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Concomitants theory and renormalization in even dimensions

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The complete set of concomitants is given of the metric, a scalar function, and their derivatives in six dimensions without imposing conditions on the order of the derivatives and their properties are studied under conformal transformations. These results are useful in several physical problems, like finding $\langle\varphi^2\rangle^{\rm ren}$ and $\langle T^{\mu\nu}\rangle^{\rm ren}$ in any geometry.

I. INTRODUCTION

Conformal invariance plays an important role in the theory of massless fields. It is well known that in the quantum domain this invariance is violated due to the presence of divergences that are to be removed by a renormalization procedure. Among the most important examples are conformal trace anomalies, i.e., arising of the nonzero values of $\langle T^{\mu}{}_{\mu}\rangle^{\rm ren}$ for conformally invariant theories. To study conformal anomalies in a conformal scalar massless field theory it is often instructive to consider a "poor man" version of this effect and to discuss the behavior of the quantity $\langle\varphi^2\rangle^{\rm ren}$, which describes the quantum fluctuations of the field. According to the classical theory $g^{1/4}\varphi^2$ is invariant under the conformal transformations while $g^{1/4}\langle\varphi^2\rangle^{\rm ren}$ does not possess this property.

In order to find out the necessary transformation law, it is possible to consider the behavior of the renormalization procedure under the conformal transformations. The corresponding calculations were done in four-dimensional space by Page.[1] In higher dimensions, these direct calculations become highly complicated. On the other hand, the terms that are to be added to $g^{1/4}\langle\varphi^2\rangle^{\rm ren}$ in order to restore its conformal invariance are local and are to be constructed from the metric and its derivatives up to a given order which can be easily defined by dimensional arguments. In other words, under the conformal transformation

$$g_{\mu\nu}\to\tilde g_{\mu\nu}=\omega^2 g_{\mu\nu},$$

the transformation of the $\langle\varphi^2\rangle^{\rm ren}$ must be of the form

$$\omega^{2\beta}\langle\varphi^2\rangle^{\rm ren}-\langle\tilde\varphi^2\rangle^{\rm ren}=\mathscr{F}(g_{\mu\nu},\nabla,\omega),$$

where $\beta=1-n/2$ and n is the space-time dimension.

A powerful method for obtaining the most general structure of $\mathscr{F}$ is provided by the concomitants theory. The main aim of this paper is to demonstrate the application of the concomitants theory to the particular problem of the study of the conformal anomalies of the conformal massless field in six dimensions. The main result is the complete list of the invariants that can be used for the construction of $\mathscr{F}$.

This paper is organized as follows: first we briefly summarize the properties of conformal spaces, conformal transformations (CT), and conformal theories (Sec. II). Then we establish the problem we are interested in (Sec. III). Section IV is a comment on the case in four dimensions. In Sec. V we demonstrate the theorem from which the complete set of concomitants follow; in Sec. VI conditions on them to represent $\langle\varphi^2\rangle$ are given. Section VII is a discussion.

II. CONFORMAL SPACES, CONFORMAL TRANSFORMATIONS, AND CONFORMAL THEORIES

Two spaces, V_n and $\tilde V_n$, are said to be conformal spaces if the metric tensors $g_{\mu\nu}$ and $\tilde g_{\mu\nu}$ are related by

$$\tilde g_{\mu\nu}(x)=\omega^2(x)g_{\mu\nu}(x), \tag{1}$$

where ω is a scalar function.[2] Of course, we assume that V_n and $\tilde V_n$ are globally hyperbolic space-times of n dimensions, one temporal, and $(n-1)$ spatial ones. We also assume that the metrics and ω^2 are C^∞.

In order to preserve the relation $g_{\mu\nu}g^{\mu\rho}=\delta^{\rho}{}_{\nu}$, the contravariant components of the metric tensor must transform like

$$\tilde g^{\mu\nu}(x)=\omega^{-2}(x)g^{\mu\nu}(x). \tag{2}$$

The following relation is obtained for the Christoffel symbols:

$$\tilde{\Gamma}^{\mu}_{\rho\sigma}=\Gamma^{\mu}_{\rho\sigma}+\omega^{-1}(\delta^{\mu}_{\rho}\omega_{\sigma}+\delta^{\mu}_{\sigma}\omega_{\rho}-g_{\rho\sigma}g^{\mu\nu}\omega_{\nu}), \tag{3}$$

where ω_σ stands for $\partial\omega/\partial x^\sigma$. With $\tilde{\Gamma}^{\mu}_{\rho\sigma}$ we define $\tilde{\nabla}$, the conformally transformed covariant derivative:

$$\tilde{\nabla}_{\mu}A_{\nu}=\nabla_{\mu}A_{\nu}-\omega^{-1}(\delta^{\lambda}_{\mu}\omega_{\nu}+\delta^{\lambda}_{\nu}\omega_{\mu}-g_{\mu\nu}g^{\lambda\sigma}\omega_{\sigma}), \tag{4}$$

$$\tilde{\nabla}^{\mu}A_{\nu}=\nabla^{\mu}A_{\nu}-\omega^{-1}(\delta^{\mu}_{\lambda}\omega_{\nu}+\delta^{\mu}_{\nu}\omega_{\lambda}-g_{\lambda\nu}g^{\mu\sigma}\omega_{\sigma}). \tag{5}$$

The covariant derivatives indicated by; or ∇ (we use the notation that is more convenient in each case) are those determinated by the untransformed metric $g_{\mu\nu}$.

