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Parabolic Maximal Functions and Potentials of Distributions in H^p

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1. NOTATION AND STATEMENT OF THE MAIN RESULTS

By $x, y,..., x = (x_1,..., x_n)$ we denote points in the n-dimensional Euclidean space R^n. Given an n-tuple $a = (a_1,..., a_n)$ of real numbers $a_i \geqslant 1$, $1 \leqslant i \leqslant n$, we will consider the multiplicative group of matrices

$$A_t = \begin{bmatrix} t^{a_1} & & 0 \\ & \ddots & \\ 0 & & t^{a_n} \end{bmatrix}, \qquad t > 0.$$

If $x \neq 0$ there exists a unique $t \in R$ such that $|A_{t^{-1}}x| = 1$ (cf. [1]); then we define $[x] = t$. If $x = 0$ we set $[x] = 0$. Therefore, the parabolic metric given by $d(x, y) = [x - y]$ is naturally attached to the group of matrices A_t.

The following properties are satisfied (cf. [1]):

(i) $[A_t x] = t[x]$, $t > 0$, $x \in R^n$,

(ii) $[x] \in C^\infty(R^n \backslash \{0\})$,

(iii) $[x + y] \leqslant [x] + [y]$, and

(iv) $|x_j| \leqslant [x]^{a_j}$ for every $x \in R^n$, $i \leqslant j \leqslant n$.

If $\alpha = (\alpha_1,..., \alpha_n)$, where the α_j are nonnegative integers, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$,

$$D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f \qquad \text{and} \qquad \alpha \cdot a = \alpha_1 a_1 + \cdots + \alpha_n a_n.$$

Let L^q_{loc}, $1 < q < \infty$, be the space of all the real functions defined in R^n that are locally in L^q. We set $B(x, \rho) = \{y \in R^n : [y - x] < \rho\}$ and it is easy to verify that the Lebesgue measure $|B(x, \rho)|$ equals $C\rho^{|a|}$ (cf. [10]), where $|a| = a_1 + \cdots + a_n$ and C is a constant depending only on a.

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We will consider in L^q_{loc} the topology given by the L^q convergence over compact sets which is induced by the family of seminorms

$$|f|_{q,B} = \left(|B|^{-1} \int_B |f(y)|^q \, dy\right)^{1/q},$$

where $B = B(x, \rho)$, $\rho > 0$, $x \in R^n$.

Let u be a positive real number. If $f \in L^q_{\mathrm{loc}}$, we define a maximal function $n_{q,u}(f, x)$ as

$$n_{q,u}(f, x) = \sup_{\rho > 0} \rho^{-u} |f|_{q.B(x,\rho)}.$$

By $\mathscr{P}_u$ we will denote the subspace of L^q_{loc} which consists of all polynomial functions of the form

$$P(y) = \sum_{\alpha \cdot a < u} a_\alpha y^\alpha.$$

This subspace has finite dimension and, therefore, is a closed subspace of L^q_{loc}. The quotient space of L^q_{loc} by $\mathscr{P}_u$ will be called E^q_u. For $F \in E^q_u$ we define the family of seminorms

$$\|F\|_{q,B} = \inf\{|f|_{q,B} : f \in F\},$$

where $B = B(x, \rho)$, $\rho > 0$, $x \in R^n$. This family of seminorms induce the quotient topology in E^q_u which is a locally convex and complete metric space. For $F \in E^q_u$, we define the maximal function

$$N_{q,u}(F, x) = \inf\{n_{q,u}(f, x) : f \in F\}.$$

This maximal function is lower semicontinuous as we can see following the proof in [4] for the elliptic case.

We will call $\mathscr{H}^p_{q,u}$, $0 < p \leqslant 1$, the set of all $F \in E^q_u$ such that its maximal function $N_{q,u}(F, x)$ belongs to L^p.

For the sake of simplicity we will denote $N = N_{q,u}$, $n = n_{q,u}$, and $\mathscr{H}^p = \mathscr{H}^p_{q,u}$, whenever this notation does not bring up any confusion.

Given $F \in \mathscr{H}^p$, we define

$$\|F\|_{\mathscr{H}^p} = \left(\int N(F, x)^p \, dx\right)^{1/p}.$$

The set $\mathscr{H}^p$ with the distance $d(F, G) = \|F - G\|^p_{\mathscr{H}^p}$ is a complete metric space.

As usual, we denote by $\mathscr{S}$ the space of all infinitely differentiable functions which are rapidly decreasing at infinity together with their derivatives. Given j, h nonnegative integers and $\phi \in \mathscr{S}$ we define

$$p_{j,h}(\phi) = \max_{\alpha \cdot a \leqslant h} \sup_{x \in R^n} |D^\alpha \phi(x)| \, (1 + [x])^k.$$

This family of norms $p_{j,h}$ defines the usual topology of the space $\mathscr{S}$. The letter C will stand for a constant, not necessarilly the same in each occurrence.

(1.1) DEFINITION. A class $A \in E_u^q$ is a p-atom in E_u^q if there exists a member b of A and a ball B such that $\operatorname{supp} b \subset B$ and $N(A, x) \leqslant |B|^{-1/p}$.

In Section 2 we will prove the following characterization of the space $\mathscr{H}^p_{q,u}$:

THEOREM 1. (i) *If $p \leqslant |a| \, (u + |a|/q)^{-1}$, then the space $\mathscr{H}^p$ reduces to* 0.

(ii) *Let p be such that $|a| \, (u + |a|/q)^{-1} < p \leqslant 1$. If $F \in E_u^q$ then $F \in \mathscr{H}^p$ if and only if there exist a numerical sequence $\{\mu_j\}$ such that $\sum_j |\mu_j|^p < \infty$ and a sequence $\{A_j\}$ of p-atoms in E_u^q such that*

$$F = \sum_j \mu_j A_j \qquad in \quad E_u^q.$$

Moreover, this series converges in $\mathscr{H}^p$ and there exist two positive constants C_1 and C_2 such that

$$C_1 \|F\|^p_{\mathscr{H}^p} \leqslant \inf \sum_j |\mu_j|^p \leqslant C_2 \|F\|^p_{\mathscr{H}^p},$$

where the infimum is taken over all decompositions of F.

Section 3 deals with the connection between $\mathscr{H}^p$ and the space H^p of Calderón–Torchisnky (cf. [1]) when $a = (a_1, ..., a_n)$ has rational components.

Let k be the smallest positive integer such that k/a_i is an even number for every i. We denote by L the differential operator associated with $P(\xi) = \xi_1^{k/a_1} + \cdots + \xi_n^{k/a_n}$, that is, $Lf = (P(\xi)\hat{f})^\vee$, where $f \in \mathscr{S}'$ and $\hat{f}, \check{f}$ stand for the Fourier transform an its inverse, respectively.

Given $\phi \in \mathscr{S}$ such that $\int \phi(x)\, dx \neq 0$ and $f \in \mathscr{S}'$, we set $f^*(x) = \sup_{[x-y]<t} |f * \phi_t(y)|$, where $\phi_t(x) = t^{-|a|} \phi(A_t^{-1} x)$. The space of all tempered distributions f such that $f^* \in L^p$ is called H^p and it is defined $\|f\|^p_{H^p} = \int f^{*p}(x)\, dx$ (cf. [1]).

We will prove

THEOREM 2. *If $|a|/p < km + |a|/q$, then the differential operator L^m is an isomorphism between $\mathscr{H}^p_{q,km}$ and H^p.*

2. Proof of Theorem 1

For the proof of this theorem we need the following lemmas:

(2.1) LEMMA. *Let f_1 and f_2 be two members of the class $F \in E_u^q$. If $P = f_1 - f_2$ then for every α there exists a constant C_α such that*

$$|D^\alpha P(y)| \leqslant C_\alpha(n(f_1, x_1) + n(f_2, x_2))([x_1 - y] + [x_2 - y])^{u-\alpha\cdot a},$$

for every $x_1, x_2, y \in R^n$.

Proof. Let $\phi \in C^\infty$ with $\operatorname{supp}\phi \subset \{[x] \leqslant 1\}$ such that if $\phi_\lambda(x) = \lambda^{|a|}\phi(A_\lambda x)$ then $Q = Q * \phi_\lambda$ for every $Q \in \mathscr{P}_u$ and every $\lambda > 0$; for the existence of such ϕ cf. [5]. Differentiating $P = P * \phi_\lambda$ we have

$$D^\alpha P(y) = \lambda^{|a|+\alpha\cdot a}\int_{[y-z]<\lambda^{-1}} (f_1(z) - f_2(z))(D^\alpha\phi)(D^\alpha\phi)(A_\lambda(y-z))\,dz.$$

If $\rho = 2[y - x_1] + 2[y - x_2] = 2\lambda^{-1}$ we have

$$|D^\alpha P(y)| \leqslant \lambda^{|a|+\alpha\cdot a}\int_{[x_1-z]<\rho} |f_1(z)|\,|(D^\alpha\phi)(A_\lambda(y-z))|\,dz$$

$$+ \lambda^{|a|+\alpha\cdot a}\int_{[x_2-z]<\rho} |f_2(z)|\,|(D^\alpha\phi)(A_\lambda(y-z))|\,dz.$$

Thus, applying Hölder's inequality to these integrals we obtain the desired result.

