HOMOGENEOUS ORTHOGONALLY ADDITIVE POLYNOMIALS ON BANACH LATTICES

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Abstract

The main result in this paper is a representation theorem for homogeneous orthogonally additive polynomials on Banach lattices. The representation theorem is used to study the linear span of the set of zeros of homogeneous real-valued orthogonally additive polynomials. It is shown that in certain lattices every element can be represented as the sum of two or three zeros or, at least, can be approximated by such sums. It is also indicated how these results can be used to study weak topologies induced by orthogonally additive polynomials on Banach lattices.

1. Introduction

A continuous scalar-valued map $P$ on a Banach space $X$ is called a homogeneous polynomial of degree $n$ (or an $n$-homogeneous polynomial) if $P(x) = \Phi(x, \ldots, x)$, where $\Phi$ is a continuous $n$-linear form on $X$. (Vector-valued polynomials are defined similarly.) We consider only continuous polynomials here, and will therefore usually omit the adjective ‘continuous’.

A polynomial is continuous if and only if it is bounded on the unit ball of $X$. We denote by $P^{(n)}(X)$ the Banach space of $n$-homogeneous scalar-valued continuous polynomials equipped with the norm

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\|.$$

We shall use standard notation and terminology; see [4] and [13] for notation and results regarding polynomials, and [10] for notation and basic theory of Banach lattices. Recall that two elements $x, y$ in a Banach lattice are called orthogonal (or disjoint) if $|x| \wedge |y| = 0$.

Definition 1.1. Let $X$ be a Banach lattice. A polynomial $P$ on $X$ is said to be orthogonally additive if $P(x + y) = P(x) + P(y)$ whenever $x, y \in X$ are orthogonal. The set of all $n$-homogeneous orthogonally additive scalar-valued polynomials on $X$ is denoted by $P_o^{(n)}(X)$.

There are various weak topologies induced on a Banach space $X$ by the polynomials on $X$. In [9], two of the authors studied the analogs of these topologies induced by the class of orthogonally additive polynomials on $L_p$ and $l_p$. Their main
tool was a theorem of Sundaresan [20] which gave an explicit representation of $n$-homogeneous orthogonally additive polynomials on these spaces.

Our main result, Theorem 2.3, is a representation theorem for polynomials in $\mathcal{P}_n(aX)$, which generalizes [20]. It turns out that $\mathcal{P}_n(aX)$ can be identified with
the linear functionals on the $n$-concavification $X(n)$ of $X$. (In fact, the representation theorem holds for vector-valued polynomials as well.)

In Section 4 we generalize the results of [9] on weak polynomial topologies, induced by the orthogonally additive polynomials on $L_p$ and $l_p$, to general Köthe function spaces. To this end we need to study the zero sets of real-valued $n$-homogeneous orthogonally additive polynomials. This is done in Section 3. Since there has recently been some growing interest in the zero sets of real-valued homogeneous orthogonally additive polynomials in $X(q)$, we analyze these zero sets in rather more detail than is really necessary for the study of the weak topologies.

Fix $P \in \mathcal{P}_n(aX)$ and denote its zero set $P^{-1}(0)$ by $Z$ and the subspace that $Z$ generates by $H$. Note that the homogeneity of the polynomial implies that $Z$ is a symmetric cone; that is, $x \in Z$ implies that $\lambda x \in Z$ for all $\lambda \in \mathbb{R}$. Put $D_k Z = \{ \sum_{i=1}^k z_i : z_i \in Z \}$, and then $H = \bigcup_{k \geq 1} D_k Z$.

We show that many Banach lattices $X$ (including the Köthe function spaces) have the property that for suitable values of $n$ the subspace $H$ is dense in $X$ for every $n$-homogeneous orthogonally additive polynomial $P$ on $X$. In fact, the sets $D_3 Z$ or even $D_2 Z$ are already either all of $X$, or are dense in it.

We shall assume throughout the paper that the scalar field is the real field. This is important in Section 3, where we study zeros of polynomials. The results of Section 2 on the representation of orthogonally additive polynomials hold, however, with essentially the same proofs, in the complex case also. (See [10, p. 43] for the definition of a complex Banach lattice.)

2. Representation of orthogonally additive polynomials

Let $X$ be a Banach lattice. To simplify the presentation, we shall assume that $X$ is a lattice of functions on some set (or a lattice of equivalence classes of measurable functions on a measure space $(\Omega, \Sigma, \mu)$) with the usual order. Theorem 2.3 below holds for general lattices without this restriction, but the assumption simplifies the presentation and makes it more intuitive. In particular, working on a function lattice simplifies the functional calculus on $X$. For example, if $f \in X$ and $\alpha > 0$, then $f^\alpha$ is defined explicitly by $f^\alpha(s) = |f^\alpha(s)| \text{sign}(f(s))$. We shall use standard lattice inequalities without further mention (see, for example, [10, Proposition 1.d.2]).