Calculation of the Riemann tensor of $\tilde{g}_{\mu\nu}$ gives

$$\tilde{\mathbf{R}}^{\mu\nu}{}_{\rho\sigma}=\omega^{-2}\mathbf{R}^{\mu\nu}{}_{\rho\sigma}+\delta^{[\mu}_{[\rho}\Omega^{\nu]}_{\sigma]}, \tag{6}$$

where Ω^{α}_{β} is given by the definition

$$\Omega^{\alpha}_{\beta}=4\omega^{-1}(\omega^{-1})_{;\beta\gamma}g^{\alpha\gamma}-2\omega^{-1}_{;\rho}\omega^{-1}_{;\sigma}g^{\rho\sigma}\delta^{\alpha}_{\beta},$$

and [ ] means antisymmetrization:

$$T^{[ab]}=\tfrac{1}{2}(T^{ab}-T^{ba}).$$

The case $n=1$ is evidently of no interest. In two dimensions, any metric is reducible to $\lambda[(dx^1)^2\pm(dx^2)^2]$. It means that any V_2 is conformal to flat space in two dimensions. So, we will suppose that $n>2$. Then, the Ricci tensor of the transformed metric is

$$\tilde{\mathbf{R}}^{\mu}{}_{\nu}=\omega^{-2}\mathbf{R}^{\mu}{}_{\nu}+(n-2)\omega^{-1}(\omega^{-1})_{;\nu\alpha}g^{\alpha\mu}-(n-2)^{-1}\omega^{-n}(\omega^{n-2})_{,\rho\sigma}g^{\rho\sigma}\delta^{\mu}_{\nu}, \tag{7}$$

and its scalar curvature is

$$\tilde{R}=\omega^{-2}R-2(n-1)\omega^{-3}\omega_{;\mu\nu}g^{\mu\nu}-(n-1)\times(n-4)\omega^{-4}\omega_{;\mu}\omega_{;\nu}g^{\mu\nu}. \tag{8}$$

Using these tensors we may write another one as

$$\mathbf{C}^{\mu}_{\nu\rho\sigma}=\mathbf{R}^{\mu}_{\nu\rho\sigma}+(n-2)^{-1}(\delta^{\mu}_{\rho}\mathbf{R}_{\nu\sigma}-\delta^{\mu}_{\sigma}\mathbf{R}_{\nu\rho}+g_{\nu\sigma}\mathbf{R}^{\mu}{}_{\rho}-g_{\nu\rho}\mathbf{R}^{\mu}{}_{\sigma}+(n-1)^{-1}(n-2)^{-1}R\times(\delta^{\mu}_{\sigma}g_{\nu\rho}-\delta^{\mu}_{\rho}g_{\nu\sigma}), \tag{9}$$

and we can see that $C^{\mu}_{\nu\rho\sigma}$ is invariant under CT,

$$\tilde{\mathbf{C}}^{\mu}_{\nu\rho\sigma}=C^{\mu}_{\nu\rho\sigma}.$$

This tensor is called conformal a curvature tensor or a Weyl tensor. It may also be proved that any space with a null Weyl tensor is conformal to a flat space $(n>3)$.

Fields with special properties under CT—Weyl fields—may be also defined via the behavior of their covariant components. When one is dealing with massless fields, the action functional S is required to be conformally invariant: $\tilde{S}=S$. The action S is

$$S=\int\mathscr{L}(\Psi,\nabla\Psi)d\eta,$$

where L is the Lagrangian density, Ψ is the field, and

$$d\eta=\sqrt{-g}d^nx=\frac{1}{n!}\sqrt{-g}\epsilon_{\mu_1\mu_2\cdots\mu_n}dx^{\mu_1}dx^{\mu_2}\cdots dx^{\mu_n}.$$

As $\tilde{g}=\omega^{2n}g$, then $d\tilde{\eta}=\omega^n\,d\eta$. So L satisfies

$$\tilde{\mathscr{L}}=\omega^{-n}\mathscr{L}, \tag{10}$$

and from this requirement Weyl fields are defined: let Ψ have weight r under CT and m indices. A Weyl field satisfies $\tilde{\Psi}=(\omega^2)^r\Psi$. To find r we write this condition as

$$\tilde{\Psi}_{\mu_1\mu_2\cdots\mu_m}=\omega^{2r}\Psi_{\mu_1\mu_2\cdots\mu_m},$$

and pay attention to the fact that in the Lagrangian density we have a kinetic term like

$$\nabla_{\mu}\Psi_{\mu_1\mu_2\cdots\mu_m}\nabla^{\mu}\Psi^{\mu_1\mu_2\cdots\mu_m},$$

