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On algebras of holomorphic functions of a given type ☆

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ABSTRACT

We show that several spaces of holomorphic functions on a Riemann domain over a Banach space, including the nuclear and Hilbert–Schmidt bounded type, are locally m-convex Fréchet algebras. We prove that the spectrum of these algebras has a natural analytic structure, which we use to characterize the envelope of holomorphy. We also show a Cartan–Thullen type theorem.

0. Introduction

Holomorphy types were defined by Nachbin [45] in order to include in a single theory the most commonly used spaces of holomorphic functions on infinite dimensional spaces, like the current, nuclear, compact or Hilbert–Schmidt type. Since Pietsch' definition of ideals of multilinear forms [47], the theory of holomorphy types began to interact with the concept of normed ideal of homogeneous polynomials (see for example [34,8,7,12]). We follow this approach in this article incorporating them to the definition of holomorphy type.

A holomorphy type $\mathfrak{A}$ at a Banach space E is a sequence $\{\mathfrak{A}_n(E)\}$ of normed spaces of n-homogeneous polynomials with the property that if a polynomial is in $\mathfrak{A}$, then all its differentials also belong to $\mathfrak{A}$ and such that their norms have controlled growth. In [32], Harris deduced a tight bound for the usual norm of the differentials of a homogeneous polynomial. Noticeably, we are able to show that every known (to us) example of holomorphy type satisfies the same bound (see the examples of the first section).

A holomorphic function on an open set U of E is of type $\mathfrak{A}$ if it has positive $\mathfrak{A}$-radius of convergence at each point of U [45,23,1]. Similarly, entire functions of bounded $\mathfrak{A}$-type are defined as functions that have infinite $\mathfrak{A}$-radius of convergence at zero [28,11,6], and it is immediate to generalize this definition to holomorphic functions of bounded $\mathfrak{A}$-type on a ball. We propose a definition of holomorphic function of bounded $\mathfrak{A}$-type on a general open set U of E and on a Riemann domain spread over E, and study some properties of the space $H_{b\mathfrak{A}}(U)$ of this class of functions. When $\mathfrak{A}$ is the sequence of all continuous homogeneous polynomials (i.e. for the current holomorphy type) we recover $H_b(U)$, the space of bounded type holomorphic functions on U. The space $H_b(U)$ is a Fréchet algebra with the topology of uniform convergence on U-bounded sets. On the other hand, the spaces of nuclear [23] (see also [25, Exercise 2.63]) and Hilbert–Schmidt [46,35] polynomials were proved to be algebras, and thus the corresponding spaces of entire functions of bounded type are also algebras. A more general approach was followed in [11,13], where *multiplicative* sequences of normed ideals of polynomials

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were studied. A sequence $\mathfrak{A}$ of normed ideals of polynomials is multiplicative if a bound for the norm of the product of two homogeneous polynomials in $\mathfrak{A}$ can be obtained in terms of the product of the norms of these polynomials. It was shown there that several spaces of entire functions of bounded type are algebras (with continuous multiplication). In Section 3, we prove that for every previously mentioned example of holomorphy type the corresponding space $H_{b\mathfrak{A}}(U)$, is actually a locally m-convex Fréchet algebra with its natural topology. We also show that algebras $H_{b\mathfrak{A}}(E)$ are test algebras for Michael's problem on the continuity of characters.

We study the spectrum of the algebra $H_{b\mathfrak{A}}(U)$ and show in Theorem 4.3 that, under fairly general conditions on $\mathfrak{A}$, it may be endowed with a structure of Riemann domain over the bidual of E, generalizing some of the results in [4,13]. We show in Section 5 that the Gelfand extensions to the spectrum are holomorphic and that the spectrum is a domain of holomorphy with respect to the set of all holomorphic functions, as it was proved in [26] for the current holomorphy type. We also characterize the $H_{b\mathfrak{A}}$-envelope of holomorphy of an open set U as a part of the spectrum of $H_{b\mathfrak{A}}(U)$. The envelope of holomorphy for the space of holomorphic functions of a given type was constructed by Hirschowitz in [33]. He also raised the question whether the holomorphic extensions to the envelope of holomorphy are of the same type, see [33, pp. 289–290]. We investigate this question for the case of holomorphic functions of bounded $\mathfrak{A}$-type in the last section of the article. We need to deal there with *weakly differentiable* sequences. The concept of weak differentiability was defined in [13] and it was proved there that it is, in some sense, dual to multiplicativity (see also Remark 6.4). The importance of this duality can be seen, for instance, in Examples 3.12 and 6.5, where we prove that the Hilbert–Schmidt norm of the product of two homogeneous polynomials is bounded by the product of their Hilbert–Schmidt norms. When the sequence $\mathfrak{A}$ is weakly differentiable we succeed to show that the extension of a function in $H_{b\mathfrak{A}}(U)$ to the $H_{b\mathfrak{A}}$-envelope of holomorphy of U is of type $\mathfrak{A}$, thus answering in this case the question of Hirschowitz positively. On the other hand, it is known (see [16, Example 2.8]) that the extension of a bounded type holomorphic function to the H_b-envelope of holomorphy may fail to be of bounded type, thus, we cannot expect the extension of every function in $H_{b\mathfrak{A}}(U)$ to be of bounded $\mathfrak{A}$-type on the $H_{b\mathfrak{A}}$-envelope. We end the article with a version of the Cartan–Thullen theorem for $H_{b\mathfrak{A}}(U)$.

We refer to [25,42] for notation and results regarding polynomials and holomorphic functions in general and to [21, 29–31] for polynomial ideals and symmetric tensor products of Banach spaces.

1. Preliminaries

Throughout this article E is a complex Banach space and $B_E(x,r)$ denotes the open ball of radius r and center x in E. We denote by $\mathcal{P}^k(E)$ the Banach space of all continuous k-homogeneous polynomials from E to $\mathbb{C}$.

We define, for each $P \in \mathcal{P}^k(E)$, $a \in E$ and $j \leqslant k$ the polynomial $P_{a^j} \in \mathcal{P}^{k-j}(E)$ by

$$P_{a^j}(x) = \overset{\vee}{P}\left(a^j, x^{k-j}\right) = \overset{\vee}{P}(\underbrace{a,\ldots,a}_{j}, \underbrace{x,\ldots,x}_{k-j}),$$

where $\overset{\vee}{P}$ is the symmetric k-linear form associated to P. For $j=1$, we write P_a instead of P_{a^1}.

A Riemann domain spread over E is a pair (X,q), where X is a Hausdorff topological space and $q: X \to E$ is a local homeomorphism. For $x \in X$, a ball of radius $r>0$ centered at x is a neighborhood of x that is homeomorphic to $B_E(q(x),r)$ through q. It will be denoted by $B_X(x,r)$. When there is no place for confusion we denote the ball of center x and radius r by $B_r(x)$ (where x can be in E or in X). We also define the distance to the border of X, which is a function $d_X: X \to \mathbb{R}_{>0}$ defined by $d_X(x) = \sup\{r>0\colon\ B_r(x) \text{ exists}\}$. For a subset A of X, $d_X(A)$ is defined as the infimum of $d_X(x)$ with x in A. A subset A of X is called an X-bounded subset if $d_X(A)>0$ and $q(A)$ is a bounded set in E.

Let us recall the definition of polynomial ideal [29,30].

Definition 1.1. A **Banach ideal of (scalar-valued) continuous k-homogeneous polynomials** is a pair $(\mathfrak{A}_k, \|\cdot\|_{\mathfrak{A}_k})$ such that:

(i) For every Banach space E, $\mathfrak{A}_k(E) = \mathfrak{A}_k \cap \mathcal{P}^k(E)$ is a linear subspace of $\mathcal{P}^k(E)$ and $\|\cdot\|_{\mathfrak{A}_k(E)}$ is a norm on it. Moreover, $(\mathfrak{A}_k(E), \|\cdot\|_{\mathfrak{A}_k(E)})$ is a Banach space.
(ii) If $T \in \mathcal{L}(E_1, E)$ and $P \in \mathfrak{A}_k(E)$, then $P \circ T \in \mathfrak{A}_k(E_1)$ with

$$\|P \circ T\|_{\mathfrak{A}_k(E_1)} \leqslant \|P\|_{\mathfrak{A}_k(E)} \|T\|^k.$$

(iii) $z \mapsto z^k$ belongs to $\mathfrak{A}_k(\mathbb{C})$ and has norm 1.

We use the following version of the concept of holomorphy type.

Definition 1.2. Consider the sequence $\mathfrak{A} = \{\mathfrak{A}_k\}_{k=1}^{\infty}$, where for each k, $\mathfrak{A}_k$ is a Banach ideal of k-homogeneous polynomials. We say that $\{\mathfrak{A}_k\}_k$ is a **holomorphy type** if for each $l<k$ there exists a positive constant $c_{k,l}$ such that for every Banach space E, the following hold:

$$\text{if } P \in \mathfrak{A}_k(E),\ a \in E \quad \text{then } P_{a^l} \text{ belongs to } \mathfrak{A}_{k-l}(E) \quad \text{and} \quad \|P_{a^l}\|_{\mathfrak{A}_{k-l}(E)} \leqslant c_{k,l}\|P\|_{\mathfrak{A}_k(E)}\|a\|^l. \tag{1}$$

Remark 1.3. (a) The difference between the above definition and the original Nachbin's definition of holomorphy type [45] is twofold. First, Nachbin did not work with polynomial ideals, a concept that was not defined until mid 80's after the work of Pietsch [47]. We think however that polynomial ideals are in the spirit of the concept of holomorphy type. Holomorphy types defined as above are global holomorphy types in the sense of [7]. Second, the constants considered by Nachbin were of the form $c_{k,l} = \binom{k}{l} C^k$ for some fixed constant C. In most of the results we require that the constants satisfy, for every k, l,

$$c_{k,l} \leqslant \frac{(k+l)^{k+l}}{(k+l)!} \frac{k!}{k^k} \frac{l!}{l^l}. \tag{2}$$

These constants are more restrictive than Nachbin's constants, but, as we will see below, the constants $c_{k,l}$ of every usual example of holomorphy type satisfy (2).

Remark 1.4. In [12] we defined and studied **coherent sequences** of polynomial ideals. Any coherent sequence is a holomorphy type. In fact, a coherent sequence is a holomorphy type which satisfies the following extra condition: for each $l, k \in \mathbb{N}$ there exists a positive constant $d_{k,l}$ such that for every Banach space E,

$$\text{if } P \in \mathfrak{A}_k(E),\ \gamma \in E' \quad \text{then } \gamma^l P \text{ belongs to } \mathfrak{A}_{k+l}(E) \quad \text{and} \quad \|\gamma^l P\|_{\mathfrak{A}_{k+l}(E)} \leqslant d_{k,l} \|P\|_{\mathfrak{A}_k(E)} \|\gamma\|^l. \tag{3}$$

The constants appearing in [12] were of the form $c_{k,l} = C^l$ and $d_{k,l} = D^l$. The extra condition asked for the coherence is very natural since conditions (1) and (3) are dual to each other in the following sense: if $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ is a sequence of polynomial ideals which satisfies (1) (respectively (3)) then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}_{k=1}^{\infty}$ satisfies (3) (respectively (1)) with the same constants (see [12, Proposition 5.1]). Thus we may think a coherent sequence as a sequence of polynomial ideals which form a holomorphy type and whose adjoint ideals are also a holomorphy type.

We present now some examples of holomorphy types with constants as in (2).

Example 1.5. *The sequence $\mathcal{P}$ of ideals of continuous polynomials* is, by [32, Corollary 4], a holomorphy type with constants as in (2). The same holds for other sequences of closed ideals as the sequence $\mathcal{P}_w$ of weakly continuous on bounded sets polynomials, and the sequence $\mathcal{P}_A$ of approximable polynomials.

Slight modifications on the results of [12, Corollaries 5.2 and 5.6] (see also [44, Section 3.1.4]) show that if a sequence $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ of ideals form a holomorphy type with constants $c_{k,l}$ the sequence of maximal ideals $\{\mathfrak{A}_k^{\max}\}_{k=1}^{\infty}$ and the sequence of minimal ideals $\{\mathfrak{A}_k^{\min}\}_{k=1}^{\infty}$ are also holomorphy types with the same constants $c_{k,l}$. Also, as already mentioned in Remark 1.4, if the sequence $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ satisfies the condition of coherence (3) with constants $d_{k,l}$ then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}_{k=1}^{\infty}$ is a holomorphy type with constants $d_{k,l}$. As a consequence we have the following.

Example 1.6. *The sequence $\mathcal{P}_I$ of ideals of integral polynomials* is a holomorphy type with constants $c_{k,l} = 1$. Indeed, since condition (3) is trivially satisfied by the sequence $\mathcal{P}$ with constants $d_{k,l} = 1$ and since $(\mathcal{P}^k)^* = \mathcal{P}_I^k$, the result follows from the above comments.

Example 1.7. *The sequence $\mathcal{P}_N$ of ideals of nuclear polynomials* is a holomorphy type with constants $c_{k,l} = 1$ because $\mathcal{P}_N^k = (\mathcal{P}_I^k)^{\min}$.

Example 1.8. *Sequences of polynomial ideals associated to a sequence of natural symmetric tensor norms.* For a symmetric tensor norm β_k (of order k), the projective and injective hulls of β_k (denoted as $\backslash\beta_k/$ and $/\beta_k\backslash$ respectively) are defined as the tensor norms induced by the following mappings (see [15]):

$$\Big(\bigotimes^{k,s} \ell_1(B_E), \beta_k\Big) \overset{1}{\twoheadrightarrow} \Big(\bigotimes^{k,s} E, \backslash\beta_k/\Big),$$
$$\Big(\bigotimes^{k,s} E, /\beta_k\backslash\Big) \overset{1}{\hookrightarrow} \Big(\bigotimes^{k,s} \ell_\infty(B_{E'}), \beta_k\Big).$$

In [15], natural tensor norms for arbitrary order are introduced and studied, in the spirit of the natural tensor norms of Grothendieck. A finitely generated symmetric tensor norm of order k, β_k is *natural* if β_k is obtained from ε_k (the injective symmetric tensor norm) and π_k (the projective symmetric tensor norms) taking a finite number of projective and injective hulls (see [15] for details). For $k \geqslant 3$, it is shown in [15] that there are exactly six non-equivalent natural tensor norms and for $k = 2$ there are only four.

