type inequality from [4].

**Proposition 2.1.** *(See [4].) Given $p\geqslant 1$, $J$ and $\Omega$, there exists $\beta_p=\beta(J,\Omega,p)>0$ such that*

$$\beta_p\int_\Omega\left|u-\frac{1}{|\Omega|}\int_\Omega u\right|^p\leqslant\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big|u(y)-u(x)\big|^p\,dy\,dx\quad\forall u\in L^p(\Omega). \tag{2.2}$$

**Proposition 2.2.** *Let $f\in L^2(\Omega)$ and $p>2$. Then the functional*

$$F_p(u)=\frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\big|u(y)-u(x)\big|^p\,dy\,dx-\int_\Omega f(x)u(x)\,dx$$

*has a unique minimizer $u_p$ in $S_p:=\{u\in L^p(\Omega):\ \int_\Omega u(x)\,dx=0\}$.*

**Proof.** Let $u_n$ be a minimizing sequence. Hence, $F_p(u_n)\leqslant C$, that is

$$\frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\big|u_n(y)-u_n(x)\big|^p\,dy\,dx-\int_\Omega f(x)u_n(x)\,dx\leqslant C.$$

Then,

$$\frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\big|u_n(y)-u_n(x)\big|^p\,dy\,dx\leqslant\int_\Omega f(x)u_n(x)\,dx+C.$$

From the Poincaré inequality (2.2) and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\big|u_n(y)-u_n(x)\big|^p\,dy\,dx\\
&\quad\leqslant\|f\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)}+C\\
&\quad\leqslant\|f\|_{L^2(\Omega)}\left(\frac{1}{2\beta_2}\int_\Omega\int_\Omega J(x-y)\big|u(y)-u(x)\big|^2\,dy\,dx\right)^{\frac{1}{2}}+C\\
&\quad\leqslant C(f)\left(\int_\Omega\int_\Omega J(x-y)\big|u_n(y)-u_n(x)\big|^p\,dy\,dx\right)^{1/p}\left(\int_\Omega\int_\Omega J(x-y)\right)^{\frac{2-p}{2p}}+C.
\end{aligned}$$

Therefore, we have that

$$\int_{\Omega}\int_{\Omega} J(x-y)\left|u_n(y)-u_n(x)\right|^p \,dy\,dx \leqslant C.$$

Then, applying again Poincaré's inequality (2.2), we have $\{u_n\colon n \in \mathbb{N}\}$ is bounded in $L^p(\Omega)$. Hence, we can extract a subsequence that converges weakly in $L^p(\Omega)$ to some $u$ (that clearly has to verify $\int_\Omega u = 0$) and we obtain

$$\liminf_{n\to+\infty} \frac{1}{2p}\int_{\Omega}\int_{\Omega} J(x-y)\left|u_n(y)-u_n(x)\right|^p \,dy\,dx \geqslant \frac{1}{2p}\int_{\Omega}\int_{\Omega} J(x-y)\left|u(y)-u(x)\right|^p \,dy\,dx$$

and

$$\lim_{n\to+\infty}\int_{\Omega} f(x)u_n(x)\,dx = \int_{\Omega} f(x)u(x)\,dx.$$

Therefore, $u$ is a minimizer of $F_p$. Uniqueness is a direct consequence of the fact that $F_p$ is strictly convex. □

**Lemma 2.3.** *Given $u \in L^1(\Omega)$ such that*

$$E := \left\{(x,y) \in \Omega\times\Omega\colon \left|u(x)-u(y)\right| > d_1(x,y)\right\}$$

*is a null set of $\Omega\times\Omega$, there exists $\hat{u} \in K_1$ such that*

$$u = \hat{u} \quad \text{a.e. in } \Omega. \tag{2.3}$$

**Proof.** We can assume that $u$ is defined everywhere in $\Omega$ and bounded. Indeed, let $A$ be the null set in $\Omega$ such that for all $x \in \Omega \setminus A$, $E_x = \{y \in \Omega\colon (x,y) \in E\}$ is null and $u(x)$ is finite. Take $x \in \Omega \setminus A$, then, for all $y \in \Omega \setminus E_x$,

$$u(x) - d_1(x,y) \leqslant u(y) \leqslant u(x) + d_1(x,y),$$

and therefore $u(y)$ is a.e. bounded by $M := |u(x)| + \sup_{z\in\Omega} d_1(x,z)$. Take now $B$ the null set in $\Omega$ where $|u| > M$ and define $\tilde{u}(x) := u(x)$ in $\Omega \setminus B$, $\tilde{u}(x) := 0$ in $B$. Then $\tilde{u} = u$ a.e. and

$$\left|\tilde{u}(x)-\tilde{u}(y)\right| \leqslant d_1(x,y) \quad \forall (x,y) \in \Omega\times\Omega \setminus \left[E \cup (B\times\Omega) \cup (\Omega\times B)\right].$$

Let us consider

$$u_\varepsilon(x) = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u(z)\,dz,$$

where $u$ is extended by 0 to $\mathbb{R}^N \setminus \Omega$. Then, for any $x \in \Omega$, we define

$$\hat{u}(x) := \limsup_{\varepsilon\to 0} u_\varepsilon(x).$$

It is clear that $\hat{u} = u$ a.e. in $\Omega$.

Let $x, y \in \Omega$ be such that $|x-y| \neq i$ for any $i = 0, 1, 2, \ldots$. Then, there exists $i \in \mathbb{N}$ such that $i-1 < |x-y| < i$ and there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x), B_{\varepsilon_0}(y) \subset \Omega$ and

$$i-1 < |z_1 - z_2| < i, \quad \text{for any } (z_1, z_2) \in B_{\varepsilon_0}(x) \times B_{\varepsilon_0}(y).$$

This implies that, for any $0 < \varepsilon \leqslant \varepsilon_0$, we have

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(y) &= \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(z)\,dz - \frac{1}{|B_\varepsilon(y)|} \int_{B_\varepsilon(y)} u(z)\,dz \\ &= \frac{1}{|B_\varepsilon(0)|^2} \iint_{B_\varepsilon(x) \times B_\varepsilon(y)} \big(u(z_1) - u(z_2)\big)\,dz_1\,dz_2 \\ &\leqslant \frac{1}{|B_\varepsilon(0)|^2} \iint_{B_\varepsilon(x) \times B_\varepsilon(y)} d_1(z_1, z_2)\,dz_1\,dz_2 \\ &= d_1(x, y). \end{aligned}$$

Then, letting $\varepsilon \to 0$, we deduce that

$$\hat{u}(x) \leqslant d_1(x, y) + \hat{u}(y) \quad \text{for any } (x, y) \in \Omega \times \Omega,\ |x-y| \neq i,\ i = 1, 2, \ldots. \tag{2.4}$$

Now, assume that $x, y \in \Omega$, $|x-y| = i$, for some $i \in \mathbb{N}$. And let $\varepsilon_0$ be such that $B_{\varepsilon_0}(x), B_{2\varepsilon_0}(y) \subset \Omega$. Let $y_n \in \Omega$ be such that $y_n \to y$, $B_{\varepsilon_0}(y_n) \subset \Omega$ and $i-1 < |x-y_n| < i$. Using the continuity of $u_\varepsilon$ and (2.4) we see that, for any $0 < \varepsilon \leqslant \varepsilon_0$,

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(y) &= \lim_{n\to\infty} \left( \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} \hat{u}(z)\,dz - \frac{1}{|B_\varepsilon(y_n)|} \int_{B_\varepsilon(y_n)} \hat{u}(z)\,dz \right) \\ &= \lim_{n\to\infty} \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \big(\hat{u}(x+z) - \hat{u}(y_n+z)\big)\,dz \\ &\leqslant \lim_{n\to\infty} d_1(x, y_n) = i = d_1(x, y). \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain that

$$\hat{u}(x) \leqslant d_1(x, y) + \hat{u}(y).$$

The proof is finished. □

Now we show that the limit as $p$ goes to $\infty$ of the sequence $u_p$ of minimizers of $F_p$ in $S_p$ gives a Kantorovich potential.

**Theorem 2.4.** *Let $f^+, f^- \in L^2(\Omega)$ be two non-negative Borel functions satisfying the mass balance condition* (1.1). *Let $u_p$ be the minimizer in Proposition* 2.2 *for $f = f^+ - f^-$, $p > 2$.*

*Then, there exists a subsequence $\{u_{p_n}\}_{n\in\mathbb{N}}$ having as weak limit a Kantorovich potential $u$ for $f^{\pm}$ and the metric cost function $d_1$, that is,*

$$\int_\Omega u(x)\big(f^+(x)-f^-(x)\big)\,dx=\max_{v\in K_1}\int_\Omega v(x)\big(f^+(x)-f^-(x)\big)\,dx.$$

**Proof.** For $1\leqslant q$, we set

$$|||u|||_q:=\left(\int_\Omega\int_\Omega J(x-y)\big|u(y)-u(x)\big|^q\,dx\,dy\right)^{\frac{1}{q}}.$$

By Hölder's inequality, for $r\geqslant q$:

$$|||u|||_q\leqslant\left(\int_\Omega\int_\Omega J(x-y)\big|u(y)-u(x)\big|^r\,dx\,dy\right)^{\frac{1}{r}}\left(\int_\Omega\int_\Omega J(x-y)\,dx\,dy\right)^{\frac{r-q}{rq}},$$

that is, for $(r,q)$, $r\geqslant q$,

$$|||u|||_q\leqslant|||u|||_r\left(\int_\Omega\int_\Omega J(x-y)\,dx\,dy\right)^{\frac{r-q}{rq}}. \tag{2.5}$$

Since $F_p(u_p)\leqslant F_p(0)=0$ and Poincaré's inequality (2.2),

$$|||u_p|||_p^p\leqslant 2p\int_\Omega f(x)u_p(x)\,dx\leqslant 2p\|f\|_2\|u_p\|_2\leqslant\frac{2p\|f\|_2}{(2\beta_2)^{1/2}}|||u_p|||_2.$$

Then, for $2\leqslant q<p$, using (2.5) twice (for $(p,q)$ and for $(q,2)$),

$$\begin{aligned}|||u_p|||_q^p&\leqslant|||u_p|||_p^p\left(\int_\Omega\int_\Omega J(x-y)\,dx\,dy\right)^{\frac{p-q}{q}}\\&\leqslant\frac{2p\|f\|_2}{(2\beta_2)^{1/2}}|||u_p|||_2\left(\int_\Omega\int_\Omega J(x-y)\,dx\,dy\right)^{\frac{p-q}{q}}\\&\leqslant\frac{2p\|f\|_2}{(2\beta_2)^{1/2}}|||u_p|||_q\left(\int_\Omega\int_\Omega J(x-y)\,dx\,dy\right)^{\frac{p-q}{q}+\frac{q-2}{2q}}.\end{aligned}$$

Consequently,

$$|||u_p|||_q\leqslant\left(\frac{2p\|f\|_2}{(2\beta_2)^{1/2}}\right)^{\frac{1}{p-1}}\left(\int_\Omega\int_\Omega J(x-y)\,dx\,dy\right)^{\frac{1}{q}-\frac{1}{2(p-1)}}. \tag{2.6}$$

Then, $\{|||u_p|||_q\colon\ p > q\}$ is bounded. Hence, by Poincaré's inequality (2.2), we have that $\{u_p\colon\ p > q\}$ is bounded in $L^q(\Omega)$. Therefore, we can assume that $u_p \rightharpoonup u$ weakly in $L^q(\Omega)$. By a diagonal process, we have that there is a sequence $p_n \to \infty$, such that $u_{p_n} \rightharpoonup u$ weakly in $L^m(\Omega)$, as $n \to +\infty$, for all $m \in \mathbb{N}$. Thus, $u \in L^\infty(\Omega)$. Since the functional $v \mapsto |||v|||_q$ is weakly lower semi-continuous, having in mind (2.6), we have

$$|||u|||_q \leqslant \left(\int_\Omega \int_\Omega J(x-y)\,dx\,dy\right)^{\frac{1}{q}}.$$

Therefore, $\lim_{q\to+\infty} |||u|||_q \leqslant 1$, from where it follows that $|u(x) - u(y)| \leqslant d_1(x, y)$ a.e. in $\Omega \times \Omega$. Now, thanks to Lemma 2.3 we can suppose, that $u \in K_1$. Let us see that $u$ is a Kantorovich potential associated with the metric $d_1$. Fix $v \in K_1$. Then,

$$\begin{aligned}
-\int_\Omega f u_p &\leqslant \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\big|u_p(y)-u_p(x)\big|^p\,dx\,dy - \int_\Omega f(x)u_p(x)\,dx \\
&= F_p(u_p) \leqslant F_p\left(v - \frac{1}{|\Omega|}\int_\Omega v\right) \\
&= \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\big|v(y)-v(x)\big|^p\,dx\,dy - \int_\Omega f(x)v(x)\,dx \\
&\leqslant \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)\,dx\,dy - \int_\Omega f(x)v(x)\,dx,
\end{aligned}$$

where we have used $\int_\Omega f = 0$ for the second equality and the fact that $v \in K_1$ for the last inequality. Hence, taking limit as $p \to \infty$, we obtain that

$$\int_\Omega u(x)\big(f^+(x) - f^-(x)\big)\,dx \geqslant \int_\Omega v(x)\big(f^+(x) - f^-(x)\big)\,dx. \qquad \square$$

Let us now study a special class of Kantorovich potentials. We begin with the following lemma.

