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Decomposition of the Hochschild and cyclic homology of commutative differential graded algebras

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Abstract

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We obtain an expression for the Hochschild and cyclic homology of a commutative differential graded algebra under a suitable hypothesis.

Introduction

In Theorem 2.4 of [1] the authors show that the Hochschild and cyclic homology of a free commutative differential graded k-algebra over a characteristic zero field are the corresponding homologies of a bigraded S^1-chain complex which is simpler than the canonical one. This result allows them to compute the Hochschild and cyclic homology of an arbitrary commutative differential graded k-algebra (A, d) taking a free model $\mu : (\wedge(V), d') \to (A, d)$ of (A, d) and applying Theorem 2.4 of [1] to $(\wedge(V), d')$. Using this technique they obtain Hodge decompositions of the Hochschild and cyclic homology of (A, d) which coincide with the ones obtained by Gerstenhaber and Schack in [3] and Loday in [5], as Vigué-Poirrier showed in [7]. So, these decompositions do not depend on

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the choice of the model. Moreover, when the chosen model $(\wedge(V), d')$ is simple enough they can make explicit computations. This happens, for instance, when

$$A = \frac{k[X_1, \ldots, X_n]}{\langle f_1, \ldots, f_r \rangle}$$

with $f_1, \ldots, f_r$ a regular sequence of elements of $k[X_1, \ldots, X_n]$. Nevertheless, in general the free models are too complex and hard to construct. For instance, with this method, it is impossible to compute the cyclic homology of a localization of the k-algebra A mentioned above. At the beginning of this investigation, our purpose was precisely to solve this problem. With this in mind we prove in this work that Theorem 2.4 of [1] remains valid for algebras of the form $(A_0 \otimes_k \wedge(V), d)$, with A_0 homologically regular over a characteristic zero field (see Definition 2.1) and $V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots$ a graded k-vector space. This allows us to obtain an elementary and self-contained proof of Theorem 5 of [2]. In fact, we study the more general case of a k-algebra $\frac{A}{I}$, with A homologically regular and I an ideal which is locally a complete intersection (Corollary 3.4). As an example of these algebras consider the localization of the ring of regular functions of an affine variety that is locally a complete intersection.

The paper is divided in four sections. In the first one, a quick review of some basic notions of differential graded algebras and S^1-chain complexes is given. In Sections 2 and 3 we generalize the result of Burghelea and Vigué-Poirrier, mentioned in the beginning of this Introduction, and Theorem 5 of [2] to homologically regular k-algebras. Finally, in Section 4, we give a theorem that unifies the previous ones.

1. Preliminaries

In this section we recall some general definitions and properties about commutative differential graded algebras and S^1-chain complexes, that we are going to use later. All mentioned definitions and properties are in [1].

Definition 1.1. Let k be a field of characteristic zero; a *commutative differential graded algebra* (A, d) over k (k-CDGA) is an associative graded algebra over k, $A = \bigoplus_{n \geq 0} A_n$, with unit $1 \in A_0$, equipped with a differential d of degree -1, satisfying

(a) $a_n a_m = (-1)^{nm} a_m a_n$ if $a_n \in A_n$ and $a_m \in A_m$,

(b) $d(A_0) = 0$ and $1 \notin \operatorname{Im}(d)$,

(c) $d(ab) = (da)b + (-1)^j a(db)$ if $a \in A_j$.

Let $V = \bigoplus_{n \geq 0} V_n$ be a graded k-vector space; the *free commutative graded algebra generated by* V, that we denote by $\wedge(V)$, is

$$\wedge(V) = S\Big(\bigoplus_{n \geq 0} V_{2n}\Big) \otimes E\Big(\bigoplus_{n \geq 0} V_{2n+1}\Big),$$

where S is the symmetric algebra and E is the exterior algebra. Now, a k-CDGA (A, d) is called *free* if:

(a) $A = \wedge(V)$ for some graded k-vector space V,

(b) $dV \subseteq \wedge^{+}(V)$, where $\wedge^{+}(V)$ is the ideal in $\wedge(V)$ generated by the elements of V.

The following result is proved in [1, Proposition 1.1].

Proposition 1.2. *For any k-CDGA,* (A, d^A) *there is a free k-CDGA* $(\wedge(V), d)$ *and a quasi-isomorphism* $(\wedge(V), d) \to (A, d^A)$. *Such an algebra is called a model of* (A, d^A). $\square$

Definition 1.3. An S^1-*chain complex* $\tilde{C} = (C_n, d_n, \beta_n)_{n\geq 0}$ is a chain complex of k-vector spaces $(C_*, d_*) = (C_n, d_n)_{n\geq 0}$ equipped with linear maps $\beta_n : C_n \to C_{n+1}$ $(n \geq 0)$ such that $\beta_n \circ \beta_{n-1} = 0$ and $\beta_{n-1} \circ d_n + d_{n+1} \circ \beta_n = 0$.

To $\tilde{C}$, one associates the chain complex $({}_{\beta}C_*, {}_{\beta}d_*)$ defined by

$$ {}_{\beta}C_n = C_n \oplus C_{n-2} \oplus \cdots $$

and

$$ {}_{\beta}d_n(x_n, x_{n-2}, \ldots) = (dx_n + \beta x_{n-2}, dx_{n-2} + \beta x_{n-4}, \ldots) . $$

Definition 1.4. The *cyclic* and the *Hochschild* homology $\mathrm{HC}_*(\tilde{C})$ and $\mathrm{HH}_*(\tilde{C})$ of $\tilde{C} = (C_n, d_n, \beta_n)_{n\geq 0}$ are the homologies of $({}_{\beta}C_*, {}_{\beta}d_*)$ and (C_*, d_*), respectively.

One sees immediately that $({}_{\beta}C_*, {}_{\beta}d_*)$ is related to the chain complex (C_*, d_*) by the following exact sequence of complexes

$$ 0 \to (C_*, d_*) \to ({}_{\beta}C_*, {}_{\beta}d_*) \xrightarrow{S} ({}_{\beta}C_{*-2}, {}_{\beta}d_{*-2}) \to 0 , $$

where S is obtained by dividing $({}_{\beta}C_*, {}_{\beta}d_*)$ by its first factor. This short exact sequence gives rise to the long exact sequence

$$ \cdots \to \mathrm{HH}_n(\tilde{C}) \to \mathrm{HC}_n(\tilde{C}) \xrightarrow{S} \mathrm{HC}_{n-2}(\tilde{C}) $$
$$ \to \mathrm{HH}_{n-1}(\tilde{C}) \to \mathrm{HC}_{n-1}(\tilde{C}) \to \cdots $$

Let $\tilde{C} = (C_n, d_n, \beta_n)_{n\geq 0}$ and $\tilde{C}' = (C'_n, d'_n, \beta'_n)_{n\geq 0}$ be S^1-chain complexes. By a morphism from $\tilde{C}$ to $\tilde{C}'$ we mean a family $\tilde{f} := (f_n : C_n \to C'_n)_{n\geq 0}$ of k-morphisms such that $d'_n f_n = f_{n-1} d_n$ and $\beta'_n f_n = f_{n+1}\beta_n$ $\forall n \geq 0$. Each $\tilde{f} : \tilde{C} \to \tilde{C}'$ induces maps $f_* : (C_*, d_*) \to (C'_*, d'_*)$ and $\bar{f}_* : ({}_{\beta}C_*, {}_{\beta}d_*) \to ({}_{\beta'}C'_*, {}_{\beta'}d'_*)$. It is clear that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (C_*, d_*) & \longrightarrow & ({}_\beta C_*, {}_\beta d_*) & \xrightarrow{S} & ({}_\beta C_{*-2}, {}_\beta d_{*-2}) & \longrightarrow & 0 \\
 & & \downarrow f_* & & \downarrow \bar{f}_* & & \downarrow \bar{f}_{*-2} & & \\
0 & \longrightarrow & (C'_*, d'_*) & \longrightarrow & ({}_{\beta'} C'_*, {}_{\beta'} d'_*) & \xrightarrow{S} & ({}_{\beta'} C'_{*-2}, {}_{\beta'} d'_{*-2}) & \longrightarrow & 0
\end{array}$$

commutes.

