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On the Multiplicity of Darlington Realizations of Contractive Matrix-Valued Functions

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We obtain a unique expression for a Darlington realization of a contractive matrix-valued function $S(z) \in \mathscr{S}\pi$, valid for the three following cases: (a) $S(z)$ is inner; (b) $S(z)$ is not inner and $\det[I_n - S^*(\xi) S(\xi)] \neq 0$ a.e.; (c) $S(z)$ is not inner and $\det[I_n - S^*(1/\bar{z}) S(z)] = 0$ $(z \in D)$. On the basis of this result we examine the problem of multiplicity of realizations for the case (c).

1. INTRODUCTION AND SUMMARY OF KNOWN RESULTS

We shall denote by $L^2(C^n)$ the class of measurable functions $h(\xi)$ $(\xi = e^{it}, 0 \leqslant t \leqslant 2\pi)$, with values in C^n such that

$$\|h\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|h(e^{it})\|^2 \, dt < \infty.$$

It consists of the functions whose Fourier series is (in the sense of convergence in the mean)

$$h(\xi) = \sum_{-\infty}^{\infty} h_k \xi^k, \quad h_k \in C^n \qquad \text{and} \qquad \|h\|^2 = \sum_{-\infty}^{\infty} \|h_k\|^2 \quad \text{(cf. [1])}.$$

We shall denote by $L^2_+(C^n)$ the subspace of $L^2(C^n)$ which consists of those functions for which $h_k = 0$, $k < 0$. $H^2(C^n)$ denotes the Hilbert space of

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functions $h(z)=\sum_{k=0}^{\infty} h_k z^k$ $(z=re^{it})$, $h_k \in C^n$, holomorphic in the unit disk $D=\{z; |z|<1\}$, such that

$$\frac{1}{2\pi}\int_0^{2\pi} \|h(re^{it})\|^2\,dt \qquad (0<r<1)$$

has a bound independent of r.

A function $u(z)$ holomorphic in D is called inner if $|u(z)|<1$ $(z\in D)$ and $|u(\xi)|=1$ a.e. A function $\Phi(z)$, holomorphic in D is called outer if

$$\Phi(z)=\chi \exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\ln k(t)\,dt \qquad (z\in D),$$

where $k(t)\geqslant 0$, $\ln k(t)\in L^1$ and χ is a complex number of modulus 1. For a function $w(z)$, meromorphic in D, the characteristic of Nevanlinna is defined by the expression

$$T(w,r)=\frac{1}{2\pi}\int_0^{2\pi}\ln^+ w(re^{it})\,dt + \int_0^r \frac{n(t,w)-n(0,w)}{t}\,dt + n(0,w)\ln r,$$

where

$$\ln^+ a=\begin{cases}\ln a & \text{if } a\geqslant 1\\ 0 & \text{if } 0\leqslant a<1,\end{cases}$$

and $n(t,w)$ is the number of poles of $w(z)$, each one with its multiplicity, inside the circle $|z|<t$.

We say that $w(z)$ is of bounded characteristic if $\sup_{|r|<1}|T(w;r)|<\infty$. According to a theorem of Nevanlinna [2] the class of functions of bounded characteristic coincides with the class of functions that can be written as the ratio of two bounded holomorphic functions $(z\in D)$. Then, these functions are uniquely defined by their boundary values a.e. in the unit circle.

We design with N_0 the class of functions $f(z)$ of bounded characteristic $(z\in D)$ that can be written as a product of an inner function and an outer function. For functions of N_0 the maximum principle holds. A matrix S is called contractive iff $I-S^*S\geqslant 0$, where I is the unit matrix and the symbol $*$ denotes hermitian conjugation. We use J to design a matrix for which $J^*=J$ and $J^2=I$. A matrix A is called J-expansive iff $A^*JA-J\geqslant 0$, and J-unitary iff $A^*JA-J=0$.

We will design by $\mathscr{S}$ the class of contractive matrix-valued functions, i.e., the matrix-valued functions holomorphic in D for which $\|S(z)\| \leqslant 1$. $S(z) \in \mathscr{S}$ is inner if $I - S^*(\xi)\, S(\xi) = 0$ a.e. We say that a matrix-valued function is of bounded characteristic if all its elements possess that property, and that belongs to the class N_0 if all its elements are functions of N_0. A matrix-valued functions $S(z) \in \mathscr{S}$ belongs to the class $\mathscr{S}\pi$ if it has the additional property that its boundary values a.e. on the unit circle are, simultaneously, boundary values of a matrix-valued function $\tilde{S}(z)$ meromorphic in $D_- = \{z; |z| > 1\}$ with elements of bounded characteristic there [3], i.e.,

$$\lim_{|z| \downarrow 1} \tilde{S}(z) = \lim_{|z| \uparrow 1} S(z) \qquad \text{a.e.}$$

A meromorphic matrix-valued function $A(z)$ is J-expansive $(z \in D)$ if it assumes J-expanding values at each point of holomorphicity z, i.e.,

$$A^*(z)\, JA(z) - J \geqslant 0,$$

and a J-expansive matrix-valued function $A(z)$ is J-inner if it is J-unitary a.e. in the unit circle, i.e.,

$$A^*(\xi)\, JA(\xi) - J = 0.$$

An arbitrary J-expansive matrix-valued function is of bounded characteristic [3].

Of importance for us is the

Basic Lemma [3]. *Let* $A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$ *be a matrix-valued function of bounded characteristic of order* $n + m$, *with diagonal elements* $\alpha(z)$ *of order* n *and* $\delta(z)$ *of order* m, *where* det $\delta(z) \not\equiv 0$ $(z \in D)$, *that satisfies the condition*

$$A^*(\xi)\, jA(\xi) - j \geqslant 0 \qquad \textit{a.e.},$$

where

$$j = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} \tag{1.1}$$

and

$$\begin{aligned} a(z) &= \alpha(z) - \beta(z)\, \delta^{-1}(z)\, \gamma(z) \in N_0; \\ b(z) &= \beta(z)\, \delta^{-1}(z) \in N_0; \\ c(z) &= \delta^{-1}(z)\, \gamma(z) \in N_0; \\ d(z) &= \delta^{-1}(z) \in N_0. \end{aligned} \tag{1.2}$$

Then $A(z)$ *is J-expansive* $(z \in D)$ *and* $\|a(z)\| \leqslant 1$; $\|b(z)\| \leqslant 1$; $\|c(z)\| \leqslant 1$; $\|d(z)\| \leqslant 1$.

We will use also

THEOREM [4]. *Let* $S(z)$ *be a matrix-valued function of order n, of the class* $\mathscr{S}\pi$. *Then its boundary values* a.e. $S(\xi)$ *can be represented in the form*

$$S(\xi) = U_2^{-1}(\xi)\, S'(\xi)\, U_1(\xi), \tag{1.3}$$

where $U_1(\xi)$ *and* $U_2(\xi)$ *are* a.e. *limiting values of inner matrix-valued functions* $U_1(z)$ *and* $U_2(z)$, *and* $S'(\xi)$ *has a diagonal form with blocks* $S_1(\xi)$ *of order k and* $S_2(\xi)$ *of order* $n-k$, *i.e.,*

$$S'(\xi) = \operatorname{diag}(S_1(\xi), S_2(\xi)) \tag{1.4}$$

verifying a.e. *the relations* $\|S_1(\xi)\| < 1$ *and* $\|S_2(\xi)\| = 1$.