Let $\mathfrak{A}_k$ be an ideal of k-homogeneous polynomials associated to a finitely generated symmetric tensor norm α_k. Small variations in Lemma 3.1.34 of [44] show that if $\{\mathfrak{A}_k\}$ is a holomorphy type with constants $c_{k,l}$ then so are the sequences of maximal (or minimal) ideals associated to the projective and injective hulls of α_k. In particular, any of the sequences of maximal (or minimal) ideals associated to any of the sequences of natural norms is a holomorphy type with constants as in (2).

Example 1.9. *The sequence $\mathcal{P}_e$ of ideals of extendible polynomials.* Since the ideal of extendible polynomials $\mathcal{P}_e^k$ is the maximal ideal associated to the tensor norm $\backslash\varepsilon_k/$, we have by the previous example that the sequence $\mathcal{P}_e$ is a holomorphy type with constants as in (2).

Example 1.10. *The sequence $\mathcal{M}_r$ of ideals of multiple r-summing polynomials.* It was shown in [12, Example 1.13] that it is a coherent sequence with constants equal to 1 thus, in particular, it is a holomorphy type with constants $c_{k,l}=1$.

Example 1.11. *The sequence $\mathcal{S}_2$ of ideals of Hilbert–Schmidt polynomials.* It was shown in [27, Proposition 3] that it is a holomorphy type with constants $c_{k,l}=1$.

Example 1.12. *The sequence $\mathcal{S}_p$ of p-Schatten–von Neumann polynomials.* Let H be a Hilbert space. Recall that for $1<p<\infty$, the p-Schatten–von Neumann k-homogeneous polynomials on H may be defined, using the complex interpolation method, interpolating nuclear and approximable polynomials on H [18,11] as follows:

$$\mathcal{S}_p^k(H):=\big[\mathcal{P}_N^k(H),\mathcal{P}_A^k(H)\big]_\theta,$$

where $p(1-\theta)=1$. The space of Hilbert–Schmidt polynomials coincide (isometrically) with the space of 2-Schatten–von Neumann polynomials. Since interpolation of holomorphy types is a holomorphy type (see [13, Proposition 1.2]), we can conclude that $\{\mathcal{S}_p^k\}$ is a holomorphy type with constants

$$c_{k,l}\leqslant\left(\frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\right)^{1-\frac{1}{p}}.$$

Moreover, using the Reiteration theorem [5, 4.6.1], we have that, for $1<p<2$, $\mathcal{S}_p^k(H):=[\mathcal{P}_N^k(H),\mathcal{S}_2^k(H)]_{2\theta}$, with $p(1-\theta)=1$. Thus, for $1<p<2$, we can obtain $c_{k,l}=1$. Similarly, for $2<p<\infty$ we have

$$c_{k,l}\leqslant\left(\frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\right)^{1-\frac{2}{p}}.$$

2. Holomorphic functions of $\mathfrak{A}$-bounded type

There is a natural way to associate to a holomorphy type $\mathfrak{A}$ a class of holomorphic functions on a Riemann domain (X,q) spread over a Banach space E. This space, denoted by $H_{\mathfrak{A}}(X)$, consists on all holomorphic functions that have positive $\mathfrak{A}$-radius of convergence at each point of X, see for example [23, Definition 2]. To give the precise definition, let us recall that if f is a holomorphic function on X, then its k-th differential is defined by

$$\frac{d^kf(x)}{k!}:=\frac{d^k[f\circ(q|_{B_s(x)})^{-1}]}{k!}\big(q(x)\big).$$

Definition 2.1. Let $\mathfrak{A}=\{\mathfrak{A}_k\}_k$ be a holomorphy type; E a Banach space, and (X,q) a Riemann domain spread over a Banach space E. A holomorphic function f is of type $\mathfrak{A}$ on X if for each $x\in X$, $d^kf(x)$ belongs to $\mathfrak{A}_k(E)$ and

$$\lim_{k\to\infty}\left\|\frac{d^kf(x)}{k!}\right\|_{\mathfrak{A}_k(E)}^{1/k}<\infty.$$

We denote by $H_{\mathfrak{A}}(X)$ the space of type $\mathfrak{A}$ functions on X.

We may also define a space of entire functions of bounded $\mathfrak{A}$-type [11,28,6] as the set of entire functions with infinite $\mathfrak{A}$-radius of convergence at zero (and hence at every point). Similarly we can define the holomorphic functions of $\mathfrak{A}$-bounded type on a ball of radius r as the holomorphic functions which have $\mathfrak{A}$-radius of convergence equal r.

Definition 2.2. Let $\mathfrak{A}=\{\mathfrak{A}_k\}_k$ be a holomorphy type; E a Banach space, $x\in E$, and $r>0$. We define the space of **holomorphic functions of $\mathfrak{A}$-bounded type** on $B_r(x)$ by

$$H_{b\mathfrak{A}}\big(B_r(x)\big)=\left\{f\in H\big(B_r(x)\big):\ \frac{d^kf(x)}{k!}\in\mathfrak{A}_k(E)\text{ and }\limsup_{k\to\infty}\left\|\frac{d^kf(x)}{k!}\right\|_{\mathfrak{A}_k}^{1/k}\leqslant\frac{1}{r}\right\}.$$

We consider in $H_{b\mathfrak{A}}(B_r(x))$ the seminorms p_s, for $0<s<r$, given by

$$p_s(f)=\sum_{k=0}^{\infty}\left\|\frac{d^kf(x)}{k!}\right\|_{\mathfrak{A}_k}s^k,$$

for all $f\in H_{b\mathfrak{A}}(B_r(x))$. Then it easy to show that $(H_{b\mathfrak{A}}(B_r(x),F),\{p_s\}_{0<s<r})$ is a Fréchet space.

The following examples of spaces of holomorphic functions of bounded type on the unit ball B_E were already defined in the literature and can be seen as particular cases of the above definition.

Example 2.3.

(a) If $\mathfrak{A}$ is the sequence of ideals of continuous homogeneous polynomials, then $H_{b\mathfrak{A}}(B_E) = H_b(B_E)$.
(b) If $\mathfrak{A}$ is the sequence of ideals of weakly continuous on bounded sets polynomials, then it is not difficult to see that $H_{b\mathfrak{A}}(B_E)$ is the space $H_{wu}(B_E)$ of weakly uniformly continuous holomorphic functions on B_E-bounded sets.
(c) If $\mathfrak{A}$ is the sequence of ideals of nuclear polynomials then $H_{b\mathfrak{A}}(B_E)$ is the space of holomorphic functions of nuclear bounded type $H_{Nb}(B_E)$ defined by Gupta and Nachbin (see also [38]).
(d) If $\mathfrak{A}$ is the sequence of ideals of extendible polynomials, then by [10, Proposition 14], $H_{b\mathfrak{A}}(B_E)$ is the space of all $f \in H(B_E)$ such that, for any Banach space $G \supset E$, there is an extension $\tilde{f} \in H_b(B_G)$ of f.
(e) If $\mathfrak{A}$ is the sequence of ideals of integral polynomials, then $H_{b\mathfrak{A}}(B_E)$ is the space of integral holomorphic functions of bounded type $H_{bI}(B_E)$ defined in [22].

Remark 2.4. In general, we have that $H_{b\mathfrak{A}} \subsetneq H_{\mathfrak{A}} \cap H_b$. Indeed, Dineen found in [23, Example 9] an entire function of bounded type on a Hilbert space E, $f \in H_b(E)$, such that f is of nuclear type on E, $f \in H_N(E)$, but f is not an entire function of nuclear bounded type because $\lim_{n\to\infty} \| \frac{d^n f(0)}{n!} \|_N^{\frac{1}{n}} = 1$.

We now define holomorphic functions of $\mathfrak{A}$-bounded type on a Riemann domain (X, q) spread over a Banach space. If f is of type $\mathfrak{A}$ on X, and it has $\mathfrak{A}$-radius of convergence greater than $s > 0$ at $x \in X$, then we define

$$p_s^x(f) = \sum_{k=0}^{\infty} s^k \left\| \frac{d^k f(x)}{k!} \right\|_{\mathfrak{A}_k(E)}.$$

The holomorphic functions of $\mathfrak{A}$-bounded type on X are the holomorphic functions on X which are of the class $H_{b\mathfrak{A}}$ on every ball contained in X and, on each open X-bounded set A, the seminorms p_s^x (with $x \in A$ and $B_s(x) \subset A$) are uniformly bounded.

Definition 2.5. A holomorphic function f is of $\mathfrak{A}$**-bounded type** on (X, q) if:

(i) $f \circ (q|_{B_s(x)})^{-1} \in H_{b\mathfrak{A}}(q(B_s(x)))$ for every $s \leqslant d_X(x)$.
(ii) For each open X-bounded set A,

$$p_A(f) := \sup\{p_s^x(f):\ B_s(x) \subset A\} < \infty.$$

We denote by $H_{b\mathfrak{A}}(X)$ the space of all holomorphic functions of $\mathfrak{A}$-bounded type on (X, q).

When $\mathfrak{A}$ is the sequence $\mathcal{P}$ of ideals of continuous homogeneous polynomials, by the Cauchy inequalities, $H_{b\mathfrak{A}}(X) = H_b(X)$. If $\mathfrak{A}$ is the sequence of ideals of weakly continuous on bounded sets polynomials and U is a balanced open set then $H_{b\mathfrak{A}}(U) = H_{wu}(U)$ (see, for example [9, Proposition 1.1]).

Remark 2.6. Condition (i) in above definition states that f is a holomorphic function of $\mathfrak{A}$-bounded type on each ball contained in X. The space of holomorphic functions on X that satisfy this condition is denoted by $H_{d\mathfrak{A}}(X)$ in [44, Section 3.2.6]. This resembles much the definition given in [26, Section 3] of the space $H_d(X)$ (indeed $H_{d\mathfrak{A}}(X) = H_d(X)$ when $\mathfrak{A} = \mathcal{P}$). The seminorms $\{p_s^x:\ 0 < s < d_X(x),\ x \in X\}$ define a topology on $H_{d\mathfrak{A}}(X)$ which is always complete but not necessarily a Fréchet space topology unless E is separable. In that case we may follow the proof of [26, Proposition 3.2] to show that $H_{d\mathfrak{A}}(X)$ is a Fréchet space. Most of the results in this article remain true if we replace $H_{b\mathfrak{A}}$ by $H_{d\mathfrak{A}}$.

Proposition 2.7. *The seminorms* $\{p_A:\ A$ *open and* X*-bounded*$\}$ *define a Fréchet space topology on* $H_{b\mathfrak{A}}(X)$.

Proof. It is clear that the topology may be described with the countable set of seminorms $\{p_{X_n}\}_{n\in\mathbb{N}}$, where $X_n = \{x \in X:\ \|q(x)\| < n$ and $d_X(x) > \frac{1}{n}\}$, so we only need to prove completeness. Let $(f_k)_k$ be a Cauchy sequence in $H_{b\mathfrak{A}}(X)$, then it is a Cauchy sequence in $H_b(X)$, so there exists a function $f \in H_b(X)$ which is limit (uniformly in X-bounded sets) of the f_k's.

Let $x \in X$ and $r \leqslant d_X(x)$. Then $(f_k \circ (q|_{B_r(x)})^{-1})_k$ is a Cauchy sequence in $H_{b\mathfrak{A}}(B_r(q(x)))$ which converges pointwise to $f \circ (q|_{B_r(x)})^{-1}$. Since $H_{b\mathfrak{A}}(B_r(q(x)))$ is complete we have that $f \circ (q|_{B_r(x)})^{-1}$ belongs to $H_{b\mathfrak{A}}(B_r(q(x)))$, and thus f satisfies (i) of Definition 2.5. Moreover, for each k, $p_s^x(f - f_k) \leqslant \limsup_j p_s^x(f_j - f_k)$ for every $s < d_X(x)$. Thus, if A is an X-bounded set,

$$p_A(f - f_k) = \sup_{B_s(x)\subset A} p_s^x(f_j - f_k) \leqslant \limsup_j \sup_{B_s(x)\subset A} p_s^x(f_j - f_k) = \limsup_j p_A(f_j - f_k),$$

which goes to 0 as $k \to \infty$. Therefore, f is in $H_{b\mathfrak{A}}(X)$ and (f_k) converges to f in $H_{b\mathfrak{A}}(X)$. $\square$

3. Multiplicative sequences

In this section we show that under a condition on $\mathfrak{A}$ which is satisfied by most of the commonly used polynomial ideals, the space $H_{b\mathfrak{A}}(X)$ is a locally m-convex Fréchet algebra.

Definition 3.1. Let $\{\mathfrak{A}_k\}_k$ be a sequence of polynomial ideals. We say that $\{\mathfrak{A}_k\}_k$ is **multiplicative** at E if there exist constants $c_{k,l} > 0$ such that for each $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have that $PQ \in \mathfrak{A}_{k+l}(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leqslant c_{k,l}\|P\|_{\mathfrak{A}_k(E)}\|Q\|_{\mathfrak{A}_l(E)}.$$

If $\{\mathfrak{A}_k\}$ is a multiplicative sequence then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}$ is a holomorphy type with the same constants, and it is moreover a weakly differentiable sequence (see Remark 6.4).

In [13] we studied multiplicative sequences with constants $c_{k,l} \leqslant M^{k+l}$ for some constant $M \geqslant 1$ and proved that in this case the space of entire functions of $\mathfrak{A}$-bounded type is an algebra. To obtain algebras of holomorphic functions on balls or on Riemann domains we need to have more restrictive bounds for $c_{k,l}$. Actually, we impose the constants $c_{k,l}$ to satisfy the inequality (2) for every $k, l \in \mathbb{N}$.

Remark 3.2. Stirling's Formula states that $e^{-1}n^{n+1/2} \leqslant e^{n-1}n! \leqslant n^{n+1/2}$ for every $n \geqslant 1$, so we have that

$$\frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l} \leqslant e^2\left(\frac{kl}{k+l}\right)^{1/2}. \tag{4}$$

As a consequence, if $\mathfrak{A}$ is multiplicative with $c_{k,l}$ as in (2) then, for each $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that for every $k, l \in \mathbb{N}$, $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have,

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leqslant c_\varepsilon(1+\varepsilon)^{k+l}\|P\|_{\mathfrak{A}_k(E)}\|Q\|_{\mathfrak{A}_l(E)}.$$

We will show below that every example of holomorphy type mentioned in Section 1 is a multiplicative sequence with constants that satisfy (2). Let us see before that in this case $H_{b\mathfrak{A}}(X)$ is a locally m-convex Fréchet algebra, that is, the topology may be given by a sequence of submultiplicative seminorms. By a theorem by Mitiagin, Rolewicz and Zelazko [40], it suffices to show that they are (commutative) B_0-algebras[1] and that functions in $H(\mathbb{C})$ operate on $H_{b\mathfrak{A}}$ (that is, if $g(z) = \sum_k a_k z^k$ belongs to $H(\mathbb{C})$ and $f \in H_{b\mathfrak{A}}$, then $\sum_k a_k f^k$ belongs to $H_{b\mathfrak{A}}$). We consider first the case of a ball and the whole space.

