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NP-completeness results for edge modification problems

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Abstract

The aim of edge modification problems is to change the edge set of a given graph as little as possible in order to satisfy a certain property. Edge modification problems in graphs have a lot of applications in different areas, and many polynomial-time algorithms and NP-completeness proofs for this kind of problems are known. In this work we prove new NP-completeness results for these problems in some graph classes, such as interval, circular-arc, permutation, circle, bridged, weakly chordal and clique-Helly graphs.

Keywords: Computational complexity; Edge modification problems; Graph classes; NP-completeness

1. Introduction

Edge modification problems are concerned with making small changes to the edge set of an input graph in order to obtain a graph with a desired property. They include completion, deletion and editing problems. Let $G=(V,E)$ be a graph. Consider a graph property Π (for example, whether the graph belongs to a certain class). For a given integer number k, the *Π-editing decision problem* consists in deciding the existence of a set F of unordered pairs of different vertices of V, with $|F| \leqslant k$, such that the graph $G'=(V, E \triangle F)$ satisfies Π (here $E \triangle F$ denotes the symmetric difference between E and F). The *Π-deletion decision problem* only allows edge deletions (i.e., $F \subseteq E$). This problem is equivalent to finding a maximum generator subgraph of G (a subgraph with the same set of vertices than G) with the property Π. On the other hand, the *Π-completion decision problem* only allows the addition of edges (i.e., $F \cap E = \emptyset$). Equivalently, we seek for a minimum generator supergraph of G with the property Π.

Given a graph property Π, an instance of Π-deletion (completion or editing) is a pair $\langle G=(V,E), k\rangle$, composed by a graph and an integer.

Let Π be a graph property. If F is a set of non-edges such that $G'=(V, E \cup F)$ satisfies Π and $|F| \leqslant k$, then F is a k-completion set with respect to Π, or a *Π k-completion* set. Similarly, it is possible to define a *Π k-deletion* set and a *Π k-editing* set.

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Table 1
Summary of complexity results for some edge modification problems

Graph Classes	Completion	Deletion	Editing
Perfect	NPC [26,6]	NPC [26,6]	NPC [26,6]
Chordal	NPC [35]	NPC [30]	NPC [30]
Interval	NPC [14,35]	NPC [16]	**NPC**
Unit Interval	NPC [35]	NPC [16]	**NPC**
Circular-Arc	NPC [30]	NPC [30]	**NPC**
Unit Circular-Arc	NPC [30]	NPC [30]	**NPC**
Proper Circular-Arc	NPC [30]	NPC [30]	**NPC**
Chain	NPC [35]	NPC [30]	?
Comparability	NPC [20]	NPC [36]	NPC [26]
Cograph	NPC [12]	NPC [12]	?
AT-Free	?	NPC [30]	?
Threshold	NPC [23]	NPC [23]	?
Bipartite	irrelevant	NPC [15]	NPC [15]
Split	NPC [26]	NPC [26]	P [21]
Cluster	P [30]	NPC [12]	NPC [30]
Trivially Perfect	NPC [35]	NPC [30]	?
Permutation	**NPC**	**NPC**	**NPC**
Circle	**NPC**	**NPC**	**NPC**
Weakly Chordal	**NPC**	**NPC**	?
Bridged	?	**NPC**	?
Clique-Helly Circular-Arc	**NPC**	**NPC**	**NPC**
Clique-Helly Chordal	**NPC**	**NPC**	**NPC**
Clique-Helly Perfect	**NPC**	**NPC**	**NPC**
Clique-Helly Comparability	**NPC**	**NPC**	**NPC**
Clique-Helly Permutation	**NPC**	**NPC**	**NPC**

Boldfaced results are obtained in this work, “NPC” indicates an NP-complete problem, “P” a polynomial problem, and “?” an open problem.

Edge modification problems have applications in several areas, such as molecular biology and numerical algebra (see, for example, [1,16,18,29]).

Physical Mapping is a central problem in molecular biology and the human genome project. It comprises the recovery of the relative position of fragments of DNA along the genome from information on their pairwise overlaps. A simplified model of this problem considering false negative and positive errors in the experimental data leads to edge-modification problems in interval or unit interval graphs [16].

Bacterial DNA and cytoplasmic DNA in animals exist in closed circular form. Furthermore, giant DNA molecules in higher organisms form loop structures held together by protein fasteners in which each loop is largely analogous to closed circular DNA. In this case, similar models lead to edge-modification problems in circular-arc graphs.

The computational complexity of editing, deletion and completion problems in graph classes has been widely studied in the literature. In this work we prove new NP-completeness results for these problems in some classes of graphs, such as interval, circular-arc, permutation, circle, bridge, weakly chordal and clique-Helly graphs. Table 1 summarizes the known complexities of edge modification problems in different graph classes, including those obtained in this work (which are boldfaced). Some preliminary results of this work appear in [5].

2. Notation and definitions

Let G be a finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$ the complement of G and by $N(v)$ the set of neighbors of $v \in V(G)$. A vertex v is *universal* if $N(v) \cup \{v\} = V(G)$.

A *path* in a graph G is a sequence of pairwise different vertices $P = v_1, \ldots, v_k$, where $(v_i, v_{i+1}) \in E(G)$, for $1 \leqslant i \leqslant k-1$. If $(v_1, v_k) \in E(G)$, the path P is called a cycle. A *chord* of a path (or cycle) is an edge which joins two non-consecutive vertices from the path (or cycle). We denote the chordless or the induced path (or cycle) with k vertices by P_k (or C_k). A *bridge* of a cycle C is a shortest path in G joining non-consecutive vertices of C, which is shorter than both paths of C joining those vertices. This definition implies that a chord of a cycle is a bridge. Three vertices in G form an *asteroidal triplet* in G if they are pairwise non-adjacent, and any two of them are connected by a path which does not pass through the neighborhood of the third vertex.

A *stable set* of a graph G is a subset of pairwise non-adjacent vertices. We say that a vertex set M of a graph G is a *complete* of G if the subgraph induced by M is complete. A *clique* is a maximal complete of G (observe that some authors use the term "clique" in order to denote "complete"). Denote K_j the complete graph on j vertices.

Given two disjoint sets of vertices A and B, we say that A is *complete to* B if every vertex in A is adjacent to every vertex in B. Analogously, A is *anticomplete to* B if there are not adjacencies between vertices of A and vertices of B.

Given two sets of vertices V and V' (not necessarily disjoint), we denote by $V \times V'$ the set of all unordered pairs of vertices of the form (u, v), $u \in V$, $v \in V'$, $u \neq v$. In particular, $V \times V$ denotes the edges of a complete graph with vertex set V.

Given two disjoint graphs $G = (V, E)$ and $H = (V', E')$, their *union* and their *sum* are defined as follows: $G \cup H = (V \cup V', E \cup E')$ and $G + H = (V \cup V', E \cup E' \cup V \times V')$.

A graph property Π is *hereditary* if every induced subgraph of G satisfies Π whenever G satisfies Π. A property Π is *hereditary on subgraphs* if every subgraph of G satisfies Π whenever G satisfies Π. The *complement property* of Π, denoted by co-Π or $\overline{\Pi}$, is defined as the property Π in the complement graph (i.e., a graph G satisfies $\overline{\Pi}$ if and only if its complement $\overline{G}$ satisfies Π).