We will decribe briefly the way we have constructed in [4] the matrix-valued functions $U_1(\xi)$ and $U_2(\xi)$, because of their importance in the development that follows. For each fixed ξ where $T(\xi) =^{\text{def}} S^*(\xi)\, S(\xi)$ is defined, we arranged the eigenvalues of the nonnegative matrix $I_n - T(\xi)$ in nonincreasing order, and considered the orthonormal base of C^n formed by the corresponding eigenvectors $\{\Phi_m(\xi)\}_{m-1}^n$. Introducing the notation

$$\Phi_n(\xi) = (\Phi_{m1}(\xi), \Phi_{m2}(\xi), \dots, \Phi_{mn}(\xi))^t,$$

where the symbol $(\ )^t$ denotes transposition, we construct the matrix

$$U_1(\xi) = \{\Phi_{ij}(\xi)\}_{i,j=1}^n.$$

Due to the orthonormality of $\{\Phi_m(\xi)\}_1^n$, $U_1^*(\xi)\, U_1(\xi) = I_n$ for each fixed ξ in the unit circle except, possibly, a set of zero measure. According to the way we ordered the eigenvalues, the last $(n-k)$ rows of $U_1(\xi)$ are the eigenvectors corresponding to the eigenvalue 0. We applied a similar procedure to construct $U_2(\xi)$, which $T'(\xi) =^{\text{def}} S(\xi)\, S^*(\xi)$ playing the role of $T(\xi)$.

Arov has proved in [3] that a necessary and sufficient condition for a matrix-valued function $S(z)$ to be Darlington realizable is that $S(z)$ belongs to the class $\mathscr{S}\pi$, and considered separately three cases:

(a) $S(z)$ is inner;

(b) $S(z)$ is not inner and $\det[I_n - S^*(\xi)\, S(\xi)] \neq 0$ a.e.;

(c) $S(z)$ is not inner and $\det[I_n - S^*(1/z)\, S(z)] = 0$ $(z \in D)$.

Using results from [3, 4] we obtain in this article a unique expression, valid for the three cases for a Darlington realization of a matrix function

$S(z) \in \mathscr{S}\pi$. This result allows us to examine the problem of multiplicity of realizations not only in Case (b), solved in [3], but also in Case (c) proposed in [3] as an open problem.

2. Darlington Realizations

By Darlington realization of a matrix-valued function $S(z) \in \mathscr{S}$, of order $n \times m$, we mean the representation of $S(z)$ as the linear fractional transformation [3],

$$S(z) = [\alpha(z)\varepsilon + \beta(z)][\gamma(z)\varepsilon + \delta(z)]^{-1}, \tag{2.1}$$

over a constant matrix $\varepsilon \in \mathscr{S}$, of order $n \times m$, with a j-inner matrix of coefficients

$$A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}.$$

Theorem II.1. *Let $S(z) \in \mathscr{S}\pi$ be a matrix-valued function of order n. Then $S'(\xi)$, defined by* (1.3), *can be written as the linear fractional transformation*

$$S'(\xi) = [\alpha'(\xi)\varepsilon + \beta'(\xi)][\gamma'(\xi)\varepsilon + \delta'(\xi)]^{-1} \tag{2.2}$$

over a constant matrix $\varepsilon \in \mathscr{S}$, with a matrix of coefficients

$$A'(\xi) = \begin{pmatrix} \alpha'(\xi) & \beta'(\xi) \\ \gamma'(\xi) & \delta'(\xi) \end{pmatrix}$$

that is j-unitary a.e.

Proof. As an illustration we describe briefly the steps of the proof, which consist of

(1) making a convenient selection of ε (formula (2.5)) and

(2) carry out the demonstration of the thesis with the particular choice of $A'(\xi)$ specified in (2.6).

Let us observe that, since $S(z) \in \mathscr{S}\pi$, the nonnegative matrix-valued functions

$$F_1(\xi) \stackrel{\text{def}}{=} I_k - S_1^*(\xi)\, S_1(\xi) \qquad \text{a.e.},$$

$$F_1'(\xi) \stackrel{\text{def}}{=} I_k - S_1(\xi)\, S_1^*(\xi) \qquad \text{a.e.}$$

are boundary values of matrix-valued functions $F_1(z)$ and $F_1'(z)$ of bounded characteristic in D. Hence there exist [5, 6] solutions $\theta(\xi)$ and $\psi(\xi)$ of the factorization problems

$$F_1(\xi) = \theta^*(\xi)\,\theta(\xi) \qquad \text{a.e.,} \tag{2.3}$$

$$F_1'(\xi) = \psi(\xi)\,\psi^*(\xi) \qquad \text{a.e.} \tag{2.4}$$

that are a.e. boundary values of bounded holomorphic functions $\theta(z)$ and $\psi(z)$. These solutions are uniquely defined by the following normalization conditions: $\det \theta(z)$ and $\det \psi(z)$ are outer functions, $\theta(0) \geqslant 0$ and $\psi(0) \geqslant 0$. We will construct the representation (2.2) with

$$\varepsilon = \begin{pmatrix} 0_k & 0 \\ 0 & I_{n-k} \end{pmatrix}, \tag{2.5}$$

here $n-k$ is the dimension of the subspace

$$N = \{h \in C^n;\ S^*(\xi)\,S(\xi)h = h\}$$

for each value of ξ where $S(\xi)$ is defined [4]; and the matrix of coefficients $A'(\xi)$ with elements

$$\alpha'(\xi) = \{\operatorname{diag}(\psi^*(\xi), 0_{n-k}) + \varepsilon[U_2^{-1}(\xi) + U_1^{-1}(\xi)\,S'^*(\xi)]\}^{-1} + \tfrac{1}{2}U_2(\xi)\varepsilon,$$

$$\beta'(\xi) = S'(\xi)\{\operatorname{diag}(\theta(\xi), 0_{n-k}) + \varepsilon[U_1^{-1}(\xi) + U_2^{-1}(\xi)\,S'(\xi)]\}^{-1} - \tfrac{1}{2}U_2(\xi)\varepsilon,$$

$$\gamma'(\xi) = S'^*(\xi)\{\operatorname{diag}(\psi^*(\xi), 0_{n-k}) + \varepsilon[U_2^{-1}(\xi) + U_1^{-1}(\xi)\,S'^*(\xi)]\}^{-1} - \tfrac{1}{2}U_1(\xi)\varepsilon,$$

$$\delta'(\xi) = \{\operatorname{diag}(\theta(\xi), 0_{n-k}) + \varepsilon[U_1^{-1}(\xi) + U_2^{-1}(\xi)\,S'(\xi)]\}^{-1} + \tfrac{1}{2}U_1(\xi)\varepsilon, \tag{2.6}$$

where $U_1(\xi)$ and $U_2(\xi)$ are the unitary matrix-valued functions a.e. involved in (1.3) and defined in [4]. With this choice of ε and $A'(\xi)$, (2.2) holds. In fact

$$\begin{aligned} S'(\xi)[\gamma'(\xi)\varepsilon + \delta'(\xi)] - [\alpha'(\xi)\varepsilon + \beta'(\xi)] &= \operatorname{diag}(S_1^*(\xi)\,S_1(\xi) - I_k, 0_{n-k})\{\operatorname{diag}(\psi^*(\xi), 0_{n-k}) \\ &\quad + \varepsilon[U_2^{-1}(\xi) + U_1^{-1}(\xi)\,S'^*(\xi)]\}^{-1}\varepsilon = 0 \qquad \text{a.e.} \end{aligned}$$