Proposition 3.3. *Suppose that $\mathfrak{A}$ is multiplicative with $c_{k,l}$ as in* (2)*, and E a Banach space. Then,*

(i) *for each $x \in E$ and $r > 0$, $H_{b\mathfrak{A}}(B_r(x))$ is a locally m-convex Fréchet algebra.*
(ii) *$H_{b\mathfrak{A}}(E)$ is a locally m-convex Fréchet algebra.*

Proof. We just prove (i) because (ii) follows similarly. We will show this for $r = 1$ and $x = 0$, that is for $B_r(x) = B_E$. The general case follows by translation and dilation. We already know that $H_{b\mathfrak{A}}(B_E)$ is a Fréchet space. Let us first show that it is a B_0-algebra.

Let $f = \sum_k P_k$ and $g = \sum_k Q_k$ be functions in $H_{b\mathfrak{A}}(B_E)$. We must show that $\frac{d^n fg(0)}{n!}$ belongs to $\mathfrak{A}_n(E)$ and that $p_s(fg) = \sum_{n=0}^\infty s^n \|\frac{d^n fg(0)}{n!}\|_{\mathfrak{A}_n(E)} < \infty$ for every $s < 1$. Since $\frac{d^n fg(0)}{n!} = \sum_{k=0}^n P_k Q_{n-k}$ and $\mathfrak{A}$ is multiplicative, $\frac{d^n fg(0)}{n!}$ belongs to $\mathfrak{A}_n(E)$. On the other hand, by (4),

$$\begin{aligned}\sum_{n=0}^\infty s^n\left\|\frac{d^n fg(0)}{n!}\right\|_{\mathfrak{A}_n(E)} &\leqslant e^2\sum_{n=0}^\infty s^n\sum_{k=0}^n\left(\frac{k(n-k)}{n}\right)^{1/2}\|P_k\|_{\mathfrak{A}_k(E)}\|Q_{n-k}\|_{\mathfrak{A}_{n-k}(E)}\\ &= e^2\sum_{k=0}^\infty \sqrt{k}s^k\|P_k\|_{\mathfrak{A}_k(E)}\sum_{n=k}^\infty s^{n-k}\left(\frac{n-k}{n}\right)^{1/2}\|Q_{n-k}\|_{\mathfrak{A}_{n-k}(E)}\\ &\leqslant e^2 p_s(g)\sum_{k=0}^\infty \sqrt{k}s^k\|P_k\|_{\mathfrak{A}_k(E)}.\end{aligned}$$

[1] Recall that a B_0-algebra is a complete metrizable topological algebra such that the topology is given by means of an increasing sequence $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \cdots$ of seminorms satisfying that $\|xy\|_j \leqslant C_i\|x\|_{j+1}\|y\|_{j+1}$ for every x, y in the algebra and every $j \geqslant 1$, where C_i are positive constants. It is possible to make $C_i = 1$ for all i [52].

Therefore, for each $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, s) > 1$ such that

$$p_s(fg) = \sum s^n \left\| \frac{d^n fg(0)}{n!} \right\|_{\mathfrak{A}_n(E)} \leqslant c p_s(g) p_{s+\varepsilon}(f). \tag{5}$$

Define, for each $n \geqslant 1$, $s_n = 1 - \frac{1}{2^n}$ and $c_n = c(\frac{1}{2^{n+1}}, 1 - \frac{1}{2^n})$. Then, for every $f, g \in H_{b\mathfrak{A}}(B_E)$, we have

$$p_{s_n}(fg) \leqslant c_n p_{s_n}(g) p_{s_{n+1}}(f) \leqslant c_n p_{s_{n+1}}(f) p_{s_{n+1}}(g). \tag{6}$$

Since the seminorms p_{s_n} determine the topology of $H_{b\mathfrak{A}}(B_E)$, we conclude that $H_{b\mathfrak{A}}(B_E)$ is a B_0-algebra. Note also that (6) implies that $p_{s_n}(f^k) \leqslant c_n^{k-1} p_{s_{n+1}}(f)^{k-1} p_{s_n}(f) \leqslant c_n^k p_{s_{n+1}}(f)^k$. Take now an entire function $h \in H(\mathbb{C})$, $h(z) = \sum a_k z^k z$. Then for $f \in H_{b\mathfrak{A}}(B_E)$,

$$p_{s_n}\left(\sum_{k=N}^{M} a_k f^k\right) \leqslant \sum_{k=N}^{M} a_k p_{s_n}\big(f^k\big) \leqslant \sum_{k=N}^{M} a_k \big(c_n p_{s_{n+1}}(f)\big)^k,$$

which tends to 0 as N, M increase because h is an entire function. This means that entire functions operate in $H_{b\mathfrak{A}}(B_E)$. Therefore [40, Theorem 1] implies that $H_{b\mathfrak{A}}(B_E)$ is locally m-convex. □

Remark 3.4. Michael's conjecture [39] states that on any Fréchet algebra, every character is continuous. Adapting some of the ideas in [20,17,51], Mujica showed (see [42, Section 33] or [43]) that if every character on $H_b(E)$ is continuous for some infinite dimensional Banach space E then the conjecture is true for every commutative Fréchet algebra. As a corollary of his results we may deduce that the same is true for the Fréchet algebra $H_{b\mathfrak{A}}(E)$ for any multiplicative sequence $\mathfrak{A}$ with constants as in (2). We would like however to sketch an alternative proof of this fact as consequence of a result by Ryan [48] on the convergence of monomial expansions for entire functions on ℓ_1. Theorem 3.3 in [48] states that for each $f \in H_b(\ell_1)$ there exist unique complex coefficients $(a_m)_{m \in \mathbb{N}^{(\mathbb{N})}}$ such that $f(z) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m z^m$, where a multi-index $m \in \mathbb{N}^{(\mathbb{N})}$ is a sequence of are non-negative integers such that only a finite number of them are non-zero and where $z^m = \Pi_{j \in \mathbb{N}} z_j^{m_j}$. The convergence of the monomial expansion of f is absolute for every $z \in \ell_1$ and uniform on bounded sets of ℓ_1. Moreover, the coefficients satisfy

$$\lim_{|m| \to \infty} \big(|a_m| m^m / |m|^{|m|}\big)^{1/|m|} = 0. \tag{7}$$

Conversely, any such coefficients define a function in $H_b(\ell_1)$. Let $\mathcal{A}$ be a commutative, complete, Hausdorff locally m-convex algebra which has an unbounded character ψ. We may suppose that $\mathcal{A}$ has unit e. Let $\mathbf{x} = (x_n)$ be a sequence in $\mathcal{A}$ such that $\sum_n p(x_n) < \infty$ for each continuous seminorm p on $\mathcal{A}$, and that $(\psi(x_n))$ is unbounded. For $m = (m_1, m_2, \ldots) \in \mathbb{N}^{(\mathbb{N})}$, let $\mathbf{x}^m = \Pi_{j \in \mathbb{N}} x_j^{m_j}$, where $x^0 = e$ for every $x \in \mathcal{A}$. Let $T : H_b(\ell_1) \to \mathcal{A}$ be defined by $Tf = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m \mathbf{x}^m$, where $(a_m)_{m \in \mathbb{N}^{(\mathbb{N})}}$ are the coefficients of f. Note that $g(z) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m| z^m$ defines a function in $H_b(\ell_1)$ because its coefficients satisfy (7). Thus, for a continuous seminorm p on $\mathcal{A}$,

$$\sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m| p\big(\mathbf{x}^m\big) \leqslant \sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m| \Pi_{j \in \mathbb{N}} p(x_j)^{m_j} = g\big(\big(p(x_j)\big)_j\big),$$

which implies that T is well defined. Clearly T is an algebra homomorphism.

Note also that $T(e'_j) = x_j$, where e'_j is the j-th coordinate functional in ℓ_1. Therefore $\psi \circ T$ is a discontinuous character on $H_b(\ell_1)$.

Let now E be any Banach space and let (y_j), (y'_j) be a biorthogonal sequence in E with $\|y_j\| < 1$ and (y'_j) bounded. Let M be the closed space spanned by the y_j's and let (z_k) be a dense sequence in the unit ball of M that contains the sequence (y_j). Say $y_j = z_{n_j}$. Then the linear map which sends each $e_k \in \ell_1$ to z_k induces an isomorphism from a quotient of ℓ_1 to M. Consider the following mapping $R = R_4 R_3 R_2 R_1 : H_b(E) \to H_b(\ell_1)$

$$H_{b\mathfrak{A}}(E) \xrightarrow{R_1} H_b(E) \xrightarrow{R_2} H_b(M) \xrightarrow{R_3} H_b(\ell_1) \xrightarrow{R_4} H_b(\ell_1),$$

where R_1 is the inclusion, R_2 is the restriction from E to M, R_3 is the composition with the quotient map and R_4 is the restriction to the closed space spanned by the e_{n_j}'s (we identify this space with ℓ_1). Then $R(y'_j)$ is a linear functional on ℓ_1. Moreover, if $\bar{z}$ denotes the class of $z \in \ell_1$ in the quotient of ℓ_1 isomorphic to M, then $\overline{e_{n_k}} = z_{n_k} = y_k$. Thus $R(y'_j)(e_k) = R_3 R_2 R_1(y'_j)(e_{n_k}) = R_2 R_1(y'_j)(\overline{e_{n_k}}) = y'_j(y_k) = \delta_{kj}$, that is, $R(y'_j) = e'_j$.

Since (y'_j) is a bounded sequence in $H_{b\mathfrak{A}}(E)$ and R is an algebra homomorphism, $\psi \circ T \circ R$ is a discontinuous character on $H_{b\mathfrak{A}}(E)$.

We will now prove that $H_{b\mathfrak{A}}(X)$ is a locally m-convex Fréchet algebra, for (X, q) an arbitrary Riemann domain over E. We first need the following.

Lemma 3.5. *Let A be an X-bounded set with $d_X(A) \geqslant \delta$. If $B_s(x) \subset A$ then $B_{s+\delta}(x)$ exists.*

Proof. Suppose that we can show that $B_{s+\frac{\delta}{4}}(x)$ exists. Then $B_{s+\frac{\delta}{4}}(x)$ is contained in the X-bounded set $A_{\frac{\delta}{4}} = \bigcup_{x\in A} B_{\frac{\delta}{4}}(x)$, and $d_X(A_{\frac{\delta}{4}}) \geqslant \frac{3\delta}{4}$. Applying the result proved to $B_{s+\frac{\delta}{4}}(x)$ and $A_{\frac{\delta}{4}}$ we have that $B_{s+\frac{\delta}{4}(1+\frac{3}{4})}(x)$ exists. Applying the same process $n+1$ times, we have that $B_{s+\frac{\delta}{4}(1+\frac{3}{4}+\cdots+(\frac{3}{4})^n)}(x)$ exists. Clearly $\bigcup_{n\in\mathbb{N}} B_{s+\frac{\delta}{4}(1+\frac{3}{4}+\cdots+(\frac{3}{4})^n)}(x)$ is $B_{s+\delta}(x)$. Thus, it suffices to prove that $B_{s+\frac{\delta}{4}}(x)$ exists.

Let $C = \bigcup_{y\in B_s(x)} B_{\frac{\delta}{4}}(y)$. Then $q(C) = \bigcup_{y\in B_s(x)} B_{\frac{\delta}{4}}(q(y)) = B_{s+\frac{\delta}{4}}(q(x))$. If we show that $q|_C$ is injective then $C = B_{s+\frac{\delta}{4}}(x)$.

Take $x_0 \neq x_1$ in C. Then there exist $y_0, y_1 \in B_s(x)$ such that $x_j \in B_{\frac{\delta}{4}}(y_j)$, $j = 0, 1$. If $B_{\frac{\delta}{4}}(y_0) \cap B_{\frac{\delta}{4}}(y_1) \neq \emptyset$, then x_0 and x_1 are in $B_\delta(y_0)$. Since q is injective on $B_\delta(y_0)$, we have that $q(x_0) \neq q(x_1)$. On the other hand, if $B_{\frac{\delta}{4}}(y_0) \cap B_{\frac{\delta}{4}}(y_1) = \emptyset$, then $B_s(q(x)) \cap B_{\frac{\delta}{4}}(q(y_0)) \cap B_{\frac{\delta}{4}}(q(y_1)) = \emptyset$, because q is injective on $B_s(x)$. But, since $q(y_0)$ and $q(y_1)$ are in $B_s(q(x))$, we can conclude that $B_{\frac{\delta}{4}}(q(y_0)) \cap B_{\frac{\delta}{4}}(q(y_1)) = \emptyset$ and thus $q(x_0) \neq q(x_1)$. $\square$

Theorem 3.6. *Suppose that $\mathfrak{A}$ is a multiplicative sequence with constants as in* (2) *and let (X, q) be a Riemann domain over E. Then $H_{b\mathfrak{A}}(X)$ is a locally m-convex Fréchet algebra.*

Proof. We know from Proposition 2.7 that $H_{b\mathfrak{A}}(X)$ is a Fréchet.

Let X_n denote the set $\{x \in X\colon \|q(x)\| < n \text{ and } d_X(x) > \frac{1}{n}\}$. If $B_s(x)$ is contained in X_n and $\varepsilon_n < \frac{1}{n} - \frac{1}{n+1}$ then $B_{s+\varepsilon_n}(x) \subset X_{n+1}$ (note that $B_{s+\varepsilon_n}(x)$ exists by Lemma 3.5). Proceeding as in Proposition 3.3, we can show that for every $f, g \in H_{b\mathfrak{A}}(X)$, $p_s^x(fg) \leqslant c_n p_{s+\varepsilon_n}^x(f) p_s^x(g)$, which implies that $p_{X_n}(fg) \leqslant c_n p_{X_{n+1}}(f) p_{X_n}(g)$. Thus, $H_{b\mathfrak{A}}(X)$ is a commutative B_0-algebra. Moreover, we also have that $p_{X_n}(f^k) \leqslant c_n^k p_{X_{n+1}}(f)^k$, which implies that entire functions operate on $H_{b\mathfrak{A}}(X)$. Therefore by [40, Theorem 1] we conclude that $H_{b\mathfrak{A}}(X)$ is a locally m-convex algebra. $\square$

To finish this section we present some examples of multiplicative sequences with constants $c_{k,l}$ as in (2).

