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VECTOR FIELDS ON R^R IN WELL ADAPTED MODELS OF SYNTHETIC DIFFERENTIAL GEOMETRY

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Introduction

Vector fields in infinite-dimensional manifolds play an important role in differential topology-geometry. In particular the case when the manifolds are C^∞-map spaces. The well developed theory modeled in Banach spaces does not apply here. Instead a theory modeled in Frechet spaces is being considered. This is a theory which seems to be a much less straightforward generalization of the finite-dimensional case. The well adapted models of S.D.G. lead naturally to treat these spaces. We investigate here the case of the 'manifold' R^R, whose space of global sections is $C^\infty(\mathbb{R})$. We prove that to integrate a vector field in R^R is equivalent to a certain differential problem in $C^\infty(\mathbb{R})$. To do this, we previously characterize the maps $R^R \to R^R$, $R^R \times R \to R^R$ in the topos by means of the functions they induce in the respective spaces of global sections.

Let $\mathscr{B}$ denote the category dual to that of finitely generated C^∞-rings $C^\infty(\mathbb{R}^n)/I$ presented by an ideal of local character. We recall that $I \subseteq C^\infty(\mathbb{R}^n)$ is of local character (or of local nature) if, $f \in C^\infty(R^n)$ implies: $f \in I$ iff there exists an open covering U_α of R^n such that $f|_{U_\alpha} \in I|_{U_\alpha}$ = ideal generated by $\{h|_{U_\alpha} : h \in I\}$ in $C^\infty(U_\alpha)$. We consider in $\mathscr{B}$ the open cover (Grothendieck) topology that defines the Dubuc topos, which we denote $\mathscr{D}$ (see [2–4]). This topology is sub-canonical, i.e. $\mathscr{B} \subseteq \mathscr{D}$. Call $R = \overline{C^\infty(\mathbb{R})} \in \mathscr{B} \subseteq \mathscr{D}$. R is a ring object of line type called the line. Let D denote the subobject of R which consists of those elements of square zero:

$$D = [\![x \in R : x^2 = 0]\!] = \overline{C^\infty(\mathbb{R})/(x^2)}.$$

As is well known, a vector field over a 'manifold' M is, from the Synthetic Differential Geometry point of view, an arrow

$$v : M \times D \to M$$

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such that $v(x,0)=x$. To integrate such a vector field means to get an arrow

$$u=M\times R\to M$$

that makes the following diagram commutative:

$$\begin{tikzcd} & M\times R \arrow[dr, "u"] & \\ M\times R\times D \arrow[ur, "(\mathrm{id}_M\times +)"] \arrow[dr, "(u\times \mathrm{id}_D)"'] & & M \\ & M\times D \arrow[ur, "v"'] & \end{tikzcd}$$

and verifying $u(x,0)=x$. In other words, an arrow u such that

1. $$\begin{cases} u(x,t+d)=v(u(x,t),d), \\ u(x,0)=x. \end{cases}$$

We will deal here with this problem in the case that the 'manifold' is $M=R^R$.

The question of existence and uniqueness in the case $M=R$ is easily solved since it is equivalent to a problem of ordinary differential equations. As a matter of fact, an arrow

$$R\times D\xrightarrow{v}R$$

corresponds (in a well known way) to a $v\in C^\infty(\mathbb{R}^2)/(y^2)$, that is, v_1, v_2, $\in C^\infty(\mathbb{R})$ such that $v=v_1(x)+v_2(x)[y]$. Since $v(x,0)=x$, $v_1(x)=x$, thus

$$v=x+v_2(x)\cdot[y].$$

On the other hand, u corresponds to a $u\in C^\infty(\mathbb{R}^2)$. The validity of eqs. 1 is equivalent to

2. $$\begin{cases} u'(x,t)=v_2(u(x,t)), \\ u(x,0)=x \end{cases}$$

where u' means differentiation with respect to the time t. In this situation we know that a local solution exists: The Cauchy theorem of existence and smoothness with respect to both variables implies that there exists an open neighborhood U, $\mathbb{R}\times\{0\}\subseteq U\subseteq\mathbb{R}\times\mathbb{R}$ and u defined in U verifying eqs. 2. It follows that there is a Penon open U, $R\times\{0\}\subseteq R\times R$ in $\mathscr{D}$, and a $v:U\to R$ verifying eqs. 1 (see [6, 1]).

In order to analyze the case $M=R^R$ we get a correspondence similar to the one described above for arrows $\mathbb{R}\times D\to R$, etc., but this time for arrows of the type

$$R^R\times D\to R^R,\qquad R^R\times R\to R^R,\qquad R^R\times R\times D\to\mathbb{R}^{\mathbb{R}}.$$

Since R (and so R^R) is of line type, i.e. $\mathbb{R}^D=R\times R$ (see [2–4]), it suffices to study arrows of the type

$$R^R\to R^{R^p}\quad(p\in\mathbb{N}).$$

Let us take a map $U: R^R \to R^{R^p}$, and let Γ be the global sections functor: $\Gamma(F) = \mathrm{Hom}(1, F)$. Since $\Gamma(R^R) = C^\infty(\mathbb{R})$, $\Gamma(R^{R^p}) = C^\infty(\mathbb{R}^p)$, Γ induces a function (in general not a morphism)

$$\Gamma(u): C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^p)$$

such that for $h \in C^\infty(\mathbb{R})$, $h: 1 \to R^R$, the diagram

$$\begin{tikzcd} & R^R \arrow[r] & R^{R^p} \\ 1 \arrow[ur, "h"] \arrow[urr, "\Gamma(u)(h)"'] & & \end{tikzcd}$$

commutes. Now take a function

$$G: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^p).$$

We want to see that, if G is 'conveniently smooth' it 'comes from the topos', i.e., $G = \Gamma(u)$ for some u that we will show to be unique.