**Lemma 2.5.** *Assume that $v \in K_1$ takes a finite number of values. Then, there exists $u \in K_1$ that also takes a finite number of values but with jumps of length* 1*, the number of points in its image is less or equal than the number of points in the image of $v$ and improves in the maximization problem, that is,*

$$\int_\Omega u(x)\big(f^+(x) - f^-(x)\big)\,dx \geqslant \int_\Omega v(x)\big(f^+(x) - f^-(x)\big)\,dx.$$

**Proof.** The proof runs by induction in the number of nonempty level sets of $v$. Take $f := f^+ - f^-$ and suppose that $v \in K_1$ is given by, without loss of generality, $v(x) = a_0\chi_{A_0} + a_1\chi_{A_1} + \cdots + a_k\chi_{A_k}$, $a_0 = 0$, $|A_i| > 0$, $A_i \cap A_j = \emptyset$ for any $i \neq j$.

Set $s := \mathrm{Sign}(\int_{A_0} f)$, where

$$\mathrm{Sign}(r) = \begin{cases} 1 & \text{if } r \geqslant 0, \\ -1 & \text{if } r < 0, \end{cases}$$

and consider $t_0 = \max\{t \geqslant 0\colon\ u_t := (a_0 + st)\chi_{A_0} + a_1\chi_{A_1} + \cdots + a_k\chi_{A_k} \in K_1\}$. So, $t_0$ is such that $\exists i \neq 0$, $\mathrm{dist}(A_i, A_0) \leqslant 1$ and $|a_0 + st_0 - a_i| = 1$ and

$$\int_\Omega f(x)v(x)\,dx \leqslant \int_\Omega f(x)u_t(x)\,dx.$$

Hence, replacing $v$ by $u_{t_0}$, we can assume that $A_i$ are disjoint sets, $\mathrm{dist}(A_0, A_1) \leqslant 1$ and $|u_0 - u_1| = 1$.

Now, we set $s := \mathrm{Sign}(\int_{A_0 \cup A_1} f)$ and we consider

$$t_0 = \max\bigl\{t \geqslant 0;\ u_t := (a_0 + st)\chi_{A_0} + (a_1 + st)\chi_{A_1} + a_2\chi_{A_2} + \cdots + a_k\chi_{A_k} \in K_1\bigr\}.$$

So, $t_0$ is such that $\exists i \in \{0, 1\}$ and $\exists j_i \notin \{0, 1\}$ such that $\mathrm{dist}(A_i, A_{j_i}) \leqslant 1$, $|a_i + st_0 - a_{j_i}| = 1$ and

$$\int_\Omega f(x)v(x)\,dx \leqslant \int_\Omega f(x)u_t(x)\,dx.$$

Hence, replacing $v$ by $u_{t_0}$, we can assume that $A_i$ are disjoint sets and $|u_i - u_j| \in \{0, 1, 2\}$, for any $i, j \in \{0, 1, 2\}$.

Now, by induction assume that we have $u = a_0\chi_{A_0} + \cdots + a_l\chi_{A_l} + \cdots + a_k\chi_{A_k}$, where $A_i$ are disjoint sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j = 0, 1, \ldots, l$, and let us prove that we can assume that $A_i$ are disjoint compact sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j \in \{0, 1, \ldots, l + 1\}$. We set

$$s := \mathrm{Sign}\biggl(\int_{A_0 \cup \cdots \cup A_l} f\biggr),$$

and we consider

$$t_0 = \max\bigl\{t \geqslant 0;\ u_t := (a_0 + st)\chi_{A_0} + \cdots + (a_l + st)\chi_{A_l} + a_{l+1}\chi_{A_{l+1}} + \cdots + a_k\chi_{A_k} \in K_1\bigr\}.$$

So, $t_0$ is such that $\exists i \in \{0, 1, \ldots, l\}$ and $\exists j_i \notin \{0, 1, \ldots, l\}$ for which

$$\mathrm{dist}(A_i, A_{j_i}) \leqslant 1, \qquad |u_i + st_0 - u_{j_i}| = 1 \quad \text{and} \quad \int_\Omega f(x)u(x)\,dx \leqslant \int_\Omega f(x)u_t(x)\,dx.$$

Hence, replacing $u$ by $u_{t_0}$, we can assume that the sets $A_i$ are disjoint and $|a_i - a_j| \in \mathbb{N}$, for any $i, j \in \{0, 1, \ldots, l + 1\}$.

Finally, by induction, we deduce that we can assume that $A_i$ are disjoint compact sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j \in \{0, 1, \ldots, k\}$. □

Now we find the special Kantorovich potentials.

**Theorem 2.6.** *Let $f^+, f^- \in L^\infty(\Omega)$ be two non-negative Borel functions satisfying the mass balance condition* (1.1) *and such that* $\operatorname{supp}(f^+) \cap \operatorname{supp}(f^-)$ *is a null set. Then there exists a Kantorovich potential $u^*$ for $f^\pm$, associated with the metric $d_1$, such that $u^*(\Omega) \subset \mathbb{Z}$ and takes a finite number of values.*

**Proof.** Take $f := f^+ - f^-$. By density, we have that there exists a maximizing sequence $v_n \in K_1$ such that $v_n$ takes a finite number of values and

$$\int_\Omega v_n f \to \max_{w \in K_1} \int_\Omega wf.$$

Thanks to the previous lemma, there exists $u_n \in K_1$,

$$u_n = 0\chi_{C_0^n} + 1\chi_{C_1^n} + \cdots + k_n \chi_{C_{k_n}^n}, \quad k_n \in \mathbb{N} \cup \{0\},$$
$$\left|C_i^n\right| > 0, \qquad C_i^n \cap C_j^n = \emptyset, \quad \text{if } i \neq j,$$

a new maximizing sequence, that is,

$$\int_\Omega u_n f \to \max_{w \in K_1} \int_\Omega wf. \tag{2.7}$$

Notice now that the sequence $\{k_n\}$ is uniformly bounded by a constant that only depends on $\Omega$. Indeed, if $u \in K_1$ is of the form $u(x) = 0\chi_{C_0} + 1\chi_{C_1} + \cdots + k\chi_{C_k}$, with $|C_i| > 0$, $C_i \cap C_j = \emptyset$ for $i \neq j$, then $|x - y| > 1$ for every $(x, y) \in (C_{i-1} \times C_{i+1})$ for all $i$, otherwise $u \notin K_1$. Therefore, since $\Omega$ has finite diameter, this provides a bound $m_0 \in \mathbb{N}$ for the number of possible sets $k$, and consequently, $0 \leqslant k_n \leqslant m_0$ for all $n \in \mathbb{N}$.

By Fatou's Lemma and having in mind (2.7), we get

$$\max_{w \in K_1} \int_\Omega wf \leqslant \int_\Omega \limsup_{n \to \infty} (u_n f).$$

Now, since $\operatorname{supp}(f^+) \cap \operatorname{supp}(f^-)$ is a null set and having in mind that $u_n(x) \in \{0, 1, \ldots, m_0\}$ for all $n \in \mathbb{N}$, it is easy to see that

$$\limsup_{n \to \infty} (u_n f) \leqslant f^+ \limsup_{n \to \infty} u_n - f^- \liminf_{n \to \infty} u_n = f^+ \sum_{i=0}^{m_0} i\chi_{A_i} - f^- \sum_{i=0}^{m_0} i\chi_{B_i} = f \sum_{i=0}^{m_0} i\chi_{C_i},$$

where $C_i = (A_i \cap \{f^+(x) > 0\}) \cup (B_i \cap \{f^-(x) > 0\})$ for $i > 0$ and $C_0 = \Omega \setminus \bigcup_{i=0}^{m_0} C_i$.

Therefore, setting $u^* = \sum_{i=0}^{m_0} i\chi_{C_i}$, we have

$$\max_{w \in K_1} \int_\Omega wf \leqslant \int_\Omega fu^*.$$

To finish the proof let us see that $u^* \in K_1$. Take $x, y \in \Omega$. Let us suppose that

$$x \in A_i \cap \{f^+ > 0\} \quad \text{and} \quad y \in B_j \cap \{f^- > 0\}$$

(the other cases being similar), then we have

$$\left|u^*(x) - u^*(y)\right| = |i - j| \leqslant d_1(x, y).$$

If not, that is, if $|i - j| > d_1(x, y)$, assuming for instance that $i < j$, we have that there exists $0 < \epsilon < 1$ such that $i < i + \epsilon < j - \epsilon < j$ and there exists $n \in \mathbb{N}$ such that $u_n(x) \in [i, i + \epsilon]$, and $u_n(y) \in [j - \epsilon, j]$, that is, $u_n(x) = i$ and $u_n(y) = j$, which contradicts that $|u_n(x) - u_n(y)| \leqslant d_1(x, y)$. $\square$

**Remark 2.7.** Let us remark that the results we have obtained are also true if in the definition of the metric $d_1$ we change the Euclidean norm by any norm $\|\cdot\|$ of $\mathbb{R}^N$. Especially interesting is the case in which we consider the $\|\cdot\|_\infty$ norm since in this case it counts the maximum of steps moving parallel to the coordinate axes. That is, in this case we measure the distance cost as the number of blocks that the taxi has to cover going from $x$ to $y$ in a city.

**Remark 2.8.** If we assume that $u^*$ takes only the values $\{j, j+1, j+2, \dots, j+k\}$, $j \in \mathbb{Z}$, that is, $u^* = j\chi_{A_0} + (j+1)\chi_{A_1} + (j+2)\chi_{A_2} + \cdots. + (j+k)\chi_{A_k}$, then,

$$\left|A_k \cap \operatorname{supp}(f^-)\right| = 0 \quad \text{and} \quad \left|A_0 \cap \operatorname{supp}(f^+)\right| = 0. \tag{2.8}$$

In fact, if not, just redefine $u^*$ to be

$$\tilde{u}^*(x) = \begin{cases} j + k - 1 & \text{in } A_k \cap \operatorname{supp}(f_-), \\ u^*(x) & \text{otherwise,} \end{cases}$$

and we get that $\tilde{u}^* \in K_1$ with

$$\int_\Omega u^* f < \int_\Omega \tilde{u}^* f,$$

a contradiction. We also observe that

$$\int_{A_k} f^+ \geqslant \int_{A_{k-1}} f^-. \tag{2.9}$$

In fact, if not, we define

$$\tilde{u}^*(x) = \begin{cases} j + k - 1 & \text{in } A_k, \\ j + k - 2 & \text{in } A_{k-1} \cap \operatorname{supp}(f^-), \\ u^*(x) & \text{otherwise,} \end{cases}$$

and we get that $\tilde{u}^* \in K_1$ with

$$\int_\Omega u^* f < \int_\Omega \tilde{u}^* f,$$

a contradiction. Properties (2.8) and (2.9) will be of special interest in the next sections.

Let us finish this section by proving, working as in the proof of Lemma 6 in [9], the following *Dual Criteria for Optimality*.

**Lemma 2.9.**

1. *If $u^* \in K_1$ and $T^* \in \mathcal{A}(f^+, f^-)$ satisfy*

$$u^*(x) - u^*\big(T^*(x)\big) = d_1\big(x, T^*(x)\big) \quad \text{for almost all } x \in \operatorname{supp}\big(f^+\big), \tag{2.10}$$

*then*:
   (i) *$u^*$ is a Kantorovich potential for the metric $d_1$,*
   (ii) *$T^*$ is an optimal map for the Monge problem associated to the metric $d_1$,*
   (iii) $\inf\{\mathcal{F}_{d_1}(T)\colon T \in \mathcal{A}(f^+, f^-)\} = \sup\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_1\}$.
2. *Under* (iii)*, every optimal map $\hat{T}$ for the Monge problem associated to the metric $d_1$ and Kantorovich potential $\hat{u}$ for the metric $d_1$ satisfy* (2.10).