A family of subsets S satisfies the *Helly property* when every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is *clique-Helly* when its cliques satisfy the Helly property.

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. Intersection graphs receive much attention in the study of algorithmic graph theory and its applications (see for example [17]). Well-known special classes of intersection graphs include interval graphs, chordal graphs, circular-arc graphs, permutation graphs, and circle graphs.

A *circle graph* is the intersection graph of chords in a circle. A *circular-arc graph* is the intersection graph of a family of arcs on a circle. A graph G is *unit circular-arc* if there is a circular-arc representation of G such that all arcs have the same length. A graph G is *proper circular-arc* if there is a circular-arc representation of G such that no arc is properly contained in another.

An *interval graph* is the intersection graph of a family of intervals in the real line. A graph G is *unit interval* if there is an interval representation of G such that all intervals have the same length. A graph G is *proper interval* if there is an interval representation of G such that no interval is properly contained in any other one. A graph G is *interval containment* if its vertices can be mapped to intervals on the real line such that vertices x and y are adjacent if and only if one of the corresponding intervals contains the other.

We define $\mathscr{L} = (\mathscr{L}_1, \mathscr{L}_2)$ to be an *intersection model* in the following way. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two parallel lines in the plane with n points labelled by $1, 2, \ldots, n$ on $\mathscr{L}_1$ as well as on $\mathscr{L}_2$. For each $1 \leqslant i \leqslant n$, let L_i be the straight line segment connecting point i on $\mathscr{L}_1$ with point i on $\mathscr{L}_2$. Let $G_{\mathscr{L}} = (\{1, 2, \ldots, n\}, E_{\mathscr{L}})$ with $(i, j) \in E_{\mathscr{L}}$ if and only if L_i and L_j intersect. A graph G is a *permutation graph* if there is an intersection model $\mathscr{L}$ such that $G = G_{\mathscr{L}}$. Permutation graphs and interval containment graphs are the same class of graphs [11]. Another useful characterization of permutation graphs is the following: a graph G is a permutation graph if and only if G is the intersection graph of a family of chords in a circle that admits an equator (an additional chord that intersects any other chord of this family) [17].

A graph G is *bridged* if every cycle C of length at least 4 contains two vertices that are connected by a bridge. A graph G is *chordal* if it contains no induced cycle of length greater than 3. Clearly, chordal graphs are a subclass of bridged graphs. A graph G is *weakly chordal* if G and $\overline{G}$ contain no induced cycle C_k, $k \geqslant 5$. A graph G is *cubic* if all its vertices have degree 3. A graph is a *comparability graph* if it has a transitive orientation of its edges, that is, an orientation F for which $\overrightarrow{(a, b)}, \overrightarrow{(b, c)} \in F$ implies $\overrightarrow{(a, c)} \in F$.

A *bipartite graph* $G = (P, Q, E)$ is defined by two finite and disjoint sets of vertices P and Q and a set of edges $E \subseteq P \times Q$. A bipartite graph $G = (P, Q, E)$ is a *chain graph* if there is an ordering σ of the vertices in P, $\sigma : P \to \{1, \ldots, |P|\}$ such that $N(\sigma^{-1}(1)) \subseteq N(\sigma^{-1}(2)) \subseteq \cdots \subseteq N(\sigma^{-1}(|P|))$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ holds for every induced subgraph H of G. Here $\omega(G)$ denotes the cardinality of a maximum clique of G, and $\chi(G)$, the *chromatic number* of G, denotes the minimum number of colors needed to color the vertices of G so that no two adjacent vertices receive the same color.

Graph classes and graph theory properties not defined here can be found in [4,14,17].

3. Basic results

The following results will be frequently used in the proofs of this work.

Proposition 1 (*Natanzon et al. [26]*). *If property Π is hereditary on subgraphs then Π-deletion and Π-editing are polynomially equivalent, and Π-completion is not meaningful.*

Proposition 2 (*Natanzon et al. [26]*). *If Π and Π' are graph properties such that for every graph G and every stable set S, G satisfies Π if and only if $G \cup S$ satisfies Π', then Π-deletion is polynomially reducible to Π'-deletion. If in addition Π is hereditary, then Π-completion (Π-editing) is polynomially reducible to Π'-completion (Π'-editing).*

Proposition 3 (*Natanzon et al. [26]*). *If Π and Π' are graph properties such that for every graph G and every complete graph K, G satisfies Π if and only if $G+K$ satisfies Π', then Π-completion is polynomially reducible to Π'-completion. If in addition Π is hereditary, then Π-deletion (Π-editing) is polynomially reducible to Π'-deletion (Π'-editing).*

Since permutation graphs are an hereditary class of graphs, and a graph G is a permutation graph if and only if the graph $G+K$ is a circle graph for every complete K [17], we have the following corollary.

Corollary 4. *Permutation modification problems are polynomially reducible to the corresponding circle modification problems.*

Furthermore, it is easy to see that for every graph property Π, Π-deletion and $\overline{\Pi}$-completion are polynomially equivalent, and the same is true for Π-editing and $\overline{\Pi}$-editing.

4. NP-completeness results

Given a cubic graph $G=(V,E)$ with $V=\{v_1,\ldots,v_n\}$ ($|V|=n$), we define its *triangle graph* $\widehat{G}=(\widehat{V},\widehat{E})$ by blowing each vertex up into a triangle, and maintaining the cubic property of the graph. More precisely, the vertex set is

$$\widehat{V}=\{v_{ij},v_{ik},v_{il}\mid v_i\in V \text{ and } v_j,v_k,v_l \text{ are adjacent to } v_i \text{ in } G\},$$

where we call v_{ij},v_{ik},v_{il} the *representatives* of v_i. Note that $|\widehat{V}|=3n$.

The edge set $\widehat{E}$ consists of two kinds of edges,

$$\widehat{E}=E^{\text{new}}\cup E^{\text{old}},$$

where

(1) $E^{\text{new}}=\{(v_{ij},v_{ik}),(v_{ij},v_{il}),(v_{ik},v_{il})|v_i\in V \text{ and } v_j,v_k,v_l \text{ are the vertices adjacent to } v_i \text{ in } G\}$. Hence, the three representatives of each vertex $v_i\in V$ form a triangle in $\widehat{G}$.
(2) $E^{\text{old}}=\{(v_{ij},v_{ji})|(v_i,v_j)\in E\}$. Note than E^{old}-edges are pairwise nonincident edges. In other words, the edge (v_{ij},v_{ji}) connects representatives of its original endpoints, and for every $v\in V$, each of the three original edges that are incident to v in G is incident to a different representative of v in $\widehat{G}$.

If $e\in E^{\text{new}}$ we say that e is a New edge. On the other hand, if $e\in E^{\text{old}}$ we say that e is an Old edge.

The reduction from G to $\widehat{G}$ is clearly polynomial. It is important to note that $\widehat{G}$ is cubic and has $9n/2$ edges, $3n$ New edges and $3n/2$ Old edges. Moreover, if G is planar, then $\widehat{G}$ is also planar: starting from a planar representation of G, we can replace every $v_i\in V(G)$ by v_{ij}, v_{ik} and v_{il} while maintaining planarity. An example of this sort of reduction is shown in Fig. 1.

For a vertex $v\in G$, let S_v be the subgraph of $\widehat{G}$ induced by the three representatives of v together with their three neighbors. If v_i has v_j, v_k and v_l as neighbor vertices in G, then S_{v_i} is the subgraph induced by $\{v_{ij},v_{ik},v_{il},v_{ji},v_{ki},v_{li}\}$.