Definition 1.5. A *bigraded S^1-chain complex* $\tilde{\tilde{C}} = (C_{pq}, d^{\mathrm{I}}_{pq}, d^{\mathrm{E}}_{pq}, \beta_{pq})_{p,q\geq 0}$ is a collection of k-vector spaces $C_{p,q}$ ($p \geq 0$, $q \geq 0$), and k-linear maps

$$d^{\mathrm{I}}_{p,q} : C_{p,q} \to C_{p,q-1}\,, \quad d^{\mathrm{E}}_{p,q} : C_{p,q} \to C_{p-1,q}\,, \quad \beta_{p,q} : C_{p,q} \to C_{p+1,q}$$

such that

$$(d^{\mathrm{I}})^2 = 0\,, \quad (d^{\mathrm{E}})^2 = 0\,, \quad \beta^2 = 0\,,$$
$$\beta \circ d^{\mathrm{E}} + d^{\mathrm{E}} \circ \beta = 0\,, \quad \beta \circ d^{\mathrm{I}} + d^{\mathrm{I}} \circ \beta = 0\,, \quad d^{\mathrm{I}} \circ d^{\mathrm{E}} + d^{\mathrm{E}} \circ d^{\mathrm{I}} = 0\,.$$

For any such bigraded S^1-chain complex, one has the total S^1-chain complex

$$(\mathrm{Tot}\ \tilde{\tilde{C}}) = \Big(\bigoplus_{p+q=n} C_{p,q},\, d^{\mathrm{I}} + d^{\mathrm{E}},\, \beta \Big)\,.$$

Definition 1.6. The *cyclic* and the *Hochschild* homology of $\tilde{\tilde{C}}$ are the cyclic and Hochschild homologies of (Tot $\tilde{\tilde{C}}$).

Let (A, d) be a k-CDGA and $\bar{A} = A/k$. We define:

$$T(A, d)_{p,q} := \bigoplus_{i_0+\cdots+i_p=q} A_{i_0} \otimes \bar{A}_{i_1} \otimes \cdots \otimes \bar{A}_{i_p} \quad \text{for } p,q \geq 0\,,$$

$$d^{\otimes}_{p,q}(a_{i_0} \otimes \cdots a_{i_p}) := \sum_{j=0}^{p} (-1)^{i_0+\cdots+i_{j-1}} a_{i_0} \otimes \cdots \otimes d(a_{i_j}) \otimes \cdots \otimes a_{i_p}\,,$$

$$b_{p,q}(a_{i_0} \otimes \cdots \otimes a_{i_p}) := \sum_{j=0}^{p-1} (-1)^j a_{i_0} \otimes \cdots \otimes a_{i_j} a_{i_{j+1}} \otimes \cdots \otimes a_{i_p}$$
$$+ (-1)^{p+i_p(i_0+\cdots+i_{p-1})} a_{i_p} a_{i_0} \otimes a_{i_1} \otimes \cdots \otimes a_{i_{p-1}}\,,$$

and

$$B_{p,q}(a_{i_0} \otimes \cdots \otimes a_{i_p}) := \sum_{j=0}^{p} (-1)^{e(j)} 1 \otimes a_{i_j} \otimes \cdots a_{i_p} \otimes a_{i_0} \otimes \cdots \otimes a_{i_{j-1}}\,,$$

with $e(j) = jp + \sum_{h=j}^{p} i_h (\sum_{k \neq h} i_k)$.

One can check that $\tilde{\tilde{T}}(A, d) := (T(A, d)_{p,q}, d^{\otimes}_{p,q}, b_{p,q}, B_{p,q})_{p,q\geq 0}$ is a bigraded S^1-chain complex.

Remark 1.7. If $A = A_0$, then $T(A)_{p,q} = 0$ $\forall q > 0$ and the complex $({}_B\mathrm{Tot}(\tilde{\tilde{T}}(A))_*, {}_Bb_*)$ becomes the total complex of the double complex $B(A)_{\mathrm{norm}}$ defined in [6].

Definition 1.8. The *cyclic* and *Hochschild* homologies $\mathrm{HC}_n(A, d)$ and $\mathrm{HH}_n(A, d)$ of (A, d) are the cyclic and Hochschild homologies of $\tilde{\tilde{T}}(A, d)$.

2. The cyclic homology of a homologically regular k-CDGA

In [1], the authors show that the cyclic homology of a free k-CDGA $(\wedge(V), d)$ can be computed as the cyclic homology of a bigraded S^1-chain complex simpler than the one given in Definition 1.6, which can be identified with the algebra of differential forms of $(\wedge(V), d)$. So Burghelea and Vigué-Poirrier's result can be seen as a version of the Loday–Quillen Theorem [6, Theorem 2.9] for free k-CDGA's. Here we generalize both results; namely, we prove Burghelea and Vigué-Poirrier's result for homologically regular k-CDGA's.

Definition 2.1. (1) A k-algebra A is called *homologically regular* if the map $\theta^A_* : (A \otimes \bar{A}^*, b) \to (\Omega^*(A), 0)$ (see [6]) is a quasi-isomorphism and $\Omega^1(A)$ is flat.

(2) A k-CDGA (A, d) is *homologically regular* if $A = A_0 \otimes_k \wedge(V)$ with A_0 homologically regular and $V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots$ is a graded k-vector space.

Example 2.2. *If A' is homologically regular, then so is $A = S^{-1}(A'[X_i \colon i \in I])$ for each multiplicative subset S of $A'[X_i \colon i \in I]$.*

Proof. Let $A = S^{-1}(A'[X_i \colon i \in I])$. We must prove that the map θ^A_* is a quasi-isomorphism. Since $\mathrm{HH}_*(S^{-1}(A'[X_i \colon i \in I])) = S^{-1}(\mathrm{HH}_*(A'[X_i \colon i \in I]))$, we can assume $S = \{1\}$. Now the proof is immediate by observing that θ^A_* is the tensor product of $\theta^{A'}_*$ and $\theta^{k[x_i \colon i \in I]}_*$, which are quasi-isomorphisms by hypothesis and [1, Theorem 2.4]. □

Definition 2.3. To any homologically regular k-CDGA $(A_0 \otimes \wedge(V), d)$ we associate the k-CDGA $(\Omega^*(A_0) \otimes \wedge(V \oplus \bar{V}), \delta^d)$, defined as follows:

(1) $\bar{V}_{n+1} = V_n$ $(n \geq 1)$,

(2) δ^d is the unique derivation of degree -1 such that

$$\delta^d|_{A_0 \otimes \wedge(V)} = d \quad \text{and} \quad \delta^d \circ \beta + \beta \circ \delta^d = 0\,,$$

where β is the derivation of degree $+1$ verifying

(i) $\beta(\omega) = d_{\mathrm{DR}}(\omega)$ for $\omega \in \Omega^i(A_0)$, where $d_{\mathrm{DR}}(\omega)$ is the de Rham differential of ω,

(ii) $\beta(v) = \bar{v}$ for $v \in V_n$ $(n \geq 1)$,

(iii) $\beta \circ \beta = 0$.