3. Definition. (i) A function $c: \mathbb{R}^k \to C^\infty(\mathbb{R}^p)$ is said to be C^∞ iff the function

$$c(t_1, \ldots, t_k)(x_1, \ldots, x_p)$$

is a $(k+p)$-variables C^∞-function.

(ii) A function $G: C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^p)$ is said to be C^∞ iff for each $k \in \mathbb{N}$ and for every $c: \mathbb{R}^k \to C^\infty(\mathbb{R}^n)$ C^∞ (in the sense of (i)) $G \circ c$ is C^∞ (in the sense of (i)).

Frölicher has already worked on C^∞-functions in the sense of (i). (He calls them path-smooth. See [5].)

Some examples of C^∞-functions $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^p)$ are the following:

(1) Linear differential operators with C^∞-coefficients.

(2) Integral operators with a C^∞-kernel and of compact support.

(3) Morphisms of C^∞-rings induced by C^∞-mappings $l: \mathbb{R}^p \to \mathbb{R}^n$ by composition.

(4) Maps of the form $h_*: C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ induced by a C^∞-mapping $h: \mathbb{R} \to \mathbb{R}$.

Moreover, any composite of C^∞-functions is a C^∞-function, thus any operator constructed using (1), (2), (3) and (4) is so.

Recall now that for any ideal $I \subseteq C^\infty(\mathbb{R}^n)$ there exists the local nature closure $\hat{I}$ of I, i.e., the smallest local nature ideal which contains I. In fact, $f \in \hat{I}$ iff there exists an open covering U_α of $\mathbb{R}^n$ such that $f|_{U_\alpha} \in I|_{U_\alpha}$.

We adopt now the following

4. Notation. (i) $C^\infty(\mathbb{R}^n)$ will be understood as the ring of all C^∞-mappings of the variable $t = (t_1, \ldots, t_n)$ and $C^\infty(\mathbb{R}^{n+p})$ as the ring of C^∞-mappings of the variables $(t, x) = (t_1, \ldots, t_n, x_1, \ldots, x_p)$.

(ii) Let $I \subseteq C^\infty(\mathbb{R}^n)$ be an ideal of local character. We will denote $I(t) = I$ and $\hat{I}(t, x) =$ local nature closure of the ideal generated by I in $C^\infty(\mathbb{R}^{n+p})$.

(Notice that $I \subseteq C^\infty(\mathbb{R}^{n+p})$ since an n-variables C^∞-function is an $(n+p)$-variables C^∞-function which does not depend on the last p variables.)

If $u(t,s) \in C^\infty(\mathbb{R}^{n+1})$ and $G: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^p)$, for any fixed $t \in \mathbb{R}^n$ we may compute G on u considered as a function of s:

$$G(u(t,-)) \in C^\infty(\mathbb{R}^p).$$

Since G is C^∞, we actually have

$$G(u(t,-))(x) \in C^\infty(\mathbb{R}^{n+p})$$

as a function of (t,x). This kind of abuse of notation will frequently occur, so we warn the reader to be aware of the context in order to avoid misunderstandings. Of course, $C^\infty(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}^p))$ is the set of all C^∞-functions $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^p)$.

5. Theorem. *Γ induces a bijection*

$$\Gamma: \mathrm{Hom}(R^R, R^{R^p}) \to C^\infty(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}^p)).$$

Proof. Let us first show that if $u \in \mathrm{Hom}(R^R, R^{R^p})$, then $\Gamma(u)$ is C^∞, i.e., if $c: \mathbb{R}^n \to C^\infty(\mathbb{R})$ is C^∞, then $\Gamma(u)(c(t))(x) \in C^\infty(\mathbb{R}^{n+p})$. Since $c(t)(s) \in C^\infty(\mathbb{R}^{n+1})$, it defines a $\hat{c}: R^n \to R^R$. We have $\Gamma(u) \circ \Gamma(u \circ \hat{c}): \mathbb{R}^n \to C^\infty(\mathbb{R}^p)$. At this point we need the following.

6. Lemma. *If $w: R^n \to R^{R^p}$, then $\Gamma(w): \mathbb{R}^n \to C^\infty(\mathbb{R}^p)$ equals the exponential adjoint to w, say $\tilde{w} \in C^\infty(\mathbb{R}^{n+p})$ (regarded as $(n+p)$-variables functions).*

Proof. Straightforward, using naturality of exponential adjunction. □

Returning to the proof of Theorem 5, $\Gamma(u \circ \hat{c})$ equals the exponential adjoint of $(u \circ \hat{c})$ and then it is in $C^\infty(\mathbb{R}^{n+p})$. But also by Lemma 6, $\Gamma(\hat{c}) = c$; thus $\Gamma(u) \circ c = \Gamma(u) \circ \Gamma(\hat{c}) = \Gamma(u \circ \hat{c}) \in C^\infty(R^{n+p})$.