(2.2) LEMMA. *The following properties are satisfied.*

(i) *Given $F \in E_u^q$ and $x_0 \in R^n$ such that $N(F, x_0) < \infty$, there exists a unique $f \in F$ such that $n(f, x_0) < \infty$ and then $n(f, x_0) = N(F, x_0)$.*

(ii) *If $\{F_j\}$ is a sequence of elements of E_u^q and F_j converges to F in $\mathscr{H}^p$ for some p, $0 < p \leqslant 1$, then F_j converges to F in E_u^q.*

(iii) *If $\{F_j\}$ is a sequence of elements of E_u^q and there exists $x_0 \in R^n$ such that $\sum N(F_j, x_0) < \infty$ then $\sum F_j$ converges in E_u^q to an element F and $N(F, x_0) \leqslant \sum_j N(F_j, x_0)$. Moreover, if $f_j \in F_j$ is such that $n(f_j, x_0) = N(F_j, x_0)$ then $\sum f_j$ converges in L_{loc}^q to the function $f \in F$ which satisfies $n(f, x_0) = N(F, x_0)$.*

(iv) *The space $\mathscr{H}^p$ is complete.*

For the proof of this lemma cf. [2].

(2.3) LEMMA. *Let f be a function with compact support such that for*

$|\alpha| < u+1$, $D^\alpha f$ is a continuous function. Let us denote by F the class of f in E^q_u. Then there exists a real number λ such that λF is a p-atom in E^q_u.

Proof. First, we prove that $N(F,x) \in L^\infty$. This follows immediately if we prove first the inequality

$$\left| f(y) - \sum_{\alpha \cdot a < u} D^\alpha f(x)(y-x^\alpha)/\alpha! \right| \leqslant C[y-x]^u.$$

If $[y-x] \leqslant 1$, this inequality is obtained by applying Taylor's formula. In fact,

$$\left| f(y) - \sum_{\alpha \cdot a < u} D^\alpha f(x)(y-x)^\alpha/\alpha! \right|$$

$$= \left| \sum_{\substack{|\alpha| < u \\ \alpha \cdot a \geqslant u}} D^\alpha f(x)(y-x)^\alpha/\alpha! + \sum_{u \leqslant |\alpha| < u+1} D^\alpha f(x+\Theta(y-x))(y-x)^\alpha/\alpha! \right|$$

$$\leqslant C[y-x]^u.$$

On the other hand, if $[y-x] \geqslant 1$, we have

$$\left| f(y) - \sum_{\alpha \cdot a < u} D^\alpha f(x)(y-x)^\alpha/\alpha! \right| \leqslant \|f\|_\infty + \sum_{\alpha \cdot a < u} \|D^\alpha f\|_\infty [y-x]^{\alpha \cdot a}/\alpha!$$

$$\leqslant C[y-x]^u.$$

Let B be a ball such that $\operatorname{supp} f \subset B$ and let C_1 be a constant such that $N(F,x) \leqslant C_1$. If $\lambda = |B|^{-1/p} C_1^{-1}$ then it follows easily that λF is a p-atom in E^q_u.

(2.4) Lemma (Partition of unity). *Let Ω be a proper subset of R^n. There exists a sequence $\{\phi_k\}$ of functions C^∞ with compact support which satisfies:*

(i) *$0 \leqslant \phi_k(x) \leqslant 1$ and $\sum_k \phi_k(x) = \chi_\Omega(x)$;*

(ii) *for every k, there is a ball $B_k = B(x_k, r_k) \subset \Omega$ such that $\operatorname{supp} \phi_k \subset B_k$ and for every $z \in B_k$, $r_k \leqslant d(z, \Omega^c) \leqslant C r_k$;*

(iii) *for every k we have $B(x_k, 2r_k) \subset \Omega$, moreover, there exists an integer M such that the number of balls $B(x_j, 2r_j)$ which intersect $B(x_k, 2r_k)$ is not greater than M;*

(iv) *for every α we have $|D^\alpha \phi_k(x)| \leqslant C_\alpha r_k^{-\alpha \cdot a}$ with c_α independent of k.*

Proof. For the existence of the family $B(x_k, r_k)$ cf. [6], and the partition of unity is obtained in the same way as in [9].

(2.5) LEMMA. *Let p be such that $|a|(u+|a|/q)^{-1}<p\leqslant 1$, and let $F\in\mathscr{H}^p$. Given $t>0$ let $\Omega=\Omega_t=\{x:N(F,x)>t\}$; Ω is an open set because $N(F,x)$ is lower semicontinuous. Let $\{\phi_k\}$ be the partition of unity associated with Ω in Lemma (2.4). For every k, let $y_k\in\Omega^c$ such that $d(B(x_k,2r_k),\Omega^c)\cong d(B(x_k,2r_k),y_k)$. Given a member f of the class F, by Lemma (2.2), there exists a polynomial $P(y_k,y)$ in $\mathscr{P}_u$ which satisfies,*

$$N(F,y_k)=n(f(y)-P(y_k,y),y_k).$$

For every k, we set

$$w_k(y)=\phi_k(y)(f(y)-P(y_k,y)),$$

and we denote by W_k the class of w_k in E_u^q. Then, the following conditions are satisfied:

(i) *$N(W_k,x)\leqslant CN(F,x)$ if $x\in B(x_k,2r_k)$;*

(ii) *$N(W_k,x)\leqslant Ct(r_k/(r_k+[x-x_k]))^{u+|a|/q}$ if $x\notin B(x_k,2r_k)$;*

(iii) *the series $\sum_k N(W_k,x)$ converges almost everywhere in R^n, moreover,*

$$\int\left(\sum_k N(W_k,x)\right)^p dx\leqslant\sum_k\int N(W_k,x)^p\,dx\leqslant C\int_\Omega N(F,x)^p\,dx;$$

(iv) *the series $\sum_k W_k=W$ converges in E_u^q and we have $N(W,x)\leqslant\sum_k N(W_k,x)$ almost everywhere;*

(v) *$\int N(W,x)^p\,dx\leqslant C\int_\Omega N(F,x)^p\,dx$; and*

(vi) *if $G=F-W$ then $N(G,x)\leqslant Ct$.*

Proof. (i) We assume $N(F,x)<\infty$, since otherwise the inequality is trivial. For every x, let $P(x,y)$ be the polynomial which satisfies

$$n(f(y)-P(x,y),x)=N(F,x).$$

We set

$$Q_k(x,y)=\sum_{\alpha\cdot a<u}D_y^\alpha[\phi_k(y)(P(x,y)-P(y_k,y))]_{y=x}(y-x)^{\alpha}/\alpha!$$

$$=\sum_{\alpha\cdot a<u}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}D_y^{\alpha-\gamma}\phi_k(y)\,D_y^\gamma[P(x,y)-P(y_k,y)]|_{y=x}(y-x)^\alpha/\alpha!.$$

Let us estimate $\rho^{-u}[\rho^{-|a|}\int_{[y-x]<\rho}|w_k(y)-Q_k(x,y)|^q\,dy]^{1/q}$. By Lemma (2.1) and taking into account that $[x_k-y_k]\leqslant Cr_k$ and that $N(F,y_k)\leqslant t<N(F,x)$ we have

$$|D_y^\alpha(P(x,y)-P(y_k,y))|\leqslant CN(F,x)(\rho+r_k)^{u-\alpha\cdot a}. \tag{2.6}$$

Assume $\rho \geqslant 2r_k$; in this case,

$$|w_k(y) - Q_k(x,y)| \leqslant |\phi_k(y)(f(y) - P(x,y))| \\ + |\phi_k(y)(P(x,y) - P(y_k,y))| + |Q_k(x,y)|.$$

By (2.6), we have

$$|\phi_k(y)(P(x,y) - P(y_k,y))| \leqslant CN(F,x)\,\rho^u.$$

On the other hand, by Lemma (2.1), we obtain

$$|D_y^\alpha(P(x,y) - P(y_k,y))|_{y=x}| \leqslant CN(F,x)[x - y_k]^{u-\alpha\cdot a} \leqslant CN(F,x)\, r_k^{u-\alpha\cdot a}.$$

