Noncommutative Landau problem for particle-vortex system

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We study the problem of a charged particle in the presence of a uniform magnetic field plus a vortex in a noncommutative plane considering two non-commutative extensions of the corresponding Hamiltonian, namely "fundamental" and "antifundamental" representations. Using a Fock space formalism we construct eigenfunctions and eigenvalues finding in each case half of the states existing in the ordinary space case. In the limit of $\theta \to 0$ we recover the two classes of states found in ordinary space, relevant for the study of anyon physics.

Keywords: Noncommutative quantum mechanics; charged particles; vortices; anyons.

Estudiamos el problema de una partícula cargada en un campo magnético uniforme y un vórtice en un plano no-conmutativo, considerando dos extensiones no-conmutativas del hamiltoniano, las representaciones "fundamental" y "antifundamental". Usando el formalismo de espacios de Fock, construimos las autofunciones y los autovalores, hallando en cada caso la mitad de los estados existentes en espacio ordinario. En el límite $\theta \to 0$, recuperamos las dos clases de estados encontrados en espacio ordinario, relevantes para el estudio de aniones.

Descriptores: Mecánica Cuántica no-conmutativa; partículas cargadas; vórtices, aniones.

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The Landau and the Aharonov-Bohm problems are the two paradigms of planar quantum mechanics of charged particles in a magnetic field [1]. Inspired by recent observations on the relevance of noncommutative geometry for describing the Quantum Hall effect [2], both problems have been considered in noncommutative space [3, 4].

Adding a uniform magnetic field plus other vortex-like, turns the problem even more interesting. This system was solved in ordinary space in Ref. 5, in the context of anyon physics, where particles with exotic statistic are represented by charges pierced by a magnetic flux. In this regard, the problem of two anyons in an external magnetic field is equivalent to that of a particle in the presence of an external magnetic field plus a vortex. In Ref. 5, eigenstates and spectrum in ordinary space are found analytically by modifying in a simple but subtle way, the usual Landau problem ladder operator formalism.

We shall consider the noncommutative generalization of this problem. Namely, we shall solve the Schrödinger equation for a charged particle in the presence of a uniform magnetic field plus a vortex, when space coordinates satisfy

$$[x^1, x^2] = i\theta \ . \tag{1}$$

It is convenient to introduce complex variables and, associated to them, annihilation and creation operators a and $a^\dagger$:

$$\frac{z}{\sqrt{\theta}} \to a = \frac{x^1 + ix^2}{\sqrt{2\theta}}\,, \qquad \frac{\bar{z}}{\sqrt{\theta}} \to a^\dagger = \frac{x^1 - ix^2}{\sqrt{2\theta}}\,, \tag{2}$$

so that (1) becomes

$$[a, a^\dagger] = 1 \ . \tag{3}$$

In this framework, the connection between operators and function is $|m\rangle\langle m| \to 2(-1)^m L_m(2r^2/\theta) e^{-\frac{r^2}{\theta}}$ and derivatives are given by

$$\partial_z = -\frac{1}{\sqrt{\theta}}[a^\dagger,\ ]\,, \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{\theta}}[a,\ ]\,. \tag{4}$$

The field strength in terms of the complex vector potential components is

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + ie[A_z, A_{\bar{z}}]\,. \tag{5}$$

As stated above, we will consider the field strength

$$F_{z\bar{z}} = iB^0 - i\frac{\alpha}{e\theta}|0\rangle\langle 0|\,. \tag{6}$$

In the $\theta \to 0$ limit, the second term goes, in configuration space, to a delta function corresponding to a singular vortex, with flux related to the real parameter α [8].

In noncommutative space, even for an Abelian gauge theory one can define covariant derivatives in "fundamental" and "anti-fundamental" representations,

$$\begin{aligned} D_\mu\Psi &= \partial_\mu\Psi + ieA_\mu\Psi && \text{fundamental} \\ D_\mu\Psi &= \partial_\mu\Psi - ie\Psi A_\mu && \text{anti-fundamental}. \end{aligned} \tag{7}$$

We shall first discuss the case of the fundamental representation and then extend the results to the anti-fundamental.

The time-independent Schrödinger equation of the wave function Ψ is given by

$$H\Psi(\bar{z}, z) = -\frac{1}{2m}\left[D_z, D_{\bar{z}}\right]_+ \Psi = E\Psi(\bar{z}, z)\,. \tag{8}$$

Concerning the angular momentum operator L, it is useful to define the "covariant" position w and $\bar{w}$ [2],

$$w = z + ie\theta A_{\bar{z}} \quad \bar{w} = \bar{z} - ie\theta A_z \,. \tag{9}$$

In noncommutative space the appropriate L-operator can be defined as [3, 6]

$$L = wD_z - \bar{w}D_{\bar{z}} - \frac{eB^0}{2\,(1 - eB^0\theta)}\,[\bar{w}, w]_+ - \frac{\theta}{2}\,[D_{\bar{z}}, D_z]_+ \,. \tag{10}$$

Notice that L in (10) is the generalization to noncommutative space of the so called "mechanical angular momentum" [7].