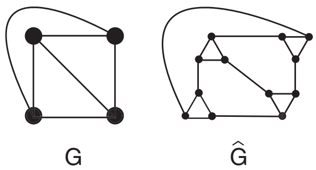

Fig. 1. Example of a cubic planar graph G and its corresponding reduction $\widehat{G}$, that is also cubic and planar.

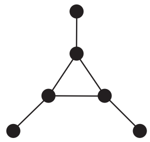

Fig. 2. The net graph, which is isomorphic to each one of the S_v.

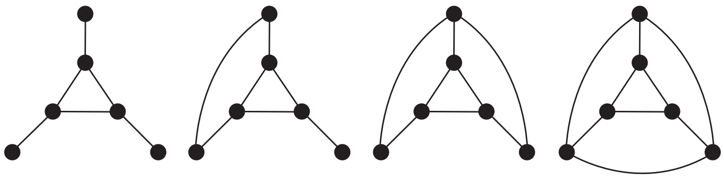

Fig. 3. Forbidden subgraphs for interval containment graphs.

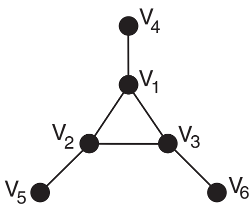

Fig. 4. Example of a forbidden induced subgraph ($\overline{S_3}$) for interval containment graphs.

The S_v are isomorphic to the net graph $\overline{S_3}$ (Fig. 2). Observe that each S_v has three New edges and three Old edges and every vertex $w \in \widehat{V}$ belongs to exactly two S_v.

4.1. Permutation deletion

In this subsection, we prove that permutation deletion is NP-complete. As corollaries, we deduce that permutation completion, circle deletion, circle completion, co-circle completion, and co-circle deletion are NP-complete, as well.

Lemma 5. *If G is an interval containment graph, then it cannot contain any of the graphs of Fig. 3 as an induced subgraph.*

Proof. Let us consider the first graph, the other cases are similar. Suppose that an interval containment graph contains $\overline{S_3}$ (see Fig. 4) as induced subgraph. The only possible representation of the triangle formed by $\{v_1, v_2, v_3\}$ is with 3 intervals X, Y, Z, where $X \subseteq Y \subseteq Z$ (see Fig. 5).

Without loss of generality, we can assume that X, Y, Z are the intervals representing v_1, v_2 and v_3, respectively. Then, the interval corresponding to v_5 must either be included in the interval corresponding to v_2 or include it. But if

Fig. 5. Interval containment representation of a triangle.

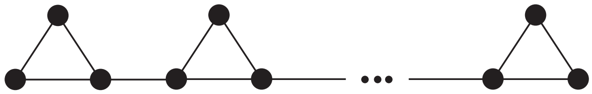

Fig. 6. The graph $\widetilde{G}$ obtained by deleting all the $\widehat{G}$-edges which do not correspond to edges in the hamiltonian path.

Fig. 7. Interval containment representation of $\widetilde{G}$.

it is included in the interval corresponding to v_2 then it is also included in the interval corresponding to v_3, and if it includes the interval corresponding to v_2 then it also includes the interval corresponding to v_1. In both cases we get a contradiction. □

Lemma 6. *If G is an interval containment graph, then it does not contain C_n $(n \geqslant 5)$ as induced subgraph.*

Proof. A graph G is a permutation graph if and only if G is the intersection graph of chords in a circle that admit an equator. Moreover, since the wheels W_j $(j \geqslant 5)$ are not circle graphs [8], then the cycles C_j $(j \geqslant 5)$ do not admit an equator. Thus they are not permutation graphs and, therefore, neither interval containment graphs, since these two classes coincide. □

Theorem 7. *Let G be a cubic planar graph. Then, G has a hamiltonian path if and only if its triangle graph $\widehat{G}$ has an interval containment subgraph with at least $4n - 1$ edges.*

Proof. $\Rightarrow$) Suppose that G contains a hamiltonian path P. Delete all the Old edges in $\widehat{G}$ which do not correspond to edges in P, and denote by $\widetilde{G}$ the resulting subgraph of $\widehat{G}$ (see Fig. 6). The graph $\widetilde{G}$ is an interval containment graph. (This can be verified by its interval containment representation as depicted in Fig. 7.) Moreover, $\widetilde{G}$ contains exactly $4n - 1$ edges, $3n$ New edges and $n - 1$ Old edges.

$\Leftarrow$) Conversely suppose that $\widetilde{G} = (V, \widetilde{E})$ is an interval containment subgraph of $\widehat{G}$ with $|\widetilde{E}| \geqslant 4n - 1$. First, we prove that $\widetilde{G}$ contains all the New edges and exactly $n - 1$ Old edges of $\widehat{G}$.

Since $\widetilde{G}$ is an interval containment graph, it cannot have $\overline{S_3}$ as an induced subgraph (by Lemma 5). Then, at least one of the edges of each S_v (there are n of them) must be missing in $\widetilde{G}$.

Suppose that two New edges from some S_x $(x \in V)$ are missing in $\widetilde{G}$. The total number of edges removed from $\widehat{G}$ to form $\widetilde{G}$ does not exceed $(n/2) + 1$ (this fact can be deduced from the calculation $3n + (3n/2) - (4n - 1)$), it turns out that at most $(n/2) - 1$ additional edges can be removed from $\widehat{G}$ in order to cancel the remaining $(n - 1)$ S_v such that $v \neq x$. Since each (New or Old) edge in $\widehat{G}$ is contained in at most two S_v, we have a contradiction (one can cancel at most $2((n/2) - 1) = (n - 2)$ S_v). Hence, out of the edge set of each S_v, at most one of the New edges is missing in $\widetilde{G}$. In particular, the representatives of each vertex induce a connected subgraph in $\widetilde{G}$.

Let H be the graph obtained from $\widetilde{G}$ by contracting all New edges. Our previous observations imply that the number of vertices in H is exactly n, one for each original vertex in G. Moreover, H is acyclic, since the existence of a chordless cycle in H would imply the existence of a chordless cycle at least twice as long in $\widetilde{G}$, contradicting Lemma 6. It follows that H contains at most $n-1$ edges. Thus, $\widetilde{G}$ contains at most $n-1$ Old edges. Since the total number of edges in $\widetilde{G}$ is at least $4n-1$, $\widetilde{G}$ must contain all the New edges ($3n$) and exactly $n-1$ Old edges.

Since H is acyclic with $n-1$ edges and n vertices, it must be connected. Suppose that H contains a vertex v with degree 3. Since we have just shown that $\widetilde{G}$ contains all the New edges, the complete S_v from $\widehat{G}$ also exists in $\widetilde{G}$, and again we reach a contradiction. Hence, the hamiltonian path in H defines a hamiltonian path in G. □

Corollary 8. *Permutation deletion is NP-complete.*

Proof. The problem belongs to NP since permutation graphs can be recognized in linear time [25]. Permutation graphs and interval containment graphs are the same class, thus permutation deletion is equivalent to the problem of finding an interval containment maximum generator subgraph. As the hamiltonian path problem restricted to cubic planar graphs is NP-complete [14], Theorem 7 implies that permutation deletion is NP-complete. □

Since permutation graphs are closed under complement [4], we have the following corollary.