Therefore, since $[y - x] < \rho$ and $\rho/r_k \geqslant 2$, we have

$$|Q_k(x,y)| \leqslant \sum_{\alpha\cdot a<u} \sum_{\gamma\leqslant\alpha} Cr_k^{-\alpha\cdot a+\gamma\cdot a}N(F,x)\, r_k^{u-\gamma\cdot a}\rho^{\alpha\cdot a} \\ \leqslant CN(F,x)\,\rho^u.$$

Then for $\rho \geqslant 2r_k$, the following inequality is satisfied:

$$|w_k(y) - Q_k(x,y)| \leqslant C\,|f(y) - P(x,y)| + CN(F,x)\,\rho^u.$$

Now we consider the case $\rho < 2r_k$. By definition of $Q_k(x,y)$, we have

$$w_k(y) - Q_k(x,y) = \phi_k(y)(f(y) - P(y_k,y)) - \sum_{\beta\cdot a<u} \Bigg[D^\beta\phi_k(x)((y-x)^\beta/\beta!) \\ \times \sum_{\gamma\cdot a<u-\beta\cdot a} D_y^\gamma(P(x,y) - P(y_k,y))|_{y=x}\,(y-x)^\gamma/\gamma! \Bigg].$$

Adding and substracting the expression

$$\phi_k(y)\,P(x,y) + \sum_{\beta\cdot a<u} D^\beta\phi_k(x)((y-x)^\beta/\beta!)(P(x,y) - P(y_k,y))$$

we obtain

$$|w_k(y) - Q_k(x,y)| \leqslant |f(y) - P(x,y)| + A_1 + A_2,$$

where

$$A_1 = \left| \phi_k(y) - \sum_{\beta\cdot a<u} D^\beta\phi_k(x)(y-x)^\beta/\beta! \right| |P(x,y) - P(y_k,y)|$$

and

$$A_2 = \left| \sum_{\beta \cdot a < u} D^\beta \phi_k(x)((y-x)^\beta/\beta!) \left[P(x,y) - P(y_k, y) \right.\right.$$

$$\left.\left. - \sum_{\gamma \cdot a < u - \beta \cdot a} D_y^\gamma (P(x,y) - P(y_k, y))|_{y=x} (y-x)^\gamma/\gamma! \right] \right| .$$

By (2.6) and applying Taylor's formula we have

$$A_1 \leqslant CN(F,x)\, r_k^u \left| \sum_{\substack{\beta \cdot a \geqslant u \\ |\beta| < u}} D^\beta \phi_k(x)(y-x)^\beta/\beta! \right.$$

$$\left. + \sum_{u \leqslant |\beta| < u+1} D^\beta \phi_k(y_0)(y-x)^\beta/\beta! \right| ,$$

where y_0 belongs to the segment joining x and y.

Since $\rho/2r_k < 1$, it follows that

$$A_1 \leqslant CN(F,x)\, r_k^u \sum_{\substack{\beta \cdot a \geqslant u \\ |\beta| < u+1}} r_k^{-\beta \cdot a} \rho^{\beta \cdot a} \leqslant CN(F,x)\, \rho^u .$$

Applying Taylor's formula in A_2 we obtain

$$A_2 \leqslant C \sum_{\beta \cdot a < u} r_k^{-\beta \cdot a} \rho^{\beta \cdot a} \left| \sum_{\substack{u - \beta \cdot a \leqslant \gamma a < u \\ |\gamma| < u - \beta \cdot a}} D_y^\gamma (P(x,y) - P(y_k, y))|_{y=x} (y-x)^\gamma/\gamma! \right.$$

$$\left. + \sum_{\substack{u - \beta \cdot a \leqslant |\gamma| < u - \beta \cdot a + 1 \\ \gamma \cdot a < u}} D_y^\gamma (P(x,y) - P(y_k, y))|_{y-y_0} (y-x)^\gamma/\gamma! \right| ,$$

where y_0 belongs to the segment joining x and y.

Since $[y_0 - x] \leqslant \rho$ and $[y_0 - y_k] \leqslant Cr_k$, then

$$A_2 \leqslant C \sum_{\beta \cdot a < u} \sum_{\substack{u - \beta \cdot a \leqslant \gamma \cdot a < u \\ |\gamma| < u - \beta \cdot a + 1}} N(F,x)(\rho/r_k)^{\gamma \cdot a + \beta \cdot a}\, r_k^u \leqslant CN(F,x)\, \rho^u .$$

Therefore, for every $\rho > 0$ and for $[y-x] < \rho$ we have

$$|w_k(y) - Q_k(x,y)| \leqslant |f(y) - P(x,y)| + CN(F,x)\, \rho^u .$$

Then

$$n(w_k(y) - Q_k(x,y), x) \leqslant CN(F,x)$$

and (i) is proved.

For the proof of (ii), (iii), (iv), and (v) cf. [2].

Now we prove (vi). Let $x_0 \notin \Omega$ such that $\sum_k N(W_k, x_0) < \infty$. Since $x_0 \notin B(x_k, 2r_k)$ for $k = 1, 2, \ldots$, we know that w_k is the unique member of the class W_k which satisfies $n(w_k, x_0) = N(W_k, x_0)$.

Then, by (iii) of Lemma (2.2) the series $\sum_k w_k$ converges in L^q_{loc} to a function w which is the member of the class $W = \sum_k W_k$ which satisfies $n(w, x_0) = N(W, x_0)$.

Therefore, the function $g = f - w$ is a member of the class $G = F - W$ and we have

$$\begin{aligned} g(y) &= f(y) && \text{if } \; y \in \Omega^c, \\ &= \sum_k \phi_k(y)\, P(y_k, y) && \text{if } \; y \in \Omega. \end{aligned}$$

We observe that g is an infinitely differentiable function in Ω. Let

$$\begin{aligned} b_\alpha(x) &= D^\alpha g(x) && \text{if } \; x \in \Omega, \\ &= D^\alpha_y P(x, y)|_{y=x} && \text{if } \; x \in \Omega^c. \end{aligned}$$

We will prove that for $\alpha \cdot a \leqslant u$, $x \in \Omega^c$, and $\bar{x} \in R^n$ we have

$$\left| b_\alpha(\bar{x}) - \sum_\beta b_{\alpha+\beta}(x)(\bar{x} - x)^\beta/\beta! \right| \leqslant Ct[\bar{x} - x]^{u - \alpha \cdot a}. \tag{2.7}$$

In fact, if $\bar{x} \in \Omega^c$ we know by Lemma (2.1) that

$$|D^\alpha_y(P(\bar{x}, y) - P(x, y))| \leqslant Ct([\bar{x} - y] + [x - y])^{u - \alpha \cdot a}$$

and, taking $y = \bar{x}$, we have

$$\left| b_\alpha(\bar{x}) - \sum_\beta b_{\alpha+\beta}(x)(\bar{x} - x)^\beta/\beta! \right| \leqslant Ct[\bar{x} - x]^{u - \alpha \cdot a}.$$

Now we consider $\bar{x} \in \Omega$. Let j be such that $\bar{x} \in \operatorname{supp} \phi_j$ and $[y_j - \bar{x}] \leqslant [y_k - \bar{x}]$ for every k such that $\bar{x} \in \operatorname{supp} \phi_k$. Then

$$\begin{aligned} &D^\alpha g(\bar{x}) - D^\alpha_y P(x, y)|_{y=\bar{x}} \\ &\quad = \sum_k \left[\sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\beta \phi_k(\bar{x})(D^\gamma_y P(y_k, y)|_{y=\bar{x}} - D^\gamma_y P(y_j, y)|_{y=\bar{x}}) \right] \\ &\qquad + [D^\alpha_y P(y_j, y)|_{y=\bar{x}} - D^\alpha_y P(x, y)|_{y=\bar{x}}]. \end{aligned}$$

Therefore, applying Lemma (2.1) and taking into account that $[y_k - \bar{x}] + [y_j - \bar{x}] \leqslant Cr_k$, $[\bar{x} - y_j] \leqslant [\bar{x} - x]$, and $r_k \leqslant [\bar{x} - x]$ we get

$$|D^\alpha g(\bar{x}) - D^\alpha_y P(x, y)|_{y=\bar{x}}| \leqslant Ct[\bar{x} - x]^{u - \alpha \cdot a}.$$

Then (2.7) is satisfied for every $\bar{x} \in R^n$.