Example 3.7. It is clear that the following sequences are multiplicative with constants $c_{k,l} = 1$.

(i) $\mathcal{P}$, of continuous homogeneous polynomials,
(ii) $\mathcal{P}_w$, of weakly continuous on bounded sets polynomials,
(iii) $\mathcal{P}_A$, of approximable polynomials,
(iv) $\mathcal{P}_e$, of extendible polynomials.

Example 3.8. *The sequence $\mathcal{P}_I$ of integral polynomials.*

It was shown in [13, Example 2.3 (c)] that if P, Q are homogeneous integral polynomials then PQ is integral with $\|PQ\|_I \leqslant \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\|P\|_I\|Q\|_I$.

Example 3.9. *The sequence $\mathcal{P}_N$ of nuclear polynomials.*

Proposition 2.6 in [13] implies that if $\{\mathfrak{A}_k\}$ is a multiplicative sequence then the sequences of maximal and minimal hulls, $\{\mathfrak{A}_k^{\max}\}$ and $\{\mathfrak{A}_k^{\min}\}$, are multiplicative with the same constants. Since nuclear polynomials are the minimal ideal associated to integral polynomials (see for example [29, 3.4]), we have that they form a multiplicative sequence with constants as in (2). See also [25, Exercise 2.63].

Note that, as a consequence of Proposition 3.3, the space of nuclearly entire functions of bounded type is a locally m-convex Fréchet algebra.

The sequences $\mathcal{P}$, $\mathcal{P}_N$, $\mathcal{P}_e$, $\mathcal{P}_I$ and $\mathcal{P}_A$ are particular cases of the following.

Example 3.10. *Let $\{\alpha_k\}_k$ be any of the sequences of natural symmetric tensor norms. Then the sequences $\{\mathfrak{A}_k^{\max}\}_k$ and $\{\mathfrak{A}_k^{\min}\}_k$ of maximal and minimal ideals associated to $\{\alpha_k\}_k$ are multiplicative with constants $c_{k,l}$ as in* (2).

This follows from the inequalities

$$\pi_{k+l}\big(\sigma(s\otimes t)\big) \leqslant \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\pi_k(s)\pi_l(t), \qquad \varepsilon_{k+l}\big(\sigma(s\otimes t)\big) \leqslant \varepsilon_k(s)\varepsilon_l(t)$$

for every $s \in \bigotimes^{k,s} E'$, $t \in \bigotimes^{l,s} E'$ together with Proposition 2.6 and Lemma 2.9 of [13].

Example 3.11. *The sequence $\mathcal{M}_r$ of multiple r-summing polynomials* is multiplicative with constants $c_{k,l} = 1$.

Proof. Let $P \in \mathcal{M}_r^k(E)$, $Q \in \mathcal{M}_r^l(E)$, then

$$(PQ)^\vee(x_1,\dots,x_{k+l}) = \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l} \overset{\vee}{P}\left(x_1, \overset{s_1\cdots s_l}{\dots}, x_{k+l}\right) \overset{\vee}{Q}\,(x_{s_1},\dots,x_{s_l})$$

where $\overset{\vee}{P}\,(x_1, \overset{s_1\cdots s_l}{\dots}, x_{k+l})$ means that coordinates $x_{s_1},\dots,x_{s_l}$ are omitted.

Take $(x_j^{i_j})_{j=1}^{m_j} \subset E$, for $j=1,\dots,k+l$, such that $w_r((x_j^{i_j}))=1$. Then, using the triangle inequality for the ℓ_r-norm,

$$\begin{aligned}
&\left(\sum_{i_1,\dots,i_{k+l}=1}^{m_1,\dots,m_{k+l}} \left|(PQ)^\vee\left(x_1^{i_1},\dots,x_{k+l}^{i_{k+l}}\right)\right|^r\right)^{\frac{1}{r}}\\
&\quad\leqslant \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l}\left(\sum_{i_1,\dots,i_{k+l}=1}^{m_1,\dots,m_{k+l}} \left|\overset{\vee}{P}\left(x_1, \overset{s_1\cdots s_l}{\dots}, x_{k+l}\right)\right|^r\left|\overset{\vee}{Q}\,(x_{s_1},\dots,x_{s_l})\right|^r\right)^{1/r}\\
&\quad\leqslant \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l}\left(\sum_{i_{s_1},\dots,i_{s_l}=1}^{m_{s_1},\dots,m_{s_l}} \left|\overset{\vee}{Q}\,(x_{s_1},\dots,x_{s_l})\right|^r \|P\|_{\mathcal{M}_r^k}^r\right)^{1/r}\\
&\quad\leqslant \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l} \|P\|_{\mathcal{M}_r^k}\|Q\|_{\mathcal{M}_r^l} = \|P\|_{\mathcal{M}_r^k}\|Q\|_{\mathcal{M}_r^l}.
\end{aligned}$$

Hence, PQ is multiple r-summing with $\|PQ\|_{\mathcal{M}_r^{k+l}} \leqslant \|P\|_{\mathcal{M}_r^k}\|Q\|_{\mathcal{M}_r^l}$. □

Example 3.12. *The sequence $\mathcal{S}_2$ of Hilbert–Schmidt polynomials.*

It was proved in [46] that the ideals of Hilbert–Schmidt polynomials form a multiplicative sequence with $c_{k,l} \leqslant 2^{k+l}$. Shortly after, Lopushansky and Zagorodnyuk showed in [35] that actually $c_{k,l}=1$. We will give an alternative proof of this fact in Example 6.5 based on the duality between multiplicativity and weak differentiability (see Remark 6.4 and [13, Proposition 3.16]).

Example 3.13. *The sequence $\mathcal{S}_p$ of p-Schatten–von Neumann polynomials.*

Using the Reiteration theorem for the complex interpolation method and the previous examples we deduce that $\{\mathcal{S}_p^k\}$ is a multiplicative sequence with constants $c_{k,l}=1$ for $2<p<\infty$ and

$$c_{k,l} \leqslant \left(\frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\right)^{\frac{2}{p}-1}$$

for $1<p<2$.

4. Analytic structure on the spectrum

Let (X,q) be a Riemann domain over a Banach space E. In this section we prove that, under fairly general conditions, the spectrum of the algebra $H_{b\mathfrak{A}}(X)$ may be endowed with a structure of Riemann domain spread over the bidual E''. This will extend some of the results in [4,13].

As in the case of H_b studied in [4] or entire functions in $H_{b\mathfrak{A}}(E)$ studied in [13], extensions to the bidual will be crucial, so we will need them to behave nicely. Indeed, we will need the following two conditions which were already defined in [13].

Definition 4.1. Let $\mathfrak{A}$ be a sequence of ideals of polynomials. We say that $\mathfrak{A}$ is **AB-closed** if for each Banach space E, $k\in\mathbb{N}$ and $P\in\mathfrak{A}_k(E)$ we have that $AB(P)$ belongs to $\mathfrak{A}_k(E'')$ and $\|AB(P)\|_{\mathfrak{A}_k(E'')} \leqslant \|P\|_{\mathfrak{A}_k(E)}$, where AB denotes the Aron–Berner extension [2].

Recall that an **Arens extension** of a k-linear form A on E is an extension to the bidual E'' obtained by w^*-continuity on each variable in some order.

Definition 4.2. We say that a sequence of ideals of polynomials $\mathfrak{A}$ is **regular** at E if, for every k and every P in $\mathfrak{A}_k(E)$, we have that every Arens extension of $\overset{\vee}{P}$ is symmetric. We say that the sequence $\mathfrak{A}$ is regular if it is regular at E for every Banach space E.

All the examples given in Section 3 are known to be AB-closed (see [13,14]). All the examples given in Section 3 but $\mathcal{P}$ and $\mathcal{M}_r$ are known to be regular at any Banach space. Any sequence of polynomial ideals is regular at a symmetrically regular Banach space.

We proved in the previous section that if $\mathfrak{A}$ is multiplicative with constants as in (2), then $H_{b\mathfrak{A}}(X)$ is a Fréchet algebra. We denote by $M_{b\mathfrak{A}}(X)$ its spectrum, that is, the set of non-zero multiplicative and continuous linear functionals on $H_{b\mathfrak{A}}(X)$. Note that evaluations at points of X are in $M_{b\mathfrak{A}}(X)$. Following [3,26], we define $\pi : M_{b\mathfrak{A}}(X) \to E''$ by $\pi(\varphi)(x') = \varphi(x' \circ q)$.

The main purpose of this section is to prove the following result.

Theorem 4.3. *Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2) *which is regular at E and AB-closed. Then $(M_{b\mathfrak{A}}(X), \pi)$ is a Riemann domain over E''.*

First we will need some preliminary lemma.

Lemma 4.4. *Let $\mathfrak{A}$ be a holomorphy type with constants as in* (2)*. Define $\delta_{(w_1^{k_1},\dots,w_h^{k_h})} \in \mathfrak{A}_k(E)'$ by,*

$$\delta_{(w_1^{k_1},\dots,w_h^{k_h})}(Q) = \overset{\vee}{Q}\left(w_1^{k_1},\dots,w_h^{k_h}\right),$$

where, $w_1,\dots,w_h \in E$ and $k_1,\dots,k_h \in \mathbb{N}$ are such that $k_1+\dots+k_h = k$. Then for any $P \in \mathfrak{A}_{k+l}(E)$, the polynomial $R(x) := \delta_{(w_1^{k_1},\dots,w_h^{k_h})}(P_{x^l})$ is in $\mathfrak{A}_l(E)$ and

$$\|R\|_{\mathfrak{A}_l(E)} \leqslant \frac{(k+l)^{k+l} k_1!\dots k_h! l!}{(k+l)! k_1^{k_1}\dots k_h^{k_h} l^l} \|w_1\|^{k_1}\cdots\|w_h\|^{k_h}\|P\|_{\mathfrak{A}_{k+l}(E)}.$$

If $\mathfrak{A}$ is also AB-closed and regular at E then the above statements hold for any $w_1,\dots,w_h \in E''$, where $\delta_{(w_1^{k_1},\dots,w_h^{k_h})} \in \mathfrak{A}_k(E)'$ is defined by $\delta_{(w_1^{k_1},\dots,w_h^{k_h})}(Q) = \overset{\vee}{AB(Q)}\,(w_1^{k_1},\dots,w_h^{k_h})$.

Proof. We proceed by induction on h. For $h=1$, this is a consequence of $\mathfrak{A}$ being a holomorphy type. Suppose that it holds for $h=n$ and let $Q(x) = \delta_{(w_1^{k_1},\dots,w_n^{k_n})}(P_{x^{k_{n+1}+l}})$. Then Q belongs to $\mathfrak{A}_{k_{n+1}+l}(E)$ and

$$\|Q\|_{\mathfrak{A}_{k_{n+1}+l}(E)} \leqslant \frac{(k+l)^{k+l} k_1!\dots k_n!(k_{n+1}+l)!}{(k+l)! k_1^{k_1}\dots k_n^{k_n}(k_{n+1}+l)^{k_{n+1}+l}} \|w_1\|^{k_1}\cdots\|w_n\|^{k_n}\|P\|_{\mathfrak{A}_{k+l}(E)}. \tag{8}$$

Thus, $x \mapsto \delta_{w_{n+1}^{k_{n+1}}}(Q_{x^l}) = \delta_{(w_1^{k_1},\dots,w_{n+1}^{k_{n+1}})}(P_{x^l})$ belongs to $\mathfrak{A}_l(E)$ and

$$\left\|x \mapsto \delta_{w_{n+1}^{k_{n+1}}}(Q_{x^l})\right\|_{\mathfrak{A}_l(E)} \leqslant \frac{(k_{n+1}+l)^{k_{n+1}+l} k_{n+1}! l!}{(k_{n+1}+l)! k_{n+1}^{k_{n+1}} l^l} \|w_{k_{n+1}}\|^{n+1}\|Q\|_{\mathfrak{A}_{k_{n+1}+l}(E)}. \tag{9}$$

Putting (8) and (9) together, we obtain our claim. The last statement follows similarly. $\square$

Lemma 4.5. *Let $\mathfrak{A}$ be a holomorphy type with constants as in* (2)*, $k \in \mathbb{N}$, $w_1,\dots,w_h \in E$ and $k_1,\dots,k_h \in \mathbb{N}$ such that $k_1+\dots+k_h = k$. Then $\delta_{(w_1^{k_1},\dots,w_h^{k_h})} \circ \frac{d^k f}{k!}$ belongs to $H_{b\mathfrak{A}}(X)$.*

If $\mathfrak{A}$ is AB-closed and regular at E then the same holds for any $w_1,\dots,w_h \in E''$.

Proof. We prove the case $w_1,\dots,w_h \in E''$. The other case is similar. By [45, §10, Proposition 2], $d^k f \in H(X, \mathfrak{A}_k(E))$. Since $\varphi = \delta_{(w_1^{k_1},\dots,w_h^{k_h})}$ is a continuous linear form on $\mathfrak{A}_k(E)$, $\varphi \circ d^k f$ is in $H(X)$.

Let $B_s(x_0) \subset X$ and denote $\frac{d^m f}{m!}(x_0)$ by Q_m. Then, for $y \in B_s(x_0)$ we have, by [45, p. 41 (1)],

$$\frac{d^k f}{k!}(y) = \sum_{m \geqslant k} \binom{m}{k} (Q_m)_{(q(y)-q(x_0))^{m-k}}.$$

This series is absolutely convergent in $\mathfrak{A}_k(E)$. Indeed, for $\delta, \varepsilon > 0$ such that $(1+\varepsilon)(\delta + \|q(y)-q(x_0)\|) < s$ and using Remark 3.2 we have,

$$\sum_{j\geqslant 0}\delta^j\sum_{m\geqslant j}\binom{m}{j}\big\|(Q_m)_{(q(y)-q(x_0))^{m-j}}\big\|_{\mathfrak{A}_j(E)}\leqslant c_\varepsilon\sum_{j\geqslant 0}\delta^j\sum_{m\geqslant j}\binom{m}{j}(1+\varepsilon)^m\|Q_m\|_{\mathfrak{A}_m(E)}\big\|q(y)-q(x_0)\big\|^{m-j}$$
$$=c_\varepsilon p^{x_0}_{(1+\varepsilon)(\delta+\|q(y)-q(x_0)\|)}(f)<\infty.$$

Then $\varphi\circ\frac{d^kf}{k!}(y)=\sum_{m\geqslant k}\binom{m}{k}\varphi\circ(Q_m)_{(q(y)-q(x_0))^{m-k}}$ and therefore,

$$\frac{d^{m-k}(\varphi\circ\frac{d^kf}{k!})}{(m-k)!}(x_0)=x\mapsto\binom{m}{k}\varphi\circ(Q_m)_{x^{m-k}}.$$

By the above lemma, the differentials of $\varphi\circ\frac{d^kf}{k!}$ are in $\mathfrak{A}$.