**Proof.** 1. By (2.10)

$$\begin{aligned}
\mathcal{F}_{d_1}\big(T^*\big) &= \int_\Omega d_1\big(x, T^*(x)\big) f^+(x)\,dx \\
&= \int_\Omega \big(u^*(x) - u^*\big(T^*(x)\big)\big) f^+(x)\,dx \\
&= \int_\Omega u^*(x) f^+(x)\,dx - \int_\Omega u^*(y) f^-(y)\,dy \\
&= \mathcal{P}_{f^+,f^-}\big(u^*\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{P}_{f^+,f^-}\big(u^*\big) = \mathcal{F}_{d_1}\big(T^*\big) &\geqslant \inf\big\{\mathcal{F}_{d_1}(T)\colon T \in \mathcal{A}\big(f^+, f^-\big)\big\} \\
&\geqslant \sup\big\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_1\big\} \\
&\geqslant \mathcal{P}_{f^+,f^-}\big(u^*\big),
\end{aligned}$$

and consequently (iii) holds. Moreover, we also get $\mathcal{P}(u^*) = \max\{\mathcal{P}(u)\colon u \in K_1\}$, from where it follows (i), and $\mathcal{F}_{d_1}(T^*) = \min\{\mathcal{F}_{d_1}(T)\colon T \in \mathcal{A}(f^+, f^-)\}$, from where (ii) follows.

2. Assume (iii) holds. Let $\hat{T}$ be an optimal map for the Monge problem associated to the metric $d_1$ and $\hat{u}$ a Kantorovich potential for the metric $d_1$. Then $\mathcal{F}_{d_1}(\hat{T}) = \mathcal{P}(\hat{u})$, that is,

$$\int_\Omega d_1\big(x, \hat{T}(x)\big) f^+(x)\,dx = \int_\Omega \big(\hat{u}(x) - \hat{u}\big(\hat{T}(x)\big)\big) f^+(x)\,dx.$$

Consequently, since $d_1(x, \hat{T}(x)) \geqslant \hat{u}(x) - \hat{u}(\hat{T}(x))$ and $f^+ \geqslant 0$, we have that $\hat{u}(x) - \hat{u}(\hat{T}(x)) = d_1(x, \hat{T}(x))$ for almost all $x \in \operatorname{supp}(f^+)$. □

**Remark 2.10.** Observe also that when $u^*$ is a Kantorovich potential for the metric $d_1$, from (1.2) and the inequality $u^*(x) - u^*(y) \leqslant d_1(x, y)$ it follows that, if $\mu^* \in \pi(f^+, f^-)$,

$$\mu^* \text{ is optimal} \quad \Longleftrightarrow \quad u^*(x) - u^*(y) = d_1(x, y), \quad \mu^*\text{-a.e. in } \Omega \times \Omega. \tag{2.11}$$

## 3. Constructing optimal transport plans. A nonlocal version of the Evans–Gangbo approach

As remarked in the introduction, although the general theory provides the existence of optimal transport plans, our objective is to give a concrete construction via an equation satisfied by the Kantorovich potentials following the approach of Evans–Gangbo.

We first begin with the one-dimensional case where some examples illustrate the difficulties of the mass transport problem with $d_1$.

### *3.1. The one-dimensional case*

#### *3.1.1. A better description of the special Kantorovich potentials*

We assume first that the functions $f^+$ and $f^-$ are $L^\infty$-functions satisfying

$$\begin{gathered} f^- = f^- \chi_{[a,0]}, \qquad f^+ = f^+ \chi_{[c,d]}, \quad c \geqslant 0, \\ \operatorname{supp}\big(f^\pm\big) \subset [-L, L], \quad \text{for some } L \in \mathbb{N}. \end{gathered} \tag{3.1}$$

Set $\Omega$ any interval containing $[-L, L]$.

By Theorem 2.6, there exists a Kantorovich potential $u^*$ associated with the metric $d_1$, such that $u^*(\Omega) \subset \mathbb{Z}$ and takes a finite number of values. It is easy to see that we can take

$$u^*(x) = \theta_\alpha(x) := \begin{cases} \vdots & \\ -1 & \text{if } \alpha - 2 < x \leqslant \alpha - 1, \\ 0 & \text{if } \alpha - 1 < x \leqslant \alpha, \\ 1 & \text{if } \alpha < x \leqslant \alpha + 1, \\ \vdots & \end{cases} \tag{3.2}$$

for some $0 < \alpha \leqslant 1$. In order to find which $\alpha$'s give the Kantorovich potential, we need to maximize

$$\int_\Omega u^*(x)\big(f^+(x)-f^-(x)\big)\,dx$$
$$= -\int_{-L}^{0} u^*(x) f^-(x)\,dx + \int_0^L u^*(x) f^+(x)\,dx$$
$$= -\sum_{j=-L}^{-1}\int_0^1 \big(\theta_\alpha(x)+j\big) f^-(x+j)\,dx + \sum_{j=0}^{L-1}\int_0^1 \big(\theta_\alpha(x)+j\big) f^+(x+j)\,dx$$
$$= -\sum_{j=-L}^{-1}\int_0^1 \theta_\alpha(x) f^-(x+j)\,dx + \sum_{j=0}^{L-1}\int_0^1 \theta_\alpha(x) f^+(x+j)\,dx$$
$$- \sum_{j=-L}^{-1}\int_0^1 j f^-(x+j)\,dx + \sum_{j=0}^{L-1}\int_0^1 j f^+(x+j)\,dx.$$

Since the last two integrals are independent of $\theta_\alpha$, we only need to maximize

$$-\sum_{j=-L}^{-1}\int_0^1 \big(\theta_\alpha(x)\big) f^-(x+j)\,dx + \sum_{j=0}^{L-1}\int_0^1 \big(\theta_\alpha(x)\big) f^+(x+j)\,dx$$
$$= \int_0^1 \theta_\alpha(x) M(x)\,dx = \int_\alpha^1 M(x)\,dx,$$

for $0 < \alpha \leqslant 1$, where

$$M(x) = -\sum_{j=-L}^{-1} f^-(x+j) + \sum_{j=0}^{L-1} f^+(x+j), \quad 0 < x \leqslant 1. \tag{3.3}$$

Observe that $\int_0^1 M(x)\,dx = \int (f^+ - f^-) = 0$. If $M(x)$ is monotone nondecreasing, it is clear that, for $0 < x \leqslant 1$,

$$\theta_\alpha(x) = \begin{cases} 0 & \text{if } M(x) < 0, \\ 1 & \text{if } M(x) > 0, \end{cases}$$

is the best choice (unique for points where $M(x) \neq 0$). If $M(x)$ is monotone nonincreasing, $\alpha = 1$ is the best choice.

**Remark 3.1.** Let us suppose now that the supports of the masses are not ordered. For example, let us search for a Kantorovich potential associated with the metric $d_1$ for $f^- = f_1 + f_2$, $f_1 = f_1^- \chi_{(a_1,a_2)}$, $f_2 = f_2^- \chi_{(c_1,c_2)}$, and $f^+ = f^+ \chi_{(b_1,b_2)}$, with $a_1 < a_2 < b_1 < b_2 < c_1 < c_2$. Let $b \in (b_1, b_2)$ be such that $\int f_1 = \int f \chi_{(b_1,b)}$ and $\int f_2 = \int f \chi_{(b,b_2)}$. Let us call $f_1^+ := f \chi_{(b_1,b)}$

and $f_2^+ := f\chi_{(b,b_2)}$. By the previous example we construct a monotone nondecreasing stair-shaped function, $\theta_1$, as Kantorovich potential for $f_1^+$ and $f_1^-$ with value at $b$ equals to some $\lambda$ fixed, and a monotone nonincreasing stair function, $\theta_2$, as Kantorovich potential for $f_2^+$ and $f_2^-$ with the same value $\lambda$ at $b$. Then, $\theta = \theta_1\chi_{(a_1,b)} + \theta_2\chi_{(b,c_2)}$ gives a Kantorovich potential for $f^+$ and $f^-$. This construction can be done for any configuration $f^+ = \sum_{i=1}^m \chi_{(b_{1,i},b_{2,i})}$ and $f^- = \sum_{i=1}^n \chi_{(c_{1,i},c_{2,i})}$.

*3.1.2. Nonexistence of optimal transport maps*

Here we see with a simple example that, in general, an optimal transport map does not exist for $d_1$ as cost function. Let us point out that for the Euclidean distance it is well known (see for instance [1] or [19]) the existence of an optimal transport map in the case $f^\pm \in L^1(a,b)$, even more, there exists a unique optimal transport map in the class of monotone nondecreasing functions:

$$T_0(x) := \sup\left\{ y \in \mathbb{R}\colon \int_a^y f^-(t)\,dt \leqslant \int_a^x f^+(t)\,dt \right\} \quad \text{if } x \in (a,b). \tag{3.4}$$

Let $f^+ = L\chi_{[0,1]}$ and $f^- = \chi_{[-L,0]}$ with $L \in \mathbb{R}$. Set $\Omega$ an interval containing $[-L,L]$. Let us see that if $L \in \mathbb{N}$, $L \geqslant 2$, then there is no optimal transport map $T$ with distance $d_1$ pushing $f^+$ to $f^-$, nevertheless we will see later in Example 3.4 that if $L \notin \mathbb{N}$ then there is an optimal transport map pushing $f^+$ to $f^-$.

A Kantorovich potential for this configuration of masses $f^+$ and $f^-$ is given by

$$u^*(x) = \begin{cases} 0, & x \in (0,1),\\ -1, & x \in (-1,0],\\ \vdots & \\ -L, & x \in (-L,-L+1], \end{cases}$$

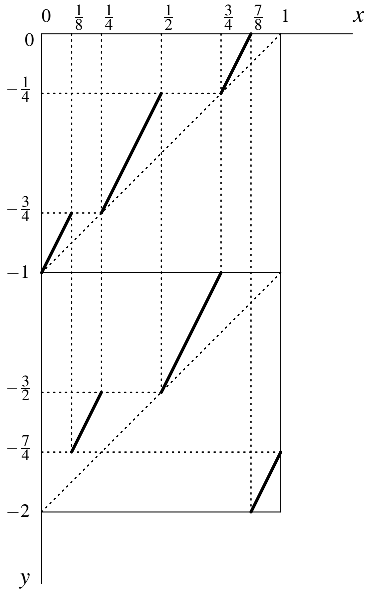
and hence we have

$$\sup\bigl\{\mathcal{P}(u)\colon u \in K_1\bigr\} = \int_\Omega u^*(x)\bigl(f^+(x) - f^-(x)\bigr)\,dx = 1 + 2 + 3 + \cdots + L = \frac{L(L+1)}{2}.$$

Let us see first that the Monge infimum and the Kantorovich minimum are the same by finding $t_n \in \mathcal{A}(f^+, f^-)$ such that

$$\mathcal{F}_{d_1}(t_n) = \int_\Omega d_1\bigl(x, t_n(x)\bigr) f^+(x)\,dx \xrightarrow{n\to 0^+} \frac{L(L+1)}{2}.$$

Consider $L = 2$ for simplicity. These $t_n$ can be constructed following the subsequent ideas. Push $f^+\chi_{[1-\frac{1}{2^{n+1}},1]}$ to $f^-\chi_{[-2,-2+\frac{1}{2^n}]}$ with a plan induced by a map as in the picture below, paying $\frac{3}{2^n}$, and $f^+\chi_{[0,1-\frac{1}{2^{n+1}}]}$ to $f^-\chi_{[-2+\frac{1}{2^n},0]}$ with a plan induced also by a map, see below, paying $3 - \frac{2}{2^n}$.




Support of $2\chi_{[0,1]}(x)\delta_{[y=t_2(x)]}$.
Observe that all the segments have slope 2.

In this way,

$$\mathcal{F}_{d_1}(t_n) = \int_{\Omega} d_1\big(x, t_n(x)\big) f^+(x)\, dx = 3 + \frac{1}{2^n} \xrightarrow{n \to 0^+} 3.$$

Arguing by contradiction assume now that there is an optimal transport map $T$ pushing $f^+$ to $f^-$. Then, since $\inf\{\mathcal{F}_{d_1}(T)\colon T \in \mathcal{A}(f^+, f^-)\} = \sup\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_1\}$, from Lemma 2.9 we have the equality $u^*(x) - u^*(T(x)) = d_1(x, T(x))$. Then,

$$A_i := \big\{x \in ]0,1[\colon d_1\big(x, T(x)\big) = i\big\} = T^{-1}\big((-i, -i+1]\big), \quad i = 1, \ldots, L.$$

Therefore, $|A_i| = |T^{-1}((-i, -i+1])| = 1/L$. Moreover, we also have $T(x) \geqslant x - i$ for all $x \in A_i$. Now, we claim that

$$T(x) = x - i \quad \text{for all } x \in A_i, \text{ for every } i = 1, \ldots, L. \tag{3.5}$$

Hence, $|T(A_i)| = 1/L$ which gives a contradiction with the fact that $|T([0,1])| = L$.

To prove (3.5) we argue as follows: assume, without lose of generality, that there is a set of positive measure $K \subset A_1$ such that $T(x) > x - 1$ in $K$. Then, it is easy to see that there exists $\theta \in (0,1)$ such that $|T^{-1}((-1, \theta - 1))| < |A_1 \cap (0, \theta)|$. Therefore, since $T^{-1}((-i, \theta - i)) \subset A_i \cap (0, \theta)$ for all $i$, we have

$$\theta = \frac{1}{L}\left|\bigcup_{i=1}^{L}(-i,\theta - i)\right| = \left|T^{-1}\left(\bigcup_{i=1}^{L}(-i,\theta - i)\right)\right|$$
$$= \left|\bigcup_{i=1}^{L} T^{-1}\big((-i,\theta - i)\big)\right| < \bigcup_{i=1}^{L}\big|A_i \cap\ (0,\theta)\big| = \theta,$$

and we arrive to a contradiction.

