The uniqueness of the energy momentum tensor in non-Abelian gauge field theories

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The uniqueness of the energy momentum tensor in non-Abelian gauge field theories is established under minimal hypothesis.

I. INTRODUCTION

In the general theory of relativity, the interaction of the gravitational field (characterized by a metric tensor g_{ij}) and a source-free gauge field (characterized by a curvature form F^{α}_{ij}; see Ref. 1 for definitions and notations) is assumed to be governed by the Einstein–Yang–Mills field equations

$$R^{ij} - \tfrac{1}{2}g^{ij}R = B_{\alpha\beta}(F^{\alpha i}{}_{k}F^{\beta jk} - \tfrac{1}{4}g^{ij}F^{\alpha hk}F^{\beta}_{hk}), \tag{1}$$

$$F^{\alpha ij}{}_{\|j} = 0, \tag{2}$$

where $B_{\alpha\beta}$ are the coefficients of a bilinear symmetric form in LG, the Lie algebra of the Lie group G, which are Ad G invariant, i.e., $B_{\alpha\beta} = \mathrm{Ad}^{\gamma}_{\alpha}(a)\mathrm{Ad}^{\eta}_{\beta}(a)B_{\gamma\eta}$ for all a in G. Besides, the covariant gauge derivative of F^{α}_{ij} is defined as

$$F^{\alpha}_{ij\|h} = F^{\alpha}_{ij,h} - \Gamma^{s}_{ih}F^{\alpha}_{sj} - \Gamma^{s}_{jh}F^{\alpha}_{is} + C^{\alpha}_{\beta\gamma}A^{\beta}_{h}F^{\gamma}_{ij}, \tag{3}$$

where $C^{\alpha}_{\beta\gamma}$ are the structure constants of the Lie group and A^{β}_{h} are the gauge potentials (see Ref. 1 or Ref. 2) related to the curvature form by

$$F^{\alpha}_{ij} = A^{\alpha}_{j,i} - A^{\alpha}_{i,j} + C^{\alpha}_{\beta\gamma}A^{\beta}_{i}A^{\gamma}_{j}. \tag{4}$$

It is easy to see that with these definitions, the following identity holds:

$$F^{\alpha}_{ij\|h} + F^{\alpha}_{jh\|i} + F^{\alpha}_{hi\|j} = 0. \tag{5}$$

Since the Einstein tensor given by the left-hand side of (1) is divergence-free, the same must be true for T^{ij}_{0}, the right-hand side of (1). This is the case because of the identity

$$T^{ij}_{0\|j} = \beta_{\alpha\beta}F^{\alpha ih}F^{\beta}{}_{h}{}^{j}{}_{\|j}, \tag{6}$$

and Eq. (2). For any T^{ij} in the right-hand side it must be true that $T^{ij}{}_{\|j} = 0$, at least when (2) holds. In other words, it must be true that

$$F^{\alpha ij}{}_{\|j} = 0 \Rightarrow T^{ij}{}_{\|j} = 0. \tag{7}$$

The uniqueness of the energy momentum tensor was established recently[1] under the restrictive hypothesis $T^{ij}{}_{\|j} = C_{\alpha\beta}H^{\beta jr}F^{\alpha i}{}_{r\|i}$. Clearly (7) is weaker and it is mandatory because of (1) and (2). In this paper we will prove that $T_0{}^{ij}$ is essentially the only solution to the following problem: to find all gauge invariant symmetric tensors $T^{ij} = T^{ij}(g_{hk};F^{\alpha}_{hk})$ such that (7) holds. Our result generalizes Ref. 3.

We want to point out that, due to the condition (7), one cannot generate energy momentum tensors by adding terms to the action.