Let A be an open X-bounded set and $\alpha<d_X(A)$. Let $B_s(x_0)\subset A$. Then, by (4) and Lemma 4.4, we have

$$\begin{aligned}\alpha^kp^{x_0}_s\left(\varphi\circ\frac{d^kf}{k!}\right)&\leqslant\alpha^k\sum_{m\geqslant k}s^{m-k}\left\|\frac{d^{m-k}(\varphi\circ\frac{d^kf}{k!})}{(m-k)!}(x_0)\right\|_{\mathfrak{A}_{m-k}(E)}\\&\leqslant e^{h+1}(k_1\dots k_h)^{\frac12}\|w_1\|^{k_1}\cdots\|w_h\|^{k_h}\alpha^k\sum_{m\geqslant k}\binom{m}{k}s^{m-k}\sqrt{\frac{m-k}{m}}\|Q_m\|_{\mathfrak{A}_m(E)}\\&\leqslant C_k\alpha^k\|w_1\|^{k_1}\cdots\|w_h\|^{k_h}\sum_{m\geqslant k}\binom{m}{k}s^{m-k}\|Q_m\|_{\mathfrak{A}_m(E)}\\&\leqslant C_k\|w_1\|^{k_1}\cdots\|w_h\|^{k_h}\sum_{j=0}^{\infty}\alpha^j\sum_{m\geqslant j}\binom{m}{j}s^{m-j}\|Q_m\|_{\mathfrak{A}_m(E)}\\&=C_k\|w_1\|^{k_1}\cdots\|w_h\|^{k_h}p^{x_0}_{s+\alpha}(f)\leqslant C_k\|w_1\|^{k_1}\cdots\|w_h\|^{k_h}p_{\tilde A}(f),\end{aligned}$$

where $\tilde A=\bigcup_{x\in A}B_\alpha(x)$ is an open X-bounded set (note that by Lemma 3.5, $B_{s+\alpha}(x_0)$ exists and is contained in $\tilde A$). Therefore $p_A(f)<\infty$ and thus $\varphi\circ\frac{d^kf}{k!}$ is in $H_{b\mathfrak{A}}(X)$. □

The following corollary states that, for $h=1$ or $h=2$ in the previous lemma, we obtain bounds which are independent of k.

Corollary 4.6. *Let $\mathfrak{A}$ be a holomorphy type with constants as in (2) and $f\in H_{b\mathfrak{A}}(X)$. Let A be an open X-bounded set and $\alpha<\rho<d_X(A)$. Define the open X-bounded set $\tilde A=\bigcup_{x\in A}B_\rho(x)$. Then there exists a positive constant C depending only on α and ρ, such that:*

(i) *for each $k\in\mathbb{N}$ and $w\in E$, $\delta_w\circ\frac{d^kf}{k!}$ belongs to $H_{b\mathfrak{A}}(X)$ and*

$$\alpha^kp_A\left(\delta_w\circ\frac{d^kf}{k!}\right)\leqslant C\|w\|^kp_{\tilde A}(f).$$

(ii) *for each $l\leqslant k\in\mathbb{N}$ and $v,w\in E$, $\delta_{(v^{k-l},w^l)}\circ\frac{d^kf}{k!}$ belongs to $H_{b\mathfrak{A}}(X)$ and*

$$\alpha^kp_A\left(\delta_{(v^{k-l},w^l)}\circ\frac{d^kf}{k!}\right)\leqslant C\|v\|^{k-l}\|w\|^lp_{\tilde A}(f).$$

If $\mathfrak{A}$ is AB-closed and regular at E then the above statements hold for any $v,w\in E''$.

Proof. We prove (ii) for $v,w\in E''$. Let $\varepsilon>0$ such that $\alpha(1+\varepsilon)<\rho$ and let $B_s(x_0)\subset A$. By the bound obtained in Lemma 4.5 for $h=2$ and $\varphi=\delta_{(v^{k-l},w^l)}$, we have

$$\begin{aligned}\alpha^kp^{x_0}_s\left(\varphi\circ\frac{d^kf}{k!}\right)&\leqslant e^3\big((k-l)l\big)^{\frac12}\|w\|^l\|v\|^{k-l}\alpha^k\sum_{m\geqslant k}\binom{m}{k}s^{m-k}\sqrt{\frac{m-k}{m}}\|Q_m\|_{\mathfrak{A}_m(E)}\\&\leqslant c_\varepsilon(1+\varepsilon)^k\|w\|^l\|v\|^{k-l}\alpha^k\sum_{m\geqslant k}\binom{m}{k}s^{m-k}\|Q_m\|_{\mathfrak{A}_m(E)}\end{aligned}$$

$$\leqslant c_\varepsilon \|w\|^l \|v\|^{k-l} \sum_{j=0}^{\infty} (1+\varepsilon)^j \alpha^j \sum_{m \geqslant j} \binom{m}{j} s^{m-j} \|Q_m\|_{\mathfrak{A}_m(E)}$$
$$= c_\varepsilon \|w\|^l \|v\|^{k-l} p^{x_0}_{s+(1+\varepsilon)\alpha}(f) \leqslant c_\varepsilon \|w\|^l \|v\|^{k-l} p_{\tilde{A}}(f),$$

where c_ε is chosen so that $e^3 j \leqslant c_\varepsilon (1+\varepsilon)^j$ for every $j \in \mathbb{N}$. $\square$

For $\varphi \in M_{b\mathfrak{A}}(X)$ and A an open X-bounded set, we will write $\varphi \prec A$ whenever there is some $c > 0$ such that $\varphi(f) \leqslant c p_A(f)$ for every $f \in H_{b\mathfrak{A}}(X)$.

Lemma 4.7. *Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2)*. Let $\varphi \in M_{b\mathfrak{A}}(X)$ and A an open X-bounded set such that $\varphi \prec A$. Let $w \in E$ with $\|w\| < d_X(A)$. Then φ^w belongs to $M_{b\mathfrak{A}}(X)$, where*

$$\varphi^w(f) := \sum_{n=0}^{\infty} \varphi\left(\frac{d^n f(\cdot)}{n!}(w)\right).$$

Moreover, $\pi(\varphi^w) = \pi(\varphi) + w$. If $\mathfrak{A}$ is also AB-closed and regular at E then the above statements hold for $w \in E''$ with $\|w\| < d_X(A)$, where,

$$\varphi^w(f) := \sum_{n=0}^{\infty} \varphi\left(AB\left(\frac{d^n f(\cdot)}{n!}\right)(w)\right).$$

Proof. We prove the case $w \in E''$. Suppose that $B_s(x_0) \subset A$. Let $\|w\| < \alpha < \rho < d_X(A)$. We can take $\varepsilon > 0$, such that $(1+\varepsilon)\alpha < \rho$. By Corollary 4.6, for the open X-bounded set $\tilde{A} = \bigcup_{x \in A} B_\rho(x)$, we have

$$\alpha^k p_A\left(AB\left(\frac{d^k f(\cdot)}{k!}\right)(w)\right) \leqslant c_\varepsilon \|w\|^k p_{\tilde{A}}(f). \tag{10}$$

Thus, the series $\sum_k p_A(AB(\frac{d^k f(\cdot)}{k!})(w))$ is convergent. Then,

$$\sum_{k=0}^{\infty} \left|\varphi\left(AB\left(\frac{d^k f(\cdot)}{k!}\right)(w)\right)\right| \leqslant c \sum_{k=0}^{\infty} p_A\left(AB\left(\frac{d^k f(\cdot)}{k!}\right)(w)\right) \leqslant \frac{\alpha c c_\varepsilon}{\alpha - \|w\|} p_{\tilde{A}}(f) < \infty.$$

Therefore φ^w is continuous and $\varphi^w \prec \tilde{A}$. The multiplicativity of φ^w and the last assertion follow as in [4, p. 551]. $\square$

In the case of entire functions φ^w may be defined translating functions by w, see [13,25]. We show next that, this is also true for arbitrary Riemann domains when we complete $H_{b\mathfrak{A}}(X)$ with respect to the topology given by the norm p_A, if $\varphi \prec A$. Given an open X-bounded set A and $w \in E''$ with $\|w\| < d_X(A)$, we define $\tilde{\tau}_w(f)$ on A as $\tilde{\tau}_w(f)(x) = AB(f \circ (q|_{B_x})^{-1})(q(x) + w)$, where B_x denotes the ball $B_{d_X(x)}(x)$.

Lemma 4.8. *Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2) *which is regular at E and AB-closed. Let A be an open X-bounded set, $\varphi \in M_{b\mathfrak{A}}(X)$ such that $\varphi \prec A$ and $w \in E''$ with $\|w\| < d_X(A)$. Then*:

(a) *the series $\sum_{n=0}^{\infty} AB(\frac{d^n f(\cdot)}{n!})(w)$ converges in $\overline{H_{b\mathfrak{A}}(X)}^{p_A}$ to $\tilde{\tau}_w(f)$ and the mapping $\tilde{\tau}_w : H_{b\mathfrak{A}}(X) \to \overline{H_{b\mathfrak{A}}(X)}^{p_A}$ is continuous,*
(b) *φ may be extended to $\overline{H_{b\mathfrak{A}}(X)}^{p_A}$ and $\varphi^w(f) = \varphi(\tilde{\tau}_w f)$.*

Proof. (a) We have already proved in Lemma 4.7 that the series $\sum_n p_A(AB(\frac{d^n f(\cdot)}{n!})(w))$ is convergent and that $p_A(\sum_n AB(\frac{d^n f(\cdot)}{n!})(w)) \leqslant \frac{\alpha c c_\varepsilon}{\alpha - \|w\|} p_{\tilde{A}}(f)$.

The equality $\tilde{\tau}_w(f)(x) = \sum_{n=0}^{\infty} AB(\frac{d^n f(x)}{n!})(w)$ is clear for each $x \in A$ since the Taylor series of f at x converges absolutely on $B_r(x)$, for each $r < d_X(A)$.

(b) The first assertion is immediate since $\varphi(f) \leqslant c p_A(f)$ for every $f \in H_{b\mathfrak{A}}(X)$. The second assertion, is a consequence of the equality $\tilde{\tau}_w(f) = \sum_{n=0}^{\infty} AB(\frac{d^n f(\cdot)}{n!})(w)$ as function in $\overline{H_{b\mathfrak{A}}(X)}^{p_A}$, the continuity of φ with respect to the norm p_A and the definition of φ^w. $\square$

Lemma 4.9. *Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2) *which is regular at E and AB-closed. Let A be an open X-bounded set, $\varphi \in M_{b\mathfrak{A}}(X)$ such that $\varphi \prec A$ and $v, w \in E''$ with $\|v\| + \|w\| < d_X(A)$. Then*:

(a) *$\tilde{\tau}_w \tilde{\tau}_v f = \tilde{\tau}_{w+v} f$ for every $f \in H_{b\mathfrak{A}}(X)$,*
(b) *$(\varphi^w)^v$ is a well-defined character in $M_{b\mathfrak{A}}(X)$ and $(\varphi^w)^v = \varphi^{w+v}$.*

Proof. (a) We must prove that $\tilde{\tau}_w \tilde{\tau}_v f(x) = \tilde{\tau}_{w+v} f(x)$ for $x \in A$. Write g_n for $AB(\frac{d^n f(\cdot)}{n!})(v)$. Then

$$\tilde{\tau}_w \tilde{\tau}_v(f) = \sum_n \tilde{\tau}_w(g_n) = \sum_n \sum_k AB\left(\frac{d^k g_n(\cdot)}{k!}\right)(w). \tag{11}$$

Since $\mathfrak{A}$ is regular at E, we may proceed as in [4, p. 552] to show that

$$\frac{d^k g_n(\cdot)}{k!} = \binom{k+n}{n} AB\left(\frac{d^{k+n} f(\cdot)}{(k+n)!}\right)_{v^n}.$$

Thus again by regularity,

$$AB\left(\frac{d^k g_n(\cdot)}{k!}\right)(w) = \binom{k+n}{n} AB\left(\frac{d^{k+n} f(\cdot)}{(k+n)!}\right)^{\vee}\left(v^n, w^k\right) = \binom{k+n}{n} \delta_{(v^n, w^k)} \circ \frac{d^{k+n} f}{(k+n)!}.$$

Let $\|v\| + \|w\| < \tilde{\alpha} < \tilde{\rho} < d_X(A)$. By Corollary 4.6, if $A^\sharp = \bigcup_{x\in A} B_{\tilde{\rho}}(x)$, then there exists a constant $C > 0$ such that,

$$\begin{aligned}\sum_{n\geqslant 0}\sum_{k\geqslant 0} \binom{k+n}{n} p_A\left(\delta_{(v^n, w^k)} \circ \frac{d^{k+n} f}{(k+n)!}\right) &\leqslant C \sum_{n\geqslant 0}\sum_{k\geqslant 0} \binom{k+n}{n} \frac{\|v\|^n \|w\|^k}{\tilde{\alpha}^{n+k}} p_{A^\sharp}(f) \\ &= C p_{A^\sharp}(f) \sum_{m\geqslant 0}^{\infty} \left(\frac{\|v\| + \|w\|}{\tilde{\alpha}}\right)^m < \infty.\end{aligned}$$

Therefore we may reverse the order of summation in (11) to obtain

$$\begin{aligned}\tilde{\tau}_w \tilde{\tau}_v(f) &= \sum_{n\geqslant 0} \sum_{l\geqslant n} \binom{l}{n} AB\left(\frac{d^l f(\cdot)}{l!}\right)^{\vee}\left(v^n, w^{l-n}\right) = \sum_{l\geqslant 0} \sum_{n=0}^{l} \binom{l}{n} AB\left(\frac{d^l f(\cdot)}{l!}\right)^{\vee}\left(v^n, w^{l-n}\right) \\ &= \sum_{l\geqslant 0} AB\left(\frac{d^l f(\cdot)}{l!}\right)(v+w) = \tilde{\tau}_{v+w}(f).\end{aligned}$$

(b) We continue using the notation of part (a). First note that $\tilde{\alpha} - \|v\|$ (resp. $\tilde{\rho} - \|v\|$) may play the role of α (resp. ρ) in the proof of Lemma 4.7, thus if $\tilde{A} = \bigcup_{x\in A} B_{\tilde{\rho}-\|v\|}(x)$, then φ^w and $\tilde{\tau}_w$ may be continuously extended to $(\overline{H_{b\mathfrak{A}}(X)}^{p_{\tilde{A}}}, p_{\tilde{A}})$. Moreover, the formula $\varphi^w(f) = \varphi(\tilde{\tau}_w f)$ holds for every f in $\overline{H_{b\mathfrak{A}}(X)}^{p_{\tilde{A}}}$.

