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Equivalence and *s*-equivalence of vector-tensor Lagrangians

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It will be proven that if a gauge-invariant Lagrangian density having the local form $L = L(g_{ij};A_i;A_{i,j})$ is such that its Euler–Lagrange equations $E^i(L) = 0$ have the same set of solutions as $E^i(L_0) = 0$, where $L_0 = g^{1/2}F^{ij}F_{ij}$, then L and cL_0 are equivalent for same constant c, i.e., $E^i(L) = E^i(cL_0)$. From a previous result it follows that $L = cL_0 + D + eg^{1/2}$, where D is a divergence and e is a constant.

I. INTRODUCTION

In recent years, much attention has been paid to the study of the relation between Lagrangians such that their Euler–Lagrange equations have the same set of solutions, e.g., Refs. 1–7. In this paper we study Lagrangians that are concomitants of a metric, a covector, and its first partial derivatives, i.e.,

$$L = L(g_{ij};A_i;A_{i,j}), \tag{1}$$

and their relation with the usual L_0 giving rise to Maxwell field equations, i.e.,

$$L_0 = g^{1/2}F^{ij}F_{ij}, \tag{2}$$

where $g = \det(g_{ij})$ and $F_{ij} = A_{i,j} - A_{j,i}$. We use the summation convention and indices are raised or lowered with g^{ij} and g_{ij}. This g_{ij} is a Lorentz metric on a four-dimensional space-time.

In a general situation, there are three notions of equivalence between two Lagrangians L_1 and L_2:

(a) L_1 and L_2 are *s-equivalent* if, for any given metric $g_{ij}(x^h)$, a field $F_{ij} = F_{ij}(x^h)$ is a solution of $E^i(L_1) = 0$ if and only if it is a solution of $E^i(L_2) = 0$;

(b) L_1 and L_2 are *equivalent* if $E^i(L_1) = E^i(L_2)$;

(c) L_1 and L_2 are *completely equivalent* if $L_1 = L_2 + D$, where D is a divergence, i.e., $D = D^i{}_{,i}$, where a comma stands for partial differentiation.

For Lagrangians of the form (1) that are scalar densities, Lovelock[8] has proved a result that can be rephrased in the following terms: if L and L_0 are equivalent and L is a scalar density, then there exist constants c and e such that L and $cL_0 + eg^{1/2}$ are completely equivalent. In this paper we will prove that, for L of the form (1), if $E^i(L)$ is a gauge invariant tensorial density (which is mandatory for field equations) and L and L_0 are *s*-equivalent, then L and L_0 are equivalent. In other words, under the above-mentioned hypothesis, *s*-equivalence implies equivalence, i.e., (a) implies (b).

The importance of this result lies in the fact that *s*-equivalence is the really significant identification of Lagrangians from a physical point of view. The essential uniqueness of L_0 follows from our result, which reinforces the choice of the usual Maxwell equations for the determination of the electromagnetic field in a four-dimensional space-time.

Before dealing with the proof, we remark that, due to the local character of L, the notion (a) means that every local solution of $E^i(L) = 0$ is a local solution of $E^i(L_0) = 0$ and vice versa. Besides, the local form (1) is restrictive; otherwise, relation (10) below could be weaker, such as, for instance,

$$E^i(L,x) = \int d^3x'\, \Lambda^i_j(x,x')E^j(L_0,x').$$

The crucial point is that it avoids the Lagrangian from being explicitly dependent on position (see Ref. 6 for the some restriction in mechanics, i.e., the Lagrangian not being explicitly dependent on time).

II. *s*-EQUIVALENCE IMPLIES EQUIVALENCE

For L of the form (1), its Euler–Lagrange expressions are

$$E^i(L) = \frac{\partial L}{\partial A_i} - \frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial A_{i,j}}\right), \tag{3}$$

or, in full expression,

$$E^i(L) = L^i - \frac{\partial L^{ij}}{\partial g_{hk}} g_{hk,j} - (L^{ij})^h A_{h,j} - (L^{ij})^{hk} A_{h,kj}, \tag{4}$$

where $L^i = \partial L/\partial A_i$ and $L^{ij} = \partial L/\partial A_{i,j}$.