(Observe that $\delta^d(\bar{v}) = -\beta(dv)$ $(v \in V)$ and $\delta^d(\omega) = 0$ $(w \in \Omega^i(A_0))$.)

Definition 2.4. Let $(A, d) = (A_0 \otimes \wedge(V), d)$ be a homologically regular k-CDGA. Let us call $\wedge^m(\bar{V})$ $(m \geq 0)$ the vector subspace in $\wedge(\bar{V})$ generated by the monomials $\bar{v}_1 \cdots \bar{v}_m$. With (A, d) we associate the bigraded S^1-chain complex

$$\tilde{\tilde{\mathscr{E}}}(A, d) := (\mathscr{E}(A, d)_{p,q}, \delta^d_{p,q}, 0, \beta_{p,q}),$$

where

$$\mathscr{E}(A, d)_{p,q} := \bigoplus_{i=0}^{p} \Omega^i(A_0) \otimes \Big(\bigwedge(V) \otimes \bigwedge^{p-i}(\bar{V})\Big)_{p+q-i}$$
$$\text{if } p \geq 0 \text{ and } q \geq 0,$$
$$\mathscr{E}(A, d)_{p,q} := 0 \quad \text{if } p < 0 \text{ or } q < 0,$$
$$\delta^d_{p,q}(\omega \otimes x) = (-1)^i \omega \cdot \delta^d(x)$$
$$\omega \in \Omega^i(A_0),\ x \in \Big(\bigwedge(V) \otimes \bigwedge^{p-i}(\bar{V})\Big)_{p+q-i},$$
$$\beta_{p,q}(\omega \otimes x) = d\omega \cdot x + (-1)^i \omega \cdot \beta(x)$$
$$\omega \in \Omega^i(A_0),\ x \in \Big(\bigwedge(V) \otimes \bigwedge^{p-i}(\bar{V})\Big)_{p+q-i}.$$

Remark 2.5. Note that if $A_0 = k[X_i : i \in I]$, then $\tilde{\tilde{\mathscr{E}}}(A, d)$ is the complex defined in [1].

The main result of this section is the following:

Theorem 2.6. *The cyclic (resp. the Hochschild) homology of a homologically regular k-CDGA (A, d) is the cyclic (resp. Hochschild) homology of the bigraded S^1-chain complex $\tilde{\tilde{\mathscr{E}}}(A, d)$.*

Proof. Let $\theta : \tilde{\tilde{T}}(A, d) \to \tilde{\tilde{\mathscr{E}}}(A, d)$ defined by

$$\theta(a_{i_0} \otimes \cdots \otimes a_{i_p}) := \left(\frac{(-1)^{i_0 + i_2 + \cdots}}{p!}\right) a_{i_0} \cdot \beta(a_{i_1}) \cdots \beta(a_{i_p})$$
$$\Big(a_{i_j} \in A_0 \otimes \bigwedge(V)_{i_j}\Big).$$

As shown in [1], θ is a map of bigraded S^1-chain complexes (i.e. $\theta \circ \beta = 0$, $\theta \circ d^{\otimes} = \delta^d$ and $\theta \circ B = \beta \circ \theta$). In order to see that θ is an isomorphism we can assume $d = 0$. Now the proof is immediate by noticing that

(1) $\mathrm{Tot}(A, 0)_{p,q}, 0, b) = (A_0 \otimes \bar{A}_0^*, b) \otimes \mathrm{Tot}(T(\wedge(V), 0)_{p,q}, 0, b)$,

(2) $\mathrm{Tot}(\mathscr{E}(A, 0)_{p,q}, 0, 0)$ is the tensor product of $\mathrm{Tot}(\mathscr{E}(\wedge(V), 0)_{p,q}, 0, 0)$ with the complex $A_0 \xleftarrow{0} \Omega^1(A_0) \xleftarrow{0} \Omega^2(A_0) \xleftarrow{0} \Omega^3(A_0) \xleftarrow{0} \cdots$,

(3) θ is the tensor product of the quasi-isomorphisms $(A_0 \otimes \bar{A}_0^*, b) \to (\Omega^*(A_0), 0)$ of Definition 2.1 and $(T(\wedge(V), 0)_{p,q}, 0, b) \to (\mathscr{E}(\wedge(V), 0)_{p,q}, 0, 0)$ in [1, Section 2]. □

Corollary 2.7. (1) *The cyclic homology of (A, d) splits into the sum of the homologies $\mathrm{HC}_*^{(j)}(A, d)$ $(j \geq 0)$ of the double complexes $\mathscr{E}^{(j)}(A, d)$*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{E}_{j+2}^{(j)}(A, d) & \xleftarrow{\beta} & \mathscr{E}_{j+1}^{(j-1)}(A, d) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{E}_3^{(1)}(A, d) & \xleftarrow{\beta} & \mathscr{E}_2^{(0)}(A, d) \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{E}_{j+1}^{(j)}(A, d) & \xleftarrow{\beta} & \mathscr{E}_{j}^{(j-1)}(A, d) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{E}_2^{(1)}(A, d) & \xleftarrow{\beta} & \mathscr{E}_1^{(0)}(A, d) \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{E}_{j}^{(j)}(A, d) & \xleftarrow{\beta} & \mathscr{E}_{j-1}^{(j-1)}(A, d) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{E}_1^{(1)}(A, d) & \xleftarrow{\beta} & \mathscr{E}_0^{(0)}(A, d)
\end{array}$$

where $\mathscr{E}_{m-2h}^{(j-h)}(A, d) = \bigoplus_{i=0}^{j-h} \Omega^i(A_0) \otimes (\wedge(V) \otimes \wedge^{j-h-i}(\bar{V}))_{m-2h-i}$ *and* δ^d, β *are as in Definition* 2.3.