With a similar proof it can be proved that there is no transport map $T$ between $f^+ = L\chi_{[0,1]}$ and $f^- = \chi_{[-L,0]}$ with $L \in \mathbb{N}$ if one considers the distance $d_{1/k}$ with $k \in \mathbb{N}$.

**Remark 3.2.** Observe that it is easy to construct an optimal transport plan $\mu^* \in \pi(f^+, f^-)$ solving the Monge–Kantorovich problem. Indeed, if define the measure $\mu^*$ in $\Omega \times \Omega$ by

$$\mu^*(x, y) := L\chi_{[0,1]}(x)\left(\frac{1}{L}\delta_{[y=-1+x]} + \frac{1}{L}\delta_{[y=-2+x]} + \cdots + \frac{1}{L}\delta_{[y=-L+x]}\right),$$

then $\mu^* \in \pi(f^+, f^-)$ and, moreover, since

$$\begin{aligned}
\mathcal{K}_{d_1}\big(\mu^*\big) &= \int_{\Omega\times\Omega} d_1(x, y)\, d\mu^*(x, y) \\
&= L\int_0^1 \left(\frac{1}{L}d_1(x, -1 + x) + \frac{1}{L}d_1(x, -2 + x) + \cdots + \frac{1}{L}d_1(x, -L + x)\right) dx \\
&= \frac{L(L+1)}{2} \\
&= \sup\big\{\mathcal{P}(u):\ u \in K_1\big\} \\
&= \min\big\{\mathcal{K}_1(\mu):\ \mu \in \pi\big(f^+, f^-\big)\big\},
\end{aligned}$$

we have that $\mu^*$ is an optimal plan.

### *3.1.3. A precise construction of optimal transport plans*

Let us now see that in one dimension we can give, in a quite easy way, a construction of optimal transport plans by using the special Kantorovich potentials obtained in Section 3.1.1. This is independent of the general construction given afterward.

We will construct an optimal transport plan under the assumptions (3.1); Remark 3.1 says how to work in a more general situation. Let $u^* = \theta_\alpha$ be the Kantorovich potential given from (3.2) and construct a new configuration of equal masses as follows:

$$f_0^+(x) = \left(\sum_{j=0}^{L-1} f^+(x + j)\right)\chi_{]0,1[}(x), \qquad f_0^-(x) = \left(\sum_{j=0}^{L-1} f^-(x - j)\right)\chi_{]-1,0[}(x).$$

For these masses, the same $u^*$ is a Kantorovich potential. Moreover,

$$\int_{-L}^{L} u^*(x)\big(f^+(x) - f^-(x)\big)\,dx$$
$$= \int_{-1}^{1} u^*(x)\big(f_0^+(x) - f_0^-(x)\big)\,dx + \sum_{j=0}^{L-1}\int_0^1 jf^+(x+j)\,dx + \sum_{j=0}^{L-1}\int_{-1}^{0} jf^-(x-j)\,dx.$$

By (2.9) there exists $\beta \in [\alpha, 1]$ such that

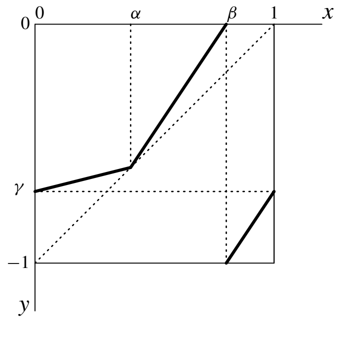
$$\int_\alpha^\beta f_0^+ = \int_{-1+\alpha}^{0} f_0^-.$$

Consider the smallest of such $\beta$. Take also the smallest $\gamma \in [-1, -1+\alpha]$ such that

$$\int_\beta^1 f_0^+ = \int_{-1}^{\gamma} f_0^-.$$

For $x \in (0, 1)$, we define $T_0$ by

$$T_0(x) = \begin{cases} \sup\{y \in \mathbb{R} : \int_{-1+\alpha}^{y} f_0^- = \int_\alpha^x f_0^+\} & \text{if } x \in (\alpha, \beta), \\ \sup\{y \in \mathbb{R} : \int_{-1}^{y} f_0^- = \int_\beta^x f_0^+\} & \text{if } x \in (\beta, 1), \\ \sup\{y \in \mathbb{R} : \int_\gamma^{y} f_0^- = \int_0^x f_0^+\} & \text{if } x \in (0, \alpha). \end{cases}$$




The straight lines are only illustrative.

It is easy to see that $T_0 \in \mathcal{A}(f^+, f^-)$ and that

$$d_1\big(x, T_0(x)\big) = u^*(x) - u^*\big(T_0(x)\big) \quad \text{a.e. } x \in \operatorname{supp}\big(f^+\big).$$

Then, by Lemma 2.9 (or a direct computation), $\mu_{00}(x,y) = f_0^+(x)\delta_{[y=T_0(x)]}$ is an optimal transport plan between $f_0^+$ and $f_0^-$ for the cost function $d_1$.

Once we have the above construction, it is also easy to see that

$$\mu_0(x,y) = \sum_{j=0}^{L-1} f^+(x)\chi_{(j,j+1)}(x)\delta_{[y=T_0(x-j)]}$$

is an optimal transport plan between $f^+$ and $f_0^-$ for the cost function $d_1$. A remarkable observation is that these $\mu_{00}$ and $\mu_0$ are induced by transport maps and that for the above configurations the Monge infimum and the Monge–Kantorovich minimum coincide.

By splitting the mass

$$f^+(x)\chi_{(j,j+1)}(x) = \sum_{i=0}^{L-1} g_{i,j}(x), \quad j = 0,1,\dots,L-1, \tag{3.6}$$

is such a way that, for $i = 0,1,\dots,L-1$,

$$\sum_{j=0}^{L-1} \int_j^{x+j} g_{i,j} = \int_{\gamma-i}^{T_0(x)-i} f^- \quad \text{if } x \in (0,\beta), \tag{3.7}$$

and

$$\sum_{j=0}^{L-1} \int_{\beta+j}^{x+j} g_{i,j} = \int_{-1-i}^{T_0(x)-i} f^- \quad \text{if } x \in (\beta,1), \tag{3.8}$$

we can finally see that

$$\mu(x,y) = \sum_{i=0}^{L-1}\sum_{j=0}^{L-1} g_{i,j}(x)\chi_{(j,j+1)}(x)\delta_{[y=-i+T_0(x-j)]}$$

is a transport plan between $f^+$ and $f^-$ for the cost function $d_1$: taking $x=\beta$ in (3.7), and $x=1$ in (3.8), respectively, we get

$$\sum_{j=0}^{L-1} \int_j^{\beta+j} g_{i,j} = \int_{\gamma-i}^{-i} f^- \quad \text{and} \quad \sum_{j=0}^{L-1} \int_{\beta+j}^{1+j} g_{i,j} = \int_{-1-i}^{\gamma-i} f^-.$$

Adding the last two equalities, we obtain

$$\sum_{j=0}^{L-1} \int_j^{1+j} g_{i,j}(x)\,dx = \int_{-1-i}^{-i} f^-(x)\,dx = \int_{-1}^{0} f^-(x-i)\,dx.$$

Hence,

$$\int_{-L}^{L} u^*(f^+ - f^-) = \int\int d_1(x,y)\mu_0(x,y) + \sum_{j=0}^{L-1}\int_{-1}^{0} j\, f^-(x-j)\,dx$$

$$= \sum_{j=0}^{L-1}\int_{j}^{j+1} d_1\big(x, T_0(x-j)\big) f^+(x) + \sum_{i=0}^{L-1} i \int_{-1}^{0} f^-(x-i)\,dx$$

$$= \sum_{j=0}^{L-1}\int_{j}^{j+1} d_1\big(x, T_0(x-j)\big)\left(\sum_{i=0}^{L-1} g_{i,j}(x)\right)dx + \sum_{i=0}^{L-1} i \sum_{j=0}^{L-1}\int_{j}^{j+1} g_{i,j}(x)\,dx$$

$$= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\int_{j}^{j+1} \big(d_1\big(x, T_0(x-j)\big) + i\big) g_{i,j}(x)\,dx$$

$$= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\int_{j}^{j+1} d_1\big(x, -i + T_0(x-j)\big) g_{i,j}(x)\,dx$$

$$= \int_{\Omega\times\Omega} d_1(x,y)\mu(x,y).$$

In the following example, $\mu(x,y) = f^+(x)\delta_{[y=T_1^*(x)]}$ illustrates the above construction.

**Example 3.3.** Set $f^- = \frac{1}{4}\chi_{]-1,0[}$ and $f^+ = \chi_{]\frac{7}{4},2[}$. Then $M = -\frac{1}{4}\chi_{]0,\frac{3}{4}[} + \frac{3}{4}\chi_{]\frac{3}{4},1[}$ and therefore $u^*(x) = \theta_{\frac{3}{4}}$ is (up to adding a constant) the unique Kantorovich potential associated with the metric $d_1$ for $f^+$ and $f^-$, moreover, $\int u^*(f^+ - f^-) = \frac{11}{16}$. Nevertheless, there exist infinitely many optimal transport maps. For example, the following two are optimal transport maps,

$$T_1^*(x) = \begin{cases} 4x - \frac{29}{4} & \text{if } \frac{28}{16} < x < \frac{29}{16}, \\ 4x - \frac{33}{4} & \text{if } \frac{29}{16} < x < 2, \\ x & \text{otherwise,} \end{cases} \qquad T_2^*(x) = \begin{cases} 4x - \frac{29}{4} & \text{if } \frac{28}{16} < x < \frac{57}{32}, \\ -4x + \frac{57}{8} & \text{if } \frac{57}{32} < x < \frac{29}{16}, \\ -4x + 7 & \text{if } \frac{29}{16} < x < 2, \\ x & \text{otherwise.} \end{cases}$$

Observe that both push the mass $f^+\chi_{]\frac{7}{4},\frac{29}{16}[}$ toward $f^-\chi_{]-\frac{1}{4},0[}$ paying, after 2 steps, $2 \times \frac{1}{16}$, and push the rest from $f^+\chi_{]\frac{29}{16},2[}$ toward $f^-\chi_{]-1,-\frac{1}{4}[}$ paying, after 3 steps, $3 \times \frac{3}{16}$. Therefore the total cost is, as known, $2 \times \frac{1}{16} + 3 \times \frac{3}{16} = \frac{11}{16}$.

We want to remark that the unique monotone nondecreasing optimal transport map, $T_0$, for the Euclidean distance as cost function that pushes $f^+$ forward to $f^-$ in this particular case is $T_0(x) = 4x - 8$. Now, $T_0$ is not an optimal transport map for $d_1$, the transport cost with this map is, in fact, $\frac{12}{16}$. However, it is well known (see [3]) that if the cost function $c(x,y)$ is equal to $\phi(|x-y|)$ with $\phi$ monotone nondecreasing and convex then $T_0$ is an optimal transport, but in our situation $\phi$ fails to be convex. On the other hand, the following simple transport plan between $f^+$ and $f^-$, not induced by a map, is optimal: $\mu = \chi_{(\frac{7}{4},2)}(x)(\frac{1}{4}\delta_{[y=x-2]} + \frac{1}{4}\delta_{[y=x-\frac{9}{4}]} + \frac{1}{4}\delta_{[y=x-\frac{10}{4}]} + \frac{1}{4}\delta_{[y=x-\frac{11}{4}]})$.

In contrast with the example given in Section 3.1.2 for which there is not optimal transport map we present the following one.

**Example 3.4.** Let $f^+ = L\chi_{[0,1]}$ and $f^- = \chi_{[-L,0]}$ with $L \notin \mathbb{N}$. Let us see that there is an optimal transport map $T$ pushing $f^+$ to $f^-$ for $d_1$. In order to simplify the exposition we take $2 < L < 3$. This particular case shows clearly how to handle the general case.