Next, we will prove that for every $x \in \Omega$ and every $\bar{x} \in R^n$ the following inequality is satisfied:

$$\left| b_0(\bar{x}) - \sum_{\alpha \cdot a < u} b_\alpha(x)(\bar{x} - x)^\alpha/\alpha! \right| \leqslant Ct|\bar{x} - x|^u. \tag{2.8}$$

In order to prove (2.8) we need the estimate

$$|D^\alpha g(x)| \leqslant Ct\, d(x, \Omega^c)^{u - \alpha \cdot a} \tag{2.9}$$

for every $x \in \Omega$ and for $\alpha \cdot a \geqslant u$. In fact, if $x' \in \Omega^c$ and $|x - x'| = d(x, \Omega^c)$ then

$$\begin{aligned} D^\alpha g(x) &= \sum_k \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\beta \phi_k(x)\, D_y^\gamma P(y_k, y)|_{y=x} - D_y^\alpha P(x', y)|_{y=x} \\ &= \sum_k \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\beta \phi_k(x) [D_y^\gamma (P(y_k, y) - P(x', y))|_{y=x}]. \end{aligned}$$

Again applying Lemma (2.1) and taking into account that $|x' - x| = d(x, \Omega^c) \leqslant Cr_k$ and $[y_k - x] \leqslant Cr_k$, we obtain (2.9).

Now we prove (2.8). We consider the cases $[x - \bar{x}] \leqslant \frac{1}{2} d(x, \Omega^c)$ and $[x - \bar{x}] > \frac{1}{2} d(x, \Omega^c)$. In the first case, applying Taylor's formula we have

$$\begin{aligned} b_0(\bar{x}) &- \sum_{\alpha \cdot a < u} b_\alpha(x)(\bar{x} - x)^\alpha/\alpha! \\ &= \sum_{\substack{|\alpha| < u \\ \alpha \cdot a \geqslant u}} b_\alpha(x)(\bar{x} - x)^\alpha/\alpha! + \sum_{u \leqslant |\alpha| < u+1} b_\alpha(x + s(\bar{x} - x))(\bar{x} - x)^\alpha/\alpha!, \end{aligned}$$

where $s \in [0, 1]$.

As $d(x + s(\bar{x} - x), \Omega^c) \geqslant \frac{1}{2} d(x, \Omega^c)$, applying (2.9) we get

$$\begin{aligned} &\left| b_0(\bar{x}) - \sum_{\alpha \cdot a < u} b_\alpha(x)(\bar{x} - x)^\alpha/\alpha! \right| \\ &\qquad \leqslant Ct \sum_{\substack{\alpha \cdot a \geqslant u \\ |\alpha| < u+1}} d(x, \Omega^c)^{u - \alpha \cdot a} [\bar{x} - x]^{\alpha \cdot a} < Ct[\bar{x} - x]^u. \end{aligned}$$

Now we consider the case $[x - \bar{x}] > \frac{1}{2} d(x, \Omega^c)$. Let $z \in \Omega^c$ be such that $[z - x] = d(x, \Omega^c)$. Adding and substracting the expressions

$$\sum_{\alpha \cdot a < u} b_\alpha(z)(\bar{x} - z)^\alpha/\alpha! \qquad \text{and} \qquad \sum_{\alpha \cdot a < u} D_y^\alpha P(z, y)|_{y=x} (\bar{x} - x)^\alpha/\alpha!,$$

and by (2.7) we have

$$\left| b_0(\bar{x}) - \sum_{\alpha \cdot a < u} b_\alpha(z)(\bar{x}-z)^\alpha/\alpha! \right|$$
$$\leqslant Ct[\bar{x}-z]^u \leqslant Ct([z-x]+[x-\bar{x}])^u \leqslant Ct[x-\bar{x}]^u,$$

$$\left| \sum_{\alpha \cdot a < u} [b_\alpha(x) - D_y^\alpha P(z,y)|_{y=x}](\bar{x}-x)^\alpha/\alpha! \right|$$
$$\leqslant Ct \sum_{\alpha \cdot a < u} [x-z]^{u-\alpha \cdot a} [\bar{x}-x]^{\alpha \cdot a} \leqslant Ct[\bar{x}-x]^u,$$

and

$$\sum_{\alpha \cdot a < u} b_\alpha(z)(\bar{x}-z)^\alpha/\alpha! = P(z,\bar{x})$$
$$= \sum_{\alpha \cdot a < u} D_y^\alpha P(z,y)|_{y=x} (\bar{x}-x)^\alpha/\alpha!.$$

Then (2.8) follows. Applying (2.7) and (2.8) and since $b_0 = g$ almost everywhere (cf. [4]), we obtain

$$N(G,x) \leqslant Ct.$$

Proof of Theorem 1. (i) Let $p \leqslant |a|\,(u+|a|/q)^{-1}$ and let $f \notin \mathscr{P}_u$. If F is the class of f in E_u^q, then $N(F,x) \notin L^p$. In fact, since $f \notin \mathscr{P}_u$, there exist a ball $B = B(0,r)$ and a real number $\delta > 0$ such that

$$\left(\int_B |f(y) - P(y)|^q \, dy \right)^{1/q} > \delta \qquad \text{for every} \quad P \in \mathscr{P}_u.$$

On the other hand,

$$n(f-P,x) = \sup_{\rho>0} \rho^{-u} \left(|B(x,\rho)|^{-1} \int_{B(x,\rho)} |f(y)-P(y)|^q \, dy \right)^{1/q}.$$

If $[x] \geqslant r$, then $B(0,r) \subset B(x, 2[x])$. Therefore, taking $\rho = 2[x]$ we have

$$n(f-P,x) \geqslant C[x]^{-(u+|a|/q)} \left(\int_{B(x,2[x])} |f(y)-P(y)|^q \, dy \right)^{1/q}$$
$$\geqslant C\delta[x]^{-(u+|a|/q)} \qquad \text{and then} \quad N(F,x) \notin L^p.$$

(ii) Let $p > |a|\,(u+|a|/q)^{-1}$. We know, by Lemma (2.3) that there exist p-atoms in E_u^q. Moreover, we know that if A is a p-atom in E_u^q, then $\int N(A,x)^p \, dx \leqslant C$, where C is a constant independent of A, (cf. [2]). Therefore, $\mathscr{H}^p$ contains nontrivial elements. If $\{A_i\}$ is a sequence of p-atoms in E_u^q and $\{\mu_i\}$ is a numerical sequence such that $\sum_i |\mu_i|^p < \infty$ then the series

$\sum_i \mu_i A_i$ converges absolutely in $\mathscr{H}^p$. Even more, if we denote by F the sum of this series we have

$$\int N(F, x)^p \, dx \leqslant C \sum_i |\mu_i|^p.$$

Following the same method as in [3] we get the second part of the proof.

3. The Proof of Theorem 2

Let $m \in N$. In the sequel, we will prove some properties of an elementary solution of L^m.

(3.1) Definition. A function f is called quasi-homogeneous of degree l if $f(A_\lambda x) = \lambda^l f(x)$ for every $\lambda > 0$ and every $x \neq 0$.

(3.2) Definition. A distribution T is called quasi-homogeneous of degree l if for every $\phi \in \mathscr{D}$ and every $\lambda > 0$, $\langle T, \phi_\lambda \rangle = \lambda^l \langle T, \phi \rangle$, where $\phi_\lambda(x) = \lambda^{-|a|} \phi(A_\lambda^{-1} x)$.

It is easy to prove that the following properties are verified:

If $T \in \mathscr{S}'$ is a quasi-homogeneous distribution of degree l, then $\hat{T}$ is a quasi-homogeneous distribution of degree $-|a| - l$. (3.3)

If T is a quasi-homogeneous distribution of degree l and there exists a function g continuous in $R^n \backslash \{0\}$ such that $\langle T, \phi \rangle = \int g(x) \phi(x) \, dx$ for every $\phi \in \mathscr{D}(R^n \backslash \{0\})$, then g is a quasi-homogeneous function of degree l. (3.4)

Let $g \in C^\infty(R^n \backslash \{0\})$ be quasi-homogeneous of degree l. Then $D^\alpha g$ is quasi-homogeneous of degree $l - \alpha \cdot a$. Moreover, $|D^\alpha g(x)| \leqslant C_\alpha [x]^{l - \alpha \cdot a}$. (3.5)

(3.6) Lemma. (a) *If $km < |a|$ then $(P(\xi))^{-m}$ is a tempered distribution and $((P(\xi))^{-m})^\vee$ is an elementary solution of L^m and*

(i) *it agrees with a function $h \in L^1_{\mathrm{loc}} \cap C^\infty(R^n \backslash \{0\})$,*

(ii) *h is quasi-homogeneous of degree $km - |a|$.*

(b) *Let $km \geqslant |a|$. We define*

$$\langle T, \phi \rangle = \int_{[\xi] \leqslant 1} \left[\phi(\xi) - \sum_{\beta \cdot a \leqslant km - |a|} D^\beta \phi(0) \, \xi^\beta / \beta! \right] (P(\xi))^{-m} \, d\xi$$
$$+ \int_{[\xi] > 1} \phi(\xi) (P(\xi))^{-m} \, d\xi.$$

Then $\check{T}$ is an elementary solution of L^m and

(i) *it agrees with a function $h \in L^1_{\text{loc}} \cap C^\infty(R^n \backslash \{0\})$,*

(ii) *if $\alpha \cdot a < km - |a| + 1$ then $D^\alpha h \in L^1_{\text{loc}}$,*

(iii) *if $\alpha \cdot a > km - |a|$ then $D^\alpha h$ is a quasi-homogeneous function of degree $km - |a| - \alpha \cdot a$.*

Proof. (a) Since $km < |a|$, $(P(\xi))^{-m} \in L^1_{\text{loc}}$; moreover, it defines a tempered distribution.