The construction of the concavification of $X$ also becomes more direct and intuitive in the case of lattices of functions. Let $q > 1$; then the $q$-concavification of $X$ is the space $X(q) = \{ f^q : f \in X \}$ with the usual algebraic operations and order, and with the natural quasi-norm $\| f \| = \| f^{1/q} \|^q$ for $f \in X(q)$. (See [8] for information on quasi-norms and quasi-normed spaces.) To see that this is a quasi-norm, fix $f, g \in X(q)$. Then

$$\| f + g \| = \| (f + g)^{1/q} \|^q 
\leq \| (|f|^{1/q} + |g|^{1/q}) \|^q 
\leq (\| f^{1/q} \| + \| g^{1/q} \|)^q 
\leq 2^{q-1} (\| f^{1/q} \|^q + \| g^{1/q} \|^q) 
= 2^{q-1} (\| f \| + \| g \|).$$
A similar proof (see also [10, p. 54]) shows that if $X$ is $q$-convex with $q$-convexity constant $M$, then
\[
|||f_1 + \ldots + f_n||| \leq M^q(|||f_1||| + \ldots + |||f_n|||)
\]
for every $f_1, \ldots, f_n \in X(q)$. It follows that when $X$ is $q$-convex with $q$-convexity constant $M = 1$, then the quasi-norm $||| \cdot |||$ is actually a norm. When $M > 1$ the quasi-norm $||| \cdot |||$ is equivalent to the norm given by
\[
|||f|||_1 = \inf \left\{ \sum |||f_i||| : f = \sum f_i \right\}.
\]

In what follows we shall not pass to the equivalent norm even when $X$ is $q$-convex, and we shall use only the quasi-norm $||| \cdot |||$.

Note that the $q$-concavification of $L_r(\mu)$ is naturally identified with $L_{r/q}(\mu)$. It follows that it is a Banach space for $q \leq r$ and a quasi-Banach space if $r < q$.

We shall need some basic facts on Baire-1 functions on compact metric spaces; we refer to Natanson’s book [14, Chapter XV] for more details. (The book treats only functions on a closed interval, but the results and proofs are the same for a general compact metric space.)

Let $K$ be a compact metric space. A function $f$ on $K$ is said to be of Baire class 1 (or a Baire-1 function) if it is the pointwise limit of a sequence of continuous functions. The space of bounded real-valued Baire-1 functions, equipped with the supremum norm, is a Banach lattice, which we denote by $B_1(K)$. In the next two lemmas we shall use Lebesgue’s characterization of Baire-1 functions (see [14, Theorem 1, p. 141]): a real-valued function $f$ on $K$ is Baire-1 if and only if the sets $\{f > \alpha\}$ and $\{f < \alpha\}$ are $F_\sigma$ for every $\alpha \in \mathbb{R}$.

The first lemma is well known.

**Lemma 2.1.** The simple functions are dense in $B_1(K)$.

**Proof.** Fix $f \in B_1(K)$ and assume, without loss of generality, that $0 \leq f(k) \leq 1$ for every $k \in K$. Fix $N$ and put $A_i = \{k \in K : (i-1)/N < f(k) < (i+1)/N\}$ for $0 \leq i \leq N$. By Lebesgue’s theorem the sets $A_i$ are $F_\sigma$ and clearly $K = \bigcup A_i$. We now find disjoint $F_\sigma$ sets $B_i \subset A_i$ such that $\bigcup B_i = K$. (This is just [14, Lemma 2, p. 140], which we reproduce for the sake of the reader.) Indeed, since the $A_i$ are $F_\sigma$, there are closed sets $C_n$ and disjoint subsets $M_0, \ldots, M_N \subset \mathbb{N}$ such that $A_i = \bigcup_{n \in M_i} C_n$.

Put $D_n = C_n \setminus \bigcup_{i<n} C_j$. Then the $D_n$ are pairwise disjoint $F_\sigma$ sets (because closed subsets of $K$ are $G_\delta$). Hence so are the disjoint sets $B_i = \bigcup_{n \in M_i} D_n$. Clearly, $B_i \subset A_i$ and $\bigcup B_i = K$.