so (10) will give

$$\tilde{\mathscr{L}}=(\omega^2)^{2r-m-1}\mathscr{L}. \tag{11}$$

Considering (10) and (11) we see that a Weyl field satisfies

$$r=2(m+1)-n/4\equiv\beta/2. \tag{12}$$

Here we are interested in scalar fields. Then we have $m=0$ in (12), i.e.,

$$\tilde{\varphi}=\omega^{\beta}\varphi,\quad\text{with }\beta=1-n/2. \tag{13}$$

We will need to consider the propagator G, G being

$$G(x,x')=\langle\varphi(x)\varphi(x')\rangle, \tag{14}$$

where $\langle\ \rangle$ means vacuum expectation value (vev). It is immediate from (13) that G transforms as

$$\tilde{G}(x,x')=\omega^{\beta}(x)G(x,x')\omega^{\beta}(x'), \tag{15}$$

or briefly

$$\tilde{G}=\omega^{\beta}G\omega'^{\beta}.$$

III. STATEMENT OF THE PROBLEM

We want to use geometrical tools for the discussion of the properties of the renormalized value of the mean squared massless scalar field $\langle\varphi^2\rangle$. The mean square $\langle\varphi^2\rangle$ is related to the propagator

$$G(x,x')=\langle\varphi(x)\varphi(x')\rangle. \tag{16}$$

When performing the coincidence limit $x\to x'$, $\langle\varphi^2\rangle$ is generally divergent and we may write its renormalized expression as

$$\langle\varphi^2\rangle^{\text{ren}}=\langle\varphi(x)\varphi(x')\rangle^{\text{ren}}=G^{\text{ren}}. \tag{17}$$

Here G^{ren} is obtained by substracting the divergent terms G^{div} from the full propagator G,

$$G^{\text{ren}}=G-G^{\text{div}}. \tag{18}$$

When one performs a CT (11) on the ST metric, the scalar field transforms as (13) says. So the transformed $\widetilde{G}^{\text{ren}}$ will be

$$\widetilde{G}^{\text{ren}}=\widetilde{G}-\widetilde{G}^{\text{div}}. \tag{19}$$

Combining (18) and (19) and recalling that $\widetilde{G}=\omega^{2\beta}G$ because $G=\langle\varphi\varphi\rangle$, we find that the transformed $\langle\widetilde{\varphi}^2\rangle^{\text{ren}}$ must satisfy:

$$\omega^{2\beta}\langle\varphi^2\rangle^{\text{ren}}-\langle\widetilde{\varphi}^2\rangle^{\text{ren}}=\widetilde{G}^{\text{div}}-\omega^{2\beta}G^{\text{div}}\equiv\mathscr{F}. \tag{20}$$

But the divergent terms are geometric objects, so $\mathscr{F}$ must be a function of them, i.e., $\mathscr{F}$ must be a function of the relevant geometrical objects of the ST: The metric and the conformal factor, plus their derivatives, because we are dealing with a conformally invariant theory. The derivatives must be taken up to the necessary order to give $\mathscr{F}$ the right units. Taking all we have said about transformation rules for $\langle\varphi^2\rangle$ into account, we see that $\mathscr{F}$ must satisfy

$$\mathscr{F}(g_{\mu\nu},\nabla,\omega)=-\omega^{2\beta}\mathscr{F}(\widetilde{g}_{\mu\nu},\widetilde{\nabla},\widetilde{\omega}), \tag{21}$$

where $\widetilde{g}_{\mu\nu}$ is defined by (1), $\widetilde{\nabla}$ by (4), (5), and $\widetilde{\omega}=1/\omega$. Equation (21) is an anticonformal transformation.

IV. $\langle\varphi^2\rangle$ IN FOUR DIMENSIONS

First we pay attention to the simple case in $n=4$. In this case, the only functions of $(g_{\mu\nu},\nabla_\mu,\omega)$ with the correct units are the scalar curvature R (this is the Weyl theorem[3]) and the contractions of two derivatives (each one gives one l^{-1}): $\nabla\omega\,\nabla\omega$ and $\nabla\nabla\omega$. We recall that we are working in natural units $c=\hbar=1$ and that we choose

$$[g_{\mu\nu}]=1,\quad\text{so}\quad[\omega]=1.$$

Requiring the action S to be dimensionless as usual implies $[\varphi^2]=l^{2-n}$ that in $n=4$ gives $[\varphi^2]=l^{-2}$.

If we want $\mathscr{F}$ to satisfy condition (21), which is the same as wanting it to be anticonformal, we get,

$$\mathscr{F}^{(4)}=\alpha(\nabla_\mu\nabla^\mu\omega/\omega^3), \tag{22}$$

where α is some coefficient that has to be obtained from comparison with a known example. This is the correct expression for $\mathscr{F}$ and α is $1/48\pi^2$.[1]

If we had wanted $\mathscr{F}$ to satisfy the CT condition, we would have obtained

$$\mathscr{F}'^{(4)}=\alpha'[\omega^{-2}R-3(\nabla_\mu\nabla^\mu\omega/\omega^3)]+\beta'(\nabla^\mu\omega\nabla_\mu\omega/\omega^4), \tag{23}$$

with α' and β' two arbitrary coefficients.