Now we try to define an inverse E for Γ. Let $G \in C^\infty(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}^p))$ and $Y = \overline{C^\infty(\mathbb{R}^n)}/I \in \mathscr{B} \subseteq \mathscr{D}$. We define

$$E(G)_Y: \mathrm{Hom}(Y, R^R) \to \mathrm{Hom}(Y, R^{R^p}).$$

Since $\mathrm{Hom}(Y, R^R) = \overline{C^\infty(\mathbb{R}^{n+1})}/\hat{I}(t,s)$ and $\mathrm{Hom}(Y, R^{R^p}) = C^\infty(R^{n+p})/I(t,x)$, this means to define a function

$$E(G)_Y: C^\infty(\mathbb{R}^{n+1})/\hat{I}(t,s) \to C^\infty(\mathbb{R}^{n+p})/\hat{I}(t,x).$$

We do this by the formula

$$E(G)_Y([c]) = [G(c(t,-))(x)]$$

where the brackets mean equivalence class modulo $\hat{I}(t,s)$ and $\hat{I}(t,x)$ respectively. Obviously we have to show that:

(i) The last definition is independent of the choice of c,
(ii) $E(G)$ actually is an arrow in the topos (i.e. is a natural transformation).
(iii) E is the inverse of Γ.

After we have proved (i), point (ii) reduces to a straightforward verification. As far as point (iii) is concerned, $\Gamma \circ E$ obviously equals the identity, and $E \circ \Gamma = \mathrm{id}$ is essentially Lemma 6. ($c: Y \to R^R$ extends to R^n

$$\begin{tikzcd} R^n \arrow[r] & R^R \\ Y \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

although not in a unique way.) Now, (i) is a property of C^∞-functions (naturality) which is proved in Theorem 8 below. □

Before, and in order to fix the notation, we recall the following fact:

7. Lemma. *For every $(n+p)$-variables C^∞-function $h: \mathbb{R}^{n+p} \to \mathbb{R}$ and for every integer $m \geq 0$ there exist C^∞-functions*

$$f_k \text{ of } n \text{ variables } \{k = (k_1, \ldots, k_p) \mid \textstyle\sum k_i \leq m\},$$

$$l_k \text{ of } n+p \text{ variables } \{k = (k_1, \ldots, k_p) \mid \textstyle\sum k_i = m+1\}$$

such that the equality

$$h(t, x) = \sum_k f_k(t) x^k + \sum_k l_k(t, x) x^k$$

holds for every $(t, x) = (t_1, \ldots, t_n, x_1, \ldots, x_p) \in \mathbb{R}^{n+p}$ where $x^k = x_1^{k_1}, \ldots, x_p^{k_p}$. Moreover, we have

$$f_k(t) = \frac{1}{k!} \left.\frac{\partial^{|k|} h}{\partial x^k}\right|_{(t,0)} \quad \text{where } |k| = \sum k_i, \; \frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x^{k_1}, \ldots, \partial x_p^{k_p}}.$$

8. Theorem (Naturality of C^∞-functions). *Let $I \subseteq C^\infty(\mathbb{R}^n)$ be any ideal and $G \in C^\infty(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}^p))$. Then, for $c, c' \subset C^\infty(\mathbb{R}^{n+1})$, we have:*

$$c \simeq c' \text{ modulo } (\hat{I}(t, s)) \text{ implies}$$

$$G(c(t, \text{-}))(x) \simeq G(c'(t, \text{-}))(x) \text{ modulo } (\hat{I}(t, x))$$

where t, x, s are variables ranging over $\mathbb{R}^n$, $\mathbb{R}^p$ and $\mathbb{R}$ respectively.

Proof. Recalling that $\hat{I}(t, x)$ is the local nature closure of $(I) \subseteq C^\infty(\mathbb{R}^{n+p})$, we must show that for any $(t_0, x_0) \in \mathbb{R}^{n+p}$, there exists an open neighborhood $U \ni (t_0, x_0)$ and a linear combination of elements of I with coefficients in $C^\infty(U)$ equal to the difference $G(c'(t,\text{-}))(x)-G(c(t,\text{-}))(x)$ for x, t in U. By hypothesis we know the latter is true for $c'(t,s)-c(t,s)$, i.e., if $(t_0,s_0)\in\mathbb{R}^{n+1}$ is given, there exists an open neighborhood W of (t_0,s_0), functions $f_i\in I\subseteq C^\infty(\mathbb{R}^n)$ and $A_i\in C^\infty(W)$ $(1\le i\le r)$ such that *in* W

$$c'(t,s)-c(t,s)=\sum_{i=1}^{r}A_i(t,s)f_i(t). \tag{1}$$

By taking an smaller W we can assume the functions A_i to be defined in all of $\mathbb{R}^{n+1}$, that is $A_i(t,s)\in C^\infty(\mathbb{R}^{n+1})$. The equation (1) makes sense in all of $\mathbb{R}^{n+1}$, but it holds only in W. Let $g\in C^\infty(\mathbb{R}^{n+r+1})$ be the function defined by

$$(*)\qquad g(t,\lambda,x)=G\left(c(t,\text{-})+\sum_{i=1}^{r}A_i(t,\text{-})\lambda_i\right).$$

$g\in C^\infty(\mathbb{R}^{n+r+1})$ since G is C^∞.