In order to prove (i), we show first, that $((P(\xi))^{-m})\check{}$ agrees with a function $h \in C^\infty(R^n \backslash \{0\})$ in the complement of the origin. Let $\Psi \in \mathscr{D}$ be such that $\Psi(\xi) = 1$ in $\{[\xi] \leqslant 1\}$ and $\Psi(\xi) = 0$ in $\{[\xi] \geqslant 2\}$, then

$$((P(\xi))^{-m})\check{} = (\Psi(\xi)(P(\xi))^{-m})\check{} + ((1 - \Psi(\xi))(P(\xi))^{-m})\check{} = h_1 + h_2.$$

Since $\Psi(\xi)(P(\xi))^{-m}$ has compact support, h_1 is an analytic function. Let us prove that h_2 agrees in the complement of the origin with a C^∞ function. Given α and β we have

$$x^\alpha D^\beta h_2 = C_{\alpha,\beta}(D^\alpha[(1 - \Psi(\xi))\, \xi^\beta (P(\xi))^{-m}])\check{}$$

and by (3.5) we obtain

$$|D^\alpha[(1 - \Psi(\xi))\, \xi^\beta (P(\xi))^{-m}]| \leqslant C_{\alpha,\beta}[\xi]^{-km + \beta \cdot a - \alpha \cdot a} \qquad \text{for} \quad [\xi] > 2.$$

If α is such that $-km + \beta \cdot a - \alpha \cdot a < -|a|$ then $D^\alpha[(1 - \Psi(\xi))\, \xi^\beta (P(\xi))^{-m}] \in L^1(R^n)$. Therefore, $x^\alpha D^\beta h_2$ is a continuous and bounded function.

Taking appropriate values of α it follows that $D^\beta h_2$ agrees in the complement of the origin with a continuous function in $R^n \backslash \{0\}$. Therefore, $((P(\xi))^{-m})\check{}$ agrees in the complement of the origin with a function $h \in C^\infty(R^n \backslash \{0\})$. Moreover, by (3.4) we obtain (ii) and by (3.5) $h \in L^1_{\text{loc}}$ and, therefore, $((P(\xi)^{-m})\check{} - h$ defines a distribution supported at $\{0\}$. Then

$$(P(\xi))^{-m} - \hat{h}(\xi) = Q(\xi),$$

where Q is a polynomial. Since $\hat{h}$ vanishes at infinity, then $Q \equiv 0$ and part (a) of the theorem follows.

(b) Let T_1 and T_2 be defined by

$$\langle T_1, \phi \rangle = \int_{[\xi] \leqslant 1} \Big(\phi(\xi) - \sum_{\beta \cdot a \leqslant km - |a|} D^\beta \phi(0)\, \xi^\beta / \beta! \Big) (P(\xi))^{-m}\, d\xi,$$

$$\langle T_2, \phi \rangle = \int_{[\xi] > 1} \phi(\xi)(P(\xi))^{-m}\, d\xi.$$

Then $T = T_1 + T_2$.

We begin with (i). Following the proof of (a), we can prove that $\check{T}$ agrees in the complement of the origin with a function $h \in C^\infty(R^n \backslash \{0\})$. Since T_1 has compact support, $\check{T}_1$ is an analytic function. On the other hand, since $km \geqslant |a|$, $T_2 \in L^2$. and, therefore, $\check{T}_2 \in L^2$. Then $\check{T}$ is a locally integrable function and we have $\check{T} = h$.

In order to prove (ii) we observe that

$$D^\alpha \check{T}_2 = C_\alpha(\xi^\alpha \chi(\xi)(P(\xi))^{-m})^\vee,$$

where $\chi(\xi)$ is the characteristic function of $\{[\xi] > 1\}$. As $\alpha \cdot a - km < 1 - |a|$ and $2 \leqslant |a|$, we obtain $\xi^\alpha \chi(\xi)(P(\xi))^{-m} \in L^2$.

Finally, if $\alpha \cdot a > km - |a|$, then $\xi^\alpha T$ agrees with the function $\xi^\alpha(P(\xi))^{-m}$ which is quasi-homogeneous of degree $\alpha \cdot a - km$. Then by (3.3) and (3.4) we obtain (iii).

(3.7) Lemma. *Let $f \in C^\infty(R^n \backslash \{0\})$ be a quasi-homogeneous function of degree $-|a| + a_j$. Then, $k = \partial f / \partial x_j$ verifies:*

(i) $k \in C^\infty(R^n \backslash \{0\})$,

(ii) *k is quasi-homogeneous of degree $-|a|$, and*

(iii) $\int_{1 \leqslant [x] < 2} k(x)\, dx = 0$.

Then k is a singular integral kernel of parabolic type (cf. [7]).

Proof. Part (i) is obvious. Part (ii) follows immediately from (3.5). In order to prove (iii) we will show first that the following limit exists and it is finite for every $\phi \in \mathscr{D}$,

$$\lim_{\varepsilon \to 0} \int_{[x] > \varepsilon} k(x)\, \phi(x)\, dx. \tag{3.8}$$

We have

$$\left\langle \frac{\partial}{\partial x_j} f, \phi \right\rangle = -\lim_{\varepsilon \to 0} \int_{[x] > \varepsilon} f(x) \frac{\partial}{\partial x_j} \phi(x)\, dx.$$

After the change of variables $A_\varepsilon y = x$, we obtain

$$\left\langle \frac{\partial}{\partial x_j} f, \phi \right\rangle = -\lim_{\varepsilon \to 0} \int_{[y] > 1} f(y) \frac{\partial}{\partial y_j} (\phi(A_\varepsilon y))\, dy.$$

Then by Green's formula we have

$$\left\langle \frac{\partial}{\partial x_j} f, \phi \right\rangle = \lim_{\varepsilon \to 0} \left[\int_{[y] > 1} k(y)\, \phi(A_\varepsilon y)\, dy \right.$$
$$\left. + \int_{[y] = 1} f(y)\, \phi(A_\varepsilon y)\, y_j\, d\sigma(y) \right].$$

Since $\lim_{\varepsilon\to 0}\int_{[y]=1} f(y)\,\phi(A_\varepsilon y)\,y_j\,d\sigma(y) = \phi(0)\int_{[y]=1} f(y)\,y_j\,d\sigma(y)$, we obtain that

$$\lim_{\varepsilon\to 0}\int_{[y]>1} k(y)\,\phi(A_\varepsilon y)\,dy$$

exists. It is easily seen that, after a change of variables, it agrees with (3.8).

Taking an appropriate ϕ it follows that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<[x]<2} k(x)\,dx$$

exists. On the other hand, after a change of variables, we have

$$\int_{1<[x]<2} k(x)\,dx = \int_{\lambda<[x]<2\lambda} k(x)\,dx$$

for every $\lambda > 0$. Taking $\lambda = 2^{-k}$, $k = 1, 2,\ldots$, we get

$$\int_{1<[x]<2} k(x)\,dx = \int_{2^{-k}<[x]<2^{-k+1}} k(x)\,dx.$$

Now

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<[x]<2} k(x)\,dx = \lim_{j\to\infty}\sum_{i=0}^{j}\int_{2^{-i}<[x]<2^{-i+1}} k(x)\,dx$$

$$= \lim_{j\to\infty}\,(j+1)\int_{1<[x]<2} k(x)\,dx;$$

since this limit is finite, (iii) follows.