II. CONSEQUENCES OF THE IMPLICATION (7)

We will work in a coordinate system for which $(g_{ij}) = \mathrm{diag}(-1,1,1,1)$ and $g_{ij,h} = 0$ (which implies $\Gamma^{i}_{jk} = 0$). Then (2) reads

$$F^{\beta}_{1k\|1} = F^{\beta}_{2k\|2} + F^{\beta}_{3k\|3} + F^{\beta}_{4k\|4}. \tag{8}$$

It is easy to see that

$$T^{ij}{}_{\|j} = \frac{\partial T^{ij}}{\partial F^{\beta}_{hk}} F^{\beta}_{hk\|j} = T^{ijhk}_{\beta}F^{\beta}_{hk\|j}, \tag{9}$$

because of the gauge invariance of T^{ij} and its tensorial character. Then $T^{ij}{}_{\|j} = 0$ written out in full in the above mentioned coordinate system is

$$\begin{aligned}(T^{i223}_{\beta} + T^{i113}_{\beta})F^{\beta}_{23\|2} + (T^{i224}_{\beta} + T^{i114}_{\beta})F^{\beta}_{24\|2} + (T^{i323}_{\beta} - T^{i112}_{\beta})F^{\beta}_{23\|3} + (T^{i334}_{\beta} + T^{i114}_{\beta})F^{\beta}_{34\|3} + (T^{i424}_{\beta} - T^{i112}_{\beta})F^{\beta}_{24\|4}\\ + (T^{i434}_{\beta} - T^{i113}_{\beta})F^{\beta}_{34\|4} + (T^{i123}_{\beta} + T^{i213}_{\beta})F^{\beta}_{23\|1} + (T^{i312}_{\beta} + T^{i213}_{\beta})F^{\beta}_{12\|3} + (T^{i313}_{\beta} - T^{i212}_{\beta})F^{\beta}_{13\|3}\\ + (T^{i124}_{\beta} + T^{i214}_{\beta})F^{\beta}_{24\|1} + (T^{i412}_{\beta} + T^{i214}_{\beta})F^{\beta}_{12\|4} + (T^{i414}_{\beta} - T^{i212}_{\beta})F^{\beta}_{14\|4} + (T^{i134}_{\beta} + T^{i314}_{\beta})F^{\beta}_{34\|1}\\ + (T^{i413}_{\beta} + T^{i314}_{\beta})F^{\beta}_{13\|4} + (T^{i234}_{\beta} + T^{i324}_{\beta})F^{\beta}_{34\|2} + (T^{i423}_{\beta} + T^{i324}_{\beta})F^{\beta}_{23\|4} = 0.\end{aligned} \tag{10}$$

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Let us choose, for arbitrary but fixed g_{hk}, F^{α}_{hk}, the derivatives $F^{\beta}_{ij,h}$ such that (8) holds.

Taking account of (8) and (5), it is clear that the $F^{\beta}_{ij\|h}$ appearing in (10) are arbitrary and independent. Then we deduce

$$\begin{aligned} T^{i313}_{\beta} - T^{i212}_{\beta} &= T^{i414}_{\beta} - T^{i212}_{\beta} = T^{i223}_{\beta} + T^{i113}_{\beta} = T^{i224}_{\beta} + T^{i114}_{\beta} = T^{i323}_{\beta} - T^{i112}_{\beta} = T^{i334}_{\beta} + T^{i114}_{\beta} \\ &= T^{i424}_{\beta} - T^{i112}_{\beta} = T^{i434}_{\beta} - T^{i113}_{\beta} = T^{i123}_{\beta} + T^{i213}_{\beta} = T^{i312}_{\beta} + T^{i213}_{\beta} = T^{i124}_{\beta} + T^{i214}_{\beta} = T^{i412}_{\beta} + T^{i214}_{\beta} \\ &= T^{i134}_{\beta} + T^{i314}_{\beta} = T^{i413}_{\beta} + T^{i314}_{\beta} = T^{i234}_{\beta} + T^{i324}_{\beta} = T^{i423}_{\beta} + T^{i324}_{\beta} = 0. \end{aligned} \tag{11}$$

Taking $i = 1,2,3,4$ in (11), a tedious but straightforward computation proves that

$$1212 = 1313 = 1414 = 2323 = 2424 = 3434 = 1234 = 1324 = 1423 = 2314 = 2413 = 3412 = 0, \tag{12}$$

$$1323 = 1424 = -2212 = -2313 = -2414 = 3312 = 4412 = 1112, \tag{13}$$

$$-1223 = 1434 = 2213 = -2312 = -3313 = -3414 = 4413 = 1113, \tag{14}$$

$$-1224 = -1334 = 2214 = -2412 = 3314 = -3413 = -4414 = 1114, \tag{15}$$

$$-1213 = 1312 = 2223 = -2434 = 3323 = 3424 = -4423 = 1123, \tag{16}$$

$$-1214 = 1412 = 2224 = 2334 = -3324 = 3423 = 4424 = 1124, \tag{17}$$

$$-1314 = 1413 = -2234 = 2324 = -2423 = 3334 = 4434 = 1134, \tag{18}$$

where, for the sake of simplicity, we have used the notation

$$T^{ijhk}_{\beta} = ijhk$$

for a fixed β.