To finish the proof we will show that

$$A'^*(\xi)\,jA'(\xi) - j = 0 \qquad \text{a.e.}$$

or, equivalently, that

$$\begin{aligned} &\alpha'^*(\xi)\,\alpha'(\xi)-\gamma'^*(\xi)\,\gamma'(\xi)=I_n \qquad \text{a.e.},\\ &\delta'^*(\xi)\,\delta'(\xi)-\beta'^*(\xi)\,\beta'(\xi)=I_n \qquad \text{a.e.},\\ &\alpha'^*(\xi)\,\beta'(\xi)-\gamma'^*(\xi)\,\delta'(\xi)=0 \qquad \text{a.e.} \end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7) we get

$$\begin{aligned} &\alpha'^*(\xi)\,\alpha'(\xi)-\gamma'^*(\xi)\,\gamma'(\xi)\\ &\quad=\operatorname{diag}(\psi^{-1}(\xi)[I_k-S_1(\xi)\,S_1^*(\xi)]\,\psi^{*-1}(\xi),0_{n-k})\\ &\qquad+\tfrac{1}{2}\{[U_2(\xi)+S'(\xi)\,U_1(\xi)]^{-1}\operatorname{diag}(\psi(\xi),0_{n-k})+\varepsilon\}^{-1}\varepsilon\\ &\qquad+\tfrac{1}{2}\varepsilon\{\operatorname{diag}(\psi^*(\xi),0_{n-k})[U_2^{-1}(\xi)+U_1^{-1}(\xi)\,S'^*(\xi)]^{-1}+\varepsilon\}^{-1}=I_n \quad \text{a.e}\\ &\delta'^*(\xi)\,\delta'(\xi)-\beta'^*(\xi)\,\beta'(\xi)\\ &\quad=\operatorname{diag}(\theta^{*-1}(\xi)[I_k-S_1^*(\xi)\,S_1(\xi)]\,\theta^{-1}(\xi),0_{n-k})\\ &\qquad+\tfrac{1}{2}\varepsilon\{\operatorname{diag}(\theta(\xi),0_{n-k})[U_1^{-1}(\xi)+U_2^{-1}(\xi)\,S'(\xi)]^{-1}+\varepsilon\}^{-1}+\varepsilon\\ &\quad=I_n \quad \text{a.e.},\\ &\alpha'^*(\xi)\,\beta'(\xi)-\gamma'^*(\xi)\,\delta'(\xi)\\ &\quad=\tfrac{1}{2}\varepsilon\{\operatorname{diag}(\theta(\xi),0_{n-k})[U_1^{-1}(\xi)+U_2^{-1}(\xi)\,S'(\xi)]^{-1}+\varepsilon\}^{-1}\\ &\qquad-\tfrac{1}{2}\{[U_2(\xi)+S'(\xi)\,U_1(\xi)]^{-1}\operatorname{diag}(\psi(\xi),0_{n-k})+\varepsilon\}^{-1}=0 \quad \text{a.e.} \end{aligned}$$

This completes the proof of Theorem II.1.

Using this theorem and (1.3) we obtain

$$\begin{aligned} S(\xi)&=U_2^{-1}(\xi)[\alpha'(\xi)\varepsilon+\beta'(\xi)][\gamma'(\xi)\varepsilon+\delta'(\xi)]^{-1}U_1^{-1}(\xi)\\ &=[U_2^{-1}(\xi)\,\alpha'(\xi)\varepsilon+U_2^{-1}(\xi)\,\beta'(\xi)][U_1^{-1}(\xi)\,\gamma'(\xi)\varepsilon\\ &\quad+U_1^{-1}(\xi)\,\delta'(\xi)]^{-1} \qquad \text{a.e.} \end{aligned}$$

This is a linear fractional transformation of $S(\xi)$ over the same constant matrix ε, given by (2.5), with a matrix of coefficients $A(\xi)=U(\xi)\,A'(\xi)$, where

$$U(\xi)=\operatorname{diag}(U_2^{-1}(\xi),U_1^{-1}(\xi)). \tag{2.8}$$

Note that $A(\xi)$ verifies

$$A(\xi)=\lim_{|z|\to 1}A(z) \qquad \text{a.e.},$$

where

$$A(z) = U(z)\, A'(z) \tag{2.9}$$

is a matrix-valued function of bounded characteristic in D. This is due to the construction of $U_1(\xi)$, $U_2(\xi)$ (cf. [4]) and $A'(\xi)$. We shall prove now

THEOREM II.2. *Let $S(z) \in \mathscr{S}\pi$ be a matrix-valued functions of order n. Then the representation*

$$S(z) = [\alpha(z)\varepsilon + \beta(z)][\gamma(z)\varepsilon + \delta(z)]^{-1} \tag{2.10}$$

with ε specified by (2.5) *and $A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$ given by* (2.9) *is a Darlington realization of $S(z)$.*

Proof. To prove the thesis we will show that $A(\xi)$ is j-unitary a.e., and $A(z)$ is j-expansive $(z \in D)$. Since $U_1(\xi)$ and $U_2(\xi)$ are unitary a.e. by construction, using (2.7), (2.8), and (2.9) we have

$$\begin{aligned} A^*(\xi)\, jA(\xi) - j &= U^*(\xi)\, A'^*(\xi)\, jA'(\xi)\, U(\xi) - j \\ &= U^*(\xi)\, jU(\xi) - j = 0 \qquad \text{a.e.} \end{aligned} \tag{2.11}$$

In view of this result, we will verify that the elements of $A(\xi)$ are a.e. boundary values of matrix-valued functions that satisfy the hypotheses of the Basic Lemma [3].

Let us consider the function

$$\begin{aligned} d(z) = {} & \{[\operatorname{diag}(\theta(z), 0_{n-k})]\, U_1(z) + \varepsilon[I_n + S(z)]\} \\ & \times \{I_n + \varepsilon[I_n + S(z)]\}^{-1} \qquad (z \in D). \end{aligned} \tag{2.12}$$

Assuming that the unity is not an eigenvalue of $-S(z)$, the function $\{I_n + \varepsilon[I_n + S(z)]\}^{-1}$ exists [3], and taking into account that $\theta(z)$, $U_1(z)$, and $S(z)$ are bounded and holomorphic $(z \in D)$, we conclude that $d(z) \in N_0$. Note that

$$d^{-1}(\xi) = \delta(\xi) = \lim_{|z| \to 1} \delta(z) \qquad \text{a.e.}$$

From (2.11) we know that

$$\alpha^*(\xi)\, \alpha(\xi) - \gamma^*(\xi)\, \gamma(\xi) = I_n \qquad \text{a.e.,}$$

$$\gamma^*(\xi)\, \delta(\xi) - \alpha^*(\xi)\, \beta(\xi) = 0 \qquad \text{a.e.}$$

Then, using (1.2) we obtain

$$\alpha^{*-1}(\xi)=a(\xi) \qquad \text{a.e.}$$

This relationship leads us to examine if $a(\xi)$ is the boundary value of a matrix-valued function $a(z)$ satisfying the hypothesis of the Basic Lemma. The expression (2.6) of $\alpha'(\xi)$ may be used to derive a formal expression for $a(z)$,

$$\begin{aligned} a(z)=2\{2I_n+[I_n+S(z)]\varepsilon\}^{-1}\{U_2^{-1}(z)\operatorname{diag}(\psi(z), O_{n-k}) \\ +[I_n+S(z)]\varepsilon\}. \end{aligned} \tag{2.13}$$

Admitting, without restriction, that unity is not an eigenvalue of $S(z)$ $(z\in D)$, it can be seen that $\{2I_n+[I_n+S(z)]\varepsilon\}^{-1}$ exists and its elements are functions of N_0 (cf. [3]).