Second, since $\varphi^w \prec \tilde{A}$ and $d_X(\tilde{A}) \geqslant d_X(A) - (\tilde{\rho} - \|v\|) > d_X(A) - \|w\| > \|v\|$, then by of Lemma 4.7, $(\varphi^w)^v$ is well defined and by part (a) of Lemma 4.8, $\tilde{\tau}_v : H_{b\mathfrak{A}}(X) \to \overline{H_{b\mathfrak{A}}(X)}^{p_{\tilde{A}}}$ is continuous.

Therefore, for every $f \in H_{b\mathfrak{A}}(X)$, we have that $(\varphi^w)^v(f) = \varphi^w(\tilde{\tau}_v f) = \varphi(\tilde{\tau}_w \tilde{\tau}_v f) = \varphi(\tilde{\tau}_{w+v} f) = \varphi^{w+v}(f)$. $\square$

The equality $(\varphi^w)^v = \varphi^{w+v}$ in the above lemma is the key property to show Theorem 4.3. Indeed, once this equality is proved, the rest of the proof can be almost entirely adapted from [4, Theorem 2.2 and Corollary 2.4]. We just point out the only difference.

Proof of Theorem 4.3. For $\varphi \in M_{b\mathfrak{A}}(X)$, $\varphi \prec A$ and $0 < \varepsilon < d_X(A)$, define $V_{\varphi,\varepsilon} = \{\varphi^w\colon\ w \in E'',\ \|w\| < \varepsilon\}$. Then the collection $\{V_{\varphi,\varepsilon}\colon\ \varphi \in M_{b\mathfrak{A}}(X),\ \varepsilon > 0\}$ define a basis for a Hausdorff topology in $M_{b\mathfrak{A}}(X)$. The fact that it is a basis of a topology follows as in [4, Theorem 2.2]. We prove that it is Hausdorff. Let $\varphi \neq \psi \in M_{b\mathfrak{A}}(X)$ and suppose that $\pi(\varphi) \neq \pi(\psi)$. Let A, D be open X-bounded sets such that $\varphi \prec A$ and $\psi \prec D$, and take $r < \min\{d_X(A), d_X(D)\}/2$. We claim that $V_{\varphi,r} \cap V_{\psi,r} = \emptyset$. Indeed, if $\|v\|, \|w\| < r$ are such that $\varphi^w = \psi^v$, then $\pi(\varphi) + w = \pi(\psi) + v$ and thus $v = w$. Moreover, by Lemma 4.9(b), $\varphi = (\varphi^v)^{(-v)} = (\psi^v)^{(-v)} = \psi$. The case $\pi(\varphi) = \pi(\psi)$ follows as in [4, Theorem 2.2]. Now, we may finish the proof of the theorem proceeding as in [4, Corollary 2.4]. $\square$

5. Holomorphic extensions

In this section and in the next one we are concerned with analytic continuation. We show first that the canonical extensions to the spectrum are holomorphic and then we characterize the $H_{b\mathfrak{A}}$-envelope of holomorphy of a Riemann domain in terms of the spectrum.

Proposition 5.1. *Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2) *which is regular at E and AB-closed. For each $f \in H_{b\mathfrak{A}}(X)$, its Gelfand transform $\tilde{f}$ is holomorphic on $M_{b\mathfrak{A}}(X)$.*

Proof. Let $\varphi \in M_{b\mathfrak{A}}(X)$, A an open X-bounded set such that $\varphi \prec A$ and $r < d_X(A)$. We prove that $\tilde{f}$ is holomorphic on $V_{\varphi,r}$, or equivalently that $\tilde{f} \circ (\pi|_{V_{\varphi,r}})^{-1}$ is holomorphic on $\pi(V_{\varphi,r}) = B_{E''}(\pi(\varphi), r)$. It suffices to show that it is uniform limit of polynomials on $rB_{E''}(\pi(\varphi))$. Note that for $\|w\| < r$,

$$\tilde{f} \circ (\pi|_{V_{\varphi,r}})^{-1}\big(\pi(\varphi) + w\big) = \tilde{f}\big(\varphi^w\big) = \varphi^w(f) = \sum_{k=0}^{\infty} \varphi\left(\delta_w \circ \frac{d^k f}{k!}\right).$$

By Lemma 4.5, for $w_1, \ldots, w_k \in E''$, $AB(\frac{d^k f}{k!})^\vee(w_1, \ldots, w_k) = \delta_{(w_1,\ldots,w_k)} \circ \frac{d^k f}{k!}$ belongs to $H_{b\mathfrak{A}}(X)$ and clearly, $AB(\frac{d^k f}{k!})^\vee$ is clearly k-linear. Thus $w \mapsto \delta_w \circ \frac{d^k f}{k!}$ is in $\mathcal{P}_a^k(E'', H_{b\mathfrak{A}}(X))$ (the space of algebraic k-homogeneous polynomials). It is also continuous since, by Corollary 4.6, for each open X-bounded set D, and $\beta < d_X(D)$, there exists $C > 0$ and an open X-bounded set $\tilde{D}$ (which do not depended on k) such that $\beta^k p_D(\delta_w \circ \frac{d^k f}{k!}) \leqslant C\|w\|^k p_{\tilde{D}}(f) < \infty$.

Therefore, if $Q_k(w) = \varphi(\delta_w \circ \frac{d^k f}{k!})$ then Q_k is in $\mathcal{P}^k(E'')$. Now, for $\|w\| < r < \alpha < d_X(A)$ and using again Corollary 4.6,

$$\begin{aligned}
\sup_{w \in rB_{E''}} \left| \varphi^w(f) - \sum_{n=0}^{m} Q_n(w) \right| &= \sup_{w \in rB_{E''}} \left| \sum_{k=m+1}^{\infty} \varphi\left(\delta_w \circ \frac{d^k f}{k!}\right) \right| \\
&\leqslant c \sup_{w \in rB_{E''}} \left| \sum_{k=m+1}^{\infty} p_A\left(\delta_w \circ \frac{d^k f}{k!}\right) \right| \\
&\leqslant cC \sum_{k=m+1}^{\infty} \left(\frac{r}{\alpha}\right)^k p_{\tilde{A}}(f) \longrightarrow 0 \quad \text{as } m \to \infty. \qquad \square
\end{aligned}$$

Remark 5.2. We know that the canonical extensions to $M_{b\mathfrak{A}}(X)$ need not be in $H_{b\mathfrak{A}}(M_{b\mathfrak{A}}(X))$ (see [16, Example 2.8] and [44, Proposition 4.3.22]). We do not know whether these extensions belong to $H_{\mathfrak{A}}(M_{b\mathfrak{A}}(X))$. When $\mathfrak{A}$ is weakly differentiable (see Section 6) and f is a polynomial, then it is possible to show that its extension to $M_{b\mathfrak{A}}(X)$ is of type $\mathfrak{A}$.

Definition 5.3. Let $\mathcal{F} \subset H(X)$ and let (Z, p) be another Riemann domain over E. An $\mathcal{F}$**-extension** is a morphism $\tau : X \to Z$ such that for each $f \in \mathcal{F}$ there exists a unique function $\tilde{f}$ holomorphic on Z, such that $\tilde{f} \circ \tau = f$. If $\mathcal{F} = H(X)$, we call it a holomorphic extension of X.

Corollary 5.4. *Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2) *which is regular at E and AB-closed. Then $M_{b\mathfrak{A}}(X)$ is a domain of holomorphy, that is, any holomorphic extension of $M_{b\mathfrak{A}}(X)$ is an isomorphism.*

Proof. We may follow the steps of [26, Proposition 2.4]. By [42, Theorem 52.6] it suffices to prove that $M_{b\mathfrak{A}}(X)$ is holomorphically separated and that for each sequence $\{\varphi_j\}$ in $M_{b\mathfrak{A}}(X)$ such that $d_{M_{b\mathfrak{A}}(X)}(\varphi_j) \to 0$, there exists a function F in $H(M_{b\mathfrak{A}}(X))$ such that $\sup_j |F(\varphi_j)| = \infty$. Since $M_{b\mathfrak{A}}(X)$ is separated by $H_{b\mathfrak{A}}(X)$, it is holomorphically separated by the above proposition. If $\{\varphi_j\} \subset M_{b\mathfrak{A}}(X)$ is such that $\sup_j |F(\varphi_j)| < \infty$ for all $F \in H(M_{b\mathfrak{A}}(X))$, then if $\tau(f) := \sup_j |\varphi_j(f)|$, τ defines a seminorm on $H_{b\mathfrak{A}}(X)$. Thus the set $V = \{f \in H_{b\mathfrak{A}}(X)\colon\ \tau(f) \leqslant 1\}$ is absolutely convex and absorbent. It is also closed because V is the intersection of the closed sets $\{f \in H_{b\mathfrak{A}}(X)\colon\ |\varphi_j(f)| \leqslant 1\}$. Since $H_{b\mathfrak{A}}(X)$ is a barreled space, V is a neighborhood of 0 and thus τ is continuous. Therefore, there are an X-bounded set D and a constant $c > 0$ such that $\tau(f) \leqslant cp_D(f)$ for every $f \in H_{b\mathfrak{A}}(X)$, which implies that $\varphi_j \prec D$ for every $j \in \mathbb{N}$. By Lemma 4.7, $d_{M_{b\mathfrak{A}}(X)}(\varphi_j) \geqslant d_X(D)$. $\square$

Let (X, q) be a connected Riemann domain over E. The envelope of holomorphy of X is an extension which is maximal in the sense that it factorizes through any other extension.

Definition 5.5. The $H_{b\mathfrak{A}}$**-envelope** of X is a Riemann domain $\mathcal{E}_{b\mathfrak{A}}(X)$ and an $H_{b\mathfrak{A}}$-extension morphism $\tau : X \to \mathcal{E}_{b\mathfrak{A}}(X)$ such that if $\nu : X \to Z$ is another $H_{b\mathfrak{A}}$-extension, then there exists a morphism $\mu : Z \to \mathcal{E}_{b\mathfrak{A}}(X)$ such that $\nu \circ \mu = \tau$.

In [33], Hirschowitz proved, using germs of analytic functions, the existence of $\mathcal{E}_{b\mathfrak{A}}(X)$ (in a more general framework) and asked whether the extended functions $\tilde{f}$ are also of type $\mathfrak{A}$ on $\mathcal{E}_{b\mathfrak{A}}(X)$, [33, p. 290]. We will give a partial positive answer to this question in the next section (Corollary 6.9).

We now characterize the $H_{b\mathfrak{A}}$-envelope of holomorphy of X in terms of the spectrum of $H_{b\mathfrak{A}}(X)$. We sketch the proof which is an adaptation of [16, Theorem 1.2]. First note that the conditions that $\mathfrak{A}$ be AB-closed and regular at E were used in Theorem 4.3 only to deal with Aron–Berner extensions. Thus, it is not difficult to show the following.

Lemma 5.6. *Let (X, p) be a Riemann domain spread over a Banach space E and let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in* (2)*. Then $(\pi^{-1}(E), \pi) \subset (M_{b\mathfrak{A}}(X), \pi)$ is a Riemann domain spread over E.*

Proposition 5.7. *Let (X, p) be a connected Riemann domain spread over a Banach space E, let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in (2) and let Y be the connected component of $\pi^{-1}(E) \subset M_{b\mathfrak{A}}(X)$ which intersects $\delta(X)$. Then $\delta : (X, p) \to (Y, \pi)$, $\delta(x) = \delta_x$ is the $H_{b\mathfrak{A}}$-envelope of X.*

Proof. Let $\sigma : X \to \mathcal{E}_{b\mathfrak{A}}(X)$ be the $H_{b\mathfrak{A}}$-extension from X to the $H_{b\mathfrak{A}}$-envelope of X. By Proposition 5.1, $\delta : X \to Y$ is an $H_{b\mathfrak{A}}$-extension. Moreover, for each point $y \in Y$, the evaluation $\delta_y : H_{b\mathfrak{A}}(X) \to \mathbb{C}$, $\delta_y(f) = \tilde{f}(y)$ is continuous. Then there is a morphism $\nu : Y \to \mathcal{E}_{b\mathfrak{A}}(X)$ such that $\sigma = \nu \circ \delta$.

We show that ν is an isomorphism. $\nu(Y)$ is open in $\mathcal{E}_{b\mathfrak{A}}(X)$ because ν is a morphism.

Let us see that $\nu(Y)$ is closed in $\mathcal{E}_{b\mathfrak{A}}(X)$. Suppose that $y \in \overline{\nu(Y)} \setminus \nu(Y)$. Let $W_n = \{\varphi \in Y\colon\ \varphi \prec X_n\}$, where $X_n = \{x \in X\colon \|p(x)\| \leqslant n,\ d_X(x) \geqslant \frac{1}{n}\}$. Then by Lemma 4.7, $d_Y(W_n) \geqslant \frac{1}{n}$. Therefore we can get a subsequence of integers $(n_k)_k$ and a sequence $(y_k)_k \subset Y$ such that $y_k \in W_{n_{k+1}} \setminus W_{n_k}$ and $y_k \to y$. Thus there are functions $f_k \in H_{b\mathfrak{A}}(X)$ such that $p_{X_{n_k}}(f_k) < \frac{1}{2^k}$ and $|\tilde{f}_k(y_k)| > k + \sum_{j=1}^{k-1} |\tilde{f}_j(y_k)|$. Then the series $\sum_{j=1}^{\infty} f_j$ converges to a function $f \in H_{b\mathfrak{A}}(X)$ and moreover $|(\sum_{j=1}^{\infty} f_j)^{\sim}(y_k)| = |\sum_{j=1}^{\infty} \tilde{f}_j(y_k)|$ because δ_{y_k} is $H_{b\mathfrak{A}}(X)$-continuous. Therefore

$$\left|\tilde{f}(y_k)\right| = \left|\sum_{j=1}^{\infty} \tilde{f}_j(y_k)\right| \geqslant \left|\tilde{f}_k(y_k)\right| - \left|\sum_{j=1}^{k-1} \tilde{f}_j(y_k)\right| - \left|\sum_{j=k+1}^{\infty} \tilde{f}_j(y_k)\right| > k - 1,$$

so we have that $|\tilde{f}(y_k)| \to \infty$ and then f cannot be extended to y. This is a contradiction since y belongs to the $H_{b\mathfrak{A}}$-envelope of X, $\mathcal{E}_{b\mathfrak{A}}(X)$. Thus $\nu(Y)$ is closed in $\mathcal{E}_{b\mathfrak{A}}(X)$. □

6. Type $\mathfrak{A}$ extensions

It is also natural to consider extensions where the extended functions are not only holomorphic but also of type $\mathfrak{A}$. As mentioned in Remark 5.2, we cannot expect in general that the extensions be of $\mathfrak{A}$-bounded type, since even for the current type, the extension of a bounded type function to the H_b-envelope of holomorphy may fail to be of bounded type. We may ask instead if, at least, they are in $H_{\mathfrak{A}}$.

