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MINIMAL WORDS IN THE FREE GROUP OF RANK TWO

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In this note is given an explicit computation method to decide about the minimality of a word in the free group of rank two. A cohomological use of this method is to yield words defining one relator duality groups.

Let F be the free group on n generators $x_1, \dots, x_n$.

Recall that a *T-transformation* of F is an automorphism τ defined on the generators by putting $\tau(x_k) = x_k$, for a fixed index k, and $\tau(x_i) =$ one of x_i, $x_k^e x_i$, $x_i x_k^e$ or $x_k^e x_i x_k^{-e}$, for $i \neq k$, where $e = \pm 1$.

A reduced word w in F is called *minimal* when $L(\tau(w)) \geqslant L(w)$ for all T-transformation τ, where L denotes the length. A *minimal form* of w is a minimal word obtained from w by applying a finite number of T-transformations. The existence of such a minimal form is clear. From Whitehead's theorem [4, Theorem N2, p. 166], Shenitzer has proved that any two minimal forms of w involve the same number of generators, and both have the same length [5, Corollary, p. 276].

From now on, we suppose $n = 2$. Let w be a word in F involving both generators. We consider the syllable expression $w = \prod_{1 \leqslant i \leqslant m} x_{v_i}^{n_i}$, where $v_i \neq v_{i+1}$, $|n_i| > 0$ and $m \geqslant 2$ (m is called the *syllable length* of w). We can suppose that w is *cyclically reduced, i.e.*, $v_1 \neq v_m$ or $v_1 = v_m$ and $\mathrm{sg}(n_1) = \mathrm{sg}(n_m)$ (sg means sign). This allows us to suppose that m is even. Indeed, if m is odd, we must have $v_1 = v_m$ and $\mathrm{sg}(n_1) = \mathrm{sg}(n_m)$. Let τ be the T-transformation of F, $\tau(z) = x_{v_1}^{-\mathrm{sg}(n_1)} z x_{v_1}^{\mathrm{sg}(n_1)}$. Applying $|n_1|$-times the transformation τ to w, we get

$$w' = x_{v_1}^{-n_1} w x_{v_1}^{n_1} = \Bigl(\prod_{2 \leqslant i \leqslant m-1} x_{v_i}^{n_i} \Bigr) x_{v_1}^{n_1 + n_m},$$

which has even syllable length $m - 1$, because $n_1 + n_m \neq 0$. But w' is minimal if, and only if, w is. For w being minimal, if w_1' is a minimal form of w' we can write

$$w_1' = \tau_s \cdots \tau_1(w') = \tau_s \cdots \tau_1 \tau^{|n_1|}(w),$$

where $\tau_1, \ldots, \tau_s$ are T-transformation, which shows that w'_1 is a minimal form of w, and since the minimal forms have the same length, we have $L(w) = L(w'_1)$. Moreover, taking into account that $\mathrm{sg}(n_1) = \mathrm{sg}(n_m)$,

$$L(w') = \sum_{2 \le i \le m-1} |n_i| + |n_1 + n_m| = \sum_{2 \le i \le m-1} |n_i| + |n_1| + |n_m| = L(w).$$

Consequently, $L(w'_1) = L(w')$, so that w' is minimal. The proof of the converse is entirely similar.

If $p = 1, 2$, we define

$$L_p(w) = \sum_{v_i = p} |n_i|,$$

$$c_p^+(w) = \#\{1 < i < m \mid v_i = p \text{ and } \mathrm{sg}(n_{i-1}) = \mathrm{sn}(n_{i+1}) = \mathrm{sg}(n_i)\},$$

$$c_p^-(w) = \#\{1 < i < m \mid v_i = p \text{ and } \mathrm{sg}(n_{i-1}) = \mathrm{sg}(n_{i+1}) \neq \mathrm{sg}(n_i)\},$$

$$c_p(w) = c_p^+(w) - c_p^-(w) + \tfrac{1}{2}(\delta_{pv_1}\, \mathrm{sg}(n_1 n_2) + \delta_{pv_m}\, \mathrm{sg}(n_{m-1} n_m)),$$

(δ is the Kronecker symbol).