V. THE CONCOMITANTS IN SIX DIMENSIONS

Now we look for $\mathscr{F}$ in six dimensions. Conditions we require are the following.

(i) Function $\mathscr{F}$ will depend only on geometrical objects, i.e., the metric $g_{\mu\nu}$, the conformal factor ω, and their derivatives.

(ii) We assume natural units and we give length units to the ST coordinates, so $[g_{\mu\nu}]=[\omega]=1$. We require the action S to be dimensionless, as it must be to be able to construct the generating functional with it, so in n dimensions- $[\mathscr{L}]=l^{-n}$ and $[\varphi^2]=l^{(-n+2)}$. In six dimensions, $[\varphi^2]=[G(x,x')]=l^{-4}$. So we are looking for a function $\mathscr{F}$,

$$\mathscr{F}=\mathscr{F}(g_{\mu\nu},g_{\mu\nu,\rho_1},g_{\mu\nu,\rho_1\rho_2},\ldots,g_{\mu\nu,\rho_1,\rho_2,\ldots,\rho_p},\omega,\omega_{\mu_1},\ldots,\omega_{\mu_1,\ldots,\mu_q}), \tag{24}$$

which has dimensions of $l-4$. Because of its dimensions, by a change of scale λ in $\mathscr{F}$, we have

$$\mathscr{F}(g_{\mu\nu},\lambda g_{\mu\nu,\rho_1},\lambda^2 g_{\mu\nu,\rho_1\rho_2},\ldots,\lambda^p g_{\mu\nu,\rho_1,\rho_2,\ldots,\rho_p},\omega,\lambda\omega_{\mu_1},\ldots,\lambda^q\omega_{\mu_1,\ldots,\mu_q})=\lambda^4\mathscr{F}(g_{\mu\nu},g_{\mu\nu,\rho_1},g_{\mu\nu,\rho_1\rho_2},\ldots,g_{\mu\nu,\rho_1,\rho_2,\ldots,\rho_p},\omega,\omega_{\mu_1},\ldots,\omega_{\mu_1,\ldots,\mu_q}). \tag{25}$$

Following the procedure developed by Aldersley,[4] we differentiate (25) four times with respect to λ and make $\lambda\to 0^+$. Thus we obtain

$$\mathscr{F}=\Lambda_1^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}g_{\mu\nu,\rho\sigma}g_{\alpha\beta,\gamma\delta}+\Lambda_2^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}g_{\mu\nu,\rho\sigma}\omega_{\alpha,\beta}\omega_{\gamma,\delta}+\Lambda_3^{\mu\nu\rho\sigma\alpha\beta\delta}g_{\mu\nu,\rho\sigma}\omega_{\alpha,\beta\delta}+\Lambda_4^{\mu\nu\rho\sigma\delta\alpha\beta}g_{\mu\nu,\rho\sigma\delta}\omega_{\alpha,\beta}+\Lambda_5^{\mu\nu\rho\sigma\alpha\beta\delta\tau}\omega_{\mu,\nu}\omega_{\rho,\sigma}\omega_{\alpha,\beta}\omega_{\delta,\tau}$$
$$+\Lambda_6^{\mu\nu\rho\sigma\alpha\beta\delta}\omega_{\mu,\nu}\omega_{\rho,\sigma}\omega_{\alpha,\beta\delta}+\Lambda_7^{\mu\nu\rho\sigma\alpha\beta}\omega_{\mu,\nu}\omega_{\rho,\sigma\alpha\beta}+\Lambda_8^{\mu\nu\rho\sigma\alpha\beta}\omega_{\mu,\nu\rho}\omega_{\sigma,\alpha\beta}+\Lambda_8^{\mu\nu\rho\sigma\alpha}\omega_{\mu,\nu\rho\sigma\alpha}+\Lambda_{10}^{\mu\nu\rho\sigma\delta\tau}g_{\mu\nu,\rho\sigma\delta\tau}+(\cdots). \tag{26}$$

Here $(\cdots)$ indicates terms containing $g_{\mu\nu,\rho}$ and $\Lambda_i^{\cdots}$ are scalar concomitants of $g_{\mu\nu}$ and ω_μ. Now we apply the replacement theorem,[5] which allows us to replace $g_{\mu\nu,\rho}$ by zero, higher derivatives of $g_{\mu\nu}$ by adequate simetrizations of the curvature tensor and its covariant derivative, and partial derivatives of ω by its covariant derivatives. On the other hand, $\Lambda_i^{\cdots}$ have recently been found.[6] Replacing all of this in (26) and performing standard calculations, we obtain

$$\mathscr{F}=a_1(\omega^\mu\omega_\mu)^2+a_2(\omega^\mu{}_\mu)^2+a_3R^2+a_4\mathbf{R}^{\mu\nu}\mathbf{R}_{\mu\nu}+a_5\mathbf{R}_{\mu\nu}\omega_{\mu\nu}+a_6\mathbf{R}^{\mu\nu}\omega_\mu\omega_\nu+a_7R\omega^\mu{}_\mu$$
$$+a_8R\omega^\mu\omega_\mu+a_9\omega^\mu{}_\mu\omega^\sigma\omega_\sigma+a_{10}\omega^{\mu\nu}\omega_{\mu\nu}+a_{11}\omega^{\mu\nu}\omega^\mu\omega_\nu+a_{12}\Box R+a_{13}R_{;\mu}\omega^\mu+a_{14}\mathbf{R}^{\mu\nu\rho\sigma}\mathbf{R}_{\mu\nu\rho\sigma}$$
$$+a_{15}\omega^\mu{}_\mu{}^\sigma{}_\sigma+a_{16}\omega^{\mu\sigma}{}_\mu\omega_\sigma+a_{17}\omega^{\mu\nu}{}_{\mu\nu}+a_{18}\omega^\mu{}_\mu{}^\sigma\omega_\sigma, \tag{27}$$