(2) *The Hochschild homology of (A, d) splits into the sum of the homologies $\mathrm{HH}_*^{(j)}(A, d)$ of the complexes $\mathscr{E}^{(j)}(A, d)_{\tau \geq 1} :=$ the first column of $\mathscr{E}^{(j)}(A, d)$ $(j \geq 0)$.*

(3) *The Gysin–Connes long exact sequence is the sum of the long exact sequences of homology associated with the short exact sequences of complexes*

$$0 \to \mathscr{E}^{(j)}(A, d)_{\tau \geq 1} \to \mathrm{Tot}(\mathscr{E}^{(j)}(A, d))_* \to \mathrm{Tot}(\mathscr{E}^{(j-1)}(A, d))_{*-2} \to 0 .$$

Proof. It follows from the fact that

$$\delta^d(\mathscr{E}_{m-2h}^{(j-h)}(A, d)) \subseteq \mathscr{E}_{m-2h-1}^{(j-h)}(A, d) \quad \text{and}$$

$$\beta(\mathscr{E}_{m-2h}^{(j-h)}(A, d)) \subseteq \mathscr{E}_{m-2h+1}^{(j-h)}(A, d) . \qquad \square$$

Remark 2.8. *Let $f : (A, d) \to (A', d')$* be a morphism of homologically regular k-CDGA's. The family of maps

$$\tilde{\tilde{\mathscr{E}}}(f) = (\mathscr{E}(f)_{p,q} : \mathscr{E}(A, d)_{p,q} \to \mathscr{E}(A', d')_{p,q})_{p,q \geq 0},$$

given by

$$\mathscr{E}(f)_{p,q}(w \cdot x \cdot \bar{v}_1 \cdots \bar{v}_{p-i}) = \Omega(f)(w) \cdot f(x) \cdot \beta(f(v_1)) \cdots \beta(f(v_{p-i}))$$
$$(\omega \in \Omega^i(A_0),\ x \in \wedge(V),\ \bar{v}_1 \cdots \bar{v}_{p-i} \in \wedge^{p-i}(\bar{V})),$$

is a morphism of bigraded S^1-chain complexes from $\tilde{\tilde{\mathscr{E}}}(A, d)$ into $\tilde{\tilde{\mathscr{E}}}(A', d')$. Moreover, the maps induced by $\tilde{\tilde{\mathscr{E}}}(f)$ between the respective Hochschild and cyclic homologies coincide with those induced by the canonical map $\tilde{\tilde{T}}(f) : \tilde{\tilde{T}}(A, d) \to \tilde{\tilde{T}}(A', d')$.

Proof. Since β and δ^d are derivations, to prove that $\beta \circ \tilde{\tilde{\mathscr{E}}}(f) = \tilde{\tilde{\mathscr{E}}}(f) \circ \beta$ and $\delta^d \circ \tilde{\tilde{\mathscr{E}}}(f) = \tilde{\tilde{\mathscr{E}}}(f) \circ \delta^d$ it is enough to verify these equalities on the elements $w \in \Omega^i(A_0)$, $v \in V$ and $\bar{v} \in \bar{V}$. But,

$$\begin{aligned}
\beta \circ \mathscr{E}(f)(\omega) &= \beta \circ \Omega(f)(\omega) = d_{\mathrm{DR}} \circ \Omega(f)(\omega) = \mathscr{E}(f) \circ \beta(\omega) \\
&= \mathscr{E}(f) \circ d_{\mathrm{DR}}(\omega) = \Omega(f) \circ d_{\mathrm{DR}}(\omega), \\
\beta \circ \mathscr{E}(f)(v) &= \beta \circ f(v) = \mathscr{E}(f)(\bar{v}) = \mathscr{E}(f) \circ \beta(v), \\
\beta \circ \mathscr{E}(f)(\bar{v}) &= \beta \circ \beta \circ f(\bar{v}) = 0 \quad \text{and} \quad \mathscr{E}(f) \circ \beta(\bar{v}) = \mathscr{E}(f)(0) = 0, \\
\delta^d \circ \mathscr{E}(f)(\omega) &= \delta^d \circ \Omega(f)(\omega) = 0 \quad \text{and} \quad \mathscr{E}(f) \circ \delta^d(\omega) = \mathscr{E}(f)(0) = 0, \\
\delta^d \circ \mathscr{E}(f)(v) &= \delta^d \circ f(v) = d \circ f(v) = f \circ d(v) \\
&= \mathscr{E}(f) \circ d(v) = \mathscr{E}(f) \circ \delta^d(v), \\
\delta^d \circ \mathscr{E}(f)(\bar{v}) &= \delta^d \circ \beta \circ f(v) = -\beta \circ \delta^d \circ f(v) = -\beta \circ d \circ f(v) \\
&= -\beta \circ f \circ d(v) = -\beta \circ \mathscr{E}(f) \circ d(v) \\
&= -\mathscr{E}(f) \circ \beta \circ d(v) = \mathscr{E}(f) \circ \delta^d(\bar{v}).
\end{aligned}$$

To finish the proof it is enough to observe that the diagram

$$\begin{array}{ccc}
\tilde{\tilde{T}}(A, d) & \xrightarrow{\tilde{\tilde{T}}(f)} & \tilde{\tilde{T}}(A', d') \\
\downarrow \theta & & \downarrow \theta \\
\tilde{\tilde{\mathscr{E}}}(A, d) & \xrightarrow{\tilde{\tilde{\mathscr{E}}}(f)} & \tilde{\tilde{\mathscr{E}}}(A', d')
\end{array}$$

commutes. □

3. Some computations

In [2, Theorem 5], the authors compute the cyclic homology for an algebra of the type $\frac{A}{I}$, where A is the ring of regular functions of a nonsingular variety and I is locally a complete intersection ideal of A. In this section we give an elementary and self-contained proof of this result. We also give a similar decomposition for the Hochschild homology. This last theorem generalizes the main result of [8] and also appears in [4].

3.1. Let (A, d) be a homologically regular k-CDGA and $I := d(A_1) \subseteq A_0$. For each $j \geq 0$ we consider the complexes

$$L^*_{(j)}(A_0/I):\quad 0 \longrightarrow \frac{I^j\Omega^0(A_0)}{I^{j+1}\Omega^0(A_0)} \xrightarrow{d_{\mathrm{DR}}} \frac{I^{j-1}\Omega^1(A_0)}{I^j\Omega^1(A_0)} \xrightarrow{d_{\mathrm{DR}}} \cdots \xrightarrow{d_{\mathrm{DR}}} \frac{\Omega^j(A_0)}{I\Omega^j(A_0)} \to 0$$

and

$$D^*_{(j)}(A_0/I):\quad 0 \longrightarrow \frac{\Omega^0(A_0)}{I^{j+1}\Omega^0(A_0)} \xrightarrow{d_{\mathrm{DR}}} \frac{\Omega^1(A_0)}{I^j\Omega^1(A_0)} \xrightarrow{d_{\mathrm{DR}}} \cdots \xrightarrow{d_{\mathrm{DR}}} \frac{\Omega^j(A_0)}{I\Omega^j(A_0)} \to 0\,,$$

where d_{DR} is induced by the de Rham differential. We define morphisms $\bar{\varphi}^{(j)}_*$ from $\mathrm{Tot}(\mathscr{E}^{(j)}(A, d))$ to $D^{2j-*}_{(j)}(A_0/I)$ and $\varphi^{(j)}_*$ from $\mathscr{E}^{(j)}(A, d)_{\tau\geq 1}$ to $L^{2j-*}_{(j)}(A_0/I)$, setting

$$\bar{\varphi}^{(j)}_m : \bigoplus_{h=0}^{m-j} \mathscr{E}^{(j-h)}_{m-2h}(A, d) \to \frac{\Omega^{2j-m}(A_0)}{I^{m-j+1}\Omega^{2j-m}(A_0)}\,,$$

$$\bar{\varphi}^{(j)}_m(\omega \cdot x \cdot \bar{v}_1 \cdots \bar{v}_{j-h-i})$$

$$= \begin{cases} 0 & \text{if } \mathrm{dg}(x) > 0 \text{ or } \mathrm{dg}(v_t) > 1 \text{ for some } 1 \leq t \leq j-h-i\,, \\ (-1)^{j-h-i}\overline{w \cdot x \cdot d(v_1) \cdots d(v_{j-h-i})} & \text{otherwise}\,, \end{cases}$$