Corollary 9. *Permutation completion is NP-complete.*

As a consequence of this corollary, the results in Section 3 and the fact that circle graphs can be recognized in polynomial time [31], we have the following further corollary.

Corollary 10. *Circle deletion, circle completion, co-circle completion and co-circle deletion are NP-complete problems.*

4.2. Interval editing

In this subsection we prove that the interval editing problem is NP-complete. As corollaries, we deduce that unit interval editing, circular-arc editing, unit circular-arc editing and proper circular-arc editing also are NP-complete.

Theorem 11. *Let $G=(V,E)$ be a cubic planar graph and let $\widehat{G}=(\widehat{V},\widehat{E})$ be its triangle graph. Then G has a hamiltonian path if and only if there exists an edge set F, $|F|\leqslant n/2+1$, such that $\widetilde{G}=(\widehat{V},\widehat{E}\triangle F)$ is an interval graph.*

Proof. $\Rightarrow$) Suppose that G contains a hamiltonian path P. Delete all Old edges in $\widehat{G}$ which do not correspond to edges in P, and denote by $\widetilde{G}$ the resulting subgraph of $\widehat{G}$ (see Fig. 6). The graph $\widetilde{G}$ is an interval graph, as it can be verified by its interval representation (Fig. 8). Moreover, $\widetilde{G}$ contains exactly $4n-1$ edges, $3n$ New edges and $n-1$ Old edges, i.e., $n/2+1$ edges have been deleted from $\widehat{G}$.

$\Leftarrow$) Now suppose that there exists an edge set F such that $\widetilde{G}=(\widehat{V},\widehat{E}\triangle F)$ is an interval graph with $|F|\leqslant n/2+1$.

Since $\widetilde{G}$ is an interval graph, it does not have $\overline{S_3}$ as an induced subgraph because v_{ji}, v_{ki} and v_{li} form an asteroidal triplet (AT), a forbidden structure for interval graphs [22]. Then, each S_v (there are n of them) contains a forbidden structure (proper of this S_v) that has to be cancelled by adding to or deleting from $\widehat{G}$ an edge that belongs to F.

Let us see what kind of edges may appear in F. We may first distinguish between edges belonging to $\widehat{G}$ (which can be deleted) and edges that not belonging to $\widehat{G}$ (which can be added). Edges belonging to $\widehat{G}$ were previously classified as New and Old edges. The deletion of a New edge of S_v cancels the AT corresponding to that S_v. The deletion of an Old edge of S_v (as it belongs to two different S_v) cancels both AT's corresponding to these S_v. We can identify three different classes of edges that do not belong to $\widehat{G}$: External edges, Good Internal edges and Bad Internal edges

Fig. 8. Interval representation of $\widetilde{G}$.

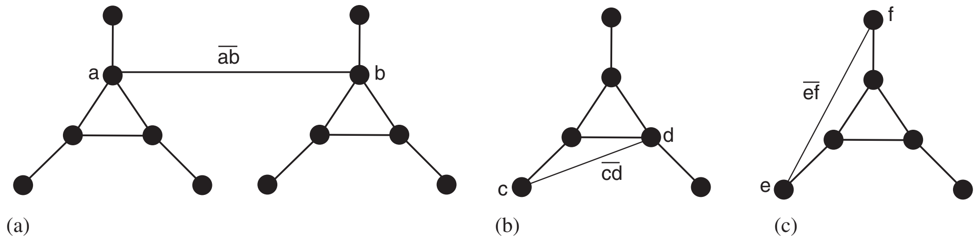

Fig. 9. (a) $\overline{ab}$ is an External edge. (b) $\overline{cd}$ is a Good Internal edge. (c) $\overline{ef}$ is a Bad Internal edge.

Table 2
Kinds of edges that may appear in F

Name	$\in \widehat{E}$	Description	Number of S_v cancelled [a]	Vertices
Old	yes	Joins a representative of v_i with one of v_j, if v_i and v_j were adjacent in V	2	(v_{ij}, v_{ji})
New	yes	Joins 2 representatives of the same $v_i \in V$	1	(v_{ij}, v_{ik}), $j \neq k$
Good	no	$\exists S_v /$ both edge endpoints belong to it and one of them is a representative of v.	1	(v_{ij}, v_{ki}) or (v_{ji}, v_{ik}), $j \neq k$
Bad	no	$\exists S_v /$ both edge endpoints belong to it and none of them is a representative of v.	0 (cancels a forbidden structure but adds another)	(v_{ki}, v_{ji}), $j \neq k$
External	no	$\nexists S_v /$ both edge endpoints belong to it	0	(v_{ij}, v_{kl}), i, j, k, l all diffrent subindices

[a]Number of S_v with a proper forbidden structure that are cancelled by adding or deleting that edge.

(see Fig. 9). External edges are the edges that join two vertices not belonging to the same S_v. Adding an External edge does not cancel any AT properly included in the vertex set of an S_v. Good Internal edges (which will be called Good edges from now on) are the edges joining two vertices x and y such that there exists an S_v which includes both x and y, and one of them is a representative of v. Adding a Good edge cancels the AT corresponding to this S_v. Bad Internal edges (which will be called Bad edges from now on) are the edges joining two vertices x and y such that there exists an S_v which includes both x and y, but neither x nor y is a representative of v. Adding a Bad edge cancels the AT corresponding to this S_v, but generates a chordless cycle of length 4, which is another forbidden structure for interval graphs and thus it cannot appear in $\widetilde{G}$. This cycle is properly included in the vertex set of the same S_v, meaning that the edge of F cancelling it must be a New, Old or Good edge of that S_v, with the same effect as if the AT had never been cancelled. These results are summarized in Table 2.

The following three remarks are very useful:

(i) There cannot be two edges e_1 and e_2 in F such that $e_1, e_2 \in$ New $\cup$ Good and both cancel forbidden structures from the same S_x.
Assume the opposite is true. As we already know, the number of proper forbidden structures cancelled by these edges is exactly 1. Then we need at most $n/2 - 1$ additional edges in F in order to cancel the remaining $n - 1$ proper forbidden structures of the other S_v's. Since each edge in F cancels at most the proper forbidden structures of two S_v's, we reach a contradiction. In particular, the representatives of each vertex induce a connected subgraph in $\widetilde{G}$.

(ii) There cannot be two edges e_1 and e_2 in F such that $e_1 \in$ External $\cup$ Bad and $e_2 \in$ External $\cup$ Bad $\cup$ Good $\cup$ New.
Assume the opposite is true. As we already know, the number of proper forbidden structures cancelled by these edges is at most 1, then we need at most $n/2 - 1$ additional edges in F in order to cancel the remaining $n - 1$ proper forbidden structures of the other S_v's. Since each edge in F cancels at most the proper forbidden structures of two S_v's, we reach a contradiction. In particular, if there exists an edge $e \in F$ such that $e \in$ External $\cup$ Bad, then the remaining edges of F must be Old edges.

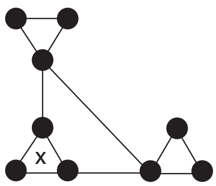

Fig. 10. Cycle formed by a Bad edge and two Old edges.

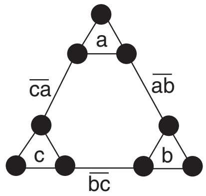

Fig. 11. Cycle formed by three Old edges.