The simple function $g_N = \sum (i/N) \chi_{B_i}$ is then a Baire-1 function which satisfies $\|f - g_N\| \leq 1/N$. \hfill \Box

**Lemma 2.2.** Let $K$ be a compact metric space, and let $P$ be a real-valued polynomial on $C(K)$. Then $P$ extends to a polynomial $Q$ on $B_1(K)$. If $P$ is orthogonally additive, then so is $Q$.

**Proof.** The extension is given explicitly as follows: fix $g \in B_1(K)$ and a bounded sequence of continuous functions $g_j$ converging pointwise to $g$. Then put $Q(g) = \lim P(g_j)$. Since $C(K)$ has the Dunford–Pettis property, and since the $g_j$ comprise a weak Cauchy sequence in $C(K)$, it follows from Pelczyński [15, Corollary 3] that the limit actually exists and is independent of the choice of the sequence $g_j$. 
We now check that when $P$ is orthogonally additive, then so is $Q$. Indeed, fix $f, g \in B_1(K)$ with disjoint supports. By Lebesgue’s characterization of Baire-1 functions, there are two increasing sequences $F_n$ and $G_n$ of closed sets such that $\{|f| > 0\} = \cup F_n$ and $\{|g| > 0\} = \cup G_n$. For each fixed $n$ the two sets $F_n$ and $G_n$ are closed and disjoint; hence there are continuous functions $\varphi_n$ and $\psi_n$ with disjoint supports, $\|\varphi_n\| = \|\psi_n\| = 1$, so that $\varphi_n \equiv 1$ on $F_n$ and $\psi_n \equiv 1$ on $G_n$.

Now pick two bounded sequences of continuous functions, $f_n$ and $g_n$, which converge pointwise to $f$ and $g$ respectively. Then, by the construction, the products $f_n \varphi_n$ and $g_n \psi_n$ also converge pointwise to $f$ and $g$ respectively, and they clearly satisfy $|f_n \varphi_n| \wedge |g_n \psi_n| = 0$. By the orthogonal-additivity of $P$, we have

$$Q(f + g) = \lim P(f_n \varphi_n + g_n \psi_n) = \lim (P(f_n \varphi_n) + P(g_n \psi_n)) = Q(f) + Q(g).$$

**Remark 1.** In [16] Pełczyński actually shows that weakly compact vector-valued polynomials on $C(K)$ extend to the space of all bounded Baire functions (and not just to $B_1(K)$). We do not need this stronger result.

Although we are mainly interested in the representation of scalar-valued polynomials, Theorem 2.3 applies to vector-valued ones as well, so we introduce the necessary notation.

Let $X$ be a Banach (or quasi-Banach) lattice, and let $E$ be a Banach space. We denote the space of bounded linear operators from $X$ to $E$ by $L(X, E)$ and the space of $E$-valued orthogonally additive polynomials by $\mathcal{P}_0(^n X, E)$.

For each linear operator $T \in L(X(n), E)$, define a continuous $n$-homogeneous orthogonally additive polynomial $P_T$ from $X$ to $E$ by the formula $P_T(f) = T(f^n)$ for $f \in X$. Then $P_T$ is induced by the continuous $n$-linear map $A(f_1, \ldots, f_n) = T(f_1 \cdot \cdots \cdot f_n)$ and it is orthogonally additive because $(f + g)^n = f^n + g^n$ whenever $f$ and $g$ have disjoint supports. It turns out that this is the general form of such a polynomial.

**Theorem 2.3.** Let $X$ be a Banach lattice of functions, and let $E$ be a Banach space. Fix $n \in \mathbb{N}$. Then the map $T \rightarrow P_T$ is a linear isometry of $L(X(n), E)$ onto $\mathcal{P}_0(^n X, E)$. In particular, when $E$ is the scalar field, the map $\varphi \rightarrow P_\varphi$ is a surjective linear isometry between $(X(n), \|\cdot\|)$ and $\mathcal{P}_0(^n X)$.

**Proof.** We denote the quasi-norm of an operator $T \in L(X(n), E)$ by $\|T\|$; that is, $\|T\| = \sup_{\|g\| \leq 1} \|Tg\|$.

The map $T \rightarrow P_T$ is clearly linear. It is an isometry by the definition of $\|\cdot\|$:

$$\|P_T\| = \sup_{\|f\| \leq 1} \|P_T(f)\| = \sup_{\|f\| \leq 1} \|T(f^n)\| = \sup_{\|g\| \leq 1} \|T(g)\| = \|T\|.$$  

To show that the map is surjective, fix $P \in \mathcal{P}_0(^n X, E)$ and put $T(f) = P(f^{1/n})$. It is clear that $T$ is 1-homogeneous and continuous. We need only to check that $T$ is linear. It is then clear that $P = P_T$.