(note that $\mathrm{dg}(x) = 0$ and $\mathrm{dg}(v_t) = 1$ $\forall 1 \leq t \leq j-p-i$ is equivalent to $2j - m = i$) and

$$\varphi^{(j)}_m : \mathscr{E}^{(j)}_m(A, d) \to \frac{I^{m-j}\Omega^{2j-m}(A_0)}{I^{m-j+1}\Omega^{2j-m}(A_0)}\,,$$

$$\varphi^{(j)}_m(\omega \cdot x \cdot \bar{v}_1 \cdots \bar{v}_{j-i}) = \bar{\varphi}^{(j)}_m(\omega \cdot x \cdot \bar{v}_1 \cdots \bar{v}_{j-i})\,.$$

It follows easily from the definitions that $\bar{\varphi}_*^{(j)}$ and $\varphi_*^{(j)}$ are morphisms of complexes.

Remark 3.2. (1) The diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathscr{C}^{(j)}(A,d)_{\tau\ge 1} & \longrightarrow & \mathrm{Tot}(\mathscr{C}^{(j)}(A,d))_* & \longrightarrow & \mathrm{Tot}(\mathscr{C}^{(j-1)}(A,d))_{*-2} & \longrightarrow 0 \\
& \downarrow{\scriptstyle \varphi_*^{(j)}} & & \downarrow{\scriptstyle \bar{\varphi}_*^{(j)}} & & \downarrow{\scriptstyle \bar{\varphi}_{*-2}^{(j-1)}} & \\
0 \longrightarrow & L^{2j-*}_{(j)}(A_0/I) & \longrightarrow & D^{2j-*}_{(j)}(A_0/I) & \longrightarrow & D^{2j-*}_{(j-1)}(A_0/I) & \longrightarrow 0
\end{array}$$

commutes.

(2) Given a morphism $f:(A,d)\to(A',d')$ of homologically regular k-CDGA's, the diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathscr{C}^{(j)}(A',d')_{\tau\ge 1} & \longrightarrow & \mathrm{Tot}(\mathscr{C}^{(j)}(A',d'))_* & \longrightarrow & \mathrm{Tot}(\mathscr{C}^{(j-1)}(A',d'))_{*-2} & \longrightarrow 0 \\
0 \longrightarrow & \mathscr{C}^{(j)}(A,d)_{\tau\ge 1} & \longrightarrow & \mathrm{Tot}(\mathscr{C}^{(j)}(A,d))_* & \longrightarrow & \mathrm{Tot}(\mathscr{C}^{(j-1)}(A,d))_{*-2} & \longrightarrow 0 \\
0 \longrightarrow & L^{2j-*}_{(j)}(A'_0/I') & \longrightarrow & D^{2j-*}_{(j)}(A'_0/I') & \longrightarrow & D^{2j-*}_{(j-1)}(A'_0/I') & \longrightarrow 0 \\
& \nearrow{\scriptstyle \alpha^{2j-*}_{(j)}(f)} & & \nearrow{\scriptstyle \gamma^{2j-*}_{(j)}(f)} & & \nearrow{\scriptstyle \gamma^{2j-*}_{(j-1)}(f)} & \\
0 \longrightarrow & L^{2j-*}_{(j)}(A_0/I) & \longrightarrow & D^{2j-*}_{(j)}(A_0/I) & \longrightarrow & D^{2j-*}_{(j-1)}(A_0/I) & \longrightarrow 0
\end{array}$$

where the upper face arrows are induced by $\mathscr{C}^{(j)}(f)$ and $\alpha^{2j-*}_{(j)}(f)$, $\gamma^{2j-*}_{(j)}(f)$, $\gamma^{2j-*}_{(j-1)}(f)$ are all induced by f, commutes.

Theorem 3.3. *Let (A,d) be a homologically regular k-CDGA satisfying: $H_i(A,d)=0$ $\forall i>0$ and $I:=d(A_1)\subseteq A_0$ is locally a complete intersection. Hence, $\bar{\varphi}_*^{(j)}$ and $\varphi_*^{(j)}$ are quasi-isomorphisms.*

Proof. From Remark 3.2 one sees (through induction on j) that it suffices to prove the theorem for $\varphi_*^{(j)}$. We shall prove that the map $\bigoplus_{j\ge 0}\varphi_*^{(j)}$, from $\mathrm{Tot}(\mathscr{C}(A,d)_{p,q},\delta^d,0)=\bigoplus_{j\ge 0}(\mathscr{C}^{(j)}(A,d)_{\tau\ge 1})_*$ into $\bigoplus_{j\ge 0}L^{2j-*}_{(j)}(A_0/I)$ is an isomorphism. Since, after localization, $(A,d)=(A_0\otimes\wedge(V),d)$ is quasi-isomorphic to a quotient $\frac{A'_0}{I'}$ where A'_0 has the same properties as A_0, and $I'=\langle P_1,\dots,P_r\rangle$ with $P_1,\dots,P_r$ a regular sequence, we can work with the Koszul complex $K^*(A_0,P_1,\dots,P_r)$ (Remark 3.2). So we assume that $(A,d)=K^*(A_0,P_1,\dots,P_r)$. Recall that this complex has the form $(A_0\otimes\wedge(V_1),d)$ with $V_1=\bigoplus_{i=1}^r k\cdot e_i$, $d(e_i)=P_i$. Now, the quasi-isomorphism $\pi:(A,d)\to\frac{A_0}{I}$ induces a quasi-isomorphism $\pi\otimes_{A_0}\mathrm{id}$ from

$$\mathrm{Tot}(\mathscr{C}(A,d)_{p,q},\delta^d,0)\cong\left(A_0\otimes\bigwedge(V_1),d\right)\otimes_{A_0}\left(\Omega^*(A_0)\otimes\bigwedge(\bar{V}_1),\delta^d\right)$$

into

$$\frac{A_0}{I} \otimes_{A_0} \Big(\Omega^*(A_0) \otimes \bigwedge (\bar{V}_1), \delta^d \Big) .$$