Theorem. *The word w is minimal if, and only if,*

$$L_q(w) \geq \tfrac{1}{2}(m-1) + |c_p(w)| + \tfrac{1}{2}|\mathrm{sg}(n_1)\, \mathrm{sg}(c_p(w)) + \mathrm{sg}(n_m)|,$$

for $p \neq q$.

Proof. To simplify the notation, we assume that $v_1 = p$, so that $v_m = q$, since m is even.

Let τ be a T-transformation satisfying $x_p \mapsto x_p$ and $x_q \mapsto x_p^e x_q$, where $e = \pm 1$. First, we calculate the length of $\tau(w)$. Since τ does not produce symbols x_q, we verify immediately that $L_q(w) = L_q(\tau(w))$. We have to calculate then $L_p(\tau(w))$. To this end, we analyze the syllable expression of $\tau(w)$:

(a) If $1 < i < m$ and $v_i = p$, the factor $x_q^{n_{i-1}} x_p^{n_i} x_q^{n_{i+1}}$ of w becomes $(x_p^e x_q)^{n_{i-1}} x_p^{n_i} (x_p^e x_q)^{n_{i+1}}$ in $\tau(w)$. The possible reductions depend upon the signs of n_{i-1} and n_{i+1}. Explicitly, if n_{i-1} and n_{i+1} are both positive, n_i transforms into $n_i + e$; when both of them are negative, into $n_i - e$; and remains unchanged when one is positive and the other is negative. We summarize the three cases saying that n_i transforms into $n_i + \frac{1}{2} e\, \mathrm{sg}(n_{i-1})(\mathrm{sg}(n_{i-1} n_{i+1}) + 1)$.

(b) If $i = 1$, we have $\tau(w) = x_p^{n_1}(x_p^e x_q)^{n_2} \cdots$. Arguing as in (a), n_1 transforms into $n_1 + \frac{1}{2} e(\mathrm{sg}(n_2) + 1)$.

(c) If $1 < i < m$ and $v_i = q$, $\tau(w) = \cdots x_p^{n_{i-1}} (x_p^e x_q)^{n_i} x_p^{n_{i+1}} \cdots$. Reducing, it follows that

$$\tau(w) = \cdots x_p^{n_{i-1}+e} (x_q x_p^e)^{n_i - 1} x_q x_p^{n_{i+1}} \cdots \quad \text{for } n_i > 0,$$

and

$$\tau(w) = \cdots x_p^{n_{i-1}} x_q^{-1} (x_p^{-e} x_q^{-1})^{-(n_i+1)} x_p^{n_{i+1} - e} \cdots \quad \text{for } n_i < 0.$$

This means that, for each $1<i<m$ verifying $v_i=q$, the symbol $x_p^{\mp e}$ appears $|n_i|-1$ times in the reduced form of $\tau(w)$.

(d) If $i=m$, $\tau(w)=\cdots x_p^{n_{m-1}}(x_p^e x_q)^{n_m}$. As in the above case, we see that the symbol $x_p^{\mp e}$ appears $|n_m|-\frac{1}{2}(\text{sg}(n_m)+1)$ times in the reduced form of $\tau(w)$.

Now, we calculate $L_p(\tau(w))$ using that $|n+t|=|n|+\text{sg}(n)\cdot t$ for integers $n\neq 0$ and $|t|\leqslant 1$.

$$\begin{aligned}
L_p(\tau(w)) &= \sum_{\substack{1<i<m\\ v_i=p}} |n_i+\tfrac{1}{2}e\ \text{sg}(n_{i-1})(\text{sg}(n_{i-1}n_{i+1})+1)|+|n_1+\tfrac{1}{2}e(\text{sg}(n_2)+1)|\\
&\quad+\sum_{\substack{1<i<m\\ v_i=q}}(|n_i|-1)+(|n_m|-\tfrac{1}{2}(\text{sg}(n_m)+1)\\
&=\sum_{\substack{1<i<m\\ v_i=p}}|n_i|+\tfrac{1}{2}e\ \text{sg}(n_{i-1}n_i)(\text{sg}(n_{i-1}n_{i+1})+1)\\
&\quad+|n_1|+\tfrac{1}{2}e\ \text{sg}(n_1)(\text{sg}(n_2)+1)+\sum_{\substack{1<i<m\\ v_i=q}}(|n_i|-1)\\
&\quad+(|n_m|-\tfrac{1}{2}(\text{sg}(n_m)+1))\\
&=L_p(w)+L_q(w)+e\cdot c_p(w)-\tfrac{1}{2}(m-1)+\tfrac{1}{2}(e\cdot\text{sg}(n_1)-\text{sg}(n_m)).
\end{aligned}$$