By Lemma 7 with $m=0$, we have

$$g(t,\lambda,x)-g(t,\mathbf{0},x)=\sum_{i=1}^{r}\lambda_iH_i(t,\lambda,x)$$

where $H_i\in C^\infty(\mathbb{R}^{n+r+p})$. If $\lambda_i=f_i(t)$, if follows that

$$G\left(c(t,\text{-})+\sum_{i=1}^{r}A_i(t,\text{-})f_i(t)\right)(x)-G(c(t,\text{-}))(x)\in I(t,x).$$

But at this point we apparently got stuck because

$$c(t,s)+\sum_{i=1}^{r}A_i(t,s)f_i(t)=c'(t,s)$$

only in W, so that, for the moment we cannot conclude what we wanted. Do not worry. We have:

$$\begin{aligned}&G(c'(t,\text{-}))(x)-G(c(t,\text{-}))(x)\\&=\left[G(c'(t,\text{-}))(x)-G\left(\sum_{i=1}^{r}A_i(t,\text{-})f_i(t)+c(t,\text{-})\right)(x)\right]\\&\quad+[G(\textstyle\sum A_i(t,\text{-})f_i(t)+c(t,\text{-}))(x)-G(c(t,\text{-}))(x)].\end{aligned}$$

The second bracket is in $\hat{I}(t,x)$ as we have shown above. We will prove two facts:

(i) Given $W\subseteq\mathbb{R}^{n+1}$ open and relatively compact there exist $f_i\in I$, $A_i\in C^\infty(\mathbb{R}^{n+1})$ such that (1) holds in W.

(ii) If (1) holds in a large enough set W open and relatively compact, then the first bracket vanishes in an open neighborhood of (t_0,x_0). Clearly this will finish the proof. □

Point (i) and a more general version of (ii) will be Lemma 9 and Corollary 15 respectively. In the proof of Corollary 15 we will need either Lemma 14 or Lemma 12 and Corollary 13.

9. Lemma. *Let $I \subseteq C^\infty(\mathbb{R}^n)$ be a local nature ideal, $c \in \hat{I}(\bar{t}, s) \subseteq C^\infty(\mathbb{R}^{n+1})$, $W \subseteq \mathbb{R}^{n+1}$ open and relatively compact. Then, there exist functions $f_i \in I$ and $A_i \in C^\infty(\mathbb{R}^{n+1})$, $1 \leq i \leq r$ such that the equality*

$$c(t, s) = \sum_{i=1}^{r} A_i(t, s) f_i(t)$$

holds in W.

Proof. Since $c \in \hat{I}(t, s)$, there exists an open covering $\{U_\alpha\}_{\alpha \in J}$ of $\mathbb{R}^{n+1}$, $f_i \in I$, $A_i^\alpha \in C^\infty(\mathbb{R}^{n+1})$, $1 \leq i \leq r_\alpha$ such that the equality

$$c(\bar{t}, s) = \sum_{i=1}^{r_\alpha} A_i^\alpha(t, s) f_i^\alpha(t)$$

holds in U_α. We may assume U_α locally finite with a subordinate partition of unity φ_α. We have the equality

$$\varphi_\alpha \cdot c = \sum_{i=1}^{r_\alpha} (A_i^\alpha \varphi_\alpha) f_i^\alpha$$

which holds in all of $\mathbb{R}^{n+1}$. Since $\bar{W}$ is compact, the set J_0 of indices such that $\operatorname{supp}(\varphi_\alpha) \cap W \neq \emptyset$ is finite. Thus, in W we have

$$c = \sum_{\alpha \in J_0} \varphi_\alpha \cdot c = \sum_{\alpha \in J_0} \sum_{i=1}^{r_\alpha} A_i^\alpha \sigma_\alpha f_i^\alpha. \qquad \square$$

10. Definition. Recall that a sequence $\{f_k\}_{k \in N} \subseteq C^\infty(\mathbb{R}^n)$ is said to converge to $f \in C^\infty(\mathbb{R}^n)$ in the *C^∞-compact open topology* (C^∞-CO) iff for every compact set K, $d \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $|D^\alpha f_k - D^\alpha f| < \varepsilon$ in K for all α such that $|\alpha| \leq d$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ and $|\alpha| = \sum \alpha_i$.

11. Definition. We will say that a sequence $\{f_k\}_{k \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^n)$ converges to $f \in C^\infty(\mathbb{R}^n)$ in the *Stone topology* iff for every compact set $K \leq \mathbb{R}^n$ there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $f_k = f$ in K.

As an example, notice that Lemma 9 says exactly that $f \in \hat{I}(t, s)$ iff there exists a sequence of linear combinations of elements of I Stone-converging to f. On the other hand, with the same idea as in Lemma 9 one may easily prove that the local nature closure of any ideal equals to its closure in the Stone topology. Then, *I is of local nature iff it is closed in the Stone topology,* as was noted by J. Penon.