Definition 6.1. A Riemann domain morphism $\tau : (X, q) \to (Y, \tilde{q})$ is an $H_{b\mathfrak{A}}$**–**$H_{\mathfrak{A}}$**-extension** if for each $f \in H_{b\mathfrak{A}}(X)$ there exists a unique $\tilde{f} \in H_{\mathfrak{A}}(Y)$ such that $\tilde{f} \circ \tau = f$.

For the current type, $H_{\mathfrak{A}}$ is the space of all holomorphic functions, thus in this case, every extension of a function in $H_{b\mathfrak{A}} = H_b$ belongs to $H_{\mathfrak{A}} = H$. On the other hand, Dineen found (see [23, Example 11]) an entire function f of bounded type that has nuclear radius of convergence $r > 0$ and such that there exists $x \in E$ for which $d^2 f(x) \notin \mathcal{P}_N^2(E)$. This means that f belongs to $H_{bN}(rB_E)$ and it extends to an entire function in $H_b(E)$, but this extension is not in $H_N(E)$. Thus, the extension of a single function in $H_{b\mathfrak{A}}$ need not be of type $\mathfrak{A}$. In this section we show, under the additional hypothesis of weak differentiability, that when all functions in $H_{b\mathfrak{A}}$ are extended simultaneously (that is, when one deals with $H_{b\mathfrak{A}}$-extensions), the extended functions are of type $\mathfrak{A}$.

Remark 6.2. There is also a corresponding notion of $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-envelope of holomorphy. This was considered by Moraes in [41], where she proved, using germs of analytic functions, that the $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-envelope of holomorphy always exists. Thus, a positive answer to the question of Hirschowitz whether the extensions to the $H_{b\mathfrak{A}}$-envelope are of type $\mathfrak{A}$ is equivalent to the coincidence of the $H_{b\mathfrak{A}}$-envelope with the $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-envelope. This will be proved in Corollary 6.9 for weakly differentiable sequences.

Definition 6.3. Let $\mathfrak{A}$ be a sequence of polynomial ideals and let E be a Banach space. We say that $\mathfrak{A}$ is **weakly differentiable** at E if there exist constants $c_{k,l} > 0$ such that, for $l < k$, $P \in \mathfrak{A}_k(E)$ and $\varphi \in \mathfrak{A}_{k-l}(E)'$, the mapping $x \mapsto \varphi(P_{x^l})$ belongs to $\mathfrak{A}_l(E)$ and

$$\left\|x \mapsto \varphi(P_{x^l})\right\|_{\mathfrak{A}_l(E)} \leqslant c_{k,l} \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Remark 6.4. Weak differentiability, a condition which is stronger than being a holomorphy type, was defined in [13] (see also [28,6], where a holomorphy type satisfying a similar condition is called a π_2-holomorphy type) and is dual to multiplicativity in the following sense: if $\{\mathfrak{A}_k\}_k$ is a weakly differentiable sequence then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}_k$ is multiplicative (with the same constants); and if $\{\mathfrak{A}_k\}_k$ is multiplicative then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}_k$ is weakly differentiable (with the same constants), see [13, Proposition 3.16].

All examples appearing in Section 1 but $\mathcal{M}_r$ and $\mathcal{S}_p$ were shown in [13] to be weakly differentiable sequences. It is not difficult to see that the constants satisfy (2) in all those cases. We don't know if the sequence of multiple r-summing polynomials is weakly differentiable. We prove now that the sequences of Hilbert–Schmidt and Schatten–von Neumann ideals of polynomials are weakly differentiable. Moreover, the duality between multiplicativity and weak differentiability allows us to show that they are also multiplicative.

Example 6.5. The sequence $\mathcal{S}_2$ of ideals of Hilbert–Schmidt polynomials is weakly differentiable and multiplicative with constants $c_{k,l}=1$.

Proof. Let H be a Hilbert space with orthonormal basis $(e_i)_i$. Recall that $\mathcal{S}_2^k(H)$ is the completion of finite type k-homogeneous polynomials on H with respect to the norm associated to the inner product

$$\langle P,Q\rangle_{\mathcal{S}_2^k(H)}=\sum_{i_1,\dots,i_k}\overset{\vee}{P}(e_{i_1},\dots,e_{i_k})\overline{\overset{\vee}{Q}(e_{i_1},\dots,e_{i_k})}.$$

Let $P\in\mathcal{P}^k(H)$. It is not difficult to deduce (see [27, Lemma 1]) that P belongs to $\mathcal{S}_2^k(H)$ if and only if it is (uniquely) expressed as a limit in the $\mathcal{S}_2^k(H)$-norm by

$$P=\sum_{i_1,\dots,i_k}a_{i_1\dots i_k}e'_{i_1}\dots e'_{i_k}, \tag{12}$$

with symmetric coefficients $a_{i_1\dots i_k}\in\mathbb{C}$ and

$$\sum_{i_1,\dots,i_k}|a_{i_1\dots i_k}|^2=\|P\|^2_{\mathcal{S}_2^k(H)}<\infty.$$

Let $\varphi\in\mathcal{S}_2^l(H)'$ and let $Q=\sum_{i_1,\dots,i_l}b_{i_1\dots i_l}e'_{i_1}\dots e'_{i_l}\in\mathcal{S}_2^l(H)$ be such that $\varphi=\langle\cdot,Q\rangle_{\mathcal{S}_2^l(H)}$. Then

$$\langle P_{x^{k-l}},Q\rangle_{\mathcal{S}_2^l(H)}=\sum_{i_1,\dots,i_l}\Bigl(\sum_{i_{l+1},\dots,i_k}a_{i_1\dots i_k}x_{i_{l+1}}\dots x_{i_k}\Bigr)\overline{b_{i_1\dots i_l}}.$$

This series is absolutely convergent, indeed

$$\begin{aligned}\Bigl(\sum_{i_1,\dots,i_l}\Bigl(\sum_{i_{l+1},\dots,i_k}|a_{i_1\dots i_k}x_{i_{l+1}}\dots x_{i_k}|\Bigr)|b_{i_1\dots i_l}|\Bigr)^2&\leqslant\|Q\|^2_{\mathcal{S}_2^l(H)}\sum_{i_1,\dots,i_l}\Bigl(\sum_{i_{l+1},\dots,i_k}|a_{i_1\dots i_k}x_{i_{l+1}}\dots x_{i_k}|\Bigr)^2\\&\leqslant\|Q\|^2_{\mathcal{S}_2^l(H)}\sum_{i_1,\dots,i_l}\Bigl(\sum_{i_{l+1},\dots,i_k}|a_{i_1\dots i_k}|^2\Bigr)\Bigl(\sum_{i_{l+1},\dots,i_k}|x_{i_{l+1}}\dots x_{i_k}|^2\Bigr)\\&\leqslant\|Q\|^2_{\mathcal{S}_2^l(H)}\|P\|^2_{\mathcal{S}_2^k(H)}\|x\|^{2(k-l)}.\end{aligned}$$

Thus reversing the order of summation we obtain,

$$x\mapsto\langle P_{x^{k-l}},Q\rangle_{\mathcal{S}_2^l(H)}=\sum_{i_{l+1},\dots,i_k}\Bigl(\sum_{i_1,\dots,i_l}a_{i_1\dots i_k}\overline{b_{i_1\dots i_l}}\Bigr)e'_{i_{l+1}}\dots e'_{i_k}.$$

Note that this representation is as in (12) and since

$$\sum_{i_{l+1},\dots,i_k}\Bigl|\sum_{i_1,\dots,i_l}a_{i_1\dots i_k}\overline{b_{i_1\dots i_l}}\Bigr|^2\leqslant\sum_{i_{l+1},\dots,i_k}\Bigl(\sum_{i_1,\dots,i_l}|a_{i_1\dots i_k}|^2\Bigr)\Bigl(\sum_{i_1,\dots,i_l}|b_{i_1\dots i_l}|^2\Bigr)=\|P\|^2_{\mathcal{S}_2^k(H)}\|Q\|^2_{\mathcal{S}_2^l(H)},$$

we conclude that $x\mapsto\langle P_{x^{k-l}},Q\rangle_{\mathcal{S}_2^l(H)}$ is in $\mathcal{S}_2^l(H)$ and has $\mathcal{S}_2^l(H)$-norm $\leqslant\|P\|_{\mathcal{S}_2^k(H)}\|Q\|_{\mathcal{S}_2^l(H)}$, that is, $\mathcal{S}_2$ is weakly differentiable with $c_{k,l}=1$.

Moreover, since the adjoint ideal of $\mathcal{S}_2^k$ (as a normed ideal of polynomials on Hilbert spaces) is again $\mathcal{S}_2^k$ and since the sequence of adjoint ideals of a weakly differentiable sequence is multiplicative with the same constants by [13, Proposition 3.16], we conclude that $\mathcal{S}_2$ is multiplicative with $c_{k,l}=1$. $\square$

Example 6.6. *The sequence $\mathcal{S}_p$ of Schatten–von Neumann polynomials.*

Using the Reiteration theorem for the complex interpolation method, or by duality with Example 3.13, we deduce that $\mathcal{S}_p$ is weakly differentiable with constants $c_{k,l}=1$ if $1<p<2$ and

$$c_{k,l}\leqslant\Bigl(\frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\Bigr)^{1-\frac{2}{p}}$$

for $2<p<\infty$.

We prove now that for weakly differentiable holomorphy types every $H_{b\mathfrak{A}}$-extension is an $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-extension.

Lemma 6.7. *Let $\mathfrak{A}$ be a weakly differentiable holomorphy type with constants as in* (2) *and $f \in H_{b\mathfrak{A}}(X)$. Then for each open X-bounded set A and $\alpha < d_X(A)$, there exist a positive constant C and an open X-bounded set $\tilde{A}$ such that for every $k \in \mathbb{N}$ and $\varphi \in \mathfrak{A}_k(E)'$, $\varphi \circ \frac{d^k f}{k!}$ belongs to $H_{b\mathfrak{A}}(X)$ and*

$$\alpha^k p_A\left(\varphi \circ \frac{d^k f}{k!}\right) \leqslant C\|\varphi\|_{\mathfrak{A}_k(E)'} p_{\tilde{A}}(f).$$

Proof. Since $\mathfrak{A}$ is weakly differentiable, the proof of Lemma 4.5 with the bound obtained in Corollary 4.6 works here for any $\varphi \in \mathfrak{A}_k(E)'$. $\square$

Proposition 6.8. *Let (X, q) be a connected Riemann domain spread over a Banach space E, let $\mathfrak{A}$ be a weakly differentiable holomorphy type with constants as in* (2). *Let $e : (X, q) \to (Y, p)$ be an $H_{b\mathfrak{A}}$-extension. Then for each $y \in Y$ there exists a connected open subset Z, $e(X) \cup \{y\} \subset Z \subset Y$ such that for every $f \in H_{b\mathfrak{A}}(X)$, the extension $\tilde{f}$ to Y is in $H_{b\mathfrak{A}}(Z)$. In particular, $\tilde{f}$ is of type $\mathfrak{A}$ on Y.*

Proof. Note first that the evaluation at each point y in Y defines a continuous character on $H_{b\mathfrak{A}}(X)$, $\delta_y(f) := \tilde{f}(y)$. Indeed, define the set $V \subset Y$ consisting in all points y_0 for which there is an open connected subset Z such that y_0 belongs to Z and every point in Z induce a continuous evaluation. Clearly V is an open nonempty set ($e(X)$ is contained in V). Moreover, if $(y_n) \subset V$ and $y_n \to y_0$, then since δ_{y_n} is continuous, the seminorm defined by $f \mapsto \sup_{n\in\mathbb{N}} |\delta_{y_n}(f)|$ is continuous (the sets $\{f \in H_{b\mathfrak{A}}(X)\colon \sup_{n\in\mathbb{N}} |\delta_{y_n}(f)| \leqslant r\}$ are closed, absolutely convex and absorbent and $H_{b\mathfrak{A}}(X)$ is a Fréchet space, thus they have nonempty interior). Therefore there are an open X-bounded set A and a constant $c > 0$ such that $|\tilde{f}(y_0)| \leqslant \sup_{n\in\mathbb{N}} |\delta_{y_n}(f)| \leqslant cp_A(f)$ for every $f \in H_{b\mathfrak{A}}(X)$. Thus V is closed and since Y is connected, we have $V = Y$.

Take a point $y \in Y$ and let $\gamma : [0, 1] \to Y$ be a curve such that $\gamma(0) \in e(X)$ and $\gamma(1) = y$. By compactness, it follows that there is some open X-bounded set A such that $\delta_{\gamma(t)} \prec A$ for every $t \in [0, 1]$, that is, for each t, there exists $c > 0$ such that $|\tilde{g}(\gamma(t))| \leqslant cp_A(g)$ for every $g \in H_{b\mathfrak{A}}(X)$.

Let I denote the set of all $t_0 \in [0, 1]$ such that there exists a connected open subset $Z \subset Y$ which contains $e(X)$ and satisfies that $\gamma(t) \in Z$ for every $t \leqslant t_0$ and that $\tilde{g}|_Z$ belongs to $H_{b\mathfrak{A}}(Z)$ for every $g \in H_{b\mathfrak{A}}(X)$. To prove the proposition it is enough to show that $I = [0, 1]$. Since I is clearly open, it suffices to prove that if $[0, t_0) \subset I$ then t_0 belongs to I. Take $t_1 < t_0$ such that $\gamma((t_1, t_0])$ is contained in some ball B of center $\gamma(t_1)$ and radius $r < d_X(A)$ in Y. Let Z be the subdomain which exists for t_1 in the definition of I. Note that $e : (X, q) \to (Z, p|_Z)$ is an $H_{b\mathfrak{A}}$-extension.