Using the procedure introduced in this subsection we have that

$$T_0(x) = \begin{cases} \frac{L}{2}x - 1 & \text{if } 0 < x < \frac{2(3-L)}{L}, \\ \frac{L}{3}(x-1) & \text{if } \frac{2(3-L)}{L} < x < 1, \end{cases}$$

is an optimal transport map pushing $f_0^+$ to $f_0^-$ ($\alpha = 1 = \beta$ and $\gamma = -1$). Now, we perform the splitting procedure (3.6) (there are many different ways) in the following adequate way. For $x < \frac{2(3-L)}{L}$ we have to distribute the mass $f^+$ in two *equiweighted* parts, so, set the rectangles with corner coordinates,

$$\text{upper-left,}\quad ul_i = (x_{i+1}, y_i), \qquad \text{upper-right,}\quad ur_i = (x_i, y_i),$$
$$\text{lower-left,}\quad ll_i = (x_{i+1}, y_{i+1}), \qquad \text{lower-right,}\quad lr_i = (x_i, y_{i+1}),$$

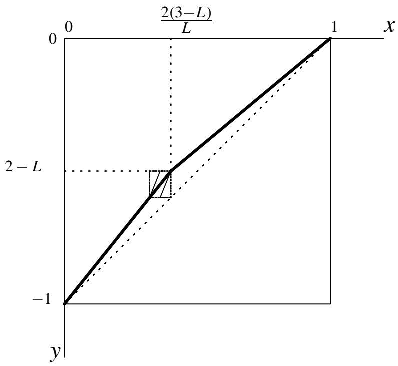
$i = 1, 2, \ldots,$ where

$$x_1 = \frac{2(3-L)}{L}, \qquad y_1 = 2 - L,$$
$$y_{i+1} = x_i - 1, \qquad x_{i+1} = x_i - \frac{2}{L}(y_i - y_{i+1}) = \frac{2}{L}(y_{i+1} + 1)$$

(observe that $lr_i \in [y = x - 1]$ and $ll_i, ur_i \in [y = \frac{L}{2}x - 1]$); in each rectangle we can trace 2 parallel segments of slope $L$ defined by the lines

$$y = L(x - x_i) + y_i \quad \text{and} \quad y = L(x - \hat{x}_i) + y_i, \quad \text{with } \hat{x}_i = x_i - \frac{x_i - x_{i+1}}{2};$$

then $T_i(x) = f^+(x)\chi_{]\hat{x}_i,x_i[}(x)\delta_{[y=L(x-x_i)+y_i]} + f^+(x)\chi_{]x_{i+1},\hat{x}_i[}(x)\delta_{[y=L(x-\hat{x}_i)+y_i-1]}$ push in an optimal way $f^+\chi_{]x_{i+1},x_i[}$ to $f^-\chi_{]y_{i+1},y_i[\cup]y_{i+1}-1,y_i-1[}$, for $i = 1, 2, \ldots.$




For $x > \frac{2(3-L)}{L}$ we have to distribute the mass $f^+$ in three *equiweighted* parts, in this case, set the rectangles with corner coordinates,

$$\text{lower-left,}\quad ll_i = (x_i, y_i),\qquad \text{lower-right,}\quad lr_i = (x_{i+1}, y_i),$$
$$\text{upper-left,}\quad ul_i = (x_i, y_{i+1}),\qquad \text{upper-right,}\quad ur_i = (x_{i+1}, y_{i+1}),$$

$i = 1, 2, \ldots,$ where now

$$x_1 = \frac{2(3-L)}{L}, \qquad y_1 = 2 - L,$$
$$x_{i+1} = y_i + 1, \qquad y_{i+1} = y_i + \frac{L}{3}(x_{i+1} - x_i) = \frac{L}{3}(x_{i+1} - 1)$$

(observe that $lr_i \in [y = x - 1]$ and $ll_i, ur_i \in [y = \frac{L}{3}(x - 1)]$); in each rectangle we can trace three parallel segments of slope $L$ defined by the lines

$$y = L(x - x_i) + y_i, \qquad y = L(x - \hat{x}_i) + y_i, \qquad \hat{x}_i = x_i + \frac{x_{i+1} - x_i}{3},$$

and

$$y = L(x - \tilde{x}_i) + y_i, \qquad \tilde{x}_i = x_i + 2\frac{x_{i+1} - x_i}{3};$$

then

$$T_i(x) = f^+(x)\chi_{(x_i,\hat{x}_i)}(x)\delta_{[y=L(x-x_i)+y_i]} + f^+(x)\chi_{(\hat{x}_i,\tilde{x}_i)}(x)\delta_{[y=L(x-\hat{x}_i)+y_i-1]}$$
$$+ f^+(x)\chi_{(\tilde{x}_i,x_{i+1})}(x)\delta_{[y=L(x-\tilde{x}_i)+y_i-2]}$$

push in an optimal way $f^+\chi_{(x_i,x_{i+1})}$ to $f^-\chi_{(y_i,y_{i+1})\cup(y_i-1,y_{i+1}-1)\cup(y_i-2,y_{i+1}-2)}$, for $i = 1, 2, \ldots.$

### *3.2. Characterizing the Euler–Lagrange equation: A nonlocal version of the Evans–Gangbo approach*

Our first objective is to characterize the Euler–Lagrange equation associated with the variational problem $\sup\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_{d_1}(\Omega)\}$, that is, characterize $f^+ - f^- \in \partial \mathbb{I}_{K_1}(u)$, where, as above, we denote for simplicity $K_1 := K_{d_1}(\Omega)$.

Let $\mathcal{M}_b^a(\Omega \times \Omega) :=$ {bounded antisymmetric Radon measures in $\Omega \times \Omega$}. And define the multivalued operator $B_1$ in $L^2(\Omega)$ as follows: $(u, v) \in B_1$ if and only if $u \in K_1$, $v \in L^2(\Omega)$, and there exists $\sigma \in \mathcal{M}_b^a(\Omega \times \Omega)$ such that

$$\sigma = \sigma \llcorner \big\{(x, y) \in \Omega \times \Omega\colon |x - y| \leqslant 1\big\},$$

$$\int_{\Omega \times \Omega} \xi(x)\, d\sigma(x, y) = \int_{\Omega} \xi(x) v(x)\, dx, \quad \forall \xi \in C_c(\Omega),$$

and

$$|\sigma|(\Omega \times \Omega) \leqslant 2 \int_{\Omega} v(x) u(x)\, dx.$$

**Theorem 3.5.** *The following characterization holds*: $\partial \mathbb{I}_{K_1} = B_1$.

**Proof.** Let us first see that $B_1 \subset \partial \mathbb{I}_{K_1}$. Let $(u, v) \in B_1$, to see that $(u, v) \in \partial \mathbb{I}_{K_1}$ we need to prove that

$$0 \leqslant \int_{\Omega} v(x)\big(u(x) - \xi(x)\big)\, dx, \quad \forall \xi \in K_1.$$

Using an approximation procedure, we can assume that $\xi \in K_1$ is continuous. Then,

$$\begin{aligned}
\int_{\Omega} v(x)\big(u(x) - \xi(x)\big)\, dx &\geqslant \frac{1}{2}|\sigma|(\Omega \times \Omega) - \int_{\Omega} v(x)\xi(x)\, dx \\
&= \frac{1}{2}|\sigma|(\Omega \times \Omega) - \int_{\Omega \times \Omega} \xi(x)\, d\sigma(x, y) \\
&= \frac{1}{2}|\sigma|(\Omega \times \Omega) - \frac{1}{2} \int_{\Omega \times \Omega} \big(\xi(x) - \xi(y)\big)\, d\sigma(x, y) \geqslant 0,
\end{aligned}$$

where in the last equality we have used the antisymmetry of $\sigma$. Therefore, we have $B_1 \subset \partial \mathbb{I}_{K_1}$. Since $\partial \mathbb{I}_{K_1}$ is a maximal monotone operator, to see that the operators are equal we only need to show that for every $f \in L^2(\Omega)$ there exists $u \in K_1$ such that

$$u + B_1(u) \ni f. \tag{3.9}$$

Let $J:\mathbb{R}^N\to\mathbb{R}$ as in (2.1). By results in [5], given $p>N$ and $f\in L^2(\Omega)$ there exists a unique solution $u_p\in L^\infty(\Omega)$ of the nonlocal $p$-Laplacian problem

$$u_p(x)-\int_\Omega J(x-y)\big|u_p(y)-u_p(x)\big|^{p-2}\big(u_p(y)-u_p(x)\big)\,dy=T_p(f)(x)\quad \forall x\in\Omega, \tag{3.10}$$

where $T_k(r):=\max\{\min\{k,r\},-r\}$. And we also know, using again Lemma 2.3, that there exists $u\in K_1$ such that

$$u_p\to u\quad \text{in } L^2(\Omega) \text{ as } p\to+\infty, \tag{3.11}$$

with $u+\partial\mathbb{I}_{K_1}(u)\ni f$, from where it follows that

$$\int_\Omega\big(f(x)-u(x)\big)\big(w(x)-u(x)\big)\,dx\leqslant 0,\quad \forall w\in K_1,$$

and consequently, $u=P_{K_1}(f)$. Multiplying (3.10) by $u_p$ and integrating, we get

$$\int_\Omega\big(T_p(f)(x)-u_p(x)\big)u_p(x)\,dx=\frac{1}{2}\int_{\Omega\times\Omega}J(x-y)\big|u_p(y)-u_p(x)\big|^p\,dx\,dy, \tag{3.12}$$

from where it follows that

$$\int_{\Omega\times\Omega}J(x-y)\big|u_p(y)-u_p(x)\big|^p\,dx\,dy+\int_\Omega\big|u_p(x)\big|^2\,dx\leqslant\|f\|^2_{L^2(\Omega)}. \tag{3.13}$$

If we set $\sigma_p(x,y):=J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))$, by Hölder's inequality,

$$\begin{aligned}
&\int_{\Omega\times\Omega}\big|\sigma_p(x,y)\big|\,dx\,dy\\
&\quad=\int_{\Omega\times\Omega}J(x-y)\big|u_p(y)-u_p(x)\big|^{p-1}\,dx\,dy\\
&\quad\leqslant\bigg(\int_{\Omega\times\Omega}J(x-y)\big|u_p(y)-u_p(x)\big|^p\,dx\,dy\bigg)^{\frac{p-1}{p}}\bigg(\int_{\Omega\times\Omega}J(x-y)\,dx\,dy\bigg)^{\frac{1}{p}}\\
&\quad=\bigg(\int_{\Omega\times\Omega}J(x-y)\big|u_p(y)-u_p(x)\big|^p\,dx\,dy\bigg)^{\frac{p-1}{p}}.
\end{aligned}$$

Now, by (3.13), we have

$$\int_{\Omega\times\Omega}\big|\sigma_p(x,y)\big|\,dx\,dy\leqslant\big(\|f\|^2_{L^2(\Omega)}\big)^{\frac{p-1}{p}}.$$

Hence, $\{\sigma_p\colon\ p \geqslant 2\}$ is bounded in $L^1(\Omega \times \Omega)$, and consequently we can assume that

$$\sigma_p(.,.) \rightharpoonup \sigma \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega \times \Omega). \tag{3.14}$$

Obviously, since each $\sigma_p$ is antisymmetric, $\sigma \in \mathcal{M}_b^a(\Omega \times \Omega)$. Moreover, since $\operatorname{supp}(J) = \overline{B_1(0)}$, we have $\sigma = \sigma \llcorner \{(x, y) \in \Omega \times \Omega\colon\ |x - y| \leqslant 1\}$. On the other hand, given $\xi \in C_c(\Omega)$, by (3.10), (3.11) and (3.14), we get

$$\begin{aligned}
\int_{\Omega\times\Omega} \xi(x)\, d\sigma(x, y) &= \lim_{p\to+\infty} \int_{\Omega\times\Omega} \xi(x)\sigma_p(x, y)\, dx\, dy \\
&= \lim_{p\to+\infty} \int_{\Omega\times\Omega} J(x - y)\big|u_p(y) - u_p(x)\big|^{p-2}\big(u_p(y) - u_p(x)\big)\xi(x)\, dx\, dy \\
&= \lim_{p\to+\infty} \int_{\Omega} \big(T_p(f)(x) - u_p(x)\big)\xi(x)\, dx \\
&= \int_{\Omega} \big(f(x) - u(x)\big)\xi(x)\, dx.
\end{aligned}$$

Then, to prove (3.9), we only need to show that $|\sigma|(\Omega \times \Omega) \leqslant 2\int_\Omega (f(x) - u(x))u(x)\, dx$. In fact, by (3.14), we have

$$|\sigma|(\Omega \times \Omega) \leqslant \liminf_{p\to+\infty} \int_\Omega \int_\Omega \big|\sigma_p(x, y)\big|\, dx\, dy.$$

Now, by (3.12),

$$\begin{aligned}
\int_{\Omega\times\Omega} \big|\sigma_p(x, y)\big|\, dx\, dy &\leqslant \bigg(\int_{\Omega\times\Omega} J(x - y)\big|u_p(y) - u_p(x)\big|^p\, dx\, dy\bigg)^{\frac{p-1}{p}} \\
&= \bigg(2\int_\Omega \big(T_p(f)(x) - u_p(x)\big)u_p(x)\, dx\bigg)^{\frac{p-1}{p}} \\
&= 2^{\frac{p-1}{p}}\bigg(\int_\Omega \big(T_p(f)(x) - u_p(x)\big)u_p(x)\, dx\bigg)^{\frac{p-1}{p}}.
\end{aligned}$$

Therefore $|\sigma|(\Omega \times \Omega) \leqslant 2\int_\Omega (f(x) - u(x))u(x)\, dx$. □

We can rewrite the operator $B_1$ as follows.