12. Lemma (Glueing Lemma, see [7, lemma below Theorem 3]). *If $\{f_k\}_{k \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^n)$ converges to f in the C^∞-CO topology, then, there exists a subsequence f_{k_r} and $F \in C^\infty(\mathbb{R}^{n+1})$ such that*

$$F(t, 1/r) = f_{k_r}; \qquad F(t, s) = f(t) \quad \textit{if } s < 0.$$

In other words, if $f_k \to f$ in the C^∞-CO topology, then there exists a subsequence f_{k_r} and a C^∞-curve $F: \mathbb{R} \to C^\infty(\mathbb{R}^n)$ that passes along f_{k_r} at $s = 1/r$ and $F = f$ for $s < 0$.

Proof. We may assume that $f = 0$. Let $K_r = [-r, r]^n \subseteq \mathbb{R}$ be the n-dimensional closed cube. By hypothesis, there exists k_r such that

$$|D^\alpha f_{k_r}| < e^{-r} \quad \text{in } K_r \ \forall \alpha \text{ s.t. } |\alpha| \le r.$$

Now take $\varphi \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\{\varphi\} \subseteq (-1, 1)$ and $\varphi(0) = 1$. Let us write $\varphi_r(s) = \varphi(2r(r+1)(s - 1/r))$.

$$\operatorname{supp} \varphi_r \subseteq \left(\frac{1}{r} - \frac{1}{2r(r+1)}, \frac{1}{r} + \frac{1}{2r(r+1)}\right) \quad \text{and} \quad \varphi_r(1/r) = 1.$$

Let us define

$$F(t, s) = \begin{cases} f_{k_r}(t) \cdot \varphi_r(s) & \text{if } s \in \operatorname{supp}(\varphi_r), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $F(t, s) = f_{k_r}$ at $s = 1/r$. The reader may easily check that $F \in C^\infty(\mathbb{R}^{n+1})$. $\square$

13. Corollary. *Any C^∞-function $G: C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^p)$ is continuous in the C^∞-CO topology.*

Proof. Suppose $\{f_k\}_{k \in \mathbb{N}} \subseteq C^\infty(\mathbb{R}^n)$ converges to $f \in C^\infty(\mathbb{R}^n)$ in the C^∞-CO topology and that $G(f_k)$ does not C^∞-CO converge to $G(f)$. This means that there exist a compact set $K \subset \mathbb{R}^p$, $\varepsilon > 0$, $\alpha \in (\mathbb{N} \cup \{0\})^p$, a subsequence f_{k_l} and $\bar{x}_l \in K$ such that

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} G(f_{k_l})(x_l) - \frac{\partial^{|\alpha|}}{\partial x^\alpha} G(f)(x_l) \right| \ge \varepsilon.$$

By compactness, we may assume $x_l \to x_0$ for some $x_0 \in K$. Now, by Lemma 12, there exists a subsequence, that we also call f_{k_l} and $F \in C^\infty(\mathbb{R}^{n+1})$ such that $F(t, 1/l) = f_{k_l}(t)$, $F(t, s) = f(t)$ for $s < 0$. For any fixed $\theta \in \mathbb{R}$ we may compute $G(F(-, \theta))(x)$ and since G is C^∞, we get a C^∞-function of both variables θ and $\bar{x}$. Thus

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} G(F(-, 1/l))(x_l) \xrightarrow[l \to \infty]{} \frac{\partial^{|\alpha|}}{\partial x^\alpha} G(F(-, 0))(x_0).$$

But this is a contradiction. $\square$

14. Lemma (Reordering Lemma). *Let $\{f_k\}_{k=1}^\infty \subseteq C^\infty(\mathbb{R}^n)$. Then $\{f_k\}$ Stone-converges to f iff for every subsequence f_{k_l} and for any decreasing sequence $\xi_l \to 0$, $\xi_l \in \mathbb{R}$, there exists an $F \in C^\infty(\mathbb{R}^{n+1})$ such that $F(t, \xi_l) = f_{k_l}(t)$ and $F(t, s) = f(t)$ for $s \le 0$.*

Proof. We may assume $f = 0$. Suppose f_k Stone-converges to 0. Take a subsequence f_k and a decreasing sequence $\xi_l \to 0$ in $\mathbb{R}$. Consider functions $\varphi_l \in C^\infty(\mathbb{R})$ such that

$\varphi_l(\xi_l)=1$, with $\operatorname{supp}(\varphi_l)\cap\operatorname{supp}(\varphi_j)=\emptyset$ for $l\neq j$, and

$$\operatorname{supp}(\varphi_l)\subseteq\left(1/l-\frac{1}{2l(l+1)},1/l+\frac{1}{2l(l+1)}\right).$$

Let

$$F(t,s)=\begin{cases}\varphi_l(s)\cdot f_{k_l}(t) & \text{if } t\in\operatorname{supp}(\varphi_l),\\ 0 & \text{otherwise.}\end{cases}$$

Since f_k Stone-converges to 0, it is trivial to see that $F(t,s)\in C^\infty(\mathbb{R}^{n+1})$.