Analogously, if τ' is a T-transformation of the type $x_p\mapsto x_p$ and $x_q\mapsto x_q x_p^e$, $e=\pm1$, we have

$$L_p(\tau'(w))=L_p(w)+L_q(w)+e\cdot c_p(w)-\tfrac{1}{2}(m-1)+\tfrac{1}{2}(-e\cdot\text{sg}(n_1)+\text{sg}(n_m)).$$

Suppose, first, that w is minimal. Since L_q remains unchanged when we apply τ or τ', we have $L_p(\tau(w))\geqslant L_p(w)$ and $L_p(\tau'(w))\geqslant L_p(w)$. Replacing in the expressions found for L_p, we obtain

$$L_q(w)+e\cdot c_p(w)-\tfrac{1}{2}(m-1)\geqslant\tfrac{1}{2}(-e\cdot\text{sg}(n_1)+\text{sg}(n_m))$$

and

$$L_q(w)+e\cdot c_p(w)-\tfrac{1}{2}(m-1)\geqslant\tfrac{1}{2}(e\cdot\text{sg}(n_1)-\text{sg}(n_m)).$$

Note that $c_p(w)\neq 0$, since $c_p(w)$ is not an integer. Then, putting $e=-\text{sg}(c_p(w))$, we get:

$$L_q(w)-|c_p(w)|-\tfrac{1}{2}(m-1)\geqslant\tfrac{1}{2}(\text{sg}(n_1)\text{sg}(c_p(w))+\text{sg}(n_m))$$

and

$$L_q(w)-|c_p(w)|-\tfrac{1}{2}(m-1)\geqslant-\tfrac{1}{2}(\text{sg}(n_1)\text{sg}(c_p(w))+\text{sg}(n_m)).$$

Thus,

$$L_q(w)-|c_p(w)|-\tfrac{1}{2}(m-1)\geqslant\tfrac{1}{2}|\text{sg}(n_1)\text{sg}(c_p(w))+\text{sg}(n_m)|,$$

as was to be proved.

Conversely, let τ be the T-transformation $x_p \mapsto x_p$ and $x_q \mapsto x_p^e x_q x_p^{-e}$. Then

$$\tau(w) = x_p^{n_1+e}\left(\prod_{2\leqslant i\leqslant m} x_{v_i}^{n_i}\right) x_p^{-e},$$

and taking lengths

$$\begin{aligned} L(\tau(w)) &= |n_1+e|+1+\sum_{2\leqslant i\leqslant m}|n_i| \\ &= |n_1| + e\cdot \mathrm{sg}(n_1)+1+\sum_{2\leqslant i\leqslant m}|n_i| \\ &= L(w)+(1+e\cdot\mathrm{sg}(n_1)) \geqslant L(w). \end{aligned}$$

Then, it suffices to prove that the length of w does not decrease for T-transformations of the type $x_p \mapsto x_p$ and $x_q \mapsto x_p^e x_q$ or $x_q \mapsto x_q x_p^e$.