**Corollary 3.6.** *$(u, v) \in B_1$ if and only if $u \in K_1$, $v \in L^2(\Omega)$, and there exists $\sigma \in \mathcal{M}_b^a(\Omega \times \Omega)$ such that*

$$\sigma^+ = \sigma^+ \llcorner \{(x,y) \in \Omega \times \Omega\colon |x-y| \leqslant 1,\ u(x) - u(y) = 1\},$$
$$\sigma^- = \sigma^- \llcorner \{(x,y) \in \Omega \times \Omega\colon |x-y| \leqslant 1,\ u(y) - u(x) = 1\},$$
$$\int_{\Omega\times\Omega} \xi(x)\,d\sigma(x,y) = \int_\Omega \xi(x)v(x)\,dx, \quad \forall \xi \in C_c(\Omega),$$

*and*

$$|\sigma|(\Omega \times \Omega) = 2\int_\Omega v(x)u(x)\,dx.$$

**Proof.** Let $(u,v) \in B_1$, then

$$\int_{\Omega\times\Omega} \xi(x)\,d\sigma(x,y) = \int_\Omega \xi(x)v(x)\,dx, \quad \forall \xi \in C_c(\Omega). \tag{3.15}$$

Hence, by approximation, we can take $\xi \in L^2(\Omega)$ in (3.15) and $\int_\Omega\int_\Omega \xi(x)\,d\sigma(x,y)$ has this sense.

Taking $\xi = u$ in (3.15) and using the antisymmetric of $\sigma$ and the previous result we get

$$\begin{aligned}
|\sigma|(\Omega\times\Omega) &\geqslant \int_{\Omega\times\Omega} \big(u(x) - u(y)\big)\,d\sigma(x,y) \\
&= 2\int_{\Omega\times\Omega} u(x)\,d\sigma(x,y) \\
&= 2\int_\Omega u(x)v(x)\,dx \\
&\geqslant |\sigma|(\Omega\times\Omega). \qquad \square
\end{aligned}$$

As consequence of the above results, we have that $u^* \in K_1$ is a Kantorovich potential for $d_1$, $f^+$, $f^-$, if and only if

$$f^+ - f^- \in B_1\big(u^*\big), \tag{3.16}$$

that is, if $u^* \in K_1$ and there exists $\sigma^* \in \mathcal{M}_b^a(\Omega\times\Omega)$, such that

$$\begin{cases}
\big[\sigma^*\big]^+ = \big[\sigma^*\big]^+ \llcorner \{(x,y) \in \Omega\times\Omega\colon u^*(x) - u^*(y) = 1,\ |x-y| \leqslant 1\}, \\
\big[\sigma^*\big]^- = \big[\sigma^*\big]^- \llcorner \{(x,y) \in \Omega\times\Omega\colon u^*(y) - u^*(x) = 1,\ |x-y| \leqslant 1\}, \\
\displaystyle\int_{\Omega\times\Omega} \xi(x)\,d\sigma^*(x,y) = \int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx,
\end{cases} \tag{3.17}$$

and

$$\left|\sigma^*\right|(\Omega \times \Omega) = 2\int_\Omega \big(f^+(x) - f^-(x)\big)u^*(x)\,dx = 2\mathcal{P}\big(u^*\big). \tag{3.18}$$

We want to highlight that (3.16) plays the role of (1.4). Moreover, we will see in the next subsection that we can construct optimal transport plans from it, more precisely, we shall see that the potential $u_1^*$ and the measure $\sigma_1^*$ encode all the information that we need to construct an optimal transport plan associated with the problem.

### *3.3. Constructing optimal transport plans*

We will use a gluing lemma (see Lemma 7.6 in [19]), which permits to glue together two transport plans in an adequate way. As remarked in [19], it is possible to state the gluing lemma in the following way (we present it for the distance $d_1$).

**Lemma 3.7.** *Let $f_1, f_2, g$ be three positive measures in $\Omega$. If $\mu_1 \in \pi(f_1, g)$ and $\mu_2 \in \pi(g, f_2)$, there exists a measure $\mathcal{G}(\mu_1, \mu_2) \in \pi(f_1, f_2)$ such that*

$$\mathcal{K}_{d_1}\big(\mathcal{G}(\mu_1, \mu_2)\big) \leqslant \mathcal{K}_{d_1}(\mu_1) + \mathcal{K}_{d_1}(\mu_2). \tag{3.19}$$

Let us now proceed with the general construction. Given $f^+, f^- \in L^\infty(\Omega)$ two non-negative Borel functions satisfying the mass balance condition (1.1) and $|\operatorname{supp}(f^+) \cap \operatorname{supp}(f^-)| = 0$, by Theorems 1.2 and 2.6, there exists a Kantorovich potential $u^*$ taking a finite number of entire values such that

$$\min\big\{\mathcal{K}_{d_1}(\mu)\colon \mu \in \pi\left(f^+, f^-\right)\big\} = \int_\Omega u^*(x)\big(f^+(x) - f^-(x)\big)\,dx.$$

Then, by Corollary 3.6, there exists $\sigma \in \mathcal{M}_b^a(\Omega \times \Omega)$ satisfying (3.17) and (3.18). We are going to give a method to obtain an optimal transport plan $\mu^*$ from the measure $\sigma$.

We divide the construction in two steps. We assume without loss of generality that

$$u^* = 0\chi_{A_0} + 1\chi_{A_1} + \cdots + k\chi_{A_k}, \quad \text{with } A_i = \big\{x \in \Omega\colon u^*(x) = i\big\}.$$

*Step 1. How the measures $\sigma^+ \llcorner (A_j \times A_{j-1})$ work.* Taking into account the antisymmetry of $\sigma$ and (3.17), we have that $\operatorname{proj}_x(\sigma^+) - \operatorname{proj}_y(\sigma^+) = f^+ - f^-$, which implies $g := \operatorname{proj}_x(\sigma^+) - f^+ = \operatorname{proj}_y(\sigma^+) - f^-$. By (2.8), $\operatorname{proj}_x(\sigma^+) \llcorner A_k = f^+\chi_{A_k}$ and $\operatorname{proj}_x(\sigma^+) \llcorner A_0 = f^+\chi_{A_0} = 0$, then

$$g \llcorner A_k = g \llcorner A_0 = 0.$$

Moreover, we have $\operatorname{proj}_x(\sigma^+ \llcorner (A_j \times A_{j-1})) = \operatorname{proj}_x(\sigma^+) \llcorner A_j$ and $\operatorname{proj}_y(\sigma^+ \llcorner (A_j \times A_{j-1})) = \operatorname{proj}_x(\sigma^+) \llcorner A_{j-1}$, then $\operatorname{proj}_x(\sigma^+ \llcorner (A_j \times A_{j-1})) = f^+\chi_{A_j} + g \llcorner A_j$ and $\operatorname{proj}_y(\sigma^+ \llcorner (A_j \times A_{j-1})) = f^-\chi_{A_{j-1}} + g \llcorner A_{j-1}$. Let us call $\mu_j := \sigma^+ \llcorner (A_j \times A_{j-1})$. Let us briefly comment what these measures do. The first one, $\mu_k$, transports $f^+\chi_{A_k}$ into $f^-\chi_{A_{k-1}}$ plus something else, that is $g\llcorner A_{k-1}$. Afterwards, $\mu_j$ transports $f^+\chi_{A_j}+g\llcorner A_j$ into $f^-\chi_{A_{j-1}}$ again plus something else, that is $g\llcorner A_{j-1}$. The last one, $\mu_1$, transports $f^+\chi_{A_1}+g\llcorner A_1$ to $f^-\chi_{A_0}$.

*Step 2. The gluing.* Now, we would like to glue this transportations, and, in order to apply the gluing lemma, we consider the measures

$$\mu_k^l(x,y) := \mu_k(x,y) + f^+(x)\chi_{A_{k-1}}(x)\delta_{[y=x]},$$

and

$$\mu_{k-1}^r(x,y) := \mu_{k-1}(x,y) + f^-(x)\chi_{A_{k-1}}(x)\delta_{[y=x]}.$$

It is easy to see that

$$\mu_k^l \in \pi\left(f^+\chi_{A_k} + f^+\chi_{A_{k-1}}, f^-\chi_{A_{k-1}} + \operatorname{proj}_x\left(\sigma^+\right)\llcorner A_{k-1}\right)$$

and

$$\mu_{k-1}^r \in \pi\left(f^-\chi_{A_{k-1}} + \operatorname{proj}_x\left(\sigma^+\right)\llcorner A_{k-1}, f^-\chi_{A_{k-1}} + f^-\chi_{A_{k-2}} + g\llcorner A_{k-2}\right).$$

Therefore, by the gluing lemma,

$$\mathcal{G}\left(\mu_k^l, \mu_{k-1}^r\right) \in \pi\left(f^+\chi_{A_k} + f^+\chi_{A_{k-1}}, f^-\chi_{A_{k-1}} + f^-\chi_{A_{k-2}} + g\llcorner A_{k-2}\right).$$

Let us now consider the measures

$$\mu_{k-1}^l(x,y) := \mathcal{G}\left(\mu_k^l, \mu_{k-1}^r\right)(x,y) + f^+(x)\chi_{A_{k-2}}(x)\delta_{[y=x]}$$

and

$$\mu_{k-2}^r(x,y) := \mu_{k-2}(x,y) + \left(f^-(x)\chi_{A_{k-1}}(x) + f^-(x)\chi_{A_{k-2}}(x)\right)\delta_{[y=x]}.$$

Then we have

$$\mu_{k-1}^l \in \pi\left(f^+\chi_{A_k} + f^+\chi_{A_{k-1}} + f^+\chi_{A_{k-2}}, f^-\chi_{A_{k-2}} + f^-\chi_{A_{k-1}} + \operatorname{proj}_x\left(\sigma^+\right)\llcorner A_{k-2}\right)$$

and

$$\begin{aligned}\mu_{k-2}^r \in \pi\big(&f^-\chi_{A_{k-2}} + f^-\chi_{A_{k-1}} + \operatorname{proj}_x\left(\sigma^+\right)\llcorner A_{k-2},\\ &f^-\chi_{A_{k-1}} + f^-\chi_{A_{k-2}} + f^-\chi_{A_{k-3}} + g\llcorner A_{k-3}\big).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{G}\left(\mu_{k-1}^l, \mu_{k-2}^r\right) \in \pi\big(&f^+\chi_{A_k} + f^+\chi_{A_{k-1}} + f^+\chi_{A_{k-2}},\\ &f^-\chi_{A_{k-1}} + f^-\chi_{A_{k-2}} + f^-\chi_{A_{k-3}} + g\llcorner A_{k-3}\big).\end{aligned}$$

Proceeding in this way we arrive to the construction of

$$\mu_2^l(x,y) = \mathcal{G}(\mu_3^l,\mu_2^r)(x,y) + f^+(x)\chi_{A_1}(x)\delta_{[y=x]},$$
$$\mu_1^r(x,y) = \mu_1(x,y) + \sum_{i=1}^{k-1} f^-(x)\chi_{A_i(x)}\delta_{[y=x]}$$

and

$$\mu^* = \mathcal{G}(\mu_2^l,\mu_1^r) \in \pi(f^+,f^-),$$

which is, in fact, an optimal transport plan since, by (3.19),

$$\begin{aligned}
\mathcal{K}_{d_1}(\mu^*) &= \mathcal{K}_{d_1}\big(\mathcal{G}(\mu_2^l,\mu_1^r)\big) \leqslant \mathcal{K}_{d_1}(\mu_2^l) + \mathcal{K}_{d_1}(\mu_1^r) \\
&= \mathcal{K}_{d_1}\big(\mathcal{G}(\mu_3^l,\mu_2^r)\big) + \mathcal{K}_{d_1}(\mu_1) \leqslant \mathcal{K}_{d_1}(\mu_3^l) + \mathcal{K}_{d_1}(\mu_2^r) + \mathcal{K}_{d_1}(\mu_1) \\
&= \mathcal{K}_{d_1}\big(\mathcal{G}(\mu_4^l,\mu_3^r)\big) + \mathcal{K}_{d_1}(\mu_2) + \mathcal{K}_{d_1}(\mu_1) \leqslant \ldots \leqslant \mathcal{K}_{d_1}(\mu_k^l) + \sum_{j=1}^{k-1}\mathcal{K}_{d_1}(\mu_j) \\
&= \sum_{j=1}^{k}\mathcal{K}_{d_1}(\mu_j) = \sum_{j=1}^{k}\int_{\Omega\times\Omega} d\sigma^+ \llcorner (A_j\times A_{j-1}) = \int_{\Omega\times\Omega} d\sigma^+ \\
&= \frac{1}{2}|\sigma|(\Omega\times\Omega) = \min\big\{\mathcal{K}_{d_1}(\mu)\colon\ \mu\in\pi(f^+,f^-)\big\}.
\end{aligned}$$

We want to remark that a similar construction works for any Kantorovich potential $u^*$, without assuming that $u^*(\Omega)\subset\mathbb{Z}$, but the above one is simpler.