$\omega^{\mu\nu}{}_\rho$ means $\nabla_\rho\nabla^\nu\nabla^\mu\omega$.

VI. PROPERTIES OF THE CONCOMITANTS UNDER ANTI-CONFORMAL TRANSFORMATIONS

In order to repeat the program that was showed in Sec. IV to obtain $\langle\varphi^2\rangle^{\text{ren}}$ in four dimensions, we first find the conformally transformed quantity that corresponds to each one of the concomitants of the previous section. The result of the long but simple computation is the following:

$$(\tilde\omega^\mu\tilde\omega_\mu)^2=\omega^{-12}(\omega^\mu\omega_\mu)^2,$$

$$(\tilde\omega^\mu{}_\mu)^2=-\omega^{-4}\Box\omega-(n-4)\omega^{-5}\omega_\mu\omega^\mu,$$

$$\tilde R^2=\omega^{-4}R^2+4(n-1)^2\omega^{-6}(\omega^\mu{}_\mu)^2+(n-1)^2(n-4)^2\omega^{-8}(\omega^\sigma\omega_\sigma)^2-4(n-1)\omega^{-5}R\omega^\mu{}_\mu$$
$$-2(n-1)(n-4)\omega^{-6}R\omega^\sigma\omega_\sigma+4(n-1)^2(n-4)\omega^{-7}\omega^\mu{}_\mu\omega^\sigma\omega_\sigma,$$

$$\tilde{\mathbf{R}}^{\mu\nu}\tilde{\mathbf{R}}_{\mu\nu}=\omega^{-4}\tilde{\mathbf{R}}^{\mu\nu}\mathbf{R}_{\mu\nu}-2\omega^{-5}R\omega^\sigma{}_\sigma-2(n-2)\omega^{-5}\mathbf{R}^{\mu\nu}\omega_{\mu\nu}+4(n-2)\omega^{-6}\mathbf{R}^{\mu\nu}\omega_\mu\omega_\nu-2(n-3)\omega^{-6}R\omega^\sigma\omega_\sigma$$
$$+(n-2)^2\omega^{-6}\omega^{\mu\nu}\omega_{\mu\nu}-4(n-2)^2\omega^{-7}\omega^{\mu\nu}\omega_\mu\omega_\nu+(3n-4)\omega^{-6}(\omega^\sigma{}_\sigma)^2+(4n^2-20n+20)\omega^{-7}\omega^\sigma{}_\sigma\omega^\mu{}_\mu$$
$$+(n^3-6n^2+13n-8)\omega^{-8}(\omega^\sigma\omega_\sigma)^2,$$

$$\tilde{\mathbf{R}}^{\mu\nu\rho\sigma}\tilde{\mathbf{R}}_{\mu\nu\rho\sigma}=\mathbf{C}^{\mu\nu\rho\sigma}\mathbf{C}_{\mu\nu\rho\sigma}+4(n-2)^{-1}\omega^{-2}\mathbf{R}^{\mu\nu}\mathbf{R}_{\mu\nu}-8\omega^{-5}\mathbf{R}^{\mu\nu}\omega_{\mu\nu}+16\omega^{-6}\mathbf{R}^{\mu\nu}\omega_\mu\omega_\nu-4\omega^{-6}R\omega^\sigma\omega_\sigma$$
$$+4(n-2)\omega^{-6}\omega^{\mu\nu}\omega_{\mu\nu}+8(n-2)^{-1}\omega^{-6}(\omega^\sigma{}_\sigma)^2-16(n-2)\omega^{-7}\omega^{\mu\nu}\omega_\mu\omega_\nu+16(n-3)$$
$$\times(n-2)^{-1}\omega^{-7}\omega^\rho{}_\rho\omega^\sigma\omega_\sigma-2(n-1)^{-1}(n-2)^{-1}\omega^{-4}R^2-[2n^3-34n^2+128n-144]$$
$$\times(n-2)^{-1}\omega^{-8}(\omega^\sigma\omega_\sigma)^2,$$

$$\tilde\omega^\mu{}_\mu{}^\sigma{}_\sigma=-\omega^{-6}\omega^\mu{}_\mu{}^\sigma{}_\sigma-2(n-4)\omega^{-7}\omega^{\rho\sigma}\omega_{\rho\sigma}+4\omega^{-7}(\omega^\alpha{}_\alpha)^2-(n-10)\omega^{-7}\omega^\sigma\omega^\mu{}_{\mu\sigma}-2(n-4)\omega^{-7}\omega_\sigma\omega^{\mu\sigma}{}_\mu$$
$$+(9n-48)\omega^{-8}\omega^\alpha{}_\alpha\omega^\sigma\omega_\sigma-2(n-4)(n-12)\omega^{-8}\omega^{\mu\nu}\omega_\mu\omega_\nu+5(n-4)(n-8)\omega^{-9}(\omega^\sigma\omega_\sigma)^2,$$