Thus fix $f_1, f_2 \in X(n)$ and assume first that they are simple functions. Passing to the algebra of sets generated by the atoms of $f_1$ and $f_2$, we may assume that the two functions are actually linear combinations of the same disjointly supported characteristic functions; that is, $f_1 = \sum a_i \chi_{E_i}$ and $f_2 = \sum b_i \chi_{E_i}$, where the sets $E_i$...
are disjoint. Then $(f_1 + f_2)^{1/n} = \sum (a_i + b_i)^{1/n} \chi_{E_i}$ and the orthogonal additivity and $n$-homogeneity of $P$ yield

$$T(f_1 + f_2) = P((f_1 + f_2)^{1/n}) = P\left(\sum (a_i + b_i)^{1/n} \chi_{E_i}\right)$$

$$= \sum P((a_i + b_i)^{1/n} \chi_{E_i}) = \sum (a_i + b_i)P(\chi_{E_i})$$

$$= P\left(\sum a_i^{1/n} \chi_{E_i}\right) + P\left(\sum b_i^{1/n} \chi_{E_i}\right)$$

$$= P(f_1^{1/n}) + P(f_2^{1/n}) = T(f_1) + T(f_2).$$

If the lattice $X$ is such that every element of $X$ is a limit of simple functions, then the theorem follows by approximating $f_1$ and $f_2$. To prove the additivity for a general lattice we may assume, by composing $P$ with linear functionals on $E$, that $P$ is real-valued. Only simple modifications are needed in the complex case.

The following standard construction enables us to pass from $X$ to a $C(K)$ space, with $K$ compact and metrizable. Fixing $f_1, f_2 \in X$, put $h = \|f_1\| + \|f_2\|$ and let

$$N = \{ f \in X : \|f\| \leq \lambda h \text{ for some } \lambda > 0 \}.$$

For $f \in N$, put $\|f\|_N = \inf\{\lambda > 0 : \|f\| \leq \lambda h\}$. Then $\|f\| \leq \|f\|_N$ for every $f \in N$. One easily checks that $(N, \|\cdot\|_N)$ is complete, and hence a Banach lattice under the order induced from $X$. Also, $h$ is a strong unit in $N$ and $N$ is an abstract $M$-space (see Lindenstrauss and Tzafriri [10, Definition 1.1b]). The same is true for the closed sublattice $(M, \|\cdot\|_N)$ of $N$ generated by $f_1$ and $f_2$. By Kakutani's representation theorem [10, Theorem 1.1b], $M$ is isometric and order-isomorphic to a $C(K)$-space, and since it is separable, it follows that $K$ is compact and metrizable.

Let $P_1$ be the composition of $P$ with the formal identity from $M$ to $X$. Then $P_1$ is a continuous $n$-homogeneous polynomial which is orthogonally additive because the orders on $X$ and $M$ coincide. By Lemma 2.2, $P_1$ extends to an orthogonally additive polynomial $Q$ on $B_1(K)$. Since by Lemma 2.1 every function in $B_1(K)$ is a limit of simple functions, the argument at the beginning of the proof gives us $Q((g_1 + g_2)^{1/n}) = Q(g_1^{1/n}) + Q(g_2^{1/n})$ for every $g_1, g_2 \in B_1(K)$. In particular, $P_1((f_1 + f_2)^{1/n}) = P_1(f_1^{1/n}) + P_1(f_2^{1/n})$ for the given functions $f_1$ and $f_2$, and then the same identity also holds for $P$.

Remarks 2. (i) Sundaresan [20] proved that for $1 \leq n \leq p < \infty$ the space $\mathcal{P}_o(nL_p)$ can be identified with $L_{p/(p-n)}$, and similarly for $l_p$. For $p > n$ he showed that $\mathcal{P}_o(nL_p) = \{0\}$ and $\mathcal{P}_o(nl_p) = l_\infty$. These are special cases of Theorem 2.3 and the known representations of the duals of $L_r$ and $l_r$ for $0 < r < \infty$.

(ii) The case $X = C(K)$ of Theorem 2.3 was recently proved independently by Pérez-García and Villanueva [17].

(iii) It should be noted that the representation theorem is trivial for discrete lattices: that is, for spaces with an unconditional basis $\{e_j\}$. Indeed, if $f = \sum a_j e_j$, then $P(f^{1/n}) = P(\sum a_j^{1/n} e_j) = \sum a_j P(e_j)$, which is clearly linear.

(iv) For earlier results on representation of orthogonally additive functions on certain classes of Banach lattices see, for example, [5, 6, 11, 12, 18, 20] and their references.