On the other hand, since I is generated by a regular sequence, $\frac{I}{I^2}$ is a free $\frac{A_0}{I}$ module generated by (cl $P_1, \ldots,$ cl P_r) where cl P_i is the class of P_i in $\frac{I}{I^2}$, and we have an isomorphism of $(\frac{A_0}{I})$-algebras between

$$S^*_{A_0/I}\Big(\frac{I}{I^2}\Big) \quad \text{and} \quad \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}} ,$$

where $S^*_{A_0/I}(\frac{I}{I^2})$ is the polynomial algebra constructed on the $\frac{A_0}{I}$ free module $\frac{I}{I^2}$. Let us consider that the elements of $\frac{I^n}{I^{n+1}}$ have degree $2n$. We have the following isomorphisms of graded algebras

$$\begin{aligned}
&\frac{A_0}{I} \otimes_{A_0} \Omega^*(A_0) \otimes \bigwedge (\bar{V}_1) \\
&\cong \Big(\frac{A_0}{I} \otimes \bigwedge (\bar{V}_1) \Big) \otimes_{A_0} \Omega^*(A_0) \\
&\cong S^*_{A_0/I}\Big(\frac{I}{I^2}\Big) \otimes_{A_0} \Omega^*(A_0) \quad \Big(\text{since } \bar{V}_1 \cong \frac{I}{I^2}\Big) \\
&\cong \Big(\bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}} \Big) \otimes_{A_0} \Omega^*(A_0) \\
&\cong \bigoplus_{n \geq 0} \frac{I^n \Omega^*(A_0)}{I^{n+1} \Omega^*(A_0)} \qquad (\text{since } \Omega^*(A_0) \text{ is } A_0\text{-flat}) .
\end{aligned}$$

It is easy to see that this map defines an isomorphism Ψ_* of k-CDGA's from $\frac{A_0}{I} \otimes_{A_0} (\Omega^*(A_0) \otimes \wedge (\bar{V}_1), \delta^d)$ onto $\bigoplus_{j \geq 0} L^{2j-*}_{(j)}(A_0/I)$. To finish the proof it is enough to check that $\bigoplus_{j \geq 0} \varphi^{(j)}_* = \psi_* \circ (\pi \otimes_{A_0} \text{id})$, which is immediate. □

Corollary 3.4. *Under the same hypothesis of Theorem* 3.3, *we have*:

(1) $$\mathrm{HC}_n(A_0/I) = \bigoplus_{i=0}^{[n/2]} H^{n-2i}(D^*_{(n-i)}(A_0/I)) .$$

(2) $$\mathrm{HH}_n(A_0/I) = \bigoplus_{i=0}^{[n/2]} H^{n-2i}(L^*_{(n-i)}(A_0/I)) .$$

(3) *The Gysin–Connes long exact sequence is the sum of the long exact sequences of homology associated with the short exact sequences of complexes*

$$0 \to L^*_{(j)}(A_0/I) \to D^*_{(j)}(A_0/I) \to D^*_{(j-1)}(A_0/I) \to 0 .$$

Proof. It follows immediately from Corollary 2.7 and Theorem 3.3. □

4. Final result

In this section we obtain a generalization of Corollary 2.7, for k-CDGA's (B, d^B) with $B = \frac{A_0}{I} \otimes_k \wedge(V)$ $(V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots)$, where A_0 is a homologically regular k-algebra and $I \subseteq A_0$ is locally a complete intersection ideal of A_0. When $I = 0$ we recover Corollary 2.7 and when $V = \{0\}$ (i.e. $\wedge(V) = k$) we recover Theorem 3.3.

Proposition 4.1. *Let A_0 be a homologically regular k-algebra, $I \subseteq A_0$ an ideal and $\gamma : (A, d^A) \to \frac{A_0}{I}$ a model of $\frac{A_0}{I}$, with $A = A_0 \otimes_k \wedge(W)$ $(W = W_1 \oplus W_2 \oplus W_3 \oplus \cdots)$. For each k-CDGA (B, d^B), with $B = \frac{A_0}{I} \otimes_k \wedge(V)$ $(V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots)$, there exists a k-CDGA (C, d^C) and a quasi-isomorphism $\bar{\gamma} : (C, d^C) \to (B, d^B)$, verifying:*
 (i) $C = A_0 \otimes_k \wedge(W) \otimes_k \wedge(V)$,
 (ii) $d^C|_A = d^A$,
 (iii) $\bar{\gamma} = \gamma \otimes \mathrm{id}_{\wedge(V)}$.

Proof. For each $j \geq 1$ we will denote with $\wedge^{(j)}(W)$ the vectorial subspace of $\wedge(W)$ formed by the elements of degree j. We have to define a differential d^C of C that extends d^A and such that $\gamma \otimes \mathrm{id}_{\wedge(V)}$:

$$\begin{array}{ccccccc}
A_0 & \xleftarrow{d^C} & \bigoplus_{i+j=1}\left(A_0 \otimes \wedge^{(i)}(W) \otimes \wedge^{(j)}(V)\right) & \xleftarrow{d^C} & \bigoplus_{i+j=2}\left(A_0 \otimes \wedge^{(i)}(W) \otimes \wedge^{(j)}(V)\right) & \xleftarrow{d^C} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\frac{A_0}{I} & \xleftarrow{d^B} & \frac{A_0}{I} \otimes \wedge^{(1)}(V) & \xleftarrow{d^B} & \frac{A_0}{I} \otimes \wedge^{(2)}(V) & \xleftarrow{d^B} & \cdots
\end{array}$$

is a quasi-isomorphism.

For each $i \geq 0$, let $C^i = A \otimes_k \wedge(V_1 \oplus \cdots \oplus V_i) = A_0 \otimes_k \wedge(W) \otimes_k \wedge(V_1 \oplus \cdots \oplus V_i)$ and let (B^i, d^{B^i}) be the differential graded subalgebra of (B, d^B) generated by $\frac{A_0}{I} \otimes_k \wedge(V_1 \oplus \cdots \oplus V_i)$. We will prove the existence of d^C by showing that each differential d^{C^i} of C^i such that

$$\gamma^i : (C^i, d^{C^i}) \xrightarrow{\gamma \otimes \mathrm{id}_{\wedge(V_1 \otimes \cdots \otimes V_i)}} (B^i, d^{B^i})$$

is a quasi-isomorphism, can be extended to a differential $d^{C^{i+1}}$ of C^{i+1} in such a way that

$$\gamma^{i+1} : (C^{i+1}, d^{C^{i+1}}) \xrightarrow{\gamma \otimes \mathrm{id}_{\wedge(V_1 \otimes \cdots \otimes V_{i+1})}} (B^{i+1}, d^{B^{i+1}})$$

is a quasi-isomorphism.

Let $(v_j)_{j \in I_{i+1}}$ be a basis of V_{i+1}. For each $j \in I_{i+1}$ there exists $\alpha_j \in C_i^i$ verifying

$$\gamma^i(\alpha_j) = \gamma \otimes \mathrm{id}_{\wedge(V_1 \otimes \cdots \otimes V_i)}(\alpha_j) = d^B(v_j) \quad \text{and} \quad d^{C^i}(\alpha_j) = 0\,.$$

In fact, since γ^i is a quasi-isomorphism, there exists $\alpha'_j \in C^i_i$ such that $d^{C^i}(\alpha'_j) = 0$ and $\gamma^i(\alpha_j) = d^B(v_j) + d^{B^i}(a)$ for some $a \in B^i_{i+1}$. Now, as γ^i is an epimorphism we can modify α'_j by taking $\alpha_j = \alpha'_j - d^{C^i}(b)$ with $b = C^i_{i+1}$ such that $\gamma^i(b) = a$.