(iii) There cannot be three edges e_1, e_2 and e_3 in F such that $e_1, e_2, e_3 \in$ Good $\cup$ New.
Assume the opposite is true. Then, the proper forbidden structures of at most three S_v's will be cancelled, leaving $n/2 - 2$ edges in F in order to cancel the remaining $n - 3$ proper forbidden structures from the other S_v's. Since each edge in F cancels at most the proper forbidden structures of two S_v's, we reach a contradiction. In particular, if there exist two edges e_1 and $e_2 \in F$ such that $e_1, e_2 \in$ Good $\cup$ New, then the remaining edges of F must be Old edges.

Let H be the graph obtained from $\widetilde{G}$ by contracting all New edges. By the observations in (i) (we cannot have in F two New edges from the same S_x), the number of vertices in H is exactly n, one for each original vertex in G.

Let us verify that H is acyclic. The existence of an induced cycle of length $\geqslant 4$ in H would imply the existence of an induced cycle of at least the same length in $\widetilde{G}$, contradicting the fact that $\widetilde{G}$ must be chordal because it is an interval graph. Then, the only chordless cycles that may appear in H must have length 3. Suppose that there is a cycle of length 3 (a triangle) in H. Remarks (i), (ii) and (iii) exclude some edge combinations that could have built this cycle. Let us analyze which of the remaining combinations may appear to conclude that none of them is possible:

- *Bad Old Old*: If both Old edges have vertex x in common in H, then they will be incident to different representatives of x in $\widetilde{G}$. This would imply that there is at least one more edge in the cycle in $\widetilde{G}$. We must verify that no chord was added to $\widehat{G}$ in order to cancel the cycle. As we have seen in (ii), whenever there is a Bad edge then the remaining edges of F must be Old edges. So, no more edges can be added to $\widehat{G}$ (see Fig. 10).
- *External Old Old*: If both Old edges have vertex x in common in H, then they will be incident to different representatives of x in $\widetilde{G}$, and this implies that there is at least one more edge in the cycle in $\widetilde{G}$. We must verify that no chord was added to $\widehat{G}$ in order to cancel the cycle. As we have seen in (ii), if there is an External edge then the remaining edges of F must be Old edges. Hence, no more edges could be added to $\widehat{G}$.
- *Old Old Old*: This would imply that there exists a cycle of length at least 6 in $\widetilde{G}$. Fig. 11 depicts this situation. If we want to cancel this cycle, then we must add at least three chords in order to avoid remaining cycles of length greater than 3. As we have seen in (ii), if there exists an edge $e \in F$ such that $e \in$ External $\cup$ Bad, then the remaining edges of F must be Old edges, and as we have seen in (iii), if there are two edges e_1 and e_2 in F such that $e_1, e_2 \in$ Good $\cup$ New, then the remaining edges of F must be Old edges. So, we cannot add three more edges.
- *Old Good Good*: Suppose that the triangle abc appears in H, with $\overline{ab}$ as an Old edge. Since there are two Good edges, as we have seen in (iii) no more edges can be added to $\widehat{G}$ in order to cancel the cycle. Edges $\overline{ac}$ and $\overline{bc}$ in $\widetilde{G}$ must be incident to the same representatives of a and b as edge $\overline{ab}$, otherwise $\widetilde{G}$ would contain a cycle with at least one more edge. Then, the edge $\overline{ac}$ must be Internal to a, because the representative belongs to both S_a and S_b, and all the remaining vertices belong to exactly two S_v's. Analogously, $\overline{bc}$ must be Internal to b. Finally, the representatives of c, where $\overline{ac}$ and $\overline{bc}$ are incident, must be different, because if they were the same then this

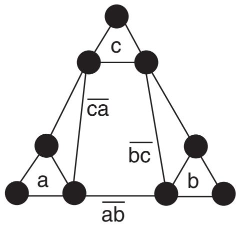

Fig. 12. Cycle formed by an Old edge and two Good edges.

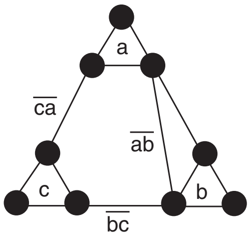

Fig. 13. Cycle formed by a Good edge and two Old edges.

representative would belong to three S_v's. Thus, there exists a cycle with at least one more edge in $\widetilde{G}$. Fig. 12 depicts this situation.

- *Old Old Good*: Suppose that the triangle abc appears in H, with $\overline{ab}$ as a Good edge. Since both Old edges have vertex c in common in H, then they are incident to different representatives of c in $\widetilde{G}$, and that implies that there is at least one more edge in the cycle in $\widetilde{G}$ (there is a cycle of length at least 4). Let us see that $\overline{ab}$ cannot be incident simultaneously to the same representatives of a and b as $\overline{ac}$ and $\overline{bc}$, respectively, (implying that there is at least one edge more in the cycle in $\widetilde{G}$, that is, the cycle has at least five edges). If $\overline{ab}$ is incident to the same representative of a as $\overline{ac}$, then $\overline{ab}$ is Internal to a, otherwise the representative belongs to three S_v's. Then, it cannot be incident to the same representative as $\overline{bc}$ because it would belong to three S_v's. The remaining case is analogous. So, there is a cycle of length greater or equal to 5 in $\widetilde{G}$. At least two chords are needed in order to cancel it, but only one can be added, because $\overline{ab}$ is a Good edge (see Fig. 13).

Since H is acyclic and has n vertices, then it contains at most $n-1$ edges, so $\widetilde{G}$ contains at most $n-1$ Old edges. Since the total number of edges in $\widetilde{G}$ is at least $4n-1$, $\widetilde{G}$ must contain all New edges ($3n$) and exactly $n-1$ Old edges, i.e., all edges in F are Old edges. Hence, if there exists an interval $(n/2+1)$-editing set F, then F must be an interval $(n/2+1)$-deletion set too.

Since H is acyclic with $n-1$ edges and n vertices, it must be connected. Suppose that H contains a vertex v with degree 3. Since we have just proved that $\widetilde{G}$ contains all New edges and no more edges were added, then the complete S_v from $\widehat{G}$ also exists in $\widetilde{G}$, and we obtain a contradiction. Hence, the hamiltonian path in H defines a hamiltonian path in G. □

Corollary 12. *Interval editing is NP-complete.*

Proof. The problem belongs to NP since interval graphs can be recognized in linear time [3]. As the hamiltonian path problem restricted to cubic planar graphs is NP-complete [14], Theorem 11 implies this corollary. □

A graph G is a (unit) interval graph if and only if $G \cup S$ is a (unit) circular-arc graph, for any stable set S. Additionally, (unit) interval is a hereditary class of graphs. So, by Proposition 2, (unit) interval edge modification problems are polynomially reducible to the corresponding (unit) circular-arc edge modification problems. Since we can recognize in polynomial time (unit) circular-arc graphs [9,24], proper circular-arc graphs [7], and circle graphs [31], the results from Section 3 imply the following corollaries.

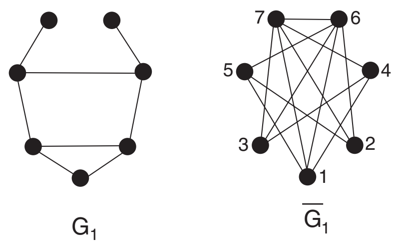

Fig. 14. Forbidden subgraphs for permutation graphs.