From now on we shall turn to real-valued polynomials, that is, to $\mathcal{P}_o(nX)$. The successful application of the theorem depends on a good description of the dual of $X_{(n)}$. To this end, we shall restrict ourselves from now on to Köthe function spaces.
Definition 2.4. A Banach lattice $X$ of equivalent classes of locally integrable measurable functions on a complete $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ is called a Köthe function space if the following characteristics hold.

- If $g \in X$ and if $f$ is measurable and $|f(\omega)| \leq |g(\omega)|$ almost everywhere, then $f \in X$ and $\|f\| \leq \|g\|$.
- $\chi_E \in X$ for every $E \in \Sigma$ with finite measure.

The class of Köthe function spaces contains many of the common Banach lattices. Moreover, by [10, Theorem 1.b.14] every order continuous Banach lattice with a weak unit is isomorphic as a Banach space and as a lattice to a Köthe function space.

We shall use the easy fact that order continuous Köthe function spaces satisfy the dominated convergence theorem: if $f_n \to 0$ in measure and if there is a $g \in X$ so that $|f_n| \leq g$ for every $n$, then $\|f_n\| \to 0$.

By the discussion in [10, p. 29] it follows that when $X$ is an order continuous Köthe function space, then its dual is given by integrals. More precisely, every continuous linear functional on $X$ is given by $\varphi(f) = \int f \xi \, d\mu$, where $\xi$ is a measurable function such that $f \xi \in L_1(\mu)$ for every $f \in X$. This representation also holds for quasi-Banach lattices, and hence for functionals on $(X_{(n)}, \|\cdot\|)$ (which is order continuous whenever $X$ is). Note, however, that when $X$ is not $n$-convex it may very well happen that $(X_{(n)}, \|\cdot\|)^* = \{0\}$. This happens, for example, when $X = L_p$ and $p < n$.

We summarize the results of this section in the manner in which they will be used in the subsequent sections.

Corollary 2.5. Let $X$ be an order continuous Köthe function space. Then every $n$-homogeneous orthogonally additive polynomial $P \in \mathcal{P}_o^{(n)}X$ can be represented as

$$ P(f) = \int f^n \xi \, d\mu $$

for some measurable function $\xi$ on $\Omega$.

3. Sums of zeros

In this section we study the zero sets of $n$-homogeneous orthogonally additive polynomials on order continuous Köthe function spaces and the subspaces that these zero sets generate. Recall that the zero set of the polynomial $P$ is denoted by $Z$, and that $D_k Z = \{\sum_{j=1}^k z_j : z_j \in Z\}$. We shall always assume that $\mu$ is a nonnegative measure.

Theorem 3.1. Let $X$ be an order continuous Köthe function space, and let $P(f) = \int f^n \xi \, d\mu$ be a $n$-homogeneous orthogonally additive polynomial on $X$.

(i) If $n$ is even and $\xi$ does not have constant sign, then $X = D_2 Z$.

(ii) If $n > 1$ is odd and $\int_A \xi \, d\mu \neq 0$ for at least three disjoint measurable subsets $A$, then $X = D_3 Z$.

Proof. Fix $f \in X$, and we first prove statement (i). Denote the restrictions of $f$ to the disjoint sets $\{\xi > 0\}$ and $\{\xi \leq 0\}$ by $f_1$ and $f_2$ respectively, and put $a_j = P(f_j)$. Then $a_1 \geq 0 \geq a_2$ (because $n$ is even). If $a_1 \neq 0$ and $a_2 \neq 0$, choose
\( \lambda \in \mathbb{R} \) such that \( a_1 + \lambda^n a_2 = 0 \). It then follows that both \( z_1 = f_1 + \lambda f_2 \) and \( z_2 = f_1 - \lambda f_2 \) are in \( Z \) and
\[
f = \frac{\lambda + 1}{2\lambda} z_1 + \frac{\lambda - 1}{2\lambda} z_2.
\]

If \( a_2 = 0 \) (say), then take any \( g \) supported in \( \{ \xi < 0 \} \) such that \( P(g) = -a_1 \), and then \( P(f \pm g) = 0 \) and \( f = (f + g)/2 + (f - g)/2 \).