III. THE UNIQUENESS OF THE ENERGY MOMENTUM TENSOR

Let us denote, for fixed α, β, and γ,

$$T^{ijhkrslm} = \frac{\partial^3 T^{ij}}{\partial F^{\alpha}_{hk}\,\partial F^{\beta}_{rs}\,\partial F^{\gamma}_{lm}}.$$

We will prove that all these derivatives are zero. From (12)–(18) it is clear that it is enough to consider the cases $ijhk = 1213, 1214, 1223, 1224, 1323, 1314$.

(i) The case ijhk=1213: It is clear that $T^{1213rslm} = 0$ except perhaps for $(r,s) \neq (1,2) \neq (l,m)$ and $(r,s) \neq (3,4) \neq (l,m)$. Since $|T^{ijhk}_{\beta}| = |T^{hkij}_{\beta}|$ for $i \neq j$ and $h \neq k$ as a consequence of (12)–(18), then in this case all the pairs commute, and so it is enough to consider $(r,s) \neq (1,3) \neq (l,m) \neq (r,s)$, which leave us with the following cases:

(a) $(r,s,l,m) = (1,4,2,3)$,

(b) $(r,s,l,m) = (1,4,2,4)$,

(c) $(r,s,l,m) = (2,3,2,4)$.

In case (a) using (16), (15), and $|T^{ijhk}_{\beta}|$ for $i \neq j$ and $h \neq k$, we have

$$|T^{12131423}| = |T^{33231423}| = |T^{33142323}| = |T^{24122323}| = |T^{23232412}| = 0. \tag{19}$$

In case (b) we have

$$|T^{12131424}| = |T^{13121424}| = |T^{13241214}| = 0. \tag{20}$$

Finally, in case (c) it is

$$|T^{12132324}| = |T^{13122324}| = |T^{13241223}| = 0, \tag{21}$$

where we have also used the equality of the cross derivatives. We conclude that

$$T^{1213rslm} = 0 \quad \text{for all } r,s,l,m. \tag{22}$$

(ii) The case ijhk=1214: It is easy to see that $T^{1214rslm} = 0$ except perhaps for $(r,s) \neq (1,2) \neq (l,m) \neq (r,s)$, $(r,s) \neq (1,4) \neq (l,m) \neq (r,s)$, $(r,s) \neq (3,4) \neq (l,m) \neq (r,s)$, and $(r,s) \neq (2,3) \neq (l,m) \neq (r,s)$, which leaves us only with the case $T^{12141324}$, and this is zero because of (20). Then

$$T^{1213rslm} = 0 \quad \text{for all } r,s,l,m. \tag{23}$$

(iii) The case ijhk=1223: As in case (ii) $T^{1223rslm} = 0$ except perhaps for $(r,s) \neq (1,2)$, $(2,3)$, $(3,4)$, $(1,4)$ $\neq (l,m)$, and $(r,s) \neq (l,m)$. This leaves us only with the case $T^{12231324}$ which is zero by (21). Then

$$T^{1223rslm} = 0 \quad \text{for all } r,s,l,m. \tag{24}$$

(iv) The case ijhk=1224: As before, it is enough to consider the cases $(r,s) \neq (1,2)$, $(2,4)$, $(3,4)$, $(1,3)$ $\neq (l,m)$, and $(r,s) \neq (l,m)$. Then there is only the case $|T^{12241423}| = |T^{14231224}| = 0$. Then

$$T^{1224rslm} = 0 \quad \text{for all } r,s,l,m. \tag{25}$$

(v) The case ijhk=1323: It is enough to consider $(r,s) \neq (1,3)$, $(2,3)$, $(2,4)$, $(1,4)$ $\neq (l,m)$, and $(r,s) \neq (l,m)$. Then $|T^{13231234}| = |T^{12341323}| = 0$. Thus