$$\tilde{\mathbf{R}}^{\mu\nu}\tilde\omega_{\mu\nu}=-\omega^{-6}\mathbf{R}^{\mu\nu}\omega_{\mu\nu}-\omega^{-7}R\omega^\sigma\omega_\sigma+(n-2)\omega^{-7}\omega_{\mu\nu}\omega^{\mu\nu}+\omega^{-7}(\omega^\sigma{}_\sigma)^2+4\omega^{-7}\mathbf{R}^{\mu\nu}\omega_\mu\omega_\nu-6(n-2)\omega^{-8}\omega^{\mu\nu}\omega_\mu\omega_\nu$$
$$+[(n-2)+(n-3)+(n-4)]\omega^{-8}\omega^\sigma\omega_\sigma\omega^\rho{}_\rho+[6(n-2)+(n-3)(n-4)]\omega^{-9}(\omega^\sigma\omega^\sigma)^2,$$

$$\tilde{\mathbf{R}}^{\mu\nu}\tilde\omega_\mu\tilde\omega_\nu=\omega^{-8}\mathbf{R}^{\mu\nu}\omega_\mu\omega_\nu+(n-1)\omega^{-10}(\omega^\mu\omega_\mu)^2-\omega^{-9}\omega^\sigma{}_\sigma\omega^\mu\omega_\mu-(n-2)\omega^{-9}\omega^{\mu\nu}\omega_\mu\omega_\nu,$$

$$\widetilde{R}\widetilde{\omega}^{\mu}{}_{\mu}= -\omega^{-6}R\omega^{\mu}{}_{\mu}-(n-4)\omega^{-7}R\omega^{\sigma}\omega_{\sigma}+2(n-1)\omega^{-7}(\omega^{\sigma}{}_{\sigma})^2+3(n-1)(n-4)\omega^{-8}\omega^{\sigma}{}_{\sigma}\omega^{\rho}\omega_{\rho}+(n-1)\times(n-4)^2\omega^{-9}(\omega^{\sigma}\omega_{\sigma})^2,$$

$$\widetilde{R}\widetilde{\omega}^{\mu}\widetilde{\omega}_{\mu}=\omega^{-8}R\omega^{\mu}\omega_{\mu}-2(n-1)\omega^{-9}\omega^{\rho}\omega_{\rho}\omega^{\mu}{}_{\mu}-(n-1)(n-4)\omega^{-10}(\omega^{\rho}\omega_{\rho})^2,$$

$$\widetilde{\omega}^{\mu}{}_{\mu}\widetilde{\omega}^{\sigma}\widetilde{\omega}_{\sigma}= -\omega^{-10}\omega^{\mu}{}_{\mu}\omega^{\sigma}\omega_{\sigma}-(n-4)\omega^{-11}(\omega^{\rho}\omega_{\rho})^2,$$

$$\widetilde{\omega}^{\mu\nu}\widetilde{\omega}_{\mu\nu}=\omega^{-8}\omega^{\mu\nu}\omega_{\mu\nu}-8\omega^{-9}\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}+2\omega^{-9}\omega^{\rho}{}_{\rho}\omega^{\sigma}\omega_{\sigma}+(n-8)\omega^{-10}(\omega^{\sigma}\omega_{\sigma})^2,$$

$$\widetilde{\omega}^{\mu\nu}\widetilde{\omega}_{\mu}\widetilde{\omega}_{\nu}= -\omega^{-10}\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}+3\omega^{-11}(\omega^{\sigma}\omega_{\sigma})^2,$$

$$\begin{aligned}\Box\widetilde{R}=&\,\omega^{-4}\Box R-2\omega^{-5}R\omega^{\rho}{}_{\rho}+(n-6)\omega^{-5}\omega^{\mu}R_{;\mu}-2(n-1)\omega^{-5}\omega^{\rho}{}_{\rho}{}^{\mu}{}_{\mu}-2(n-5)\omega^{-6}R\omega^{\sigma}\omega_{\sigma}\\&+6(n-1)\omega^{-6}(\omega^{\rho}{}_{\rho})^2-2(n^2-9n+8)\omega^{-6}\omega^{\mu}\omega^{\rho}{}_{\rho\mu}-2(n-1)(n-4)\omega^{-6}\omega^{\mu\nu}\omega_{\mu\nu}-2(n-1)\\&\times(n-4)\omega^{-6}\omega^{\mu\sigma}{}_{\mu}\omega_{\sigma}+2(5n^2-31n+26)\omega^{-7}\omega^{\rho}{}_{\rho}\omega^{\sigma}\omega_{\sigma}-2(n-1)(n-4)(n-10)\omega^{-7}\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}\\&+4(n-1)(n-4)(n-7)\omega^{-8}(\omega^{\sigma}\omega_{\sigma})^2,\end{aligned}$$