Conversely, suppose that f_k does not Stone-converge to 0, i.e., there exists a compact set K and a subsequence f_{k_l} such that f_{k_l} does not vanish identically in K. Let $\xi_l=\max_{t\in K}|f_{k_l}(t)|$. Since by hypothesis $\{f_k\}_{k=1}^\infty$ may be glued with an F, we may assume ξ_l converges decreasingly to 0 (taking another subsequence if necessary). Let us take t_l a point of K such that $|f_{k_l}(t_l)|=\xi_l\neq 0$. By compactness we may assume $t_l\to t_0$ for some $t_0\in K$ (again, taking another subsequence if needed). By hypothesis, there exists $F\in C^\infty(\mathbb{R}^{n+1})$ such that $F(t,\xi_l)=f_{k_l}(t)$, $F(t,s)=0$ for $s\leq 0$. Then

$$\frac{F(t_l,\xi_l)-F(t_l,0)}{\xi_l}=\pm\frac{\xi_l-0}{\xi_l}=\pm 1$$

and by Lagrange's mean value theorem, there exists a real number θ_l, $0<\theta_l<\xi_l$ such that

$$\frac{F(t_l,\xi_l)-F(t_l,0)}{\xi_l}=\left.\frac{\partial F}{\partial s}\right|_{(t_l,\theta_l)}$$

Then $(\partial F/\partial s)|_{(t_l,\theta_l)}=\pm 1$. Now, since $(t_l,\theta_l)\to(t_0,0)$ and since $F\in C^\infty(\mathbb{R}^{n+1})$, we get $(\partial F/\partial s)|_{(t_0,0)}=\pm 1$. But since $F=0$ for $s<0$, we actually have $(\partial F/\partial s)|_{(t_0,0)}=0$ which is a contradiction. □

15. Corollary. *Any C^∞-function $G:C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^p)$ is continuous in the Stone-topology.*

Proof. Suppose $\{f_k\}_{k=1}$ Stone-converges to f, but $G(f_k)$ does not Stone-converge to $G(f)$. Let us apply Lemma 14 to $G(f_k)$: there exists a decreasing sequence $\xi_l\to 0$, a subsequence f_{k_l} of f_k such that no $F\in C^\infty(\mathbb{R}^{p+1})$ verifies $F(x,\xi_l)=G(f_{k_l})(x)$, $F(x,s)=G(f)(x)$ for $s\leq 0$. But again by Lemma 14, since f_k Stone converges to f, there exists $H\in C^\infty(\mathbb{R}^{n+1})$ such that $H(t,\xi_l)=f_{k_l}(t)$, $H(t,s)=f(t)$ for $s\leq 0$. Let $F(x,s)=G(H(-,s))(x)$, we have: $F(x,\xi_l)=G(f_{k_l})(x)$, $F(x,s)=G(f)(x)$ for $s\leq 0$ and $F(x,s)\in C^\infty(\mathbb{R}^{p+1})$, which is a contradiction. □

Let us present an alternative proof of Corollary 15.

(i) *About the differential of a C^∞-function.* Let us define *the differential*

$$dG:C^\infty(\mathbb{R}^n)\times C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^p)$$

by

$$\mathrm{d}G_h(k)(\bar{x}) = \lim_{\lambda \to 0} \frac{G(h+\lambda k)(\bar{x}) - G(h)(\bar{x})}{\lambda}.$$

Since G is C^∞, we know that this limit exists, and, setting $H(\lambda, \boldsymbol{x}) = G(h+\lambda k)(\boldsymbol{x})$ it follows from Lemma 7 that $\mathrm{d}G_h(k) \in C^\infty(\mathbb{R}^p)$. Similarly, it may be seen that $\mathrm{d}G$ is a C^∞-function of both variables h and k (i.e. preceded by a pair of C^∞-curves as in Definition 3(i) it yields a C^∞-curve) and then, with the same proof as the one given for Corollary 13, it follows that $\mathrm{d}G$ is a C^∞-CO continuous mapping from $C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^p)$. Moreover $\mathrm{d}G_h : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^p)$ is $\mathbb{R}$-linear. To see this take $k, k' \in C^\infty(\mathbb{R}^n)$, consider the function

$$K(\lambda, \mu, \boldsymbol{x}) = G(h, \lambda k + \mu k')$$

and use Lemma 7 with $m = 0$.

The interested reader can see [5] where the definition of $\mathrm{d}G$ as well as some of the result developed here in point (i) can be found. However the context is much more general and the proofs to be found there less elementary.

(ii) *Alternative proof of Corollary* 15. Take $f, h \in C^\infty(\mathbb{R}^n)$ and let $H(\lambda, \bar{x}) = G(f+\lambda h)(\bar{x})$. By Lagrange's mean value theorem, we have

$$\begin{aligned} G(f+h)(\boldsymbol{x}) - G(f)(\boldsymbol{x}) &= H(1, \boldsymbol{x}) - H(0, \boldsymbol{x}) \\ &= \frac{\partial H}{\partial \lambda}\bigg|_{(\xi, \boldsymbol{x})} = \mathrm{d}G_{f+\xi h}(h)(\boldsymbol{x}) \quad \text{for some } 0<\xi<1,\ \xi = \xi(\boldsymbol{x}). \end{aligned}$$

For $l \in C^\infty(\mathbb{R}^k)$, $K \subseteq \mathbb{R}^k$ compact, $r \in \mathbb{N} \cup \{0\}$ denote

$$\|l\|_K^r = \sup_{\substack{|\alpha| \le r \\ \boldsymbol{x} \in K}} \left| \frac{\partial^{|\alpha|}}{\partial \boldsymbol{x}^\alpha} l(\boldsymbol{x}) \right|.$$