(3.9) Corollary. *Let h be the elementary solution of L^m which is defined in Lemma* (3.6). *If $\alpha\cdot a = km$ then $D^\alpha h$ is a singular integral kernel of parabolic type.*

(3.10) Lemma. *The differential operator L^m is well defined in E^q_{km} and is injective on $\mathscr{H}^p_{q,km}$.*

Proof. If f_1 and f_2 are two members of the class $F\in E^q_{km}$ then $L^m f_1 = L^m f_2$ because $f_1 - f_2 \in \mathscr{P}_{km}$. Therefore, we may define $L^m F = L^m f$, where f is any member of F. Given $F\in\mathscr{H}^p_{q,km}$ and f a member of F, we know that $f\in\mathscr{S}'$ (cf. [2]). Then if $L^m f = 0$, we have $(P(\xi))^m\hat{f} = 0$ and, therefore, $\hat{f}$ is supported at the origin. Now, the proof follows as in [4, Lemma 9].

(3.11) LEMMA. *Let* $g \in L^q_{\text{loc}} \cap \mathscr{S}'$ *and* $f = L^m g$. *If* $j > km + |a|$ *and* $\phi \in \mathscr{S}$ *then*

$$f^*(x) \leqslant C p_{j,km}(\phi)\, n(g, x).$$

Proof. If $h \in L^q_{\text{loc}}$ and $\Psi \in \mathscr{S}$ then

$$\int |h(u)|\, |\Psi(u)|\, du \leqslant C p_{j,0}(\Psi)\, n(h, 0). \tag{3.12}$$

For the proof of (3.12) see [2]. Now

$$(f * \phi_t)(y) = (L^m g * \phi_t)(y) = (g * L^m \phi_t)(y).$$

Since $\hat{\phi}_t(\xi) = \hat{\phi}(A_t \xi)$, we have

$$L^m \phi_t(y) = \int e^{-2\Pi i y \xi} (P(\xi))^m\, \hat{\phi}(A_t \xi)\, d\xi.$$

If we set $\eta = A_t \xi$, then

$$L^m \phi_t(y) = t^{-|a| - km} (L^m \phi)(A_t^{-1} y).$$

Therefore,

$$f * \phi_t(y) = t^{-km} \int g(z) (L^m \phi)_t\, (y - z)\, dz.$$

If $z = x + A_t u$, we get

$$(f * \phi_t)(y) = t^{-km} \int g(x + A_t u)(L^m \phi)_t\, (y - x - A_t u)\, t^{|a|}\, du.$$

Applying (3.12) with $h(u) = g(x + A_t u)$ and $\Psi(u) = (L^m \phi)_t$ $(y - x - A_t u)\, t^{|a|}$, and taking into account that $n(h, 0) = t^{km} n(g, x)$ we obtain

$$|(f * \phi_t)(y)| \leqslant C n(g, x)\, p_{j,0}((L^m \phi)(A_t^{-1}(y - x) - u)).$$

Since $[y - x] < t$, we have $1 + [u] \leqslant 2(1 + [A_t^{-1}(y - x) - u])$; then

$$\begin{aligned} &p_{j,0}((L^m \phi)(A_t^{-1}(y - x) - u)) \\ &\qquad \leqslant C \sup_{u \in R^n} |(L^m \phi)(A_t^{-1}(y - x) - u)|\, (1 + [A_t^{-1}(y - x) - u])^j \\ &\qquad = C p_{j,0}(L^m \phi) \leqslant C p_{j,km}(\phi), \end{aligned}$$

and the lemma is proved.

(3.13) LEMMA. *Let* b *be a* p*-atom with null moments up to order*

$N \geqslant km$, $\operatorname{supp} b \subset B(0,r)$, and $\|b\|_\infty \leqslant |B|^{-1/p}$. Let f be the solution of $L^m f = b$ obtained as $f = h * b$, where h is the elementary solution of L^m obtained in Lemma (3.6). *Then*

(i) *if* $[x] \geqslant 2r$, *we have*

$$|D^\alpha f(x)| \leqslant C r^{-|a|/p} [x]^{km-\alpha\cdot a} (r/[x])^{|a|+N+1} \qquad \text{for every } \alpha.$$

(ii) *if* $[x] \leqslant 2r$, $|f(x)| \leqslant C r^{-|a|/p+km}$ *holds.*

Proof. (i) Since $[x] \geqslant 2r$ and L^m is a hypoelliptic operator, f is infinitely differentiable at x and

$$\begin{aligned} D^\alpha f(x) &= \int_{[z]\leqslant r} D^\alpha h(x-z)\, b(z)\, dz \\ &= \int_{[z]\leqslant r} \sum_{|\beta|\leqslant N} D^\beta D^\alpha h(x)((-z)^\beta/\beta!)\, b(z)\, dz \\ &\quad + \int_{[z]\leqslant r} \sum_{|\beta|=N+1} D^\beta D^\alpha h(x-\lambda z)((-z)^\beta/\beta!)\, b(z)\, dz \end{aligned}$$

with $0 < \lambda < 1$. Since b has null moments up to order N, the first addend equals zero.

If $|\beta| = N+1$, then, by Lemma (3.6), $D^{\beta+\alpha} h$ is a quasi-homogeneous function. Then

$$|D^\alpha f(x)| \leqslant C \sum_{|\beta|=N+1} r^{-|a|/p} r^{\beta\cdot a} \int_{[z]\leqslant r} [x-\lambda z]^{km-|a|-\alpha\cdot a-\beta\cdot a}\, dz.$$

Since $[\lambda z] \leqslant r < [x]/2$, we have $[x-\lambda z] \geqslant [x]/2$. Therefore,

$$|D^\alpha f(x)| \leqslant C \sum_{|\beta|=N+1} r^{-|a|/p} [x]^{km-\alpha\cdot a} (r/[x])^{\beta\cdot a+|a|}.$$

As $r/[x] \leqslant \frac{1}{2}$ and $\beta\cdot a \geqslant |\beta| = N+1$, part (i) follows.

In order to prove (ii), we first assume $km < |a|$. In this case, h is quasi-homogeneous of degree $-|a| + km$ and, therefore,

$$\begin{aligned} |f(x)| &\leqslant \int_{[z]\leqslant r} |h(x-z)|\, |b(z)|\, dz \leqslant C r^{-|a|/p} \int_{[z]\leqslant r} |h(x-z)|\, dz \\ &\leqslant C r^{-|a|/p} \int_{[y]\leqslant 3r} |h(y)|\, dy \leqslant C r^{-|a|/p} \int_{[y]\leqslant 3r} [y]^{-|a|+km}\, dy \\ &= C r^{-(|a|/p)+km}. \end{aligned}$$

On the other hand, if $km \geqslant |a|$ we have $f(x) = (\hat{b}T)^\vee(x)$, where T is the Fourier transform of h.

Applying Taylor's formula to the function $e^{2\Pi iy\xi}$, we have

$$\hat{b}(\xi)=\int_{[y]\leqslant r} e^{2\Pi iy\xi}\,dy$$

$$=\sum_{|\alpha|\leqslant N}\int_{[y]\leqslant r}(2\Pi i)^{|\alpha|}\,(y^{\alpha}/\alpha!)\,b(y)\,dy+\sum_{|\alpha|=N+1}\int_{[y]\leqslant r}(\xi^{\alpha}/\alpha!)$$

$$\times\left(\int_0^1(2\Pi i)^{|\alpha|}\,y^{\alpha}\,e^{ity\cdot\xi}(1-t)^N\,(N+1)\,dt\right)b(y)\,dy.$$

Since b has null moments up to order N, the first addend equals zero. Then

$$|\hat{b}(\xi)|\leqslant C\sum_{|\alpha|=N+1}[\xi]^{\alpha\cdot a}\,r^{\alpha\cdot a}r^{-|a|/p}r^{|a|} \tag{3.14}$$

with C independent of b. Moreover, if $\beta\cdot a\leqslant km-|a|$ we have $(D^{\beta}\hat{b})(0)=0$. Then $\hat{b}T=\hat{b}(\xi)(P(\xi))^{-m}\in L^1(R^n)$. Therefore,

$$f(x)=\int e^{-2\Pi ix\xi}\hat{b}(\xi)(P(\xi))^{-m}\,d\xi.$$

Then

$$|f(x)|\leqslant\int|\hat{b}(\xi)|\,(P(\xi))^{-m}\,d\xi$$

$$=\int_{[\xi]\leqslant r^{-1}}|\hat{b}(\xi)|\,(P(\xi))^{-m}\,d\xi+\int_{[\xi]>r^{-1}}|\hat{b}(\xi)|\,(P(\xi))^{-m}\,d\xi.$$

By (3.14) we get

$$\int_{[\xi]\leqslant r^{-1}}|\hat{b}(\xi)|\,(P(\xi))^{-m}\,d\xi$$

$$\leqslant C\int_{[\xi]\leqslant r^{-1}}\sum_{|\alpha|=N+1}[\xi]^{\alpha\cdot a}\,r^{\alpha\cdot a-|a|/p+|a|}[\xi]^{-km}\,d\xi$$

$$\leqslant C\sum_{|\alpha|=N+1}r^{\alpha\cdot a-|a|/p+|a|}r^{-\alpha\cdot a+km}r^{-|a|}=Cr^{-|a|/p+km}.$$

On the other hand, applying Schwartz inequality we obtain

$$\int_{[\xi]>r^{-1}}|\hat{b}(\xi)|\,(P(\xi))^{-m}\,d\xi\leqslant C\,\|\hat{b}\|_{L^2}\left(\int_{[\xi]>r^{-1}}[\xi]^{-2km}\,d\xi\right)^{1/2}$$

$$=C\left(\int_{[y]\leqslant r}|b(y)|^2\,dy\right)^{1/2}\left(\int_{r^{-1}}^{\infty}s^{-2km+|a|-1}\,ds\right)^{1/2}$$

$$\leqslant Cr^{-|a|/p}r^{|a|/2}r^{km-|a|/2}=Cr^{-|a|/p+km}.$$

Then (ii) is proved.