Let $\varphi_k \in \mathfrak{A}_k(E)'$ and $f \in H_{b\mathfrak{A}}(X)$. By Lemma 6.7, $\varphi_k \circ \frac{d^k f}{k!} \in H_{b\mathfrak{A}}(X)$, and since the extension of f to Z, $\tilde{f}|_Z$, belongs to $H_{b\mathfrak{A}}(Z)$, we also have that $\varphi_k \circ \frac{d^k \tilde{f}|_Z}{k!} \in H_{b\mathfrak{A}}(Z)$.

Moreover, $\varphi_k \circ \frac{d^k \tilde{f}|_Z}{k!}$ is an extension of $\varphi_k \circ \frac{d^k f}{k!}$ to Z. Indeed if $x \in X$ and (V_x, q) is a chart at x in X such that $(V_{e(x)}, p) = (e(V_x), p)$ is a chart at $e(x)$ in Z, then

$$\begin{aligned}\varphi_k \circ \frac{d^k \tilde{f}|_Z}{k!}\big(e(x)\big) &= \varphi_k \circ \frac{d^k[\tilde{f} \circ (p|_{V_{e(x)}})^{-1}]}{k!}\big(p\big(e(x)\big)\big) \overset{(*)}{=} \varphi_k \circ \frac{d^k[f \circ (q|_{V_x})^{-1}]}{k!}\big(q(x)\big)\\ &= \varphi_k \circ \frac{d^k f}{k!}(x),\end{aligned}$$

where $(*)$ is true because $\tilde{f} \circ (p|_{V_{e(x)}})^{-1} = f \circ (q|_{V_x})^{-1}$ since e is an $H_{b\mathfrak{A}}$-extension.

Since $(\varphi_k \circ \frac{d^k f}{k!})^{\sim}$ is also an extension of $\varphi_k \circ \frac{d^k f}{k!}$ to Z, we must have that

$$\left(\varphi_k \circ \frac{d^k f}{k!}\right)^{\sim} = \varphi_k \circ \frac{d^k \tilde{f}|_Z}{k!}. \tag{13}$$

Therefore for $r < \alpha < d_X(A)$,

$$\begin{aligned}\sum_k r^k \left\| \frac{d^k \tilde{f}|_Z}{k!}\big(\gamma(t_1)\big)\right\|_{\mathfrak{A}_k(E)} &= \sum_k r^k \sup_{\varphi_k \in B_{\mathfrak{A}_k(E)'}} \left|\varphi_k\left(\frac{d^k \tilde{f}|_Z}{k!}\big(\gamma(t_1)\big)\right)\right|\\ &= \sum_k r^k \sup_{\varphi_k \in B_{\mathfrak{A}_k(E)'}} \left|\left(\varphi_k \circ \frac{d^k f}{k!}\right)^{\sim}\big(\gamma(t_1)\big)\right|\\ &\leqslant \sum_k r^k \sup_{\varphi_k \in B_{\mathfrak{A}_k(E)'}} cp_A\left(\varphi_k \circ \frac{d^k f}{k!}\right)\\ &\leqslant cC\frac{\alpha}{\alpha - r} p_{\tilde{A}}(f) < \infty,\end{aligned}$$

where $\tilde{A}$ and C are, respectively, the X-bounded set and the constant given by Lemma 6.7. Thus $\tilde{f}$ belongs to $H_{b\mathfrak{A}}(Z \cup B)$. Since this holds for every $f \in H_{b\mathfrak{A}}(X)$ and $[0, t_0]$ is contained in $Z \cup B$, we conclude that t_0 is in I. □

As a corollary we have the following partial answer to a question of Hirschowitz (see comments after Definition 5.5).

Corollary 6.9. *If $\mathfrak{A}$ be a weakly differentiable holomorphy type with constants as in* (2)*, then the extensions to the $H_{b\mathfrak{A}}$-envelope of holomorphy are of type $\mathfrak{A}$.*

The following result can be proved as Corollary 5.4, but using the above corollary instead of Proposition 5.1.

Corollary 6.10. *Let $\mathfrak{A}$ be a multiplicative and weakly differentiable holomorphy type with constants as in* (2)*. Then the $H_{b\mathfrak{A}}$-envelope of holomorphy is an $H_{\mathfrak{A}}$-domain of holomorphy, that is, any $H_{\mathfrak{A}}$-extension is an isomorphism.*

The Cartan–Thullen theorem characterizes domains of holomorphy in $\mathbb{C}^n$ in terms of holomorphic convexity. It was extended for bounded type holomorphic functions on Banach spaces by Dineen [24]. Shortly after, Cartan–Thullen type theorems were proved for very general classes of spaces of holomorphic functions on infinite dimensional spaces by Coeuré [19], Schottenloher [49,50] and Matos [36,37]. Despite the generality of this theorems,[2] the fact that any holomorphically convex domain is a domain of holomorphy was only proved for spaces of analytic functions which one may associate to the current holomorphy type (spaces of analytic functions which are bounded on certain families of subsets, with the topology of uniform convergence on these subsets). We guess that this may be due to the fact that the concept of holomorphic convexity considered there make use of the seminorms associated to the current type. We propose instead a concept of $H_{b\mathfrak{A}}$-convexity which uses the seminorms associated to the corresponding holomorphy type, and show then that a Riemann domain is an $H_{b\mathfrak{A}}$-domain of holomorphy if and only if it is $H_{b\mathfrak{A}}$-convex.

Definition 6.11. For each open X-bounded set A, we define its $H_{b\mathfrak{A}}(X)$**-convex hull** as

$$\hat{A}_{H_{b\mathfrak{A}}(X)} := \{x \in X\colon \text{ there exists } c > 0 \text{ such that } |f(x)| \leqslant c p_A(f) \text{ for every } f \in H_{b\mathfrak{A}}(X)\}.$$

Remark 6.12. If the seminorms p_A are submultiplicative (as in the case of H_b) then the constant in above definition may be taken $c = 1$.

Definition 6.13. We say that a Riemann domain (X, q) is $H_{b\mathfrak{A}}$**-convex** if for each open X-bounded set A, its $H_{b\mathfrak{A}}(X)$-convex hull $\hat{A}_{H_{b\mathfrak{A}}(X)}$ is X-bounded.

Definition 6.14. We say that a Riemann domain (X, q) is an $H_{b\mathfrak{A}}$**-domain of holomorphy** ($H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$**-domain of holomorphy**) if each $H_{b\mathfrak{A}}$-extension ($H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-extension) morphism is an isomorphism.

We are now ready to prove the Cartan–Thullen theorem for $H_{b\mathfrak{A}}$.

Theorem 6.15. *Let $\mathfrak{A}$ be a holomorphy type with constants as in* (2)*. Let (X, q) be a Riemann domain spread over a Banach space E. Consider the following conditions.*

(i) *X is $H_{b\mathfrak{A}}$-convex and $d_X(\hat{A}_{H_{b\mathfrak{A}}(X)}) = d_X(A)$ for each open X-bounded set A.*
(ii) *X is $H_{b\mathfrak{A}}$-convex.*
(iii) *For each sequence $(x_n) \subset X$ such that $d_X(x_n) \to 0$, there exists a function $f \in H_{b\mathfrak{A}}(X)$ such that $\sup_n |f(x_n)| = \infty$.*
(iv) *X is an $H_{b\mathfrak{A}}$-domain of holomorphy.*
(v) *For each open subset A of X which is not X-bounded there exists a function $f \in H_{b\mathfrak{A}}(X)$ such that $p_A(f) := \sup\{p_s^x(f)\colon B_s(x)$ contained in A and X-bounded$\} = \infty$.*
(vi) *X is an $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-domain of holomorphy.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi).

Moreover, if $\mathfrak{A}$ is also weakly differentiable with constants as in (2)*, then all the above conditions are equivalent.*

Proof. The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (vi) are clear. The equivalence (iii) and (iv) is contained in [19, Theorem 4.9].

[2] The natural Fréchet spaces considered by Coeuré include, by Corollary 4.6, the spaces $H_{b\mathfrak{A}}$ and so do the regular classes studied by Schottenloher when the holomorphy type is multiplicative.

Let us prove equivalence between (ii) and (iii). Suppose that $d_X(x_n) \to 0$ and that $\tau(f) = \sup_n |f(x_n)| < \infty$ for every $f \in H_{b\mathfrak{A}}(X)$. Then the set $V = \{f \in H_{b\mathfrak{A}}(X)\colon\ \tau(f) \leqslant 1\}$ is absolutely convex and absorbent. Moreover V is closed since it is the intersection of the sets $\{f \in H_{b\mathfrak{A}}(X)\colon\ |f(x_n)| \leqslant 1\}$ which are closed because evaluations at x_n are continuous in $H_{b\mathfrak{A}}(X)$. Since $H_{b\mathfrak{A}}(X)$ is a barreled space, V is a neighborhood of 0 and thus τ is a continuous seminorm. Therefore, there are an X-bounded set C and a constant $c > 0$ such that $\tau(f) \leqslant cp_C(f)$ for every $f \in H_{b\mathfrak{A}}(X)$. That is, $(x_n) \subset \hat{C}_{H_{b\mathfrak{A}}(X)}$, and thus X is not $H_{b\mathfrak{A}}$-convex. Conversely, if A is an X-bounded set such that $\hat{A}_{H_{b\mathfrak{A}}(X)}$ is not X-bounded, then there is a sequence (x_n) in $\hat{A}_{H_{b\mathfrak{A}}(X)}$ such that $d_X(x_n) \to 0$. This sequence satisfies that $\sup_n |f(x_n)| < \infty$.

We prove now that (v) implies (vi). Let $\tau : X \to Y$ be a morphism which is an $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-extension but is not surjective and take y in the border of X. Let (x_n) be a sequence contained in X converging to y and let B_n be the ball of center x_n and radius $\frac{d_X(x_n)}{2}$. Since $A = \bigcup_n B_n$ is not X-bounded, there is some $f \in H_{b\mathfrak{A}}(X)$ such that $p_A(f) = \infty$. Let $A_k = \bigcup_{n\geqslant k} B_n$, then clearly $p_{A_k}(f) = \infty$ for every $k \geqslant 1$. Since f extends to $\tilde{f} \in H_{\mathfrak{A}}(Y)$, there is some $r > 0$ such that $p_{B_r(y)}(\tilde{f}) < \infty$. Moreover, if k is large enough then $A_k \subset B_r(y)$. Thus, for k large enough, we have that $p_{A_k}(f) = p_{A_k}(\tilde{f}) \leqslant p_{B_r(y)}(\tilde{f}) < \infty$, which is a contradiction.

It remains to prove that (vi) implies (i) when $\mathfrak{A}$ is weakly differentiable.

Claim. *If $\mathfrak{A}$ be a weakly differentiable holomorphy type with constants as in* (2)*, A an open X-bounded set and $y \in \hat{A}_{H_{b\mathfrak{A}}(X)}$ and $f \in H_{b\mathfrak{A}}(X)$, then $f \circ (q|_{B_y})^{-1}$ extends to a function $\tilde{f} \in H_{b\mathfrak{A}}(B_{d_X(A)}(q(y)))$.*

Proof of the claim. Let $\alpha < \alpha_0 < d_X(A)$. Then by Lemma 6.7, there exists a constant C (independent of k) such that $\alpha_0^k \sup_{\varphi_k \in B_{\mathfrak{A}_k(E)'}} p_A(\varphi_k \circ \frac{d^k f}{k!}) \leqslant Cp_{\tilde{A}}(f)$. Thus,

$$\sum_k \alpha^k \left\| \frac{d^k f}{k!}(y) \right\|_{\mathfrak{A}_k(E)} = \sum_k \alpha^k \sup_{\varphi_k \in B_{\mathfrak{A}_k(E)'}} \left| \varphi_k \left(\frac{d^k f}{k!}(y) \right) \right| \leqslant \sum_k \alpha^k \sup_{\varphi_k \in B_{\mathfrak{A}_k(E)'}} cp_A \left(\varphi_k \circ \frac{d^k f}{k!} \right)$$
$$\leqslant c \sum_k \left(\frac{\alpha}{\alpha_0} \right)^k Cp_{\tilde{A}}(f) = cC \frac{\alpha_0}{\alpha_0 - \alpha} p_{\tilde{A}}(f) < \infty.$$

Since this is true for every $\alpha < d_X(A)$, we have that the Taylor series of $f \circ (q|_{B_y})^{-1}$ at $q(y)$ converges on $B_{d_X(A)}(q(y))$ and that $f \circ (q|_{B_y})^{-1}$ belongs to $H_{b\mathfrak{A}}(B_{d_X(A)}(q(y)))$, and the claim is proved. □

Suppose now that $d_X(\hat{A}_{H_{b\mathfrak{A}}(X)}) < d_X(A)$. Take $y \in \hat{A}_{H_{b\mathfrak{A}}(X)}$ such that $d_X(y) < d_X(A)$. We define a Riemann domain $\tilde{X}$ adjoining to X the ball $B_{d_X(A)}(q(y))$ as follows. First define a Riemann domain (X_0, q_0) as the disjoint union $X \cup B_{d_X(A)}(q(y))$, and $q_0(x_0) = q(x_0)$ if $x_0 \in X$; $q_0(x_0) = x_0$ if $x_0 \in B_{d_X(A)}(q(y))$. Then define the following equivalence relation $\sim$ on X_0: each point is related with itself and two points $x_0 \in B_{d_X(A)}(q(y))$, $x_1 \in X$ are related if and only if $q(x_1) = x_0$ and x_1 may be joined to y by a curve contained in $q^{-1}(B_{d_X(A)}(q(y)))$. Let $\tilde{X}$ be the Riemann domain $X_0/{\sim}$. By the claim the inclusion $X \hookrightarrow \tilde{X}$ is an $H_{b\mathfrak{A}}$–$H_{\mathfrak{A}}$-extension morphism, and it is not an isomorphism. □

Remark 6.16. By [19, Theorem 4.9] we also have that if $\mathfrak{A}$ is a holomorphy type with constants as in (2) and (X, q) is a domain over a separable Banach space E, then X is an $H_{b\mathfrak{A}}$-domain of holomorphy if and only if X is the domain of existence of a function $f \in H_{b\mathfrak{A}}(X)$.

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