$$\begin{aligned}\widetilde{R}^{;\mu}\widetilde{\omega}_{\mu}=&-\omega^{-6}R^{;\mu}\omega_{\mu}+2\omega^{-7}R\omega^{\sigma}\omega_{\sigma}+2(n-1)\omega^{-7}\omega^{\rho}{}_{\rho}{}^{\mu}\omega_{\mu}-6(n-1)\omega^{-8}\omega^{\sigma}\omega_{\sigma}\omega^{\rho}_{\rho}+2(n-1)\\&\times(n-4)\omega^{-8}\omega^{\sigma\mu}\omega_{\sigma}\omega_{\mu}-4(n-1)(n-4)\omega^{-9}(\omega^{\sigma}\omega_{\sigma})^2,\end{aligned}$$

$$\widetilde{\omega}^{\mu\sigma}{}_{\mu}\widetilde{\omega}_{\sigma}=\omega^{-8}\omega^{\mu\sigma}{}_{\mu}\omega_{\sigma}-5\omega^{-9}\omega^{\mu}{}_{\mu}\omega^{\sigma}\omega_{\sigma}+(n-6)\omega^{-9}\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}-(4n-19)\omega^{-10}(\omega^{\mu}\omega_{\mu})^2,$$

$$\begin{aligned}\widetilde{\omega}^{\mu\nu}{}_{\mu\nu}=&-\omega^{-6}\omega^{\mu\nu}{}_{\mu\nu}-2(n-6)\omega^{-7}\omega^{\mu}\omega^{\nu}{}_{\mu\nu}+5\omega^{-7}\omega^{\mu}\omega^{\nu}{}_{\nu\mu}-(n-6)\omega^{-7}\omega^{\mu\nu}\omega_{\mu\nu}+5\omega^{-7}(\omega^{\mu}{}_{\mu})^2\\&+9(n-6)\omega^{-8}\omega^{\mu}{}_{\mu}\omega^{\sigma}\omega_{\sigma}-(n^2-21n+80)\omega^{-8}\omega^{\mu}\omega^{\nu}\omega_{\mu\nu}+(n-8)(4n-19)\omega^{-9}(\omega^{\mu}\omega_{\mu})^2,\end{aligned}$$

$$\widetilde{\omega}^{\mu}{}_{\mu}{}^{\sigma}\widetilde{\omega}_{\sigma}=\omega^{-8}\omega^{\mu}{}_{\mu}{}^{\sigma}\omega_{\sigma}+2(n-4)\omega^{-9}\omega^{\mu}\omega^{\nu}\omega_{\mu\nu}-4\omega^{-9}\omega^{\mu}{}_{\mu}\omega^{\nu}\omega_{\nu}-5(n-4)\omega^{-10}(\omega^{\mu}\omega_{\mu})^2.$$

Imposing the fact that a linear combination with coefficients allowed by Eq. (27)—for example, a number by a power of ω, i.e., $A\omega^{\alpha}$—must be antisymmetric with respect to the corresponding conformally transformed one and also that the linear combination must be multiplied by ω^{2-n} to satisfy Eq. (21), one gets an expression like

$$\begin{aligned}&a\omega^{\mu}{}_{\mu}{}^{\sigma}{}_{\sigma}+b\omega^{\mu\nu}{}_{\mu\nu}+c\omega^{\mu}{}_{\mu}{}^{\nu}\omega_{\nu}+d\omega^{\mu\nu}{}_{\mu}\,\omega_{\nu}+e(\omega^{\mu}{}_{\mu})^2+f\omega^{\mu\nu}\omega_{\mu\nu}+g\omega^{\mu}_{\mu}\omega^{\nu}\omega_{\nu}+h\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}+jR\omega^{\mu}{}_{\mu}+k\mathbf{R}^{\mu\nu}\omega_{\mu\nu}+lR_{;\mu}\omega^{\mu}\\&\quad+mR\omega^{\mu}\omega_{\mu}+n\mathbf{R}^{\mu\nu}\omega_{\mu}\omega_{\nu}+p(\omega^{\mu}\omega_{\mu})^2+q\Box R+sR^2+t\mathbf{R}^{\mu\nu}\mathbf{R}_{\mu\nu}+u[\mathbf{C}^{\mu\nu\rho\sigma}\mathbf{C}_{\mu\nu\rho\sigma}-4(n-2)^{-1}\mathbf{R}^{\mu\nu}\mathbf{R}_{\mu\nu}\\&\quad+2(n-1)^{-1}(n-2)^{-1}R^2]\\&=-\omega^{(2-n)}[-\omega^{\alpha-6}a\omega^{\mu}{}_{\mu}{}^{\nu}{}_{\nu}-\cdots-b\omega^{\beta-6}\omega^{\mu\nu}{}_{\mu\nu}-\cdots+c\omega^{\gamma-8}\omega^{\mu}{}_{\mu\nu}\omega^{\nu}+\cdots+d\omega^{\delta-8}\,\omega^{\mu\nu}{}_{\mu}\omega_{\nu}+\cdots\\&\quad+e\omega^{\mu-8}(\omega^{\mu}{}_{\mu}{}^{2}+\cdots+f\omega^{\nu-8}\omega^{\mu\nu}\omega_{\mu\nu}+\cdots-g\omega^{\rho-10}\omega^{\sigma}\omega_{\sigma}\omega^{\mu}{}_{\mu}+\cdots-h\omega^{\sigma-10}\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}+\cdots-j\omega^{\lambda-6}R\omega^{\sigma}_{\sigma}\\&\quad+k\omega^{\kappa-6}\mathbf{R}^{\mu\nu}\omega_{\mu\nu}+\cdots-l\omega^{\theta-6}R_{;\mu}\omega_{\mu}+\cdots+m\omega^{\varphi-8}R\omega^{\mu}\omega_{\mu}+\cdots+n\omega^{\xi-8}\mathbf{R}^{\mu\nu}\omega_{\mu}\omega_{\nu}+\cdots+p\omega^{\eta-12}(\omega^{\mu}\omega_{\mu})^2\\&\quad+q\omega^{\psi-4}\Box R+\cdots+s\omega^{\zeta-4}R^2+\cdots t\omega^{\upsilon-4}\mathbf{R}^{\mu\nu}\mathbf{R}_{\mu\nu}+\cdots+u\omega^{\chi}\mathbf{C}^{\mu\nu\rho\sigma}\mathbf{C}_{\mu\nu\rho\sigma}+\cdots.\end{aligned}$$