We introduce two pairs of creation and annihilation operators, the noncommutative analogue of those developed for the usual Landau problem [5]. They are defined as

$$\begin{aligned} c_1 &= \frac{1}{\sqrt{|eB^0|}}D_z, & c_2 &= -\frac{1}{\sqrt{|eB^0|}}D_{\bar{z}}, \\ d_1 &= \frac{1}{\sqrt{|\kappa|}}(w + \kappa D_{\bar{z}}), & d_2 &= \frac{1}{\sqrt{|\kappa|}}(\bar{w} - \kappa D_z), \end{aligned} \tag{11}$$

where

$$\kappa = \frac{1 - eB^0\theta}{eB^0} \,. \tag{12}$$

One can verify that with this definition

$$\begin{aligned} [c_1, c_2] &= \mathrm{sgn}(eB^0) - \frac{\alpha}{\theta|eB^0|}|0\rangle\langle 0|, \\ [d_1, d_2] &= \mathrm{sgn}\,\kappa + \frac{\alpha}{\theta|eB^0(1 - eB^0\theta)|}|0\rangle\langle 0|, \\ [c_1, d_1] &= \frac{\alpha}{eB^0\theta\sqrt{|1 - eB^0\theta|}}|0\rangle\langle 0|, \qquad (13) \\ [c_2, d_2] &= -\frac{\alpha}{eB^0\theta\sqrt{|1 - eB^0\theta|}}|0\rangle\langle 0|, \\ [c_1, d_2] &= [c_2, d_1] = 0. \qquad (14) \end{aligned}$$

Note that for those states χ such that $|0\rangle\langle 0|\chi = 0$ operators c_i's and d_i's will play, depending on the sign of eB^0 and κ, the role of creation $(c^\dagger, d^\dagger)$ or annihilation (c, d) operators. We shall call $\mathcal{P}_0$ the subspace of states satisfying this condition. In the commutative space limit the condition becomes the "hard-core condition" since it corresponds to the vanishing of wave functions at the origin. Moreover, in the $\theta \to 0$ limit, the algebra (14) coincides with that obtained in Ref. 5.

The Hamiltonian (8) takes the form

$$H = \frac{\omega}{2}\left[c, c^\dagger\right]_+ \,, \quad \text{with}\ \omega = \frac{|eB^0|}{m} \,. \tag{15}$$

For states restricted to $\mathcal{P}_0$, the angular momentum is

$$\begin{aligned} L &= \mathrm{sgn}\kappa(c^\dagger c - d^\dagger d) && \text{for} \quad 1 - eB^0\theta > 0, \\ L &= -(c^\dagger c + d^\dagger d) && \text{for} \quad 1 - eB^0\theta < 0, \end{aligned} \tag{16}$$

and $[H, L] = 0$. Thus, one can to find common eigenfunctions of H and L

Now, we have to write an explicit expression for the vector potential leading to a field strength (6). A possible choice for $1 - eB^0\theta > 0$ and positive α is

$$A_z = \frac{i}{\sqrt{\theta}}g(N)a^\dagger \;\; , \;\; A_{\bar{z}} = -\frac{i}{\sqrt{\theta}}ag(N) \tag{17}$$

with

$$g(N) = -\frac{1}{e}\left(1 - \sqrt{1 - eB^0\theta + \frac{\alpha}{N}}\right) . \tag{18}$$

We shall now construct the Fock space associated to the operators $N_c = c^\dagger c$ and $N_d = d^\dagger d$. Let us start by considering a state χ such that

$$c\chi = 0 \,. \tag{19}$$

Then, χ is an eigenstate of the Hamiltonian. In order to make χ also an eigenstate of L, we propose the following ansatz

$$\chi(a, a^\dagger) = a^{\dagger n}h(N) \,. \tag{20}$$

Note that in the commutative limit $a^{\dagger n}$ can be connected to $\exp(-in\varphi)$, an eigenfunction of the canonical angular momentum with eigenvalue $-n$.

One can see that $\chi \in \mathcal{P}_0$ provided $n > 0$. Given L as in (16), the ansatz is consistent only for $0 < eB^0\theta < 1$; we shall study other regimes later on. One has,

$$L\chi = -\left(n + \bar{\alpha}\right)\chi \;, \quad \text{with} \quad \bar{\alpha} = \frac{\alpha}{1 - eB^0\theta} \,. \tag{21}$$

We start solving (19) for $n = 1$ and denote this eigenstate as χ_{01}. We obtain

$$\chi_{01}(z, \bar{z}) = \bar{z} * \frac{2}{\sqrt{\theta}}\sum_m (-1)^m h_m L_m(2r^2/\theta)e^{-\frac{r^2}{\theta}} \,, \tag{22}$$

where

$$h_m = \left(\frac{\Gamma(m + 2 + \bar{\alpha})}{(m + 1)!\,\Gamma(2 + \bar{\alpha})}\frac{(eB^0\theta)^{2+\bar{\alpha}}\left(1 - eB^0\theta\right)^m}{2\pi\theta}\right)^{1/2} \tag{23}$$