Corollary 13. *Unit interval editing is NP-complete.*

Proof. Consequence of Theorem 11 and Corollary 12, the same proof can be applied to this case because all the intervals in Fig. 8 may have the same length. □

Corollary 14. *Circular-arc, unit circular-arc and proper circular-arc editing are NP-complete.*

Proof. Consequence of Corollary 12, the fact that unit interval graphs are equivalent to proper interval graphs [4] and Proposition 2, because G is a (unit) (proper) interval graph if and only if $G \cup K_1$ is a (unit) (proper) circular-arc graph. □

Note that the same complexities hold for the complement classes, because Π-editing and $\overline{\Pi}$-editing are polynomially equivalent.

4.3. Permutation editing

In this subsection we prove that the permutation editing problem is NP-complete. As corollaries, we deduce that circle editing and co-circle editing are NP-complete, too.

Lemma 15. *A permutation graph does not contain the graphs in Fig. 14 as induced subgraphs.*

Proof. A graph G is a permutation graph if and only if G and $\overline{G}$ are comparability graphs [4]. Let us verify that $\overline{G_1}$ is not a comparability graph.

If $\overline{G_1}$ is a comparability graph, then it has a transitive orientation of its edges. We will see that such orientation cannot be defined.

To this end, consider the labels of Fig. 14. By symmetry, we may assume that the edge $(6, 7)$ is oriented as $\overrightarrow{(6, 7)}$. This forces the following "chain" of orientations: $\overrightarrow{(4, 7)}$, $\overrightarrow{(2, 7)}$, $\overrightarrow{(1, 7)}$, $\overrightarrow{(3, 7)}$, $\overrightarrow{(6, 5)}$, $\overrightarrow{(6, 3)}$, $\overrightarrow{(4, 3)}$, $\overrightarrow{(4, 1)}$. At this point $(1, 5)$ cannot be oriented, since $\overrightarrow{(1, 7)}$ forces $\overrightarrow{(1, 5)}$ while $\overrightarrow{(4, 1)}$ forces $\overrightarrow{(5, 1)}$, a contradiction. □

Theorem 16. *Let $G = (V, E)$ be a cubic planar graph and let $\widehat{G} = (\widehat{V}, \widehat{E})$ be its triangle graph. Then, G has a hamiltonian path if and only if there exists an edge set F such that $\widetilde{G} = (\widehat{V}, \widehat{E} \triangle F)$ is an interval containment graph and $|F| \leqslant n/2 + 1$.*

Proof. $\Rightarrow$) Theorem 7 states that there exists such an edge set F so that $\widetilde{G} = (\widehat{V}, \widehat{E} \setminus F)$ is interval containment, which is a stronger property.

$\Leftarrow$) Suppose now that there exists an edge set F such that $\widetilde{G} = (\widehat{V}, \widehat{E} \triangle F)$ is an interval containment graph with $|F| \leqslant n/2 + 1$.

Since $\widetilde{G}$ is an interval containment graph, it does not have $\overline{S_3}$ as an induced subgraph (Lemma 5). Then, each S_v (there are n of them) contains a forbidden structure (proper of this S_v) that has to be cancelled by adding to or deleting from $\widehat{G}$ an edge belonging to F.

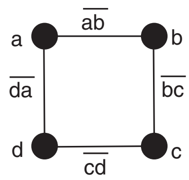

Fig. 15. Chordless cycle of length 4 in H.

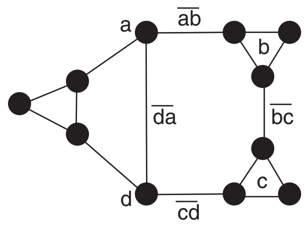

Fig. 16. Cycle composed by a Bad edge and three Old edges.

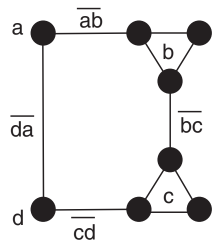

Fig. 17. Cycle composed by an External edge and three Old edges.

The kinds of edges that may appear in F are the same than those shown in Theorem 11 and are summarized in Table 2. The number of forbidden structures cancelled by each one of these edges is the same, but we must note that in this case Bad edges cancel a forbidden structure and generate the second graph of Lemma 5, which is forbidden for interval containment. So, the remarks from the proof of Theorem 11 are still valid in this case.

Let H be the graph obtained from $\widetilde{G}$ by contracting all New edges. The same arguments form the proof of Theorem 11 can be applied to this case showing that the number of vertices in H is exactly n, one for each original vertex in G.

Let us verify that H is acyclic. The existence of an induced cycle of length $\geqslant 5$ in H would imply the existence of an induced cycle of at least the same length in $\widetilde{G}$, contradicting Lemma 6. Then, the only chordless cycles that may appear in H must have length 3 or 4. Suppose that there is a chordless cycle of length 4 composed by vertices a, b, c and d in H (see Fig. 15).

Remarks (i), (ii) and (iii) from the proof of Theorem 11 exclude some edge combinations that can compose this cycle. Let us analyze the remaining combinations to conclude that none of them is possible:

- *Bad Old Old Old*: Suppose that $\overline{da}$ is the Bad edge, and $\overline{ab}$, $\overline{bc}$ and $\overline{cd}$ are Old edges. Then, a New edge belonging to S_b and another New edge belonging to S_c are added to the cycle in $\widetilde{G}$. So, the cycle in $\widetilde{G}$ has at least six edges. By remark (ii), no chord can be added to this cycle, a contradiction (see Fig. 16).
- *External Old Old Old*: Suppose that $\overline{da}$ is an External edge, and $\overline{ab}$, $\overline{bc}$ and $\overline{cd}$ are Old edges. Then, a New edge that belongs to S_b and another New edge belonging to S_c are added to the cycle in $\widetilde{G}$. So, the cycle in $\widetilde{G}$ has six edges, and by (ii), the remaining edges in F are Old. Hence, no chord can be added in $\widehat{G}$ in order to cancel the cycle, a contradiction (see Fig. 17).
- *Old Old Old Old*: This configuration would imply that there is a cycle with at least eight edges in $\widetilde{G}$, even if no New edges were deleted (Fig. 18). By remarks (ii) and (iii) at most two edges can be added to cancel the cycle. If we add

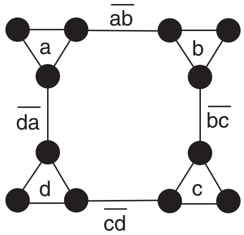

Fig. 18. Cycle composed by four Old edges.

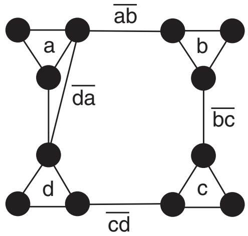

Fig. 19. Cycle composed by a Good edge and three Old edges.

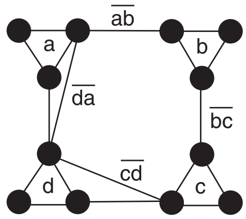

Fig. 20. Cycle composed by two Good edges and two Old edges.

only one edge, then there is a chordless cycle of length at least 5, a contradiction. If we add two edges, by remark (iii) and because the cycle in H is chordless, then both edges must be Good ones that join vertices separated by two edges in the cycle. Then, there exists a chordless cycle of length at least 6, a contradiction.