## 4. Convergence to the classical problem

The task of this section is the connection between this discrete mass transport problem and the classical transport problem for the Euclidean distance. In particular we recover the PDE formulation (1.4) of Evans–Gangbo by means of this discrete approach.

Let us begin by remarking that an equivalent result to Corollary 3.5 for $d_\varepsilon$ gives us that $(u_\varepsilon^*,\sigma_\varepsilon^*)$ is a solution of the Euler–Lagrange equation

$$f^+ - f^- \in \partial\mathbb{I}_{K_{d_\varepsilon}(\Omega)}(u), \tag{4.1}$$

that corresponds to the maximization problem

$$\max\left\{\int_\Omega u(x)\big(f^+(x)-f^-(x)\big)\,dx\colon\ u\in K_{d_\varepsilon}(\Omega)\right\},$$

if and only if $u_\varepsilon^* \in K_{d_\varepsilon}(\Omega)$ and $\sigma_\varepsilon^*$ in $\Omega$ is an antisymmetric bounded Radon measure such that

$$[\sigma_\varepsilon^*]^+ = [\sigma_\varepsilon^*]^+ \llcorner \{(x,y) \in \Omega \times \Omega \colon u_\varepsilon^*(x) - u_\varepsilon^*(y) = \varepsilon,\ |x-y| \leqslant \varepsilon\},$$
$$[\sigma_\varepsilon^*]^- = [\sigma_\varepsilon^*]^- \llcorner \{(x,y) \in \Omega \times \Omega \colon u_\varepsilon^*(y) - u_\varepsilon^*(x) = \varepsilon,\ |x-y| \leqslant \varepsilon\}, \tag{4.2}$$

$$\int_{\Omega\times\Omega} \xi(x)\, d\sigma_\varepsilon^*(x,y) = \int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\, dx, \tag{4.3}$$

and

$$|\sigma_\varepsilon^*|(\Omega \times \Omega) = \frac{2}{\varepsilon}\int_\Omega \big(f^+(x) - f^-(x)\big)u_\varepsilon^*(x)\, dx = \frac{2}{\varepsilon}\mathcal{P}\big(u_\varepsilon^*\big). \tag{4.4}$$

*4.1. Convergence to the classical problem*

Let us fix $f^+, f^- \in L^2(\Omega)$ satisfying the mass balance condition (1.1). First of all, in the following result we state the convergence to the Monge–Kantorovich problems. We will denote $K_\varepsilon = K_{d_\varepsilon}(\Omega)$ and $K_{d_{|\cdot|}} = K_{d_{|\cdot|}}(\Omega)$ for simplicity (recall that $d_{|\cdot|}$ denotes the Euclidean distance), and

$$\mathcal{W} := \sup\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_{d_{|\cdot|}}\} = \min\{\mathcal{K}_{d_{|\cdot|}}(\mu)\colon \mu \in \pi(f^+, f^-)\}$$
$$= \inf\{\mathcal{F}(T)\colon T \in \mathcal{A}(f^+, f^-)\},$$
$$\mathcal{W}_\varepsilon := \sup\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_\varepsilon\} = \min\{\mathcal{K}_\varepsilon(\mu)\colon \mu \in \pi(f^+, f^-)\}.$$

**Proposition 4.1.** *For the costs $\mathcal{W}_\varepsilon$ and $\mathcal{W}$ the following facts hold*:

$$\mathcal{W}_\varepsilon \leqslant \mathcal{W}_{\varepsilon'} \quad \text{for } \varepsilon \leqslant \varepsilon'.$$
$$0 \leqslant \mathcal{W}_\varepsilon - \mathcal{W} \leqslant \varepsilon \int_\Omega f^+(x)\, dx \quad \text{for any } \varepsilon > 0. \tag{4.5}$$

*For the primal problems, it also holds*:

$$\lim_{\varepsilon\to 0^+} \inf\{\mathcal{F}_\varepsilon(\mu)\colon \mu \in \pi(f^+, f^-)\} = \mathcal{W}. \tag{4.6}$$

**Proof.** Since

$$d_\varepsilon(x,y) - \varepsilon \leqslant d_{|\cdot|}(x,y) \leqslant d_\varepsilon(x,y), \tag{4.7}$$

given $\mu \in \pi(f^+, f^-)$, we have

$$\int_{\Omega\times\Omega} \big(d_\varepsilon(x,y) - \varepsilon\big)\, d\mu(x,y) \leqslant \int_{\Omega\times\Omega} d_{|\cdot|}(x,y)\, d\mu(x,y) \leqslant \int_{\Omega\times\Omega} d_\varepsilon(x,y)\, d\mu(x,y).$$

Then, taking the minimum over all $\mu \in \pi(f^+, f^-)$, and having in mind that

$$\int_{\Omega\times\Omega} d\mu(x,y) = \int_{\Omega} f^+(x)\,dx,$$

we obtain (4.5). Moreover, since $d_\varepsilon \leqslant d_{\varepsilon'}$ for $\varepsilon \leqslant \varepsilon'$, the sequence of costs $\{\mathcal{W}_\varepsilon\}_{\varepsilon>0}$ is monotone nonincreasing as $\varepsilon$ decreases to zero.

Let us now prove (4.6), which, by Example 1.4, is not a trivial consequence of the above statement. Precisely, this previous statement gives:

$$\lim_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon = \inf\{\mathcal{F}(T):\ T \in \mathcal{A}(f^+, f^-)\}. \tag{4.8}$$

Take now $T'$ a transport map. Thanks to (4.7),

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \inf\{\mathcal{F}_\varepsilon(T):\ T \in \mathcal{A}(f^+, f^-)\} \\
&\quad = \limsup_{\varepsilon\to 0} \inf\left\{\int_{\Omega} d_\varepsilon\big(x, T(x)\big) f^+(x)\,dx:\ T \in \mathcal{A}(f^+, f^-)\right\} \\
&\quad \leqslant \limsup_{\varepsilon\to 0} \int_{\Omega} d_\varepsilon\big(x, T'(x)\big) f^+(x)\,dx = \int_{\Omega} \big|x - T'(x)\big| f^+(x)\,dx.
\end{aligned}$$

Therefore,

$$\limsup_{\varepsilon\to 0} \inf\{\mathcal{F}_\varepsilon(T):\ T \in \mathcal{A}(f^+, f^-)\} \leqslant \inf\{\mathcal{F}(T):\ T \in \mathcal{A}(f^+, f^-)\}. \tag{4.9}$$

On the other hand,

$$\mathcal{W}_\varepsilon = \min\{\mathcal{K}_\varepsilon(\mu):\ \mu \in \pi(f^+, f^-)\} \leqslant \inf\{\mathcal{F}_\varepsilon(T):\ T \in \mathcal{A}(f^+, f^-)\}.$$

Taking now the $\liminf_{\varepsilon\to 0}$ in the above expression and taking into account (4.8) and (4.9) we obtain (4.6). $\square$

Let us now proceed with the approximation of optimal transport plans. Let us consider, for each $\varepsilon > 0$, an optimal transport plan $\mu_\varepsilon$ between $f^+$ and $f^-$ for $d_\varepsilon$, that is, $\mu_\varepsilon \in \pi(f^+, f^-)$ such that

$$\mathcal{K}_\varepsilon(\mu_\varepsilon) = \min\{\mathcal{K}_\varepsilon(\mu):\ \mu \in \pi(f^+, f^-)\}.$$

**Proposition 4.2.** *There exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and $\mu^* \in \pi(f^+, f^-)$ such that*

$$\mu_{\varepsilon_n} \rightharpoonup \mu^* \quad \textit{as measures}$$

*and*

$$\mathcal{K}(\mu^*) = \min\{\mathcal{K}(\mu):\ \mu \in \pi(f^+, f^-)\}.$$

**Proof.** To prove this we just observe that

$$d_{|\cdot|}(x,y)=|x-y|\leqslant d_\varepsilon(x,y)\leqslant |x-y|+\varepsilon$$

(note that this implies $d_\varepsilon(x,y)\to|x-y|$ uniformly as $\varepsilon\to 0$). Hence,

$$\int_{\Omega\times\Omega}|x-y|\,d\mu_\varepsilon(x,y)\leqslant\int_{\Omega\times\Omega}d_\varepsilon(x,y)\,d\mu_\varepsilon(x,y)\leqslant\int_{\Omega\times\Omega}\big(|x-y|+\varepsilon\big)\,d\mu_\varepsilon(x,y).$$

On the other hand, by Prokhorov's Theorem, we can assume that, there exists a sequence $\varepsilon_n\to 0$ as $n\to\infty$ such that $\mu_{\varepsilon_n}$ converges weakly* in the sense of measures to a limit $\mu^*$. Therefore, we conclude that

$$\int_{\Omega\times\Omega}|x-y|\,d\mu^*(x,y)=\lim_{n\to+\infty}\int_{\Omega\times\Omega}d_{\varepsilon_n}(x,y)\,d\mu_{\varepsilon_n}(x,y).$$

Finally, by Proposition 4.1 we obtain that $\mu^*$ is a minimizer for the usual Euclidean distance. □

To illustrate these results, we present an example in one dimension that shows how one can recover the unique monotone nondecreasing optimal transport map for the Euclidean distance between $f^+$ and $f^-$.

**Example 4.3.** Let $f^+=2\chi_{[0,1]}$ and $f^-=\chi_{[-2,0]}$. Set $\Omega$ an interval containing $[-2,2]$. As we set in Section 3.1.2, there is no transport map $T$ between $f^+$ and $f^-$ if one considers the distance $d_{1/k}$ with $k\in\mathbb{N}$. Nevertheless, for each $n\in\mathbb{N}$,

$$\mu_n(x,y)=\chi_{[\frac{2^n-1}{2^n},1]}(x)\delta_{[y=x-1]}+\sum_{m=1}^{2^n-1}\chi_{[\frac{2^n-m-1}{2^n},\frac{2^n-m+1}{2^n}]}(x)\delta_{[y=x-1-\frac{m}{2^n}]}+\chi_{[0,\frac{1}{2^n}]}(x)\delta_{[y=x-2]}$$

is an optimal transport plan between $f^+$ and $f^-$ for the distance $d_{\frac{1}{2^n}}$ such that

$$\mu_n\rightharpoonup f^+(x)\delta_{[y=T(x)]}\quad\text{weakly* as measures,}$$

where $T(x)=2x-2$ is the unique monotone nondecreasing optimal transport map for the Euclidean distance between $f^+$ and $f^-$.

Let us finish this subsection with a convergence result for Kantorovich potentials.

**Proposition 4.4.** *Let $u_\varepsilon^*$ be a Kantorovich potential for $f^+-f^-$ associated with the metric $d_\varepsilon$. Then, there exists a sequence $\varepsilon_n\to 0$ as $n\to\infty$ such that*

$$u_{\varepsilon_n}^*\rightharpoonup u^*\quad\text{in } L^2,$$

*where $u^*$ is a Kantorovich potential associated with the Euclidean metric $d_{|\cdot|}$.*

**Proof.** It is an obvious fact that $\{u_\varepsilon\}$ is $L^\infty$-bounded, then, there exists a sequence

$$u^*_{\varepsilon_n} \rightharpoonup v \quad \text{in } L^2.$$

Therefore,

$$\lim_{n\to+\infty} \int_\Omega u^*_{\varepsilon_n}(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_\Omega v(x)\big(f^+(x) - f^-(x)\big)\,dx.$$

Now, since

$$\int_\Omega u^*_{\varepsilon_n}(x)\big(f^+(x) - f^-(x)\big)\,dx = \sup\big\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_{\varepsilon_n}\big\},$$

by Proposition 4.1, we conclude that

$$\int_\Omega v(x)\big(f^+(x) - f^-(x)\big)\,dx = \sup\big\{\mathcal{P}_{f^+,f^-}(u)\colon u \in K_{d_{|\cdot|}}\big\}.$$

In order to have that the limit $v$ is a maximizer $u^*$ we need to show that $v \in K_{d_{|\cdot|}}$, and this follows by the Mosco-convergence of $\mathbb{I}_{K_\varepsilon}$ to $\mathbb{I}_{K_{d_{|\cdot|}}}$ (see [5]). □

## 4.2. *Approximating the Evans–Gangbo PDE*

The main task in this subsection is to show how from the solutions $(u^*_\varepsilon, \sigma^*_\varepsilon)$ of the Euler–Lagrange equation

$$f^+ - f^- \in \partial\mathbb{I}_{K_{d_\varepsilon}(\Omega)}(u),$$

that corresponds to the maximization problem

$$\max\left\{\int_\Omega u(x)\big(f^+(x) - f^-(x)\big)\,dx\colon u \in K_{d_\varepsilon}(\Omega)\right\},$$

we can recover $u^* \in K_{d_{|\cdot|}}(\Omega)$ such that

$$\int_\Omega u^*(x)\big(f^+(x) - f^-(x)\big)\,dx = \max\left\{\int_\Omega u(x)\big(f^+(x) - f^-(x)\big)\,dx\colon u \in K_{d_{|\cdot|}}(\Omega)\right\},$$

and $0 \leqslant a \in L^\infty(\Omega)$ such that

$$f^+ - f^- = -\operatorname{div}\big(a\nabla u^*\big) \quad \text{in } \mathcal{D}'(\Omega), \qquad \big|\nabla u^*\big| = 1 \quad \text{a.e. on the set } \{a > 0\}.$$

Remember that $u^*_\varepsilon \in K_{d_\varepsilon}(\Omega)$ and $\sigma^*_\varepsilon$ is an antisymmetric bounded Radon measure in $\Omega$ satisfying (4.2), (4.3) and (4.4). Moreover, by Proposition 4.4, after a subsequence,

$$u_\varepsilon^* \rightharpoonup u^* \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0,$$

where $u^*$ is a Kantorovich potential associated with the metric $d_{|\cdot|}$.