Consider now the second factor in (2.13). The matrix-valued function $U_2^{-1}(z)$ is, by its construction, of bounded characteristic in D. We can select an inner scalar function $b_1(z)$ being common denominator of all the elements of $U_2^{-1}(z)$, and construct a function $\psi(\xi)=\psi_0(\xi)\,b_1(\xi)$, where $\psi_0(\xi)$ is a solution of (2.4) uniquely defined by the normalization conditions. Therefore $\psi(\xi)$ is also solution of (2.4). Furthermore,

$$\psi(\xi)=\lim_{|z|\to 1}\psi(z) \qquad \text{a.e.,}$$

where $\psi(z)=\psi_0(z)\,b_1(z)$ is bounded and holomorphic in D. With this choice of $\psi(z)$ the elements of $U_2^{-1}(z)\operatorname{diag}(\psi(z), O_{n-k})$ are scalar functions of N_0. Since $[I_n+S(z)]$ is holomorphic and bounded $(z\in D)$, the above conclusions allow us to afirm that $a(z)\in N_0$.

Consider now the element

$$b(\xi)=\beta(\xi)\,\delta^{-1}(\xi)=\beta(\xi)\,d(\xi) \qquad \text{a.e.}$$

Using (2.6) and (2.12), $b(z)$ may be written

$$b(z)=\{S(z)-\tfrac{1}{2}\varepsilon[I_n+S(z)]\}\{I_n+\varepsilon[I_n+S(z)]\}^{-1} \qquad (z\in D).$$

When we examined the element $d(z)$, we have shown that $I_n+\varepsilon[I_n+S(z)]$ is invertible. This fact, together with property of boundedness and holomorphicity of $S(z)$, implies that $b(z)\in N_0$.

Let us now consider the block

$$\begin{aligned} c(\xi)=\delta^{-1}(\xi)\,\gamma(\xi)=d(\xi)\,U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi)) \\ \times\{\operatorname{diag}(\psi^*(\xi), O_{n-k})+\operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi)+U_1^{-1}(\xi) \\ \times\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}-\tfrac{1}{2}d(\xi)\,U_1^{-1}(\xi)\operatorname{diag}(O_k, I_{n-k}). \end{aligned}$$

Since we have proved that $d(\xi) \in N_0$, it follows immediately that the second term at the right also satisfies that condition. Hence it is left to show that the first term belongs to the class N_0. We denote this term by $c'(\xi)$. Replacing $d(\xi)$ in it and using the equality

$$\begin{aligned}\{I_n + \tfrac{1}{2}\operatorname{diag}(O_k, I_{n-k})[I_n + S(\xi)]\}^{-1} \\ = I_n - \{I_n + \tfrac{1}{2}\operatorname{diag}(O_k, I_{n-k})[I_n + S(\xi)]\}^{-1}\tfrac{1}{2}\operatorname{diag}(O_k, I_{n-k}) \\ \times [I_n + S(\xi)]\end{aligned}$$

we obtain

$$c'(\xi) = c_1(\xi) - c_2(\xi), \tag{2.14}$$

where

$$\begin{aligned}c_1(\xi) = \{&\operatorname{diag}(\theta(\xi), O_{n-k})\, U_1(\xi) + \operatorname{diag}(O_k, I_{n-k})[I_n + S(\xi)] \\ &\times U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))\{\operatorname{diag}(\psi^*(\xi), O_{n-k}) \\ &+ \operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}\end{aligned}$$

and

$$\begin{aligned}c_2(\xi) = \tfrac{1}{2}d(\xi)&\operatorname{diag}(O_k, I_{n-k})[I_n + S(\xi)]\, U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi)) \\ &\times \{\operatorname{diag}(\psi^*(\xi), O_{n-k}) + \operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi) \\ &\times \operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}.\end{aligned}$$

The term $c_1(\xi)$ can be rewritten in the form

$$c_1(\xi) = c_3(\xi) + 2c_2(\xi), \tag{2.15}$$

where

$$\begin{aligned}c_3(\xi) = \operatorname{diag}(\theta(\xi)\, S_1^*(\xi), O_{n-k})&\{\operatorname{diag}(\psi^*(\xi), O_{n-k}) \\ &+ \operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}.\end{aligned}$$

Observe that the factor between brackets is, due to its construction, a triangular matrix-valued function, with its right superior block equal to zero. Therefore

$$c_3(\xi) = \operatorname{diag}(\theta(\xi)\, S_1^*(\xi)\, \psi^{*-1}(\xi), O_{n-k}). \tag{2.16}$$

Replacing (2.16) and (2.15) in (2.14) we obtain

$$c'(\xi)=\operatorname{diag}(\theta(\xi)\, S_1^*(\xi)\, \psi^{-1*}(\xi), O_{n-k})+c_2(\xi).$$

The matrix-valued function defined by (2.16) is the boundary value a.e. of the matrix-valued function

$$c_3(z)=\operatorname{diag}(\theta(z)\, S_1^*(1/\bar{z})\, \psi^{*-1}(1/\bar{z}), O_{n-k}) \qquad (z\in D)$$

of bounded characteristic in D. Let $b_2(z)$ be an inner scalar function that is the common denominator of $S_1^*(1/\bar{z})\, \psi^{*-1}(1/\bar{z})$. If $\theta_0(\xi)$ is a solution of the factorization problem (2.3) satisfying the normalization conditions, then $\theta(\xi)=b_2(\xi)\, \theta_0(\xi)$ is also a solution of that problem. Moreover, $\theta(\xi)$ verifies

$$\theta(\xi)=\lim_{|z|\to 1} \theta(z) \qquad \text{a.e.},$$

where $\theta(z)$ is an holomorphic and bounded matrix-valued function $(z\in D)$. Therefore we can conclude that $c_3(\xi)\in N_0$. To prove that $c(\xi)\in N_0$ it is only left to show that $c_2(\xi)\in N_0$. By virtue of (1.3) the term $c_2(\xi)$ can be written as follows:

$$\begin{aligned} c_2(\xi)&=\tfrac{1}{2}d(\xi)\operatorname{diag}(O_k, I_{n-k})[U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))+U_2^{-1}(\xi)\\ &\quad\times\operatorname{diag}(S_1(\xi)\, S_1^*(\xi), I_{n-k})+U_2^{-1}(\xi)-U_2^{-1}(\xi)]\{\operatorname{diag}(\psi^*(\xi), O_{n-k})\\ &\quad+\operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi)+U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}\\ &=\tfrac{1}{2}d(\xi)\operatorname{diag}(O_k, I_{n-k})\{U_2^{-1}(\xi)\operatorname{diag}(S_1(\xi)\, S_1^*(\xi)-I_k, O_{n-k})\\ &\quad+[U_2^{-1}(\xi)+U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi)]\}\{\operatorname{diag}(\psi^*(\xi), O_{n-k})\\ &\quad+\operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi)+U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}\\ &=\tfrac{1}{2}d(\xi)\operatorname{diag}(O_k, I_{n-k})\, U_2^{-1}(\xi)\\ &\quad\times\operatorname{diag}([S_1(\xi)\, S_1^*(\xi)-I_k]\, \psi^{*-1}(\xi), O_{n-k})\\ &\quad+\tfrac{1}{2}d(\xi)\operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi)+U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\\ &\quad\times\{\operatorname{diag}(\psi^*(\xi), O_{n-k})+\operatorname{diag}(O_k, I_{n-k})[U_2^{-1}(\xi)+U_1^{-1}(\xi)\\ &\qquad\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]\}^{-1}. \end{aligned}$$

Without loss of generality we can suppose that the unity is not eigenvalue of $S(z)$, and, using the same argument we have mentioned when we consider the block $b(z)$, we can afirm that $[I+S(\xi)]$ is invertible, therefore $[U_2^{-1}(\xi)+U_1^{-1}(\xi)\operatorname{diag}(S_1^*(\xi), S_2^*(\xi))]$ is also invertible. Taking this into account and recalling that $\psi(\xi)$ is solution of the factorization problem (2.4) we arrive at the conclusion that

$$\begin{aligned} c_2(\xi) &= \tfrac{1}{2}d(\xi)\, U_2^{-1}(\xi)\,\mathrm{diag}(\psi(\xi), O_{n-k}) + \tfrac{1}{2}d(\xi)\,\mathrm{diag}(O_k, I_{n-k}) \\ &\quad \times \{\mathrm{diag}(\psi^*(\xi), O_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi) \\ &\quad \times \mathrm{diag}(S_1^*(\xi), S_2^*(\xi))]^{-1} + \mathrm{diag}(O_k, I_{n-k})\}^{-1} \\ &= \tfrac{1}{2}d(\xi)\,\mathrm{diag}(O_k, I_{n-k})\, U_2^{-1}(\xi)\,\mathrm{diag}(\psi(\xi), O_{n-k}) + \tfrac{1}{2}d(\xi) \\ &\quad \times \mathrm{diag}(O_k, I_{n-k}). \end{aligned}$$

Recalling that with the convenient construction of $\psi(\xi)$ we have set when the block $a(z)$ was examined, the elements of $U_2^{-1}(z)\,\mathrm{diag}(\psi(z), O_{n-k})$ are bounded and holomorphic scalar functions ($z \in D$), we arrive to the conclusions that $c_2(\xi) \in N_0$. Therefore, $c(\xi) \in N_0$. Hence the matrix of coefficients $A(z)$, defined by (2.9) is j-inner and the proof that (2.10) is a Darlington realization is finished.

3. Set of Realizations

The Darlington realization of a matrix-valued function $S(z) \in \mathscr{S}\pi$ is not unique [3]. It is important for a practical viewpoint (synthesis of an n-port with specified scattering matrix) to describe all the possible realizations of $S(z)$. An analogous problem has been cited by Cauer [6], in the case of reactance matrices, the equivalence problem.

Let us consider an arbitrary realization of $S(z) \in \mathscr{S}\pi$,

$$S(z) = [\alpha(z)\varepsilon + \beta(z)][\gamma(z)\varepsilon + \delta(s)]^{-1}, \tag{3.1}$$

over a constant matrix ε ($\in \mathscr{S}$) with a j-expansive matrix of coefficients

$$A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$$

since $\varepsilon \in \mathscr{S}$, the matrices

$$F = In - \varepsilon^*\varepsilon,$$

$$F' = In - \varepsilon\varepsilon^*,$$

are nonnegative, and there exist unitary matrices V_1 and V_2 diagonalizing them. If we denote by r the dimension of the range of F, and introduce the matrix

$$\varepsilon' \overset{\text{def}}{=} V_2 \varepsilon V_1^{-1}, \tag{3.2}$$

we know (cf. 4, Theorem III.1]) that ε' may be written in a diagonal form, i.e.,

$$\varepsilon' = \operatorname{diag}(\varepsilon_1, \varepsilon_2), \tag{3.3}$$

with blocks ε_1 of order r and ε_2 of order $n-r$ satisfying $\|\varepsilon_1\| < 1$ and $\|\varepsilon_2\| = 1$. Note that

$$F_1 = I_r - \varepsilon_1^* \varepsilon_1,$$
$$F_1' = I_r - \varepsilon_1 \varepsilon_1^*,$$

are positive definite matrices.

Consider now the linear fractional transformation

$$\varepsilon' = [\alpha'_\varepsilon \varepsilon_0 + \beta'_\varepsilon][\gamma'_\varepsilon \varepsilon_0 + \delta'_\varepsilon]^{-1}$$

specified by

$$\varepsilon_0 = \operatorname{diag}(O_r, I_{n-r}) \tag{3.4}$$

and the j-unitary matrix

$$U_{\varepsilon'} = \begin{pmatrix} \alpha_{\varepsilon'} & \beta_{\varepsilon'} \\ \gamma_{\varepsilon'} & \delta_{\varepsilon'} \end{pmatrix},$$

where

$$\begin{aligned} \alpha_{\varepsilon'} &= \operatorname{diag}(F_1'^{-1/2}, \varepsilon_2), & \beta_{\varepsilon'} &= \operatorname{diag}(F_1^{-1/2}, O_{n-r}), \\ \gamma_{\varepsilon'} &= \operatorname{diag}(\varepsilon_1^* F_1'^{-1/2}, O_{n-r}), & \delta_{\varepsilon'} &= \operatorname{diag}(F_1^{-1/2}, I_{n-r}). \end{aligned} \tag{3.5}$$

Using this construction we conclude that

$$\varepsilon = [\alpha_\varepsilon \varepsilon_0 + \beta_\varepsilon][\gamma_\varepsilon \varepsilon_0 + \delta_\varepsilon]^{-1}, \tag{3.6}$$

where

$$U_\varepsilon = \begin{pmatrix} \alpha_\varepsilon & \beta_\varepsilon \\ \gamma_\varepsilon & \delta_\varepsilon \end{pmatrix} = \operatorname{diag}(V_2^{-1}, V_1^{-1}) \times U_{\varepsilon'}, \tag{3.7}$$

is j-unitary. This is due to (3.2) and the fact that V_1 and V_2 are unitary. Substituting (3.6) into (3.1), the resultant expression for $S(z)$ is

$$S(z) = [\alpha_0(z)\, \varepsilon_0 + \beta_0(z)][\gamma_0(z)\, \varepsilon_0 + \delta_0(z)]^{-1} \qquad (z \in D) \tag{3.8}$$