- *Good Old Old Old*: This configuration would imply that there is a cycle with at least seven edges in $\widetilde{G}$ (even if no New edges were deleted), since the Good edge joins two vertices separated by two edges in the cycle (Fig. 19). By remarks (ii) and (iii) only a Good edge can be added to cancel the cycle. But then there exists a chordless cycle of length at least 6, a contradiction.
- *Good Good Old Old*: Suppose that $\overline{da}$ and $\overline{cd}$ are Good edges. There exist New edges that belong to S_b and S_c in the cycle in $\widetilde{G}$, implying that the cycle has length at least 6, and by (ii) and (iii), no chord may appear in $\widehat{G}$ in order to cancel the cycle, a contradiction (see Fig. 20).
- *Good Old Good Old*: Suppose that $\overline{da}$ and $\overline{bc}$ are Good edges. The edge $\overline{da}$ must be Internal to S_d or Internal to S_a. If it is Internal to S_a, then it cannot be incident to the same representative of d as $\overline{cd}$, since this vertex would belong to three different S_v's, a contradiction (the same happens if $\overline{da}$ is Internal to S_d). Then, $\overline{da}$ and $\overline{cd}$ are incident to different representatives of d. So, a New edge that belongs to S_d exists in the cycle in $\widetilde{G}$, implying that the cycle has length at least 6, and by (ii) and (iii), no chord may appear in $\widehat{G}$ in order to cancel the cycle, a contradiction (Fig. 21).

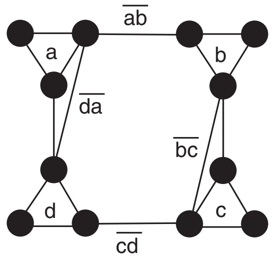

Fig. 21. Cycle composed by alternating two Good edges and two Old edges.

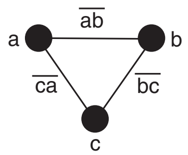

Fig. 22. Cycle of length 3 in H.

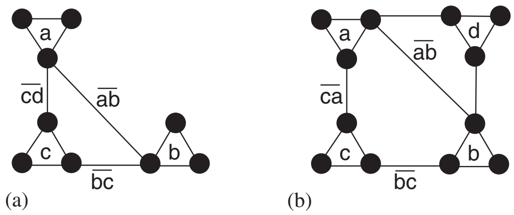

Fig. 23. Cycle composed by a Bad edge and two Old edges: (a) Edge $\overline{ab}$ is Internal to S_c. (b) Edge $\overline{ab}$ is Internal to a vertex that does not belong to the cycle C.

Now suppose that H contains a cycle of length 3 composed by vertices a, b, and c (see Fig. 22).

Remarks (i), (ii) and (iii) exclude some edge combinations for this cycle. Let us analyze the remaining combinations to conclude that none of them is possible:

- *Bad Old Old*: Suppose that $\overline{ab}$ is a Bad edge. If $\overline{ab}$ is Internal to S_c, then by (ii), any additional edge in F must be Old. Hence, we obtain G_1 (see Lemma 15) as induced subgraph of $\widetilde{G}$, contradicting that $\widetilde{G}$ is a permutation graph (see Fig. 23). If $\overline{ab}$ is Internal to S_d, where d is some other vertex adjacent to both a and b in H, then the endpoints of $\overline{ab}$ cannot be the same representatives of a and b as those that are reached by $\overline{bc}$ and $\overline{ca}$, since that would imply that these vertices belong to three different S_v. So, two New edges that belong to S_a and S_b are added to the cycle in $\widetilde{G}$, and a New edge that belongs to S_c is also added to the cycle, since two Old edges cannot be incident to the same representative of c, implying that the length of the cycle is 6. By remark (ii), no chord can exist in $\widehat{G}$ in order to cancel the cycle, a contradiction (see Fig. 23).
- *Old Old Old*: This would imply that there is a cycle with at least 6 edges in $\widetilde{G}$, even if no New edges were deleted (see Fig. 24). If only one Good edge is added in order to cancel the cycle, then there is a chordless cycle with at least five edges in $\widetilde{G}$, a contradiction. If a Bad edge is added in order to cancel the cycle, then by (ii) we have G_1, one of the forbidden subgraphs of Lemma 15 as induced subgraph of $\widetilde{G}$, contradicting that $\widetilde{G}$ is a permutation graph. It is not possible to add an External edge in order to cancel the cycle. If two Good edges are added in order to cancel the cycle, the only three possible resulting graphs are those of Fig. 25 (note that all New edges are present by (iii)). In cases (a) and (c), G_1, one of the forbidden subgraphs of Lemma 15, appears as induced subgraph of $\widetilde{G}$, contradicting the fact that $\widetilde{G}$ is a permutation graph. In case (b) there is a chordless cycle of length 5, again a contradiction.

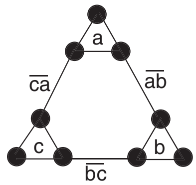

Fig. 24. Cycle composed by three Old edges.

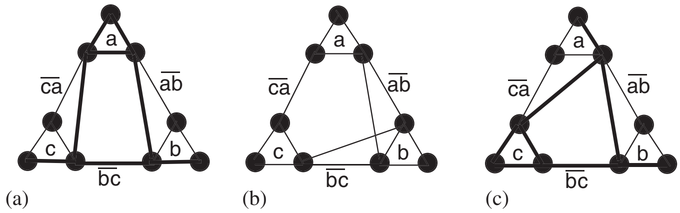

Fig. 25. The only 3 possible ways of adding 2 Good edges in order to cancel the cycle with six or more edges.

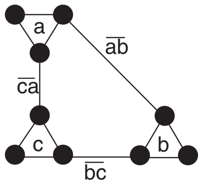

Fig. 26. Cycle composed by an External edge and two Old edges.

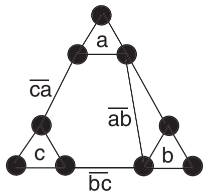

Fig. 27. Cycle composed by a Good edge and two Old edges.

- *External Old Old*: Suppose that $\overline{ab}$ is an External edge. A New edge joining the endpoints of Old edges $\overline{bc}$ and $\overline{ca}$ (that cannot be incident to the same representative of c) is added to the cycle C in $\widetilde{G}$. The endpoints of $\overline{ab}$ cannot be simultaneously the same representatives of a and b as those that are reached by $\overline{bc}$ and $\overline{ca}$, since that would imply that $\overline{ab}$ is an Internal edge of S_c, contradicting that it is External. So, at least one New edge that belongs to S_a or S_b is added to the cycle in $\widetilde{G}$, implying that the length of the cycle is at least 5. But by (ii), no chord can be added to $\widehat{G}$ in order to cancel the cycle, again a contradiction (see Fig. 26).
- *Good Old Old*: This combination would imply that there is a cycle of at least five edges in $\widetilde{G}$, or more than five edges if a New edge is deleted (see Fig. 27). By remarks (ii) and (iii), at most one Good edge can be added in order to cancel the cycle, but in this case the subgraph (a) or the subgraph (c) from Fig. 25 would be constructed, and we have just proven that it cannot happen.
- *Good Good Old*: If a cycle of length at least 5 is constructed we reach a contradiction, since by (iii) we could not have added more edges in order to cancel it. The only two ways that a cycle of less than five edges can be attained are those in Fig. 28, but these graphs coincide with subgraphs (a) and (c) from Fig. 25, a contradiction.