Let us now fix

$$\Omega' \Subset \Omega'' \Subset \Omega \tag{4.10}$$

be such that $|x-y| > r = \operatorname{diam}(\operatorname{supp}(f^+ - f^-))$ for any $x \in \operatorname{supp}(f^+ - f^-)$ and any $y \in \Omega \setminus \Omega'$. By (4.3),

$$\int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_{\Omega\times\Omega} \xi(x)\,d\sigma_\varepsilon^*(x,y), \quad \forall \xi \in C_c(\Omega). \tag{4.11}$$

Hence, for $\xi \in C_c^1(\Omega)$, by (4.11) and the antisymmetry of $\sigma_\varepsilon^*$, we have that

$$\int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_{\Omega\times\Omega} \xi(x)\,d\sigma_\varepsilon^*(x,y) = \int_{\Omega\times\Omega} \frac{\xi(x)-\xi(y)}{\varepsilon}\,d\bigg(\frac{\varepsilon}{2}\sigma_\varepsilon^*(x,y)\bigg),$$

and

$$\begin{aligned}\int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx &= \int_{\Omega\times\Omega} \xi(x)\,d\sigma_\varepsilon^*(x,y)\\ &= \int_{\Omega\times\Omega} \frac{\xi(x)-\xi(y)}{\varepsilon}\,d\big(\varepsilon\big[\sigma_\varepsilon^*\big]^+(x,y)\big).\end{aligned} \tag{4.12}$$

Now observe that for $\varphi \in C_c(\Omega\times\Omega)$, if $\phi(x,z) = \varphi(x, x+\varepsilon z)$ and $T_\varepsilon(x,y) = \frac{y-x}{\varepsilon}$, then

$$\begin{aligned}\int_{\Omega\times\Omega} \varphi(x,y)\,d\big[\sigma_\varepsilon^*\big]^+(x,y) &= \int_{\Omega\times\Omega} \phi\big((\pi_1, T_\varepsilon)(x,y)\big)\,d\big[\sigma_\varepsilon^*\big]^+(x,y)\\ &= \int_{\Omega\times\frac{\Omega-\Omega}{\varepsilon}} \phi(x,z)\,d\big((\pi_1,T_\varepsilon)\,\#\,\big[\sigma_\varepsilon^*\big]^+\big)(x,z)\\ &= \int_{\Omega\times\frac{\Omega-\Omega}{\varepsilon}} \varphi(x, x+\varepsilon z)\,d\big((\pi_1,T_\varepsilon)\,\#\,\big[\sigma_\varepsilon^*\big]^+\big)(x,z).\end{aligned}$$

Also, since

$$\big[\varepsilon\sigma_\varepsilon^*\big]^+ = \big[\varepsilon\sigma_\varepsilon^*\big]^+ \llcorner \big\{(x,y) \in \Omega\times\Omega:\ u_\varepsilon^*(x) - u_\varepsilon^*(y) = \varepsilon,\ |x-y| \leqslant \varepsilon\big\},$$

and $(\pi_1, T_\varepsilon)$ is one to one and continuous, we have that, setting $\mu_\varepsilon := (\pi_1, T_\varepsilon)\,\#\,[\varepsilon\sigma_\varepsilon^*]^+$,

$$\mu_\varepsilon = \mu_\varepsilon \llcorner (\pi_1, T_\varepsilon)\big(\big\{(x,y) \in \Omega\times\Omega:\ u_\varepsilon^*(x) - u_\varepsilon^*(y) = \varepsilon,\ |x-y| \leqslant \varepsilon\big\}\big),$$

that is,

$$\mu_\varepsilon = \mu_\varepsilon \llcorner \big\{(x,z):\ x \in \Omega,\ x+\varepsilon z \in \Omega,\ |z| \leqslant 1,\ u_\varepsilon^*(x) - u_\varepsilon^*(x+\varepsilon z) = \varepsilon\big\}.$$

Therefore, we can rewrite (4.12) as

$$\int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_{\Omega\times\overline{B}_1(0)} \frac{\xi(x) - \xi(x+\varepsilon z)}{\varepsilon}\,d\mu_\varepsilon(x,z). \tag{4.13}$$

On the other hand, by (4.4), $\mu_\varepsilon$ is bounded by a constant independent of $\varepsilon$. Therefore there exists a subsequence $\varepsilon_n \to 0$ such that

$$\mu_{\varepsilon_n} \rightharpoonup \vartheta \quad \text{weakly as measures,} \tag{4.14}$$

with

$$\vartheta = \vartheta \llcorner \big\{(x,z):\ x \in \Omega,\ |z| \leqslant 1\big\}.$$

Then, taking limit in (4.13), for $\varepsilon = \varepsilon_n$, as $n$ goes to infinity, we obtain

$$\int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_{\Omega\times\overline{B}_1(0)} \nabla\xi(x)\cdot(-z)\,d\vartheta(x,z). \tag{4.15}$$

Now, by disintegration of the measure $\vartheta$ (see [2]),

$$\vartheta = (\vartheta)_x \otimes \mu,$$

with

$$\mu = \pi_1 \,\#\, \vartheta,$$

that is a non-negative measure. Moreover, if we define

$$\nu(x) := \int_{\overline{B}_1(0)} (-z)\,d(\vartheta)_x(z), \quad x \in \Omega,$$

then, $\nu \in L^1_\mu(\Omega, \mathbb{R}^N)$ and we can rewrite (4.15) as

$$\int_\Omega \xi(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_\Omega \nabla\xi(x)\cdot\nu(x)\,d\mu(x), \quad \forall\xi \in C_c^1(\Omega). \tag{4.16}$$

Let us see that

$$\operatorname{supp}(\mu) \Subset \Omega. \tag{4.17}$$

The proof of (4.17) follows the argument of [1, Lemma 5.1] (we include this argument here for the sake of completeness). In fact, let $x_0 \in \operatorname{supp}(f^+ - f^-)$ be a minimum point for the restriction of $u^*$ to $\operatorname{supp}(f^+ - f^-)$ and define

$$w(x) := \min\{\big(u^*(x) - u^*(x_0)\big)^+,\ \operatorname{dist}\big(x, \Omega \setminus \Omega'\big)\},$$

where $\Omega'$ verifies (4.10). Then, $w(x) = u^*(x) - u^*(x_0)$ on $\operatorname{supp}(f^+ - f^-)$ and $w \equiv 0$ on $\Omega \setminus \Omega'$. On the other hand,

$$\begin{aligned}\mu(\Omega) = \vartheta(\Omega \times \mathbb{R}^N) &\leqslant \liminf_{\varepsilon\to 0} \mu_\varepsilon(\Omega \times \mathbb{R}^N) \leqslant \liminf_{\varepsilon\to 0} \varepsilon[\sigma_\varepsilon^*]^+(\Omega \times \mathbb{R}^N)\\ &= \liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon^*(x)(f^+(x) - f^-(x))\,dx\\ &= \int_\Omega u^*(x)(f^+(x) - f^-(x))\,dx, \end{aligned}\tag{4.18}$$

and, for a regularizing sequence $\{\rho_{\frac{1}{n}}\}$, on account of (4.16) and using that $|\nu(x)| \leqslant 1$, we have

$$\begin{aligned}\int_\Omega u^*(x)\big(f^+(x) - f^-(x)\big)\,dx &= \int_\Omega \big(u^*(x) - u^*(x_0)\big)\big(f^+(x) - f^-(x)\big)\,dx\\ &= \lim_n \int_\Omega (w * \rho_{\frac{1}{n}})(x)\big(f^+(x) - f^-(x)\big)\,dx\\ &= \lim_n \int_\Omega \nabla(w * \rho_{\frac{1}{n}})(x) \cdot \nu(x)\,d\mu(x) \leqslant \mu\big(\Omega''\big),\end{aligned}$$

where $\Omega''$ verifies (4.10). So, $\mu(\Omega \setminus \Omega'') = 0$, and (4.17) is satisfied.

Let us now recall some tangential calculus for measures (see [7,8]). We introduce the tangent space $\mathcal{T}_\mu$ to the measure $\mu$ which is defined $\mu$-a.e. by setting $\mathcal{T}_\mu(x) := \mathcal{N}_\mu^\perp(x)$ where:

$$\mathcal{N}_\mu(x) = \{\xi(x):\ \xi \in \mathcal{N}_\mu\} \quad \text{being}$$
$$\mathcal{N}_\mu = \{\xi \in L_\mu^\infty\big(\Omega, \mathbb{R}^N\big):\ \exists u_n \text{ smooth}, u_n \to 0 \text{ uniformly},\ \nabla u_n \rightharpoonup \xi \text{ weakly}^* \text{ in } L_\mu^\infty\}.$$

In [7], given $u \in \mathcal{D}(\Omega)$, for $\mu$-a.e. $x \in \Omega$, the tangential derivative $\nabla_\mu u(x)$ is defined as the projection of $\nabla u(x)$ on $\mathcal{T}_\mu(x)$. Now, by [8, Proposition 3.2], there is an extension of the linear operator $\nabla_\mu$ to $\operatorname{Lip}_1(\Omega, d_{|\cdot|})$ the set of Lipschitz continuous functions. Let us see that

$$\nu(x) \in \mathcal{T}_\mu(x), \quad \mu\text{-a.e. } x \in \Omega. \tag{4.19}$$

For that we need to show that

$$\int_\Omega \nu(x) \cdot \xi(x)\,d\mu(x) = 0, \quad \forall \xi \in \mathcal{N}_\mu. \tag{4.20}$$

In fact, given $\xi \in \mathcal{N}_\mu$, there exists $u_n$ smooth, $u_n \to 0$ uniformly, $\nabla u_n \rightharpoonup \xi$ weakly* in $L^\infty_\mu$. Then, taking $\xi = u_n$ in (4.16), which is possible on account of (4.17), we obtain

$$\int_\Omega u_n(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_\Omega \nabla u_n(x)\cdot \nu(x)\,d\mu(x),$$

from here, taking limit as $n \to +\infty$, we get

$$\int_\Omega v(x)\xi(x)\cdot \nu(x)\,d\mu(x) = 0, \quad \forall v \in D(\Omega),$$

from where (4.20) follows. Now, if we set $\Phi := \nu\mu$, by (4.16) we have

$$-\mathrm{div}(\Phi) = f^+ - f^- \quad \text{in } \mathcal{D}'(\Omega).$$

Then, having in mind (4.19), by [8, Proposition 3.5], we get

$$\int_\Omega u^*(x)\big(f^+(x) - f^-(x)\big)\,dx = \int_\Omega \nu(x)\nabla_\mu u^*(x)\,d\mu(x), \tag{4.21}$$

where $\nabla_\mu u^*$ is the tangential derivative. Then, since $|\nu(x)| \leqslant 1$ and $|\nabla_\mu u^*(x)| \leqslant 1$ for $\mu$-a.e. $x \in \Omega$, from (4.21) and (4.18), we obtain that $\nu(x) = \nabla_\mu u^*(x)$ and $|\nabla_\mu u^*(x)| = 1$, $\mu$-a.e. $x \in \Omega$. Therefore, we have

$$\begin{cases} -\mathrm{div}\big(\mu\nabla_\mu u^*\big) = f^+ - f^- & \text{in } \mathcal{D}'(\Omega), \\ \big|\nabla_\mu u^*(x)\big| = 1 & \mu\text{-a.e. } x \in \Omega. \end{cases}$$

Now, by the regularity results given in [12] (see also [1] and [13]), since $f^+, f^- \in L^\infty(\Omega)$, we have that the transport density $\mu \in L^\infty(\Omega)$. Consequently we conclude that the density transport of Evans–Gangbo is represented by $a = \pi_1 \# \vartheta$ for any $\vartheta$ obtained as in (4.14).

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