(3.15) LEMMA. *Let $|a|/p < km + |a|/q$ and let b be a p-atom with null moments up to order $N \geqslant km + |a|/q$. Let f be the solution of $L^m f = b$ obtained as in Lemma* (3.13). *If F is the class of f in E^q_{km} then there exists a constant C, independent of b, such that*

$$\int N(F,x)^p \, dx \leqslant C.$$

Proof. By translation, we may assume that $\operatorname{supp} b$ is centered at the origin. That is, $\operatorname{supp} b \subset B(0,r)$ and $\|b\|_\infty \leqslant |B|^{-1/p}$. In order to estimate $N(F,x)$, we first assume that $[x] > 4r$. In this case, if $[x] > 2\rho$ we have

$$\rho^{-km}\left[\rho^{-|a|}\int_{[y]<\rho} |f(x+y) - P(x,y)|^q \, dy\right]^{1/q} \leqslant Cr^{-|a|/p}(r/[x])^{|a|+N+1} \tag{3.16}$$

with $P(x,y) = \sum_{\alpha\cdot a<km} D^\alpha f(x)\, y^\alpha/\alpha!$. In fact,

$$f(x+y) - P(x,y) = \sum_{\substack{|\alpha|<km\\ \alpha\cdot a\geqslant km}} D^\alpha f(x)\, y^\alpha/\alpha! + \sum_{|\alpha|=km} D^\alpha f(x+\theta y)\, y^\alpha/\alpha! \qquad \text{with} \quad 0<\theta<1.$$

Then

$$\begin{aligned}
&\rho^{-(km+|a|/q)}\left(\int_{[y]<\rho} |f(x+y) - P(x,y)|^q \, dy\right)^{1/q}\\
&\leqslant \rho^{-(km+|a|/q)}\left[\sum_{\substack{|\alpha|<km\\ \alpha\cdot a\geqslant km}} \left(\int_{[y]<\rho} |D^\alpha f(x)\, y^\alpha/\alpha!|^q \, dy\right)^{1/q}\right.\\
&\quad \left. + \sum_{|\alpha|=km} \left(\int_{[y]<\rho} |D^\alpha f(x+\theta y)\, y^\alpha/\alpha!|^q \, dy\right)^{1/q}\right]\\
&= \rho^{-(km+|a|/q)}(I_1 + I_2).
\end{aligned}$$

Applying Lemma (3.13), we obtain

$$\rho^{-(km+|a|/q)} I_1 \leqslant C \sum_{\substack{|\alpha|<km\\ \alpha\cdot a\geqslant km}} (\rho/[x])^{\alpha\cdot a-km}\, r^{-|a|/p}(r/[x])^{|a|+N+1}.$$

Since $\rho/[x] < 1$ and $\alpha\cdot a - km \geqslant 0$, we have

$$\rho^{-(km+|a|/q)} I_1 \leqslant Cr^{-|a|/p}(r/[x])^{|a|+N+1}.$$

As $[\theta y] \leqslant \rho < [x]/2$, we have $[x+\theta y] \geqslant [x] - [\theta y] \geqslant [x]/2 > 2r$ and, therefore, we can estimate I_2 in the same way as I_1.

Following with $[x] \geqslant 4r$, we assume now $[x] \leqslant 2\rho$. Then

$$\rho^{-(km+|a|/q)} \left(\int_{[y]<\rho} |f(x+y) - P(x,y)|^q \, dy \right)^{1/q}$$
$$\leqslant \rho^{-(km+|a|/q)} \left[\left(\int_{[y]<\rho} |f(x+y)|^q \, dy \right)^{1/q} + \left(\int_{[y]<\rho} |P(x,y)|^q \, dy \right)^{1/q} \right]$$
$$= \rho^{-(km+|a|/q)}(I_1 + I_2).$$

For I_1 we have

$$I_1 \leqslant \|f\|_q \leqslant \left(\int_{[u] \leqslant 2r} |f(u)|^q \, du \right)^{1/q} + \left(\int_{[u] \geqslant 2r} |f(u)|^q \, du \right)^{1/q}.$$

By (ii) of Lemma (3.13) we get

$$\left(\int_{[u] \leqslant 2r} |f(u)|^q \, du \right)^{1/q} \leqslant Cr^{-|a|/p + km + |a|/q}.$$

On the other hand, by (i) of Lemma (3.13) we obtain

$$\left(\int_{[u]>2r} |f(u)|^q \, du \right)^{1/q} \leqslant Cr^{-|a|/p+|a|+N+1} \left(\int_{[u]>2r} [u]^{kmq-|a|q-Nq-q} \, du \right)^{1/q}$$
$$\leqslant Cr^{-|a|/p+km+|a|/q}.$$

Then,

$$\rho^{-(km+|a|/q)} I_1 \leqslant C([x]/2)^{-(km+|a|/q)} \, r^{-|a|/p+km+|a|/q}.$$

For I_2 by (i) of Lemma (3.13), we have

$$I_2 = \left(\int_{[y]<\rho} \left| \sum_{\alpha \cdot a < km} D^\alpha f(x) \, y^\alpha / \alpha! \right|^q dy \right)^{1/q}$$
$$\leqslant Cr^{-|a|/p} (r/[x])^{|a|+N+1} \sum_{\alpha \cdot a < km} [x]^{km - \alpha \cdot a} \rho^{\alpha \cdot a} \rho^{|a|/q}.$$

Therefore,

$$\rho^{-(km+|a|/q)} I_2 \leqslant Cr^{-|a|/p} (r/[x])^{|a|+N+1} \sum_{\alpha \cdot a < km} ([x]/\rho)^{km - \alpha \cdot a}.$$

Since $[x] \leqslant 2\rho$, we have

$$\sum_{\alpha \cdot a < km} ([x]/\rho)^{km - \alpha \cdot a} \leqslant C.$$

Then

$$\rho^{-(km + |a|/q)} I_2 \leqslant C r^{-|a|/p} (r/[x])^{|a| + N + 1}.$$

As $|a|/q + km \leqslant N$, it holds that

$$\rho^{-(km + |a|/q)} \left(\int_{[y] < \rho} |f(x+y) - P(x,y)|^q \, dy \right)^{1/q} \leqslant C r^{-|a|/p} (r/[x])^{km + |a|/q} \tag{3.17}$$

for $4r < [x] \leqslant 2\rho$. By (3.16) and (3.17) it follows that

$$N(F, x) \leqslant C r^{-|a|/p} (r/[x])^{km + |a|/q}.$$

Then

$$\int_{[x] > 4r} N(F, x)^p \, dx \leqslant C,$$

where C is a constant independent of b. For $[x] \leqslant 4r$ we have

$$\begin{aligned} f(x+z) &- P(x,z) \\ &= \int \Big(h(x+z-y) - \sum_{\alpha \cdot a < km} D^\alpha h(x-y) \, z^\alpha / \alpha! \Big) \, b(y) \, dy \\ &= \int_{[x-y] < 2[z]} \Big(h(x+z-y) - \sum_{\alpha \cdot a < km} D^\alpha h(x-y) \, z^\alpha / \alpha! \Big) \, b(y) \, dy \\ &\quad + \int_{[x-y] \geqslant 2[z]} \Big(h(x+z-y) - \sum_{\alpha \cdot a < km} D^\alpha h(x-y) \, z^\alpha / \alpha! \Big) \, b(y) \, dy \\ &= I_1 + I_2. \end{aligned}$$