Suppose that $\tau(x_q) = x_p^e x_q$. From the formula found for $L_p(\tau(w))$, we obtain

$$\begin{aligned} L(\tau(w)) - L(w) &= L_p(\tau(w)) - L_p(w) \\ &= L_q(w) + e\cdot c_p(w) - \tfrac{1}{2}(m-1) + \tfrac{1}{2}(e\cdot\mathrm{sg}(n_1) - \mathrm{sg}(n_m)) \\ &\geqslant |c_p(w)| + e\cdot c_p(w) + \tfrac{1}{2}|\mathrm{sg}(n_1)\mathrm{sg}(c_p(w)) + \mathrm{sg}(n_m)| \\ &\quad + \tfrac{1}{2}(e\cdot\mathrm{sg}(n_1) - \mathrm{sg}(n_m)), \qquad \text{by our assumption.} \end{aligned}$$

If $e = \mathrm{sg}(c_p(w))$, we have

$$\begin{aligned} L(\tau(w)) - L(w) &\geqslant 2|c_p(w)| + \tfrac{1}{2}|\mathrm{sg}(n_1)\mathrm{sg}(c_p(w)) + \mathrm{sg}(n_m)| \\ &\quad + \tfrac{1}{2}(\mathrm{sg}(n_1)\mathrm{sg}(c_p(w)) - \mathrm{sg}(n_m)) \geqslant 0, \end{aligned}$$

because

$$|c_p(w)| \geqslant \tfrac{1}{2} \quad \text{and} \quad \tfrac{1}{2}(\mathrm{sg}(n_1)\mathrm{sg}(c_p(w)) - \mathrm{sg}(n_m)) \geqslant -1.$$

If $e = -\mathrm{sg}(c_p(w))$,

$$\begin{aligned} L(\tau(w)) - L(w) &\geqslant \tfrac{1}{2}|\mathrm{sg}(n_1)\mathrm{sg}(c_p(w)) + \mathrm{sg}(n_m)| \\ &\quad - \tfrac{1}{2}(\mathrm{sg}(n_1)\mathrm{sg}(c_p(w)) + \mathrm{sg}(n_m)) \geqslant 0. \end{aligned}$$

The case $\tau(x_q) = x_q x_p^e$ is similar.

Example. The word w is minimal in each of the following cases:

(i) $|n_i| \geqslant 2$ $(1 \leqslant i \leqslant m)$.

(ii) $\mathrm{sg}(n_i) = \mathrm{sg}(n_j)$ $(1 \leqslant i, j \leqslant m)$ and $L_p(w) \geqslant m$ for $p = 1, 2$.

(iii) $\mathrm{sg}(n_i) \neq \mathrm{sg}(n_{i+1})$ $(1 \leqslant i \leqslant m-1)$ and $L_p(w) \geqslant m$ for $p = 1, 2$.

Minimality of words in free groups is related with the following question, posed by Johnson and Wall for Poincaré duality [2, Problem 4, p. 597]: which words define (one relator) duality groups? In fact, a one relator group defined by a minimal word, involving all generators, which is not a proper power, satisfies cohomological duality in dimension two. This is merely the conjunction of the following two known facts:

(i) ([1, Example 1, p. 121]). If $G = \langle F, w\rangle$ and w is not a proper power in F, then G is a two dimensional duality group if, and only if, G is freely indecomposable and non-cyclic.

(ii) ([3, p. 107]). With the notation above, G is such a group if, and only if, any minimal form of w involves all generators.

Therefore, we obtain:

Corollary. *If $G = \langle x, y;\ w(x, y)\rangle$, such that w is not a proper power in F and satisfies the assumptions of the theorem, then G is a duality group of dimension two.*

Example. (cf. [1, example 2, p. 121]). The group

$$G = \langle x, y;\ yxy^{-1}x^{-2}\rangle$$

is a two dimensional duality group, because the word $x_2x_1x_2^{-1}x_1^{-2}$ is minimal. For, in this case, we have $m = 4$, $c_1(w) = c_2(w) = \frac{1}{2}$, and then,

$$\tfrac{1}{2}(m-1) + |c_1(w)| + \tfrac{1}{2}|\mathrm{sg}(n_1)\mathrm{sg}(c_1(w)) + \mathrm{sg}(n_4)| = \tfrac{3}{2} + \tfrac{1}{2} = 2 \leqslant L_2(w) = 2$$

$$\tfrac{1}{2}(m-1) + |c_2(w)| + \tfrac{1}{2}|\mathrm{sg}(n_1)\mathrm{sg}(c_2(w)) + \mathrm{sg}(n_4)| = 2 \leqslant L_1(w) = 3.$$

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