To prove part (ii), choose three disjointly supported functions \( f_1, f_2, f_3 \) with \( P(f_i) \neq 0 \) such that \( f \) is a linear combination of the functions \( f_i \). (This is made possible by the assumption on the measure \( \xi d\mu \).) Put \( a_i = P(f_i)^{-1/n} \), and let
\[
\begin{align*}
z_1 &= a_1 f_1 - a_2 f_2; \\
z_2 &= a_2 f_2 - a_3 f_3; \\
z_3 &= a_1 f_1 + a_2 f_2 - 2^{1/n} a_3 f_3.
\end{align*}
\]

Clearly, \( z_i \in Z \), and since the matrix
\[
\begin{pmatrix}
a_1 & -a_2 & 0 \\
0 & a_2 & -a_3 \\
a_1 & a_2 & -2^{1/n} a_3
\end{pmatrix}
\]
is invertible, it follows that the values of \( z_i \) and \( f_i \) span the same three-dimensional subspace of \( X \). In particular, \( f \) is a linear combination of the functions \( z_i \); that is, \( f \in D_3Z \).

**Remark 3.** The conditions of the theorem are necessary for the following reasons.

(i) If \( n \) is even and \( \xi \) has constant sign, then clearly \( Z = \{ f \in X : f \xi = 0 \text{ a.e. } d\mu \} \); that is, \( f \in Z \) if and only if the support of \( f \) is disjoint from the support of \( \xi \).

(ii) If \( n > 1 \) is odd and the measure \( \xi d\mu \) consists of just two atoms, then \( Z \) is a one-codimensional subspace. Indeed, every \( f \in X \) is constant on the two atoms, and we denote these values by \( f_1 \) and \( f_2 \) respectively. Also, we denote by \( \alpha_1, \alpha_2 \) the \( \xi d\mu \) measures of the atoms. Then \( Z = \{ f \in X : f_1 \alpha_1^{1/n} = f_2 \alpha_2^{1/n} \} \).

(iii) If \( n > 1 \) is odd, then we really need to pass to \( D_3Z \), and it is no longer true that \( X = D_2Z \). A simple example is \( P(f) = \int f^3 d\mu \) (that is, \( \xi \equiv 1 \)) on \( L_p[0, 1] \) for \( p \geq 3 \). In this case \( \chi_E \notin D_2Z \) for any measurable set of positive measure. Indeed, assume for a contradiction that \( \chi_E = f_1 + f_2 \) with \( f_i \in Z \) and choose \( g \) such that \( f_1 = \frac{1}{2} \chi_E + g \) and \( f_2 = \frac{1}{2} \chi_E - g \). Then
\[
0 = P(f_1) + P(f_2) = \int \left( \frac{1}{2} \chi_E + g \right)^3 + \int \left( \frac{1}{2} \chi_E - g \right)^3 = \int \chi_E \left( \frac{1}{4} + 3g^2 \right),
\]
which is impossible because \( E \) has positive measure.

The next result shows that when \( n > 1 \) and the measure space is non-atomic, then \( D_2Z \) (which — as we saw above — is not necessarily equal to \( X \)) is at least dense in it.

**Theorem 3.2.** Let \( X \) be an order continuous Köthe function space on a non-atomic measure space. Let \( n > 1 \) be an odd integer, and let \( P(f) = \int f^n \xi d\mu \). Then the set \( D_2Z \) is dense in \( X \).
Proof. Fix \( f \in X \); we start with a few reductions. We may assume that \( f \) and \( \xi \) are nonnegative. Indeed, \( \Omega \) decomposes as the disjoint union of four sets, on each of which \( f \) and \( \xi \) have constant sign, and it suffices to approximate \( f \) on each of these sets separately. We may also assume by approximation that the support of \( f \), which we denote by \( E \), has finite measure and then, by normalizing \( \mu \) and \( \xi \), that \( \mu(E) = P(\chi_E) = 1 \).

Fix \( m \in \mathbb{N} \) and put \( t = t_m = (m + 1)^n / (2m + 1) \).

Claim. There is a partition of \( E \) to \( 2m + 2 \) disjoint sets \( B \) and \( \{ A_j \}_{j=-m}^m \) such that
\[
\mu(A_j) = \frac{1}{(2m + 1)(t + 1)} = P(f\chi_{A_j})
\]
for every \( j \), and such that
\[
\mu(B) = \frac{t}{t + 1} = P(f\chi_B).
\]

Indeed, note that \( \nu(A) = P(f\chi_A) = \int_A f^n \xi d\mu \) is a non-atomic probability measure on \( E \) (by our normalization that \( P(\chi_E) = 1 \)). By Liapounoff’s theorem (see [19, Theorem 5.5]), the range of the vector measure \( (\mu, \nu) \) is convex. Since \( \mu(\emptyset) = \nu(\emptyset) = 0 \) and \( \mu(E) = \nu(E) = 1 \), it follows that there is a subset \( A_{-m} \subset E \) such that
\[
\mu(A_{-m}) = \nu(A_{-m}) = \frac{1}{(2m + 1)(t + 1)}.
\]
The set \( A_{-m+1} \) is obtained similarly by applying Liapounoff’s theorem to \( E \setminus A_{-m} \). We continue inductively to obtain the other subsets \( A_j \), and then we take \( B = E \setminus \bigcup_{-m}^m A_j \). This proves the claim.