with a matrix of coefficients

$$A_0(z) = A(z)\, U_\varepsilon. \tag{3.9}$$

Using (1.2) together with (3.7) we obtain a linear fractional transformation of

$$S'(\xi) = [\alpha'(\xi)\,\varepsilon_0 + \beta'(\xi)][\gamma'(\xi)\,\varepsilon_0 + \delta'(\xi)]^{-1} \qquad \text{a.e.,} \tag{3.10}$$

with a matrix of coefficients, expressed in terms of A_0, $A'(\xi) = \text{diag}(U_2(\xi), U_1(\xi))\,A_0(\xi)$. $A'(\xi)$ is j-unitary a.e. because the unitarity of $U_1(\xi)$ and $U_2(\xi)$, and the j-unitarity of $A_0(\xi)$. For a matrix-valued function $S(z) \in \mathscr{S}\pi$, we define [4]

$$N_s = \{h(\xi) \in L_+^2(C^n), \qquad T(\xi)\,h(\xi) = h(\xi) \text{ a.e.}\}, \tag{3.11}$$

where $T(\xi) = S^*(\xi)\,S(\xi)$, we have proved in [4] that N_s is a closed linear manifold and, consequently, that

$$L_+^2(C^n) = N_s \oplus N_{s\perp},$$

where the symbol $\perp$ denotes orthogonal complement. For each value of ξ where $T(\xi)$ is denoted we set [4],

$$N_\xi = \{h \in C^n, T(\xi)h = h\},$$
$$N'_\xi = \{h' \in C^n, T'(\xi)\,h' = h'\},$$

where $T'(\xi) = S(\xi)\,S^*(\xi)$ a.e. These relations imply

$$C^n = N_\xi \oplus N_{\xi\perp},$$
$$C^n = N'_\xi \oplus N'_{\varepsilon\perp},$$

and we known from [4] that the subspaces N_ξ have the same dimension for almost every ξ in the unit circle.

Let us consider a vector-valued function $h(\xi) \in N_s$. From definition (3.11) it follows that

$$[I_n - T(\xi)]\,h(\xi) = 0 \qquad \text{a.e.}$$

This relationship, together with (1.3), may be used to derive

$$U_1^{-1}(\xi)[I_n - S'^*(\xi)\,S'(\xi)]\,U_1(\xi)\,h(\xi) = 0 \qquad \text{a.e.} \tag{3.12}$$

Taking into account the particular way of constructing $U_1(\xi)$ and the fact that $h(\xi) \in N_s$, we conclude that

$$x(\xi) = U_1(\xi)\,h(\xi) = (0,..., 0, x_{k+1}(\xi),..., x_n(\xi))^t, \tag{3.13}$$

Since $U_1(\xi)$ is unitary a.e. from (3.12) it follows that

$$[I_n - S'^*(\xi)\, S'(\xi)]\, x(\xi) = 0 \qquad \text{a.e.}$$

Recalling that the dimension of the range of F is r, we suppose without loss of generality, that $\dim N_\xi = n - k$, and introduce the notation

$$w'(\xi) \stackrel{\text{def}}{=} [\gamma'(\xi)\,\varepsilon_0 + \delta'(\xi)]^{-1} = \begin{pmatrix} w'_{11}(\xi) & w'_{12}(\xi) \\ w'_{21}(\xi) & w'_{22}(\xi) \end{pmatrix}, \tag{3.14}$$

where $w'_{11}(\xi)$ is a block of $k \times r$ (k columns and r rows), $w'_{12}(\xi)$ of $(n-k) \times r$, w'_{21} of $kx(n-r)$ and $w'_{22}(\xi)$ of $(n-k) \times (n-r)$. Therefore, the expression (3.13) may be written using (3.4) in the alternative form

$$w'^*(\xi)\, \mathrm{diag}(I_r, 0_{n-r})\, w'(\xi)\, x(\xi) = 0 \qquad \text{a.e.} \tag{3.15}$$

This relationship, together with (3.13) and (3.14), ensures that $w'_{12}(\xi) = 0$ a.e. Using for $\gamma'(\xi)$ and $\delta'(\xi)$ a notation consistent with (3.14), it is easy now to express the blocks of $w'(\xi)$ in terms of the blocks of $\gamma'(\xi)$ and $\delta'(\xi)$, i.e.,

$$\gamma'(\xi) = \begin{pmatrix} \gamma'_{11}(\xi) & \gamma'_{12}(\xi) \\ \gamma'_{21}(\xi) & \gamma'_{22}(\xi) \end{pmatrix}, \qquad \delta'(\xi) = \begin{pmatrix} \delta'_{11}(\xi) & \delta'_{12}(\xi) \\ \delta'_{21}(\xi) & \delta'_{22}(\xi) \end{pmatrix},$$

where the blocks $\gamma'_{11}(\xi)$ and $\delta'_{11}(\xi)$ are of $r \times k$; $\gamma'_{21}(\xi)$ and $\delta'_{21}(\xi)$ of $r \times (n-k)$; $\gamma'_{12}(\xi)$ and $\delta'_{12}(\xi)$ of $(n-r) \times k$; and $\gamma'_{22}(\xi)$ and $\delta'_{22}(\xi)$ of $(n-r) \times (n-k)$; and

$$w'(\xi) = \begin{pmatrix} \delta_{11}'^{-1}(\xi) & 0 \\ [\gamma'_{22}(\xi) + \delta'_{22}(\xi)]^{-1}\, \delta'_{22}(\xi)\, \delta_{11}'^{-1}(\xi) & [\gamma'_{22}(\xi) + \delta'_{22}(\xi)]^{-1} \end{pmatrix}. \tag{3.16}$$

On examinating the expressions (3.16), (3.15), and (1.4) it is found that

$$I_k - S_1^*(\xi)\, S_1(\xi) = \delta_{11}'^{-1*}(\xi)\, \delta_{11}'^{*}(\xi) \qquad \text{a.e.,} \tag{3.17}$$

we know [5] that there exists a solution $\theta_0(\xi)$ of the factorization problem (3.17), verifying

$$\theta_0(\xi) = \lim_{|z| \to 1} \theta_0(z),$$

where $\theta_0(z)$ is bounded and holomorphic in D, uniquely defined among the infinite set of solutions by the normalization conditions $\theta_0(0) > 0$ and $\det \theta_0(z)$ is an outer function. We also know [5] that any solution of (3.17) satisfies $\theta(\xi) = V(\xi)\, \theta_0(\xi)$ a.e., where $V(\xi)$ is a unitary matrix valued function a.e. This fact and (3.17) imply

$$\delta'_{11}(\xi) = \theta^{-1}(\xi) \qquad \text{a.e.,} \tag{3.18}$$

where $\theta(\xi)$ is the boundary value a.e. of a bounded holomorphic matrix-valued function $(z \in D)$ and solution of the factorization problem (3.17).