Let us assume that $E^i(L)$ is a gauge invariant tensorial density. Then it is known[9,10] that L is equivalent to a gauge invariant scalar density. So we can assume this last property for L from the start. Then, from the replacement theorem,[11]

$$L(g_{ij};A_i;A_{i,j}) = L(g_{ij};0;-\tfrac{1}{2}F_{ij}) = L_1(g_{ij};F_{ij}). \tag{5}$$

Denoting $L_1^{\overline{ij}} = \partial L_1/\partial F_{ij}$, we see from (5) that

$$L_1^{\overline{ij}} = \tfrac{1}{2}(L^{ij} - L^{ji}). \tag{6}$$

But the invariance identities that L has to fulfill[12] imply that L^{ij} is skew symmetric in i,j. So, from (6)

$$L_1^{\overline{ij}} = L^{ij}. \tag{7}$$

Then it is easy to prove that

$$E^i(L) = (L^{ij})^{hk} F_{h(k|j)}, \tag{8}$$

where a bar stands for covariant derivative with respect to the Christoffel symbols associated to g_{ij}, and a parenthesis means symmetrization.

We remark that

$$E^i(L_0) = g^{1/2} F^{ij}{}_{|j}. \tag{9}$$

Now we suppose L and L_0 *s*-equivalent. We first prove two

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facts: (i) for any given point with coordinates $(\mathring{x}^k)$ and for any given set of numbers $\mathring{F}_{ij}$ and $\mathring{F}_{ij,h}$ skew symmetric in i,j, it holds that

$$E^i(L_0)(g_{ij}(\mathring{x}^k);g_{ij,h}(\mathring{x}^k);\mathring{F}_{ij};\mathring{F}_{ij,h})=0$$
$$\Rightarrow E^i(L)(g_{ij}(\mathring{x}^k);g_{ij,h}(\mathring{x}^k);\mathring{F}_{ij};\mathring{F}_{ij,h})=0.$$

(ii) There exists a concomitant $\Lambda^i_j=\Lambda^i_j(g_{ij};F_{ij})$ such that $E^i(L)=\Lambda^i_jE^j(L_0)$.

To prove (i) we consider a point $(\mathring{x}^k)$ and $\mathring{F}_{ij}$, $\mathring{F}_{ij,h}$ such that

$$0=\mathring{F}^{ij}{}_{|j}=\mathring{g}^{ih}\mathring{g}^{jk}\mathring{F}_{hk\,|j},$$

where $\mathring{g}^{ih}=g^{ih}(\mathring{x}^k)$ and

$$\mathring{F}_{hk\,|j}=\mathring{F}_{hk,j}-\mathring{\Gamma}^s_{hj}\mathring{F}_{sk}-\mathring{\Gamma}^s_{jk}\mathring{F}_{hs}.$$

We can choose the coordinate system such that $\mathring{g}_{ij,h}=0$ and so $\mathring{\Gamma}^s_{ij}=0$; we can also assume $\mathring{x}^k=0$.

Let us consider the local field defined by

$$F^{ij}=\mathring{g}^{1/2}g^{-1/2}(\mathring{F}^{ij}+\mathring{F}^{ij}{}_{,r}x^r).$$

A straightforward computation proves that

(1) $F_{ij}(\mathring{x}^k)=\mathring{F}_{ij}$,

(2) $F_{ij,h}(\mathring{x}^k)=\mathring{F}_{ij,h}$,

(3) $F^{ij}{}_{|j}=0$.

Then we have proved that any point which is a solution of $E^i(L_0)=0$ can be extended locally to a field which is a solution of $E^i(L_0)=0$ in a neighborhood of $\mathring{x}^k$. By s-equivalence, $E^i(L)=0$ in that neighborhood, and so, making $x^k=\mathring{x}^k$, we have the implication proved.

To prove (ii) we use the following fact from linear algebra which is easy to prove: if every solution of the linear system $Ax=a$ is a solution of the system $Bx=b$ (A and B being $n\times m$ matrices and a and b vectors in R^n), then there is an $n\times n$ matrix C such that $B=CA$ and $b=Ca$. It means, in our case

$$E^i(L)=\Lambda^i_jE^j(L_0) \tag{10}$$

for some matrix Λ^i_j. Differentiating (10) with respect to $A_{h,kj}$ and taking (4) into account, we deduce

$$(L^{ij})^{hk}+(L^{ik})^{hj}=\Lambda^i_s(g^{sk}g^{hj}+g^{sj}g^{hk}-2g^{sh}g^{jk})g^{1/2}$$
$$=(\Lambda^{ik}g^{hj}+\Lambda^{ij}g^{hk}-2\Lambda^{ih}g^{jk})g^{1/2}. \tag{11}$$