Since $\mathrm{d}G$ is C^∞-CO continuous, we have that, given $\varepsilon > 0$ and a compact set $K' \subseteq \mathbb{R}^p$ there exist $\delta > 0$, a compact set $K = K(\varepsilon, K')$ (that may be assumed to be an n-cube) and $r \in \mathbb{N} \cup \{0\}$ such that $\|h\|_K^r < \delta$ and $\|l - f\|_K^r < \delta$ imply $\|\mathrm{d}G_l(h)\|_{K'}^0 < \varepsilon$. Now, if h vanishes in the cube $K = K(\varepsilon, K')$, then $\|h\|_K^r = 0 < \delta$ and $\|f + \xi h - f\|_K^r = 0 < \delta$ for all ξ. Thus, if h vanishes in K, we have

$$\lambda \|\mathrm{d}G_{f+\xi h}(h)\|_{K'}^0 = \|\mathrm{d}G_{f+\xi h}(\lambda h)\|_{K'}^0 < \varepsilon \quad \text{for all } \xi, \lambda.$$

It follows that $\|\mathrm{d}G_{f+\lambda h}(h)\|_{K'}^0 = 0$ for all ξ, so $G(f+h)(\boldsymbol{x}) - G(f)(\boldsymbol{x}) = \mathrm{d}G_{f+\xi h}(h)(\boldsymbol{x})$ vanishes for $\boldsymbol{x} \in K'$.

In short: given a compact set $K' \subseteq \mathbb{R}^p$ and $f \in C^\infty(\mathbb{R}^n)$ there exists a compact set $K \subseteq \mathbb{R}^n$ such that $h|_K = 0$ implies $G(f+h)|_{K'} = G(f)|_{K'}$.

This exactly means that G is Stone-continuous. □

Remark. Theorem 5 says that an arrow $R^R \to R^{R^p}$ in the topos is actually the same thing as a C^∞-function $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^p)$.

At this point we are able to deal with the main problem presented at the start. Setting $M=R^R$, and given an arrow

$$v: R^R \times D \to R^R$$

such that $v(h,0)=h$, we must find $u: R^R \times R \to R^R$ which makes the following diagram commutative:

$$\begin{tikzcd} & R^R \times R \arrow[dr, "u"] & \\ R^R \times R \times D \arrow[ur, "(\mathrm{id}_R \times +)"] \arrow[dr, "(u \times \mathrm{id}_D)"'] & & R^R \\ & R^R \times D \arrow[ur, "v"'] & \end{tikzcd} \qquad (*)$$

and verifying $u(h,0)=h$. In other words, an arrow such that

$$\begin{cases} u(h,t+d)=v(u(h,t),d), \\ u(h,0)=h. \end{cases} \qquad (1)$$

Let us explore what the commutativity of diagram $(*)$ means. Notice that since $R^D \simeq R \times R$, $(R^R)^D \simeq R^R \times R^R$, we have that the exponential adjoint to v, say $\tilde{v}: R^R \to (R^R)^D \simeq R^R \times R^R$ is a pair (v_1, v_2) of arrows $R^R \to R^R$. Call $\tilde{u}: R^R \to R^{R \times R}$ the exponential adjoint of u.

16. Lemma. (i) *If $h: 1 \to R^R$, then the exponential adjoint of the composite map*

$$R \times D = 1 \times R \times D \xrightarrow{(h \times \mathrm{id}_{R \times D})} R^R \times R \times D \xrightarrow{(u \times \mathrm{id}_D)} R^R \times D \xrightarrow{\quad v \quad} R^R$$

(getting R down) is the map $R \times R \times D \to R$ that corresponds to

$$\Gamma(v_1)(\Gamma(\tilde{u})(h)(-,y))(x) + \Gamma(v_2)(\Gamma(\tilde{u})(h)(-,y))(x) \cdot [z] \in C^\infty(\mathbb{R}^\infty)/(Z^2).$$

(ii) *If $h: 1 \to R^R$, then the exponential adjoint of the composite map*

$$R \times D = 1 \times R \times D \xrightarrow{(h \times \mathrm{id}_{R \times D})} R^R \times R \times D \xrightarrow{(\mathrm{id}_R \times +)} R^R \times R \xrightarrow{\quad u \quad} R^R \qquad (2)$$

(getting R down) is the map $R \times R \times D \to R$ that corresponds to

$$\Gamma(\tilde{u})(h)(x,y) + \frac{\partial \Gamma(\tilde{u})(h)}{\partial y}(x,y) \cdot [z] \in C^\infty(\mathbb{R}^3)/(Z^2).$$

(iii) *The condition $v(h,0)=h$ is equivalent to $v_1 = \mathrm{id}_{R^R}$, or, by Theorem 5, to $\Gamma(v_1) = \mathrm{id}_{C^\infty(\mathbb{R})}$.*

(iv) *The condition $u(h,0)=h$ is equivalent to $\Gamma(\tilde{u})(h)(x,0)=h(x)$.*

Proof. As an illustration we prove (ii). The other statements follow in a similar way. The arrow (2) equals the composite

$$R\times D \xrightarrow{+} R \xrightarrow{(h\times \mathrm{id}_R)} R^R\times R \xrightarrow{u} R^R. \tag{3}$$