Joining together coefficients multiplying each concomitant, the set of equations that the coefficients and the exponents of ω must satisfy is obtained. When considering them, it may trivially be seen that there are many possible sets of solutions for the equations, for example,

function $\mathscr{F}=g\omega^{14}[\omega^{\mu}{}_{\mu}\omega^{\sigma}{}_{\sigma}+\frac{2}{3}\omega^{\mu\nu}{}_{\mu\nu}]$, function $\mathscr{F}=a\omega^{10}\omega^{\mu}{}_{\mu}{}^{\sigma}{}_{\sigma}+b\omega^{10}\omega^{\mu\nu}{}_{\mu\nu}+[-2a-\frac{5}{2}b-5l]\omega^{9}\omega^{\mu}{}_{\mu}{}^{\sigma}\omega_{\sigma}+[-2a+(n-6)b]\omega^{9}\omega^{\mu\sigma}_{\mu}\omega_{\sigma}+[-2a-\frac{5}{2}b$

$$+\tfrac{3}{4}(n-6)b-5j]\omega^{9}(\Box\omega)^{2}+[2a+\tfrac{7}{2}(n-6)b]\omega^{9}\omega^{\mu\nu}\omega_{\mu\nu}+g\omega^{14}\omega^{\mu}{}_{\mu}\omega^{\sigma}\omega_{\sigma}+h\omega^{14}\omega^{\mu\nu}\omega_{\mu}\omega_{\nu}$$

$$+j\omega^{10}R(\Box\omega)+k\omega^{10}\mathbf{R}^{\mu\nu}\omega_{\mu\nu}+l\omega^{10}R^{;\mu}\omega_{\mu}+3(n-6)b\omega^{9}\mathbf{R}^{\mu\nu}\omega_{\mu}\omega_{\nu}+[j-l-\tfrac{3}{4}(n-6)b]\omega^{9}R\omega^{\mu}\omega_{\mu}$$

$$+(g-\tfrac{3}{2}h)\omega^{15}(\omega^{\mu}\omega_{\mu})^{2}+[-3a+\tfrac{5}{2}b-\tfrac{65}{4}(n-6)b-10l+5j-15k]\omega^{13}(\omega^{\mu}\omega_{\mu})^{2}[\text{plus}2a(n-6)$$

$$+b(n-6)^{2}-4b(n-6)=0].$$

One may also admit that the coefficients are of the type $A\omega^{\alpha}\ln\omega$ and obtain another set of equations and their respective set of solutions. Morover, one may impose that coefficients were of a different type than $A\omega^{\alpha}$ or $A\omega^{\alpha}\ln\omega$.

VII. DISCUSSION

We have given the complete set of concomitants of the metric, a scalar function, and their derivatives. We have not imposed the order of the derivatives because it turns out in turn from dimensional considerations. Then we have studied their properties under conformal and anticonformal transformations. We have given the equations that the coefficients must satisfy and some examples of the possible solutions. We have shown that—though the coefficients for $\mathscr{F}^{(4)}$ that give the divergent terms of $\langle\varphi^2\rangle$ in four dimensions can be easily obtained (Sec. IV)—this is not the case in six dimensions. Comparison with the results found for different examples in the literature[7,8] does not single out a set of equations for the coefficients that determines them completely. We have obtained mathematical results on the problem that may be useful in several physical problems concerning the computing of $T^{\mu\nu}$ or $\langle\varphi^2\rangle$, even the actual applications are not inmediate because finding the constraints that another physical requirements—for example, the correct value for the trace anomaly—would impose on $T^{\mu\nu}$ or $\langle\varphi^2\rangle$ in any geometry, obviously involves extremely long calculations.

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