We define two functions by
\[
g_m(\omega) = \begin{cases} 
  f(\omega) & \text{if } \omega \in B, \\
  jf(\omega) & \text{if } \omega \in A_j, \ j \geq 1, \\
  (j-1)f(\omega) & \text{if } \omega \in A_j, \ j \leq 0,
\end{cases}
\]
and
\[
h_m(\omega) = \begin{cases} 
  0 & \text{if } \omega \in B, \\
  (j+1)f(\omega) & \text{if } \omega \in A_j, \ j \geq 1, \\
  0 & \text{if } \omega \in A_0, \\
  (j-1)f(\omega) & \text{if } \omega \in A_j, \ j \leq -1,
\end{cases}
\]
and we check that they are in \( Z \). To show that \( P(g_m) = 0 \), write (by the orthogonal additivity of \( P \))
\[
P(g_m) = \sum_{j=1}^m \left( P(g_m\chi_{A_j}) + P(g_m\chi_{A_{1-j}}) \right) + P(g_m\chi_{A_{-m}}) + P(g_m\chi_B).
\]
For each \( 1 \leq j \leq m \) the \( j \)th term in the sum vanishes because
\[
P(g_m\chi_{A_j}) = j^n P(f\chi_{A_j}) = \frac{j^n}{(2m + 1)(t + 1)} = -P(g_m\chi_{A_{1-j}}).
\]
The remaining term also vanishes. Indeed, \( P(g_m \chi_B) = P(f \chi_B) = t/(t+1) \) and the choice of \( t \) and of \( A_{-m} \) gives

\[
P(g_m \chi_{A_{-m}}) = -(m+1)^n P(f \chi_{A_{-m}}) = \frac{-(m+1)^n}{(2m+1)(t+1)} = \frac{-t}{1+t}.
\]

We omit the computation (similar to the first computation above) which shows that \( P(h_m) = 0 \).

Since

\[
(g_m - h_m)(\omega) = \begin{cases} f(\omega) & \omega \in B, \\ -f(\omega) & \omega \in A_j, j \geq 0, \\ 0 & \omega \in A_j, j \leq -1,
\end{cases}
\]

it follows that \( f - (g_m - h_m) = f \cdot (2\chi_{\cup_{j \geq 0} A_j} + \chi_{\cup_{j \leq -1} A_j}) \). But

\[
\mu(\cup A_j) = \frac{1}{t+1} < \frac{2m+1}{(m+1)^n} \to 0 \quad \text{as } m \to \infty.
\]

Thus \( g_m - h_m \to f \) by the dominated convergence theorem in the order continuous lattice \( X \).

4. Weak polynomial topologies

In analogy with the weak topology \( \omega \) on a Banach space, it is natural to define the weak polynomial topology \( wp \), where a net \( x_\alpha \) converges to \( x \) if and only if \( P(x_\alpha) \to P(x) \) for every polynomial \( P \) on \( X \) (see Carne, Cole and Gamelin [3]). It turns out that \( wp \) is not a vector space topology: addition, although clearly continuous in each variable separately, is not necessarily continuous as a function of two variables. This led Garrido, Jaramillo and Llavona [7] to introduce and study the maximal locally convex topology, \( \tau_p \), weaker than \( wp \). It is given by the seminorms

\[
d_P(x) = \inf\{ |P(y_1 - y_0)|^{1/n} + |P(y_2 - y_1)|^{1/n} + \ldots + |P(x - y_k)|^{1/n} \}
\]

where \( P \) is a \( n \)-homogeneous polynomial, and the infimum is taken over all \( k \)-chains \( \{0 = y_0, y_1, \ldots, y_k = x\} \).

Note that the seminorm associated with a linear functional \( \varphi \in X^* \) is given by \( d_\varphi(x) = |\varphi(x)| \); hence the weak topology \( \omega \) satisfies \( \omega \subset \tau_p \). One can also easily check that \( \tau_p \subset wp \subset \| \cdot \| \), where \( \| \cdot \| \) denotes the norm topology on \( X \).

Lassalle and Llavona [9] introduced analogous topologies on Banach lattices. These topologies are defined similarly by using only the orthogonally additive polynomials; they are denoted by \( wp_a \) and \( \tau \) respectively. As before \( \tau \subset wp_a \).