Now we define $d^{C^{i+1}}$ as the unique derivative of degree -1 of C^{i+1} verifying

$$d^{C^{i+1}}(v_j) = \alpha_j \ \ \forall j \in I_{i+1} \quad \text{and} \quad d^{C^{i+1}}|_{C^i} = d^{C^i}.$$

It is clear that $(d^{C^{i+1}})^2 = 0$. It remains to prove that $\gamma \otimes \mathrm{id}_{\wedge(V_1 \otimes \cdots \otimes V_{i+1})}$ is a quasi-isomorphism, which follows immediately from the following statements:

(1) C^{i+1}_* is the total complex of the double complex

$$C^i_* \leftarrow C^i_* \otimes \textstyle\bigwedge^{(i+1)}(V_{i+1}) \leftarrow C^i_* \otimes \bigwedge^{(2i+2)}(V_{i+1}) \leftarrow \cdots,$$

(2) B^{i+1}_* is the total complex of the double complex

$$B^i_* \leftarrow B^i_* \otimes \textstyle\bigwedge^{(i+1)}(V_{i+1}) \leftarrow B^i_* \otimes \bigwedge^{(2i+2)}(V_{i+1}) \leftarrow \cdots,$$

and

(3) γ^{i+1} is the morphism from C^{i+1}_* to B^{i+1}_* induced by

$$\begin{array}{ccccccc}
C^i_* & \longleftarrow & C^i_* \otimes \bigwedge^{(i+1)}(V_{i+1}) & \longleftarrow & C^i_* \otimes \bigwedge^{(2i+2)}(V_{i+1}) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \gamma^i} & & \downarrow{\scriptstyle \gamma^i \otimes \mathrm{id}_{\wedge^{(i+1)}(V_{i+1})}} & & \downarrow{\scriptstyle \gamma^i \otimes \mathrm{id}_{\wedge^{(2i+2)}(V_{i+1})}} & & \\
B^i_* & \longleftarrow & B^i_* \otimes \bigwedge^{(i+1)}(V_{i+1}) & \longleftarrow & B^i_* \otimes \bigwedge^{(2i+2)}(V_{i+1}) & \longleftarrow & \cdots
\end{array}$$

and the vertical arrows $\gamma^i \otimes \mathrm{id}_{\wedge^{(si+s)}(V_{i+1})}$ are quasi-isomorphisms. □

Theorem 4.2. *Let A_0 be a homologically regular k-algebra. $I \subseteq A_0$ an ideal which locally is a complete intersection and (B, d^B) a k-CDGA, with $B = \frac{A_0}{I} \otimes_k \wedge(V)$ $(V = V_1 \oplus V_2 \oplus V_3 \oplus \cdots)$. Then we verify that*:

(1) *The cyclic homology of (B, d^B) splits into the direct sum of the homologies $\mathrm{HC}^{(j)}_*(B, d^B)$ $(j \geq 0)$ of the double complexes $\mathscr{C}^{(j)}(B, d^B)$*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{C}^{(j)}_{j+2}(B, d^B) & \xleftarrow{\beta} & \mathscr{C}^{(j-1)}_{j+1}(B, d^B) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{C}^{(1)}_3(B, d^B) & \xleftarrow{\beta} & \mathscr{C}^{(0)}_2(B, d^B) \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{C}^{(j)}_{j+1}(B, d^B) & \xleftarrow{\beta} & \mathscr{C}^{(j-1)}_{j}(B, d^B) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{C}^{(1)}_2(B, d^B) & \xleftarrow{\beta} & \mathscr{C}^{(0)}_1(B, d^B) \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{C}^{(j)}_{j}(B, d^B) & \xleftarrow{\beta} & \mathscr{C}^{(j-1)}_{j-1}(B, d^B) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{C}^{(1)}_1(B, d^B) & \xleftarrow{\beta} & \mathscr{C}^{(0)}_0(B, d^B)
\end{array}$$

where

$$\mathscr{E}^{(j-h)}_{m-2h}(B, d^B) = \bigoplus_{i=0}^{j-h} D^i_{(h+i)}(A_0/I) \otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-h-i}(\bar{V})\Big)_{m-2h-i}$$
$$= \bigoplus_{i=0}^{j-h} \frac{\Omega^i(A_0)}{I^{h+1}\Omega^i(A_0)} \otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-h-i}(\bar{V})\Big)_{m-2h-i},$$

$\delta^d(\omega \otimes x) = (-1)^i \omega \cdot \delta^d(x)$ *and*

$$\beta(\omega \otimes x) = d_{\mathrm{DR}}(\omega) \cdot x + (-1)^i \omega \cdot \beta(x) \quad \omega \in \frac{\Omega^i(A_0)}{I^{p+1}\Omega^i(A_0)}$$

(*with the same notations as in Definition* 2.3).

(2) *The Hochschild homology of* (B, d^B) *splits into the direct sum of the homologies* $\mathrm{HH}^{(j)}_*(B, d^B)$ *of the double complexes* $\mathscr{L}^{(j)}(B, d^B)$

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{L}^{(j)}_{j+2}(B, d^B) & \xleftarrow{\beta} & \mathscr{L}^{(j-1)}_{j+1}(B, d^B) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{L}^{(1)}_{3}(B, d^B) & \xleftarrow{\beta} & \mathscr{L}^{(0)}_{2}(B, d^B) \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{L}^{(j)}_{j+1}(B, d^B) & \xleftarrow{\beta} & \mathscr{L}^{(j-1)}_{j}(B, d^B) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{L}^{(1)}_{2}(B, d^B) & \xleftarrow{\beta} & \mathscr{L}^{(0)}_{1}(B, d^B) \\
\downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} & & & & \downarrow{\scriptstyle \delta^d} & & \downarrow{\scriptstyle \delta^d} \\
\mathscr{L}^{(j)}_{j}(B, d^B) & \xleftarrow{\beta} & \mathscr{L}^{(j-1)}_{j-1}(B, d^B) & \xleftarrow{\beta} & \cdots & \xleftarrow{\beta} & \mathscr{L}^{(1)}_{1}(B, d^B) & \xleftarrow{\beta} & \mathscr{L}^{(0)}_{0}(B, d^B)
\end{array}$$

where

$$\mathscr{L}^{(j-h)}_{m-2h}(B, d^B) = \bigoplus_{i=0}^{j-h} L^i_{(h+i)}(A_0/I) \otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-h-i}(\bar{V})\Big)_{m-2h-i}$$
$$= \bigoplus_{i=0}^{j-h} \frac{I^h\Omega^i(A_0)}{I^{h+1}\Omega^i(A_0)} \otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-h-i}(\bar{V})\Big)_{m-2h-i},$$

$\delta^d(\omega \otimes x) = (-1)^i \omega \cdot \delta^d(x)$ *and*

$$\beta(\omega \otimes x) = d_{\mathrm{DR}}(\omega) \cdot x + (-1)^i \omega \cdot \beta(x) \quad \omega \in \frac{I^h\Omega^i(A_0)}{I^{h+1}\Omega^i(A_0)}\,.$$

(3) *The Gysin–Connes long exact sequence is the sum of the long exact sequences of homology associated with the short exact sequences of complexes*

$$0 \to \mathrm{Tot}(\mathscr{L}^{(j)}(B, d^B))_* \to \mathrm{Tot}(\mathscr{E}^{(j)}(B, d^B))_*$$
$$\to \mathrm{Tot}(\mathscr{E}^{(j-1)}(B, d^B))_{*-2} \to 0\,.$$

