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Addendum to "The equivariant inverse problem in gauge field theories and the uniqueness of the Yang–Mills equations" [J. Math. Phys. 30, 2382 (1989)]

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The main theorem of the previous paper is extended to the case where $L = L(g_{ij};A_i^\alpha;A_{i,j}^\alpha)$. References and notations are the same.

Let $T = T(g_{ij};F_{ij}^\alpha) = L(g_{ij};0; -\frac{1}{2}F_{ij}^\alpha)$. Since L^{hk} is gauge invariant, by the replacement theorem[1] $L^{hk} = T^{hk}$. Then T^{hk} is gauge invariant and T has the form required in Theorem 1. Hence, $T = L_1 + L_2 + K$. Since $(L - T)^{hk} = 0$, it follows that $L = L_1 + L_2 + S$, where $S = S(A_i^\alpha;A_{i,j}^\alpha)$. Since $E_\alpha^i(L)$ is a gauge tensorial density, $L_\alpha^{i,j;}{}_\beta^{h,k} + L_\alpha^{i,k;}{}_\beta^{h,j}$ is a gauge tensorial density. Now, $\partial L_1/\partial A_{i,j}^\alpha = -2\partial L_2/\partial F_{ij}^\alpha$. Then

$$4L_{1\alpha\beta}^{ij;hk} + 4L_{1\alpha}{}^{ik;hj}{}_\beta + 4L_{2\alpha\beta}^{ij;hk} + 4L_{2\alpha}{}^{ik;hj}{}_\beta + S_\alpha^{i,j;}{}_\beta^{h,k} + S_\alpha^{i,k;}{}_\beta^{h,j}$$

is a gauge tensorial density. This is also true for the sum of the first two terms, and the sum of the following two terms is null. Thus $S_\alpha^{i,j;}{}_\beta^{h,k} + S_\alpha^{i,k;}{}_\beta^{h,j}$ is a tensorial density. As we proved in the original paper, it has the form $a_{\alpha\beta}\epsilon^{ijhk}$ and it is symmetric in k,j. Thus it is null and $S_\alpha^{i,j;}{}_\beta^{h,k} = -S_\alpha^{i,k;}{}_\beta^{h,j}$, from where it follows that

$$E_\alpha^i(S) = S_\alpha^i - S_\alpha^{i,j;h}{}_\beta A_{h,j}^\beta.$$

We deduce easily that

$$E_\alpha^i(S)(0;-\tfrac{1}{2}F) = -E_\alpha^i(S)(0;-\tfrac{1}{2}F)$$

(making $\bar{x}^i = -x^i$). Then, by the replacement theorem,[1]

$$E_\alpha^i(L) = E_\alpha^i(L_1)(g;0;-\tfrac{1}{2}F;-\tfrac{2}{3}F').$$

Since this equation is tensorial, it is valid for all coordinate systems. Then

$$E_\alpha^i(L_2+S)(g;0;-\tfrac{1}{2}F;-\tfrac{2}{3}F') = 0,$$

and so

$$5g^{lm}g_{lm,s}l_{\gamma\alpha}\epsilon^{sihk}F_{hk}^\gamma + E_\alpha^i(S)(0;-\tfrac{1}{2}F) = 0.$$

It follows easily that $l_{\alpha\beta} = 0$, i.e., $L_2 = 0$. Thus $E_\alpha^i(S)$ is a tensorial density, and so the same is true for $E_\alpha^i(S)_\beta^{h,k;r,s}{}_\alpha$. Then it is null, which means that $E_\alpha^i(S)$ is a polynomial of degree $\leqslant 1$ in $A_{i,j}^\alpha$. Then

$$E_\alpha^i(S) = d_{\alpha\beta\gamma}\epsilon^{ijhk}A_{j,h}^\gamma A_k^\beta + C_{\alpha\beta\gamma\theta}\epsilon^{ijhk}A_j^\beta A_h^\gamma A_k^\theta,$$

where $c_{\alpha\beta\gamma\theta}$ is skew symmetric in β, γ, θ. If $B_\alpha^i = E_\alpha^i(S)$, then B_α^i must satisfy

$$B_{\alpha\beta}^{i;j,h} = -B_{\beta\alpha}^{j;i,h},$$

$$B_{\alpha\beta}^{i;j} = B_{\beta\alpha}^{j;i} + \frac{\partial}{\partial x^h}(B_{\alpha\beta}^{i;j,h})$$

(see Ref. 2). We deduce

$$d_{\alpha\beta\gamma} + d_{\beta\alpha\gamma} + d_{\alpha\gamma\beta} = 0, \quad c_{\alpha\beta\gamma\theta} + c_{\beta\alpha\gamma\theta} = 0.$$

If

$$S_1 = \tfrac{1}{3}d_{\alpha\beta\gamma}\epsilon^{ijhk}A_{j,h}^\gamma A_k^\beta A_i^\alpha + \tfrac{1}{4}c_{\alpha\beta\gamma\theta}\epsilon^{ijhk}A_i^\alpha A_j^\beta A_h^\gamma A_k^\theta,$$

we have $E_\alpha^i(S) = E_\alpha^i(S_1)$, and S_1 is a scalar density. Then

$$E_\alpha^i(L) = E_\alpha^i(L_1 + S_1),$$

and besides

$$E^{ij}(L) = E^{ij}(L_1 + S_1).$$

Being that $L_1 + S_1$ is a scalar density, we are in the same situation as the one studied in the original paper. Now, the theorem follows for $L = L(g_{ij};A_i^\alpha;A_{i,j}^\alpha)$.

[1] G. W. Horndeski, Utilitas Math. **19**, 215 (1981).
[2] I. M. Anderson and T. Duchamp, Am. J. Math. **102**, 781 (1980).

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