We introduce now the notation

$$\alpha'(\xi)=\begin{pmatrix}\alpha'_{11}(\xi) & \alpha'_{12}(\xi)\\ \alpha'_{21}(\xi) & \alpha'_{22}(\xi)\end{pmatrix}, \qquad \beta'(\xi)=\begin{pmatrix}\beta'_{11}(\xi) & \beta'_{12}(\xi)\\ \beta'_{21}(\xi) & \beta'_{22}(\xi)\end{pmatrix},$$

and consider again a vector-valued function $h(\xi)\in N_s$. We know [4] that for each fixed value of ξ except, possibly, a set of zero measure it holds that

$$S(\xi)\,h(\xi)=h'(\xi)\in N'_\xi$$

which may be written in the equivalent form

$$S'(\xi)\,x(\xi)=[\alpha'(\xi)\,\varepsilon_0+\beta'(\xi)]\,w'(\xi)\,x(\xi)=x'(\xi), \tag{3.19}$$

where $x(\xi)$ is defined by (3.13) and

$$x'(\xi)=U_2(\xi)\,h'(\xi)=(0,..., 0, x'_{k+1}(\xi),..., x'_n(\xi))^t. \tag{3.20}$$

To derive this last expression we use analogous arguments to those we mentioned in obtaining (3.13). A vector-valued function $g(\xi)\in N_{s\perp}$ verifies the following expression [4] for almost every fixed ξ on the unit circle

$$S(\xi)\,g(\xi)=g'(\xi)\in N'_{\xi\perp},$$

which may be written in the alternative form

$$S'(\xi)\,y(\xi)=y'(\xi), \tag{3.21}$$

where

$$\begin{aligned} y(\xi)&=U_1(\xi)\,g(\xi)=(y_1(\xi),..., y_k(\xi), 0,..., 0)^t,\\ y'(\xi)&=U_2(\xi)\,g'(\xi)=(y'_1(\xi),..., y'_k(\xi), 0,..., 0)^t. \end{aligned} \tag{3.22}$$

From (3.19), (3.20), (3.21), and (3.22) we conclude, after a simple calculation, that

$$[\alpha'_{22}(\xi)+\beta'_{22}(\xi)][\gamma'_{22}(\xi)+\delta'_{22}(\xi)]^{-1}=S_2(\xi) \qquad \text{a.e.,} \tag{3.23}$$

$$\beta'_{11}(\xi)\,\delta'^{-1}_{11}(\xi)=S_1(\xi) \qquad \text{a.e.,} \tag{3.24}$$

$$\alpha'_{12}(\xi)+\beta'_{12}(\xi)=0 \qquad \text{a.e.,} \tag{3.25}$$

$$S_2(\xi)\,\delta'_{21}(\xi)\,\delta'^{-1}_{11}(\xi)-\beta'_{21}(\xi)\,\delta'^{-1}_{11}(\xi)=0 \qquad \text{a.e.,}$$

where the last equality may be immediately rewritten as

$$\beta'_{21}(\xi) = S_2(\xi)\,\delta'_{21}(\xi) \qquad \text{a.e.} \tag{3.26}$$

The fact that $A'(\xi)$ is j-unitary a.e. is equivalent to the following system for the blocks of $A'(\xi)$,

$$\alpha'^*(\xi)\,\alpha'(\xi) - \gamma'^*(\xi)\,\gamma'(\xi) = I_n \qquad \text{a.e.,}$$

$$\delta'^*(\xi)\,\delta'(\xi) - \beta'^*(\xi)\,\beta'(\xi) = I_n \qquad \text{a.e.,}$$

$$\alpha'^*(\xi)\,\beta'(\xi) - \gamma'^*(\xi)\,\delta'(\xi) = 0 \qquad \text{a.e.}$$

Let us develop these equations using the notation we introduce for the block of $A'(\xi)$,

$$\alpha'^*_{11}(\xi)\,\alpha'_{11}(\xi) + \alpha'^*_{21}(\xi)\,\alpha'_{21}(\xi) - \gamma'^*_{11}(\xi)\,\gamma'_{11}(\xi) - \gamma'^*_{21}(\xi)\,\gamma'_{21}(\xi) = I_k \qquad \text{a.e.,} \tag{3.27}$$

$$\alpha'^*_{22}(\xi)\,\alpha'_{22}(\xi) + \alpha'^*_{12}(\xi)\,\alpha'_{12}(\xi) - \gamma'^*_{22}(\xi)\,\gamma'_{22}(\xi) - \gamma'^*_{12}(\xi)\,\gamma'_{12}(\xi) = I_{n-k} \qquad \text{a.e.,}$$

$$\alpha'^*_{11}(\xi)\,\alpha'_{12}(\xi) + \alpha'^*_{21}(\xi)\,\alpha'_{22}(\xi) - \gamma'^*_{11}(\xi)\,\gamma'_{12}(\xi) - \gamma'^*_{12}(\xi)\,\gamma'_{22}(\xi) = 0 \qquad \text{a.e.,} \tag{3.28}$$

$$\delta'^*_{11}(\xi)\,\delta'_{11}(\xi) + \delta'^*_{21}(\xi)\,\delta'_{21}(\xi) - \beta'^*_{11}(\xi)\,\beta'_{11}(\xi) - \beta'^*_{21}(\xi)\,\beta'_{21}(\xi) = I_k \qquad \text{a.e.,}$$

$$\delta'^*_{22}(\xi)\,\delta'_{22}(\xi) + \delta'^*_{12}(\xi)\,\delta'_{12}(\xi) - \beta'^*_{22}(\xi)\,\beta'_{22}(\xi) - \beta'^*_{12}(\xi)\,\beta'_{12}(\xi) = I_{n-k} \qquad \text{a.e.,}$$

$$\delta'^*_{11}(\xi)\,\delta'_{12}(\xi) + \delta'^*_{21}(\xi)\,\delta'_{22}(\xi) - \beta'^*_{11}(\xi)\,\beta'_{12}(\xi) - \beta'^*_{21}(\xi)\,\beta'_{22}(\xi) = 0 \qquad \text{a.e.,}$$

$$\alpha'^*_{11}(\xi)\,\beta'_{11}(\xi) + \alpha'^*_{21}(\xi)\,\beta'_{21}(\xi) - \gamma'^*_{11}(\xi)\,\delta'_{11}(\xi) - \gamma'^*_{21}(\xi)\,\delta'_{21}(\xi) = 0 \qquad \text{a.e.,} \tag{3.29}$$

$$\alpha'^*_{12}(\xi)\,\beta'_{12}(\xi) + \alpha'^*_{21}(\xi)\,\beta'_{22}(\xi) - \gamma'^*_{11}(\xi)\,\delta'_{12}(\xi) - \gamma'^*_{21}(\xi)\,\delta'_{22}(\xi) = 0 \qquad \text{a.e.,} \tag{3.30}$$

$$\alpha'^*_{12}(\xi)\,\beta'_{11}(\xi) + \alpha'^*_{22}(\xi)\,\beta'_{21}(\xi) - \gamma'^*_{12}(\xi)\,\delta'_{11}(\xi) - \gamma'^*_{22}(\xi)\,\delta'_{12}(\xi) = 0 \qquad \text{a.e.,}$$

$$\alpha'^*_{12}(\xi)\,\beta'_{12}(\xi) + \alpha'^*_{22}(\xi)\,\beta'_{22}(\xi) - \gamma'^*_{12}(\xi)\,\delta'_{12}(\xi) - \gamma'^*_{11}(\xi)\,\delta'_{22}(\xi) = 0 \qquad \text{a.e.}$$