Proof. Let $\gamma:(A,d^A)\to\frac{A_0}{I}$ be a model of $\frac{A_0}{I}$, with $A=A_0\otimes_k\wedge(W)$ ($W=W_1\oplus W_2\oplus W_3\oplus\cdots$). Let (C,d^C) and $\bar\gamma:(C,d^C)\to(B,d^B)$ be as in Proposition 4.1. We define morphisms $\bar\psi_*^{(j)}$ from $\operatorname{Tot}(\mathscr{E}^{(j)}(C,d^C))$ to $\operatorname{Tot}(\mathscr{E}^{(j)}(B,d^B))$ and $\psi_*^{(j)}$ from $\mathscr{E}^{(j)}(C,d^C)_{\tau\geq 1}$ to $\operatorname{Tot}(\mathscr{L}^{(j)}(B,d^B))$, setting:

$$\bar\psi_m^{(j)}:\bigoplus_{h=0}^{m-j}\mathscr{E}_{m-2h}^{(j-h)}(C,d^C)\to\bigoplus_{h=0}^{m-j}\mathscr{E}_{m-2h}^{(j-h)}(B,d^B)\,,$$
$$\bar\psi_m^{(j)}(\omega\cdot x_w\cdot\bar w_1\cdots\bar w_{i-r}\cdot x_v\cdot\bar v_1\cdots\bar v_{j-h-i})$$
$$=\bar\varphi_{2h+\alpha}^{(h+i)}(\omega\cdot x_w\cdot\bar w_1\cdots\bar w_{i-r})\cdot x_v\cdot\bar v_1\cdots\bar v_{j-h-i}\,,$$

where $\bar\varphi$ is the morphism of 3.1, $\omega\cdot x_w\cdot\bar w_1\cdots\bar w_{i-r}\cdot x_v\cdot\bar v_1\cdots\bar v_{j-h-i}\in\Omega^r(A_0)\otimes((\wedge(W)\otimes\wedge^{i-r}(\bar W))\otimes(\wedge(V)\otimes\wedge^{j-h-i}(\bar V)))_{m-2h-r}$ and $\alpha=\operatorname{dg}(\omega\cdot x_w\cdot\bar w_1\cdots\bar w_{i-r})$, and

$$\psi_m^{(j)}:\mathscr{E}_m^{(j)}(C,d^C)\to\bigoplus_{h=0}^{m-j}\mathscr{L}_{m-2h}^{(j-h)}(B,d^B)\,,$$
$$\psi_m^{(j)}(\omega\cdot x_w\cdot\bar w_1\cdots\bar w_{i-r}\cdot x_v\cdot\bar v_1\cdots\bar v_{j-i})$$
$$=\bar\psi_m^{(j)}(\omega\cdot x_w\cdot\bar w_1\cdots\bar w_{i-r}\cdot x_v\cdot\bar v_1\cdots\bar v_{j-i})\,.$$

Since the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{E}^{(j)}(C,d^C)_{\tau\geq 1} & \longrightarrow & \operatorname{Tot}(\mathscr{E}^{(j)}(C,d^C))_* & \longrightarrow & \operatorname{Tot}(\mathscr{E}^{(j-1)}(C,d^C))_{*-2} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \psi_*^{(j)}} & & \downarrow{\scriptstyle \bar\psi_*^{(j)}} & & \downarrow{\scriptstyle \bar\psi_{*-2}^{(j-1)}} & & \\
0 & \longrightarrow & \operatorname{Tot}(\mathscr{L}^{(j)}(B,d^B))_* & \longrightarrow & \operatorname{Tot}(\mathscr{E}^{(j)}(B,d^B))_* & \longrightarrow & \operatorname{Tot}(\mathscr{E}^{(j-1)}(B,d^B))_{*-2} & \longrightarrow & 0
\end{array}$$

commutes, it is enough to see that $\psi_*^{(j)}$ is a quasi-isomorphism. Now, $\mathscr{E}^{(j)}(C,d^C)_{\tau\geq 1}$ is the total complex of the double complex

$$M^{(j)}\qquad\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \\
M_{0,j+2} & \xleftarrow{\partial^{\mathrm h}} & M_{1,j+2} & \xleftarrow{\partial^{\mathrm h}} & M_{2,j+2} & \xleftarrow{\partial^{\mathrm h}} & M_{3,j+2} & \xleftarrow{\partial^{\mathrm h}} & \cdots\\
\downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \\
M_{0,j+1} & \xleftarrow{\partial^{\mathrm h}} & M_{1,j+1} & \xleftarrow{\partial^{\mathrm h}} & M_{2,j+1} & \xleftarrow{\partial^{\mathrm h}} & M_{3,j+1} & \xleftarrow{\partial^{\mathrm h}} & \cdots\\
\downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \downarrow{\scriptstyle\partial^{\mathrm v}} & & \\
M_{0,j} & \xleftarrow{\partial^{\mathrm h}} & M_{1,j} & \xleftarrow{\partial^{\mathrm h}} & M_{2,j} & \xleftarrow{\partial^{\mathrm h}} & M_{3,j} & \xleftarrow{\partial^{\mathrm h}} & \cdots
\end{array}$$

where

$$M_{p,q} = \bigoplus_{i=2j-q}^{q} \mathscr{E}^{(i)}_{i+p}(A, d^A) \otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-1}(\bar{V})\Big)_{q-i}$$

$$= \bigoplus_{r=i-p}^{i} \bigoplus_{i=2j-q}^{q} \Omega^r(A_0) \otimes \Big(\bigwedge(W) \otimes \bigwedge^{i-r}(\bar{W})\Big)_{i+p-r}$$

$$\otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-i}(\bar{V})\Big)_{q-i},$$

$$\partial^{\mathrm{h}}(\omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r} \cdot x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i})$$

$$= \delta^{d^A}(\omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r}) \cdot x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i},$$

$$\partial^{\mathrm{v}}(\omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r} \cdot x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i})$$

$$= (-1)^{i+p} \omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r} \cdot \delta^{d^C}(x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i}),$$

for

$$\omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r} \cdot x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i}$$

$$\in \mathscr{E}^{(i)}_{i+p}(A, d^A) \otimes \Big(\bigwedge(V) \otimes \bigwedge^{j-i}(\bar{V})\Big)_{q-i}$$

and $\psi^{(j)}_*$ is the morphism induced by the double complex morphism $\psi^{(j)}_{*,*} : M^{(j)} \to \mathscr{L}^{(j)}(B, d^B)$, defined by

$$\psi^{(j)}_{p,q}(\omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r} \cdot x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i})$$

$$= \psi^{(j)}_{p+q}(\omega \cdot x_w \cdot \bar{w}_1 \cdots \bar{w}_{i-r} \cdot x_v \cdot \bar{v}_1 \cdots \bar{v}_{j-i}).$$

In order to finish the proof it is enough to observe that, from Theorem 3.3, $\psi^{(j)}_{*,q}$ is a quasi-isomorphism for each $q \geq 0$. □

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