Multiplying (11) by g_{jk} and summating over k and j, we obtain

$$g^{-1/2}2(L^{ij})^{hk}g_{jk}=\Lambda^{ih}+\Lambda^{ih}-8\Lambda^{ih}=-6\Lambda^{ih}.$$

Then

$$g^{1/2}\Lambda^{ih}=-\tfrac{1}{3}(L^{ij})^{hk}g_{jk}=-\tfrac{1}{3}(L^{ik})^{hj}g_{jk}$$
$$=-\tfrac{1}{3}(L^{hj})^{ik}g_{jk}=\Lambda^{hi}g^{1/2}, \tag{12}$$

i.e., Λ^{ih} is a symmetric tensor concomitant of g_{ij} and F_{ij}. In this case it is known[13] that

$$\Lambda^{ih}=\alpha g^{ih}+\beta F^i_{\ s}F^{sh}$$

for some scalars α, β concomitants of g_{ij} and F_{ij}. Substitution of (12) in (9) gives

$$g^{-1/2}E^i(L)=\alpha F^{ij}{}_{|j}+\beta F^{hj}{}_{|j}F^i_{\ l}F^l_{\ h}. \tag{13}$$

Now we will prove that $\beta=0$. Since $E^i(L)$ is a Euler–Lagrange expression, it has to fulfill certain identities.[14] One of them is

$$\frac{\partial E^i(L)}{\partial A_{r,tl}}=\frac{\partial E^r(L)}{\partial A_{i,tl}}. \tag{14}$$

But $F^{ij}{}_{|j}$ certainly fulfills (14). Then, from (13), we obtain

$$\beta(g^{rl}F^i_{\ s}F^{st}+g^{rt}F^i_{\ s}F^{sl})=\beta(g^{il}F^r_{\ s}F^{st}+g^{it}F^r_{\ s}F^{sl}). \tag{15}$$

If $\beta\neq 0$, we can cancel β in (15). Multiplying by g_{il}, differentiating the resulting expression with respect to F_{mk} and then with respect to F_{ls}, and multiplying the identity obtained by $g_{rl}g_{sm}g_{tk}$, we deduce $160=40$. Then it must be $\beta=0$, and so

$$E^i(L)=\alpha F^{ij}{}_{|j}g^{1/2}. \tag{16}$$

Differentiating (16) with respect to $A_{h,jk}$, we obtain

$$(L^{ij})^{hk}+\alpha(g^{ih}g^{jk}-g^{ik}g^{jh})g^{1/2}$$
$$=-((L^{ik})^{hj}+\alpha(g^{ih}g^{jk}-g^{ij}g^{hk})g^{1/2}). \tag{17}$$

Now, the left-hand side of (17) is skew symmetric in i,j and h,k, while the right-hand side is skew symmetric in i,k and h,j. Then the left-hand side is skew symmetric in all of its indices. Since we are working in a four-dimensional spacetime, it follows that

$$(L^{ij})^{hk}+g^{1/2}\alpha(g^{ih}g^{jk}-g^{ik}g^{jh})=\lambda\epsilon^{ijhk}. \tag{18}$$

Differentiating (18) with respect to F_{rs} and using the commutativity of partial derivatives, it follows that

$$\alpha^{rs}(g^{ih}g^{jk}-g^{ik}g^{hj})=\alpha^{hk}(g^{ir}g^{js}-g^{is}g^{rj}). \tag{19}$$

Multiplying (19) by $g_{jk}g_{ih}$, we deduce $\alpha^{rs}=0$. Then $\alpha=(g_{ij})$; it is known[15] that α must be a constant, say c. Then

$$E^i(L)=cE^i(L_0)=E^i(cL_0).$$

This means that L and cL_0 are equivalent. In this case we have[18]

$$L=cL_0+d\epsilon^{ijhk}F_{ij}F_{hk}+eg^{1/2} \tag{20}$$

for some constants d and e. Since $\epsilon^{ijhk}F_{ij}F_{hk}$ is a divergence, we see that L and $cL_0+eg^{1/2}$ are completely equivalent, which is the result we asserted in the Introduction.

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