When \( X \) is an order continuous Köthe function space, every linear functional is given by an integral, and hence is orthogonally additive. It follows that \( \omega \subset \tau \).

The paper [9] is devoted to the study of the special case of \( l_p \) and \( L_p \). Their main result in this direction is as follows.

**Theorem 4.1.** A net \( \{x_\alpha\} \) in \( l_p \) is \( \tau \)-convergent to \( x \) if and only if it is weakly convergent to \( x \) and \( \|x_\alpha - x\|_{2k} \to 0 \), where \( k \) is the largest integer with \( 2k \leq p \). Similarly, a net \( \{x_\alpha\} \) in \( l_p \) is \( \tau \)-convergent to \( x \) if and only if it is weakly convergent to \( x \) and \( \|x_\alpha - x\|_{2k} \to 0 \), where \( k \) is the smallest integer with \( 2k \geq p \).
The main tool in [9] is Sundaresan’s identification of orthogonally additive \( n \)-homogeneous polynomials on \( L_p \) as the dual of \( L_{p/n} \) (or, respectively, \( l_p/n \)). By Theorem 2.3 this tool is now available in general lattices.

The representation of orthogonally additive polynomials gives explicit formulas for the seminorms \( d_P \). This makes it possible to analyze their zero sets and the subspaces generated by these zeros, as we did in Section 3. This is important for the analysis of the \( \tau \) topology: it was observed in [7] and [9] that if \( P \) is a homogeneous polynomial on \( X \) and \( z \in H = \text{span}\{P^{-1}(0)\} \), then \( d_P(x) = d_P(x + z) \) for every \( x \in X \). Indeed, assume that \( z = \sum_{j=1}^m z_j \) with \( P(z_j) = 0 \), and put \( y_i = \sum_{j=1}^i z_j \). Extending any chain from 0 to \( x \) by adjoining successively the \( y_i \) at the end of the given chain gives a new chain from 0 to \( x + z \) so that \( P \) is zero on the new differences. Hence \( d_P(x + z) \leq d_P(x) \), and \( d_P(x) \leq d_P(x + z) \) similarly. It follows in particular that

\[
d_P \equiv 0 \quad \text{for every polynomial } P \text{ for which } H \text{ is dense in } X.
\]

It follows that in the analysis of \( \tau \) we need only consider polynomials whose zero sets do not span a dense subspace of \( X \).

**Corollary 4.2.** Let \( X \) be an order continuous Köthe function space on a measure space \( (\Omega, \Sigma, \mu) \), and let \( x_\alpha \) be a net in \( X \) which converges weakly to \( x \). Then \( x_\alpha \xrightarrow{\tau} x \) if and only if \( d_P(x_\alpha - x) \to 0 \) for every \( n \)-homogeneous orthogonally additive polynomial \( P(f) = \int f^n \xi \, d\mu \) on \( X \) with \( n \) even and \( \xi \) nonnegative.

**Proof.** By Theorem 3.1, \( d_P \equiv 0 \) unless \( P \) is either as above, or when \( n = 1 \), or when \( n > 1 \) is odd and \( \xi d\mu \) has just two nonzero atoms. When \( n = 1 \), then \( d_P(x) = |P(x)| \) and convergence in \( d_P \) is just weak convergence. In the third case, write \( P(f) = f_1^n \alpha_1 + f_2^n \alpha_2 \) and consider the linear functional \( \varphi(f) = f_1 \alpha_1 + f_2 \alpha_2 \). Then one can directly check that \( d_P(f_\alpha - f) \to 0 \) if and only if \( \varphi(f_\alpha - f) \to 0 \). \( \square \)

Theorem 4.1 for \( L_p \), say, follows immediately, because by Remark 2(i) the only non-zero continuous homogeneous orthogonally additive polynomials on \( L_p \) are of degree \( n \leq p \), and by Corollary 4.2 only the even-degree ones give nontrivial seminorms and influence the topology. The fact that \( L_p \) is a rearrangement invariant function space and the density of simple functions in \( L_{p/n}^* = L_p/(n-p) \) easily implies that we may assume that \( \xi \equiv 1 \). Finally, Hölder’s inequality implies that we need only to consider the largest admissible even \( n \).

The same procedure can, of course, be used for other lattices, although we shall not give detailed examples here. We just mention that one can explicitly identify the concavifications of, say, Orlicz or Lorentz spaces, and then use them to give analogous results on \( \tau \) for these lattices.

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ORTHOGONALLY ADDITIVE POLYNOMIALS

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