A tower of states with increasing energy can be constructed from χ_{01} by acting with $c^\dagger$ and $d^\dagger$,

$$\begin{aligned} \chi_{kl} &= \left(c^\dagger\right)^k \left(d^\dagger\right)^{l-1}\chi_{01}, && l > k, \\ E_k &= \omega\left(k + \frac{1}{2}\right) && 0 < eB^0\theta < 1, \\ L_{kl} &= (k - l - \bar{\alpha}). \end{aligned} \tag{24}$$

Being $l > k$, one can see that $\chi_{kl} \in \mathcal{P}_0$. In the $\theta \to 0$ limit, states (24), coincide with the corresponding "class II" states found in Ref. 5.

Inspired by the ordinary space results, one could try to construct a second class of states using, the condition $d\eta = 0$. In contrast with what happens in ordinary space, there is no solution belonging to $\mathcal{P}_0$ in the region $0 < eB^0\theta < 1$.

Region $eB^0\theta < 0$ can be studied analogously. In this case $\kappa < 0$ and then c_1, c_2 (and also d_1, d_2) interchange their roles of creation and annihilation operators. Now, one starts solving $d\eta_{10} = 0$ with the ansatz $\eta_{10} = a^\dagger f(N)$. The tower of solutions with increasing energy is now given by

$$\begin{aligned} \eta_{kl} &= \left(c^\dagger\right)^{k-1} \left(d^\dagger\right)^{l} \eta_{10}, \qquad k > l, \\ E_k &= \omega\left(k + \frac{1}{2} - \bar{\alpha}\right), \qquad eB^0\theta < 0, \\ L_{kl} &= (l - k - \bar{\alpha}). \end{aligned} \tag{25}$$

Again, eigenstates and eigenvalues coincide, in the $\theta \to 0$ limit with those called "class I" in ordinary space [5].

We have not found states in the $eB^0\theta > 1$ region. There, one has to modify the ansatz for the vector potential, adding to A_z a term proportional to z. In that case we were not able to construct the tower of eigenstates belonging to $\mathcal{P}_0$. Remarkably, this region does not exist in the $\theta \to 0$ limit.

The analysis for the anti-fundamental representation follows the same steps. One defines operators c_i and d_i as in (11) but with the covariant position and derivative in the anti-fundamental representation. Now, we restrict states to a subspace $\tilde{\mathcal{P}}_0$ such that, on it, the operator algebra reduces to the canonical one. The condition reads $\tilde{\chi} \in \tilde{\mathcal{P}}_0 \Leftrightarrow \tilde{\chi}|0\rangle\langle 0 = 0$ One finds the lowest state of the $\tilde{\chi}_{01} = s(N)a$ adjusting $c\tilde{\chi}_{01} = 0$, and from it one construct a tower of states through

$$\begin{aligned} \tilde{\chi}_{kl} &= \left(c^\dagger\right)^{k} \left(d^\dagger\right)^{l-1} \tilde{\chi}_{01}, \qquad l > k, \\ E_k &= \omega\left(k + \frac{1}{2}\right), \qquad 0 < eB^0\theta < 1, \\ L_{kl} &= -(k - l - \bar{\alpha}). \end{aligned} \tag{26}$$

Analogously, from $\tilde{\eta}_{10} = t(N)a$, one has

$$\begin{aligned} \tilde{\eta}_{kl} &= \left(c^\dagger\right)^{k-1} \left(d^\dagger\right)^{l} \tilde{\eta}_{10}, \qquad k > l, \\ E_k &= \omega\left(k + \frac{1}{2} - \bar{\alpha}\right) \qquad eB^0\theta < 0, \\ L_{kl} &= -(l - k - \bar{\alpha}). \end{aligned} \tag{27}$$

Note that these results coincide with those for the fundamental representation except for the sign in the angular momentum eigenvalues.

It is interesting to connect our results with those in ordinary space [5] in the discussion of the physics of two anyons in a uniform magnetic field. Using the analogue of operators c_i and d_i, in Ref. 5 two classes of eigenstates are constructed, one for which the energy is α-dependent (class I) and the other which exhibits an α-independent energy (class II). Concerning the angular momentum, the two classes have opposite sign eigenvalues. Class I states (classically "incorrect" circulation) have an energy which is shifted by α.

In noncommutative space only one class of states can be constructed both for the fundamental and the antifundamental representations. Indeed, in the fundamental, for uniform magnetic field in the range $0 < eB^0 < 1/\theta$, only class II (α-independent energy) states exist while for a range $eB^0 < 0$ the only possible states are class I type with α-dependent energy; concerning the antifundamental representation, the same phenomenon happens. Now, in the $\theta \to 0$ limit the fundamental representation (with charge e) and the anti-fundamental representation (with charge $-e$) merge, so that we have both classes of solutions in the whole range of values of eB^0, recovering Johnson and Canright result.

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