After the change of variables $x - y = u$, we have

$$\begin{aligned} |I_1| \leqslant & \int_{[u] < 2[z]} \Big| h(u+z) - \sum_{\alpha \cdot a \leqslant km - |a|} D^\alpha h(u) \, z^\alpha / \alpha! \Big| \, |b(x-u)| \, du \\ & + \int_{[u] < 2[z]} \Big| \sum_{km - |a| < \alpha \cdot a < km} D^\alpha h(u) \, z^\alpha / \alpha! \Big| \, |b(x-u)| \, du. \end{aligned} \tag{3.18}$$

As $D^\alpha h$ is quasi-homogeneous for $\alpha \cdot a > km - |a|$, the second part of the sum is bounded by $Cr^{-|a|/p}[z]^{km}$. If $km < |a|$, then the first addend reduces to

$$\int_{[u]<2[z]} |h(u+z)|\,|b(x-u)|\,du,$$

and since h is quasi-homogeneous, it holds the same estimate. On the other hand, if $km \geqslant |a|$ we have

$$\int_{[u]<2[z]} \left| h(u+z) - \sum_{\alpha\cdot a \leqslant km-|a|} D^\alpha h(u)\, z^\alpha/\alpha! \right| |b(x-u)|\,du$$
$$\leqslant Cr^{-|a|/p} \int_{[u]<2[z]} \left| h(u+z) - \sum_{\alpha\cdot a \leqslant km-|a|} D^\alpha h(u)\, z^\alpha/\alpha! \right| du.$$

Applying Taylor's formula we have

$$\int_{[u]<2[z]} \left| h(u+z) - \sum_{\alpha\cdot a \leqslant km-|a|} D^\alpha h(u)\, z^\alpha/\alpha! \right| du$$
$$= \int_{[u]<2[z]} \sum_{\substack{|\alpha| \leqslant km-|a| \\ \alpha\cdot a > km-|a|}} D^\alpha h(u)\, z^\alpha/\alpha! \qquad (3.19)$$
$$+ \sum_{km-|a|<|\alpha|\leqslant km-|a|+1} (z^\alpha/\alpha!) \int_0^1 D^\alpha h(u+tz)(1-t)^{s-1}\, s\,dt\,du,$$

where s is the integral part of $km - |a| + 1$.

If we set $u = A_{[z]}v$ and $\bar{z} = A_{[z]}^{-1} z$, then (3.19) equals

$$\int_{[v]<2} [z]^{|a|} \left| \sum_{\substack{|\alpha| \leqslant km-|a| \\ \alpha\cdot a > km-|a|}} [z]^{-|a|+km-\alpha\cdot a} D^\alpha h(v)\, z^\alpha/\alpha! \right.$$
$$+ \sum_{km-|a|<|\alpha|\leqslant km-|a|+1} (z^\alpha/\alpha!) \int_0^1 [z]^{-|a|+km-\alpha\cdot a}$$
$$\left. \times D^\alpha h(v+t\bar{z})(1-t)^{s-1}\, s\,dt \right| dv$$

$$= [z]^{km} \int_{[v]<2} \left| \sum_{\substack{|\alpha| \leqslant km-|a| \\ \alpha\cdot a > km-|a|}} D^\alpha h(v)\, \bar{z}^\alpha/\alpha! \right.$$
$$\left. + \sum_{km-|a|<|\alpha|\leqslant km-|a|+1} (\bar{z}^\alpha/\alpha!) \int_0^1 D^\alpha h(v+t\bar{z})(1-t)^{s-1}\, s\,dt \right| dv$$
$$= [z]^{km} \int_{[v]<2} \left| h(v+\bar{z}) - \sum_{\alpha\cdot a \leqslant km-|a|} D^\alpha h(v)\, \bar{z}^\alpha/\alpha! \right| dv.$$

By Lemma (3.6) we know that $D^\alpha h \in L^1_{\text{loc}}$ for $\alpha \cdot a \leqslant km - |a|$, then

$$\int_{[u]<2[z]} \left| h(u+z) - \sum_{\alpha \cdot a \leqslant km - |a|} D^\alpha h(u)\, z^\alpha/\alpha! \right| du$$

$$= [z]^{km} \int_{[v]<2} \left| h(v+\bar{z}) - \sum_{\alpha \cdot a \leqslant km-|a|} D^\alpha h(v)\, \bar{z}^\alpha/\alpha! \right| dv$$

$$\leqslant [z]^{km} \left(\int_{[v]<2} |h(v+\bar{z})|\, dv + \sum_{\alpha \cdot a \leqslant km - |a|} \int_{[v]<2} |D^\alpha h(v)|\, [\bar{z}]^{\alpha \cdot a}/\alpha!\, dv \right)$$

$$\leqslant C[z]^{km},$$

where C is a constant which depends on h and its derivatives of order α, with $\alpha \cdot a \leqslant km - |a|$. Therefore,

$$|I_1| \leqslant Cr^{-|a|/p}[z]^{km}.$$

For I_2 we have

$$I_2 = \int_{[x-y] \geqslant 2[z]} \left(h(x+z-y) - \sum_{\alpha \cdot a < km} D^\alpha h(x-y)\, z^\alpha/\alpha! \right) b(y)\, dy$$

$$= \int_{[x-y] \geqslant 2[z]} \left(h(x+z-y) - \sum_{\alpha \cdot a \leqslant km} D^\alpha h(x-y)\, z^\alpha/\alpha! \right) b(y)\, dy$$

$$+ \int_{[x-y] \geqslant 2[z]} \sum_{\alpha \cdot a = km} D^\alpha h(x-y)(z^\alpha/\alpha!)\, b(y)\, dy = J_1 + J_2.$$

By Taylor's formula we get

$$J_1 = \int_{[x-y] \geqslant 2[z]} \left(\sum_{\substack{|\alpha| \leqslant km \\ \alpha \cdot a > km}} D^\alpha h(x-y)\, z^\alpha/\alpha! \right.$$

$$\left. + \sum_{|\alpha| = km+1} D^\alpha h(x-y+\theta z)\, z^\alpha/\alpha! \right) b(y)\, dy$$

with $0 < \theta < 1$. Since $[x-y+\theta z] \geqslant [x-y]/2$ and as $D^\alpha h$ is quasi-homogeneous for $\alpha \cdot a > km$, we obtain

$$|J_1| \leqslant Cr^{-|a|/p}[z]^{km}.$$

On the other hand, by Corollary (3.9), $D^\alpha h$ is a singular integral kernel of parabolic type for $\alpha \cdot a = km$. Therefore, the maximal operator

$$K^*_\alpha g(x) = \sup_{\varepsilon > 0} \left| \int_{[x-y] > \varepsilon} D^\alpha h(x-y)\, g(y)\, dy \right|$$

is bounded in L^2 (cf. [7]). Moreover,

$$|J_2| \leqslant C[z]^{km} \sum_{\alpha \cdot a = km} K_\alpha^* b(x).$$

Then, we have obtained

$$|f(x+z) - P(x,z)| \leqslant Cr^{-|a|/p}[z]^{km} + C[z]^{km} \sum_{\alpha \cdot a = km} K_\alpha^* b(x).$$

Therefore, for $[x] \leqslant 4r$ we have

$$N(F,x) \leqslant C\left(r^{-|a|/p} + \sum_{\alpha \cdot a = km} K_\alpha^* b(x)\right).$$

Then

$$\int_{[x] \leqslant 4r} N(F,x)^p \, dx \leqslant C + C \sum_{\alpha \cdot a = km} \int_{[x] \leqslant 4r} (K_\alpha^* b(x))^p \, dx.$$

Applying Hölder's inequality with $r = 2/p$ and taking into account that K_α^* is bounded in L^2 we get

$$\int_{[x] \leqslant 4r} N(F,x)^p \, dx \leqslant C.$$

Then the lemma is proved.

Proof of Theorem 2. Given $F \in \mathscr{H}^p_{q,km}$, by Lemma (3.11) we have

$$(L^m F)^*(x) \leqslant CN(F,x).$$

Then

$$\|L^m F\|_{H^p} \leqslant C \|F\|_{\mathscr{H}^p_{q,km}}.$$

Moreover, by Lemma (3.10) we know that L^m is injective. On the other hand, given $f \in H^p$, there exist a sequence $\{b_j\}$ of p-atoms with null moments up to order $N \geqslant km + |a|/q$ and a numerical sequence $\{\lambda_j\}$ such that $f = \sum_j \lambda_j b_j$ and $\sum_j |\lambda_j|^p \leqslant C \|f\|_{H^p}$ (cf. [8]). The proof will be finished if we show that L^m is surjective. This follows from Lemma (3.15) in the same way as in [2].

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