$$T^{1323rslm} = 0 \quad \text{for all } r,s,l,m. \tag{26}$$

(vi) The case ijhk=1314: It is enough to consider $(r,s) \neq (1,3)$, $(1,4)$, $(2,4)$, $(2,3)$, $\neq (l,m)$, and $(r,s) \neq (l,m)$. But then $|T^{13141234}| = |T^{12341314}| = 0$, and so

$$T^{1314rslm} = 0 \quad \text{for all } r,s,l,m. \tag{27}$$

From (22)–(27) we conclude that

$$T^{ijhkrslm} = 0 \quad \text{for all } i,j,h,k,r,s,l,m,$$

and so T^{ij} is a polynomial in F^{α}_{ij} of degree not greater than two.

Consequently

$$T^{ij} = A^{ijhkrs}_{\epsilon\gamma}(g_{lm}) F^{\epsilon}_{hk} F^{\gamma}_{rs} + B^{ijhk}_{\epsilon}(g_{lm}) F^{\epsilon}_{hk} + C^{ij}(g_{lm}). \tag{28}$$

The tensorial concomitants of g_{lm} were recently found[4,5] for any valence of the tensor. Taking account of the fact that we are dealing with all coodinate systems, and not merely with those belonging to an oriented atlas, then it follows that

$$T^{ij} = (d_{\alpha\beta}\, F^{\alpha hk}F^{\beta}_{hk} + \lambda)g^{ij} + \tfrac{1}{2}a_{\alpha\beta}(F^{\alpha i}{}_{t}F^{\beta jt} + F^{\alpha j}{}_{t}F^{\beta it}), \tag{29}$$

where $d_{\alpha\beta}$, λ, and $a_{\alpha\beta}$ are real numbers and $a_{\alpha\beta} = a_{\beta\alpha}$, $d_{\alpha\beta} = d_{\beta\alpha}$. Then

$$T^{ij} = (d_{\alpha\beta}\, F^{\alpha hk}F^{\beta}_{hk} + \lambda)g^{ij} + a_{\alpha\beta}F^{\alpha i}{}_{t}F^{\beta jt}. \tag{30}$$

Assuming $F^{\alpha ij}{}_{\|j} = 0$, it follows that $T^{ij}{}_{\|j} = 0$, and so, using the identity (5) to change indices, we have

$$[2d_{\alpha\beta}g^{ij}F^{\alpha hk} + \tfrac{1}{2}a_{\alpha\beta}g^{ij}F^{\alpha hk}]F^{\beta}_{hk\,\|j} = 0. \tag{31}$$

It is easy to see that if S^{ijhk}_{β} is the term within brackets in (31) then, because of (12)–(18), we have

$$S^{ijhk}_{\beta}g_{ij} = 0. \tag{32}$$

From (32) and the definition of S^{ijhk}_{β} it follows that

$$2\cdot(4\, d_{\alpha\beta} + a_{\alpha\beta})F^{\alpha hk} = 0. \tag{33}$$

Differentiating (33) with respect to F^{α}_{hk} we obtain

$$(4d_{\alpha\beta} + a_{\alpha\beta})(g^{hr}g^{ks} - g^{kr}g^{hs}) = 0. \tag{34}$$

Multiplying (34) by $g_{hr}\cdot g_{ks}$ we deduce

$$d_{\alpha\beta} = -\tfrac{1}{4}a_{\alpha\beta}, \tag{35}$$

and so, replacing (33) in (30), it follows that

$$T^{ij} = a_{\alpha\beta}(F^{\alpha i}{}_{t}F^{\beta jt} - \tfrac{1}{4}g^{ij}F^{\alpha hk}F^{\beta}_{hk}) + \lambda g^{ij}. \tag{36}$$

It follows easily from the gauge invariance of T^{ij} that the $a_{\alpha\beta}$ are Ad G invariant.

In summary, we have proved the following.

Theorem: If $T^{ij} = T^{ij}(g_{hk}\colon F^{\alpha}_{hk})$ is a gauge invariant tensor whose divergence vanishes when the divergence of $F^{\alpha ij}$ is zero, and if $T^{ij} = T^{ji}$, then

$$T^{ij} = T_0{}^{ij} + \lambda g^{ij},$$

where T^{ij}_{0} is the usual energy momentum tensor.

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