Now the exponential adjoint of the composite

$$R \xrightarrow{(h\times \mathrm{id}_R)} R^R\times R \xrightarrow{u} R^R$$

is

$$1 \xrightarrow{h} R^R \xrightarrow{\tilde{u}} R^{R\times R}$$

or, in other words, $\Gamma(\tilde{u})(h)$. Then, getting $R\times R$ down, we obtain precisely $\Gamma(\tilde{u})(h)(x,y)\in C^\infty(\mathbb{R}^2)$. So, the exponential adjoint of (3) corresponds to

$$\Gamma(\tilde{u})(h)(x, y+[z])\in C^\infty(\mathbb{R}^3)/(Z^2).$$

Now, by Lemma 7, since $[Z]^2=0$, this equals to

$$\Gamma(\tilde{u})(h)(x,y)+\frac{\partial(\Gamma(\tilde{u})(h))}{\partial y}(x,y)\cdot[z]. \qquad \square$$

17. Theorem. (i) *Let* $v: R^R\times D\to R^R$ *be such that* $v(h,0)=h$ *and* $u: R^R\times R\to R^R$. *Then we have*

$$\text{(1)} \qquad \begin{cases} u(h, t+d)=v(u(h,t),d), \\ u(h,0)=h. \end{cases}$$

iff

$$\begin{cases} \dfrac{\partial\Gamma(\tilde{u})(h)}{\partial y}(x,y)=\Gamma(v_2)(\Gamma(\tilde{u})(h)(-,y))(x), \\ \Gamma(\tilde{u})(h)(x,0)=h(x). \end{cases}$$

(ii) *Conversely, let* $U: C^\infty(\mathbb{R})\to C^\infty(\mathbb{R}^2)$ *be a* C^∞*-function which satisfies the differential problem*

$$\text{(2)} \qquad \begin{cases} \dfrac{\partial U(h)}{\partial y}(x,y)=\Gamma(v_2)(U(h)(-,y))(x), \\ U(h)(x,0)=h(x). \end{cases}$$

Let $\tilde{u}=E(U)\in \mathrm{Hom}(R^R, R^{R\times R})$ *(see Theorem* 5) *and* $u: R^R\times R\to R^R$ *be its exponential adjoint. Then* u *verifies* (1).

Remark. Theorems 17 and 5 say that to integrate a vector field in R^R is actually the same as to solve the differential problem (2).

Proof of Theorem 17. (i) Let A and B be the composite maps

$$R^R \times R \times D \xrightarrow{(\mathrm{id}_{R^R} \times +)} R^R \times R \xrightarrow{\ u\ } R^R$$

and

$$R^R \times R \times D \xrightarrow{(u \times \mathrm{id}_D)} R^R \times D \xrightarrow{\ v\ } R^R$$

respectively. With this notation, (1) means $A = B$. Now, this happens iff the exponential adjoints of A and B (getting $R \times D$ up), say $\tilde{A}$ and $\tilde{B}$, coincide.

But, since $(R^R)^{R \times D} \cong (R^D)^{R \times R} \cong R^{R \times R} \times R^{R \times R}$, $\tilde{A}$ and $\tilde{B}$ are pairs of arrows from R^R to $R^{R \times R}$. So, by Theorem 5, they coincide iff preceded by all possible global sections they are equal. So take a global section $h: 1 \to R^R$ and consider $\tilde{A} \circ h$, $\tilde{B} \circ h$. They are the same iff their exponential adjoints (getting $R \times D$ down again) are the same.

But the exponential adjoints of $\tilde{A} \circ h$ and $\tilde{B} \circ h$ are $A \circ (h \times \mathrm{id}_{R \times D})$ and $B \circ (h \times \mathrm{id}_{R \times D})$. Now use Lemma 16.

(ii) Use Theorem 5 and (i). $\square$

Easy examples and discussion

By Lemma 16 (iii), an arrow $v: R^R \times D \to R^R$ such that $v(h, 0) = h$ is defined by an arrow $v_2: R^R \to R^R$, and, by Theorem 5 this v_2 can be identified with a C^∞-function

$$G: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),$$

$$G = \Gamma(v_2), \qquad v_2 = E(G).$$

(1) Take for $G: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ the C^∞-function $G(h) = h$. The solution of the problem

$$\begin{cases} \dfrac{\partial U(h)}{\partial y}(x, y) = G(U(h)(-, y))(x), \\ U(h)(x, 0) = h(x) \end{cases}$$

is $U(h)(x, y) = \mathrm{e}^y h(x)$, which is C^∞. In this case, the corresponding arrow v is given by: $v(h, d) = h + hd$. So the problem:

$$(1) \qquad \begin{cases} u(h, t + d) = v(u(h, t), d), \\ u(h, 0) = h \end{cases}$$

has the (unique) solution

$$u = \text{exponential adjoint to } E(U).$$

(2) Suppose now that $G(h)(x) = h'(x)$. G is C^∞. In this case, the solution of the corresponding problem (1) is given by $u =$ exponential adjoint of $E(U)$, where $U(h)(x, y) = h(x + y)$.

We do not know yet a theorem of existence for our problem. It should be a local existence, for some topological notion that must be found. For example, considering the C^∞-compact open topology, this theorem seems to be false in general. For the moment, we content ourselves by saying that, as we did in (1) and (2), and in cases even less simple than these, one can find some examples where global existence and uniqueness hold.

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