From (3.28) and (3.30), we get

$$\gamma'_{21}(\xi) = S_2^*(\xi)\alpha'_{21}(\xi) \qquad \text{a.e.} \tag{3.31}$$

and replacing the above equation, together with (3.24) and (3.26), in (3.29) we find that

$$\gamma'_{11}(\xi) = S_1^*(\xi)\,\alpha'_{11}(\xi) \qquad \text{a.e.} \tag{3.32}$$

Let us consider now the factorization problem

$$I_k - S_1(\xi)\, S_1^*(\xi) = \psi(\xi)\, \psi^*(\xi) \qquad \text{a.e.}$$

From among the infinite set of solutions we uniquely define a function $\psi_0(\xi)$, which is the boundary value of a bounded and holomorphis matrix-valued function $\psi_0(z)$ $(z \in D)$ satisfying the normalization conditions $\psi_0(0) > 0$ and det $\psi(z)$ is an outer function. Any solution of this problem is obtained by means of

$$\psi(\xi) = \psi_0(\xi)\, V_2(\xi),$$

where $V_2(\xi)$ is an isometric matrix-valued function [5]. Substituting (3.31) and (3.32) into (3.27) we obtain

$$\alpha'_{11}(\xi) = \psi^{-1*}(\xi) \qquad \text{a.e.}$$

$$\gamma'_{11}(\xi) = S_1^*(\xi)\, \psi^{-1*}(\xi) \qquad \text{a.e.,}$$

and recalling that the functions $\theta(\xi)$ and $\psi(\xi)$ may be expressed in terms of the solutions $\theta_0(\xi)$ and $\psi_0(\xi)$, i.e.,

$$\theta(\xi) = V_1(\xi)\, \theta_0(\xi), \tag{3.33}$$

$$\psi(\xi) = \psi_0(\xi)\, V_2(\xi), \tag{3.34}$$

using (3.18) and (3.24) we arrive at the following results:

$$\delta'_{11}(\xi) = \theta_0^{-1}(\xi)\, V_1^{-1}(\xi),$$

$$\beta'_{11}(\xi) = S_1(\xi)\, \theta_0^{-1}(\xi)\, V_1^{-1}(\xi),$$

$$\alpha'_{11}(\xi) = \psi^{-1*}(\xi)\, V_2(\xi),$$

$$\gamma'_{11}(\xi) = S_1^*(\xi)\, \psi^{*-1}(\xi)\, V_2(\xi),$$

where $V_1(\xi)$ and $V_2(\xi)$ are unitary matrix-valued functions a.e. We know also from (3.16) that

$$\gamma'_{12}(\xi) = -\delta'_{12}(\xi) \qquad \text{a.e.}$$

Finally, by virtue of the above results, we get the following expression for the matrix-valued function $A'(\xi)$,

$$A'(\xi) = \begin{pmatrix} \psi_0^{*-1}(\xi)\, V_2(\xi) & \alpha'_{12}(\xi) & S_1(\xi)\, \theta_0^{-1}(\xi)\, V_1^{-1}(\xi) & -\alpha'_{12}(\xi) \\ \alpha'_{21}(\xi) & \alpha'_{22}(\xi) & S_2(\xi)\, \delta'_{21}(\xi) & \beta_{22}(\xi) \\ S_1^*(\xi)\, \psi_0^{*-1}(\xi)\, V_2(\xi) & \gamma'_{12}(\xi) & \theta_0^{-1}(\xi)\, V_1^{-1}(\xi) & -\gamma'_{12}(\xi) \\ S_2^*(\xi)\, \alpha'_{21}(\xi) & \gamma'_{22}(\xi) & \delta'_{21}(\xi) & \delta'_{22}(\xi) \end{pmatrix}$$

and, recalling that $S(z) \in \mathscr{S}\pi$,

$$A'(z) = \begin{pmatrix} \psi_0^{*\,-1}(1/\bar{z})\, V_2(z) & \alpha'_{12}(z) & S_1(z)\,\theta_0^{-1}(z)\, V_1^{-1}(z) & -\alpha'_{12}(z) \\ \alpha'_{21}(z) & \alpha'_{22}(z) & S_2(z)\,\delta'_{21}(z) & \beta'_{22}(z) \\ S_1^*(1/\bar{z})\,\psi_0^{*\,-1}(1/\bar{z})\, V_2(z) & \gamma'_{12}(z) & \theta_0^{-1}(z)\, V_1^{-1}(z) & -\gamma'_{12}(z) \\ S_2^*(1/\bar{z})\,\alpha'_{21}(z) & \gamma'_{22}(z) & \delta'_{21}(z) & \delta'_{22}(z) \end{pmatrix} \quad (z \in D). \tag{3.35}$$

The relationships (3.8) and (3.9) allows us to conclude that the matrix of coefficients of any Darlington realization of a matrix-valued function $S(z)(\in \mathscr{S}\pi)$, may be written in the form

$$A(z) = \operatorname{diag}(U_1^{-1}(z),\, U_2^{-1}(z))\, A'(z)\, U_\varepsilon^{-1}, \tag{3.36}$$

where $A'(z)$ is specified by (3.3) and U_ε by (3.5); and verify the hypothesis of the Basic Lemma [3]. The formula (3.36) is a solution to the problem of multiplicity of Darlington realizations. Varying $V_1(\xi)$ and $V_2(\xi)$ [3] we obtain all the possible realizations of $S(z)$.

To complete this development it is convenient to show that (3.36) contains, as a particular case, formula (4.14) obtained by Arov in [3]. When the condition $I - T(\xi) > 0$ a.e. is satisfied, $\dim N_S = 0$ and (3.35) is reduced to

$$A'(z) = \begin{pmatrix} \psi_0^{*\,-1}(1/\bar{z}) & S_1(z)\,\theta_0^{-1}(z) \\ S_1^*(1/\bar{z})\,\psi_0^{*\,-1}(1/\bar{z}) & \theta_0^{-1}(z) \end{pmatrix} \begin{pmatrix} V_2(z) & 0 \\ 0 & V_1^{-1}(z) \end{pmatrix}$$

and, in addition, it holds that

$$S(\xi) = U_2^{-1}(\xi)\, S_1(\xi)\, U_1(\xi) \qquad \text{a.e.}$$

The above relationship, together with

$$I_n - T(\xi) = U_1^{-1}(\xi)\, \theta_0^*(\xi)\, \theta_0(\xi)\, U_1(\xi) \qquad \text{a.e.,}$$

$$I_n - T'(\xi) = U_2^{-1}\psi_0(\xi)\, \psi_0^*(\xi)\, U_2(\xi) \qquad \text{a.e.}$$

Introducing the notation

$$P(\xi) \stackrel{\text{def}}{=} \theta_0(\xi)\, U_1(\xi),$$

$$\Omega(\xi) \stackrel{\text{def}}{=} U_2^{-1}(\xi)\, \psi_0(\xi),$$

we conclude that

$$A(z)=\begin{pmatrix} Q^{*-1}(1/\bar{z}) & S(z)\,P^{-1}(z) & V_2(z) & 0 \\ S^*(1/\bar{z})\,\Omega^{*-1}(1/\bar{z}) & P^{-1}(z) & 0 & V_1^{-1}(z) \end{pmatrix} U_\varepsilon^{-1}$$

which coincides with formula (4.14) from [3] obtained by Arov as a description of the set of realizations of $S(z)$ when $I_n - T(\xi) > 0$ a.e.

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