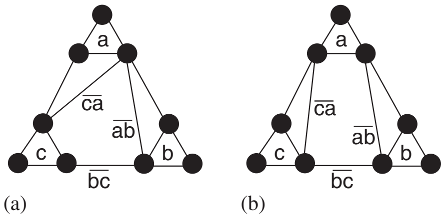

Fig. 28. Cycle composed by two Good edges and an Old edge.

Since H is acyclic and has n vertices, then it contains at most $n-1$ edges, so $\widetilde{G}$ contains at most $n-1$ Old edges. Since the total number of edges in $\widetilde{G}$ is at least $4n-1$, then $\widetilde{G}$ must contain all New edges ($3n$) and exactly $n-1$ Old edges, i.e., all edges in F are Old edges. This implies that if there exists an interval containment $(n/2+1)$-editing set F, F must be an interval containment $(n/2+1)$-deletion set too.

Since H is acyclic with $n-1$ edges and n vertices, then it must be connected. Suppose that H contains a vertex v with degree 3. Since we have just verified that $\widetilde{G}$ contains all New edges and no more edges were added, this implies that the complete S_v from $\widehat{G}$ also exists in $\widetilde{G}$, and we obtain a contradiction. Hence, the hamiltonian path in H defines a hamiltonian path in G. □

Corollary 17. *Permutation editing is NP-complete.*

Proof. The problem belongs to NP since permutation graphs can be recognized in linear time [25]. Since interval containment and permutation graphs are the same class of graphs, and the hamiltonian path problem restricted to cubic planar graphs is NP-complete [14], Theorem 16 implies that permutation editing is NP-complete. □

Again, the results of Section 3 imply the following.

Corollary 18. *Circle and co-circle editing are NP-complete.*

Note that the reductions involved in the proofs of Theorems 7, 11 and 16 imply that the NP-completeness results obtained in this work hold even when the input graphs are restricted to be cubic planar graphs.

4.4. Weakly chordal deletion

In this subsection, we prove that weakly chordal deletion and completion are NP-complete.

We first need a characterization of chain graphs. Two edges e_1 and e_2 in a graph G are independent if their endpoints induce a $2K_2$, i.e., if their four endpoints are different vertices and it does not exist another edge e_3 in G sharing an endpoint with e_1 and the other one with e_2.

Theorem 19 (*Yannakakis [35]*). *A bipartite graph G is a chain graph if and only if G does not contain an independent pair of edges.*

To obtain the NP-completeness result, we will need the following easy lemma.

Lemma 20. *Let G be a graph and let A, B be two disjoint sets of vertices of G such that A induces a complete graph and A is complete to B. Let C be an induced cycle in G of length at least 5. Then*:

(1) $|C \cap A| \leqslant 2$.
(2) *If* $|C \cap A| = 2$, *then these vertices are adjacent in* C *and* $|C \cap B| = \emptyset$.
(3) *If* $|C \cap A| = 1$, *then* $|C \cap B| \leqslant 2$.

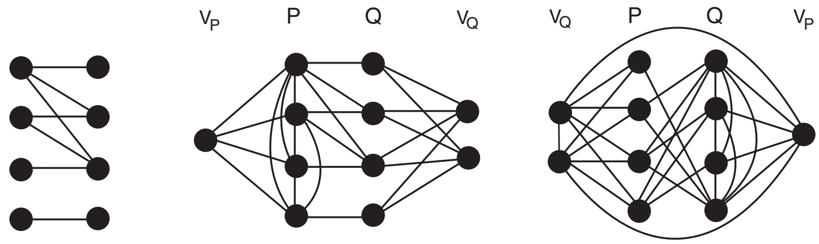

Fig. 29. (a) The bipartite graph G is an example of an instance of chain deletion, setting $k=1$. (b) The graph $C(G)$ is its corresponding instance of weakly chordal deletion, keeping $k=1$. (c) The graph $\overline{C(G)}$ is the complement of $C(G)$.

Theorem 21. *Weakly chordal deletion is NP-complete.*

Proof. The problem belongs to NP since weakly chordal graphs can be recognized in polynomial time [32].

We present a reduction from chain deletion, which is an NP-complete problem [26].

Let $\langle G=(P,Q,E),k\rangle$ be an instance of chain deletion. Construct the following instance $\langle C(G)=(V',E'),k\rangle$ of weakly chordal deletion (see Fig. 29):

- Define $V'=P\cup Q\cup V_P\cup V_Q$, where $V_P=\{v_1,\ldots,v_k\}$ and $V_Q=\{v_{k+1},\ldots,v_{2k},v_{2k+1}\}$.
- Define $E'=E\cup(P\times P)\cup(P\times V_P)\cup(Q\times V_Q)$.

We will show that this chain deletion instance is an affirmative instance if and only if the weakly chordal deletion instance is too.

$\Rightarrow$) Suppose that F is a chain k-deletion set for G. We claim that F is also a weakly chordal k-deletion set for $C(G)$. Let $H=(V',E'\backslash F)$ the resulting graph. Assume the opposite is true, so H is not weakly chordal. This implies that there exists an induced cycle of length greater than 4 in H or in $\overline{H}$. Suppose first that there exists such a cycle C in $\overline{H}$. Note that Q, V_P and V_Q each induces a complete graph in $\overline{H}$, and Q is complete to V_P; V_P is complete to V_Q and V_Q is complete to P in $\overline{H}$. Lemma 20 implies that C has at most two vertices in each of Q, V_P and V_Q. Moreover, the facts that V_Q is complete to P, V_P is anticomplete to P, Q induces a complete graph and the length of C is at least 5, imply that $|C\cap P|\leqslant 2$.

Applying Lemma 20 accordingly, we have the following:

- If $|C\cap V_P|=2$ then $|C\cap Q|=0$ and $|C\cap V_Q|=0$, a contradiction with $|C|\geqslant 5$.
- If $|C\cap V_P|=1$ then $|C\cap Q|=1$, $|C\cap V_Q|=1$ and $|C\cap P|=1$, a contradiction with $|C|\geqslant 5$.

It follows that $|C\cap V_P|=0$. If $|C\cap P|=0$, then $|C|\leqslant 4$, a contradiction. If $|C\cap P|=1$ then $|C\cap V_Q|\leqslant 1$, and thus $|C|\leqslant 4$, again a contradiction. Hence $|C\cap P|=2$ and, by Lemma 20, $|C\cap V_Q|\leqslant 1$. Moreover, $|C\cap Q|\leqslant 2$. As C has at least 5 vertices, these last two inequalities must hold as equalities.

Therefore, an induced cycle without chords of length greater than or equal to 5 in $\overline{H}$ must be of the form $\{p_1,q_2,q_1,p_2,vq_i\}$, where $p_1,p_2\in P$, $q_1,q_2\in Q$ and $vq_i\in V_Q$ (see Fig. 30). This implies that the edges (p_1,q_2) and (p_2,q_1) do not appear in H and the edges (p_1,q_1) and (p_2,q_2) appear in H. But then the edges (p_1,q_1) and (p_2,q_2) are independent in the chain graph $G'=(P,Q,E\backslash F)$, a contradiction.

Now, let C be an induced cycle in H of length greater than 4. If C contains a vertex $v\in V_P$, then both neighbors of v in C are vertices from P, a contradiction (by Lemma 20 two vertices from P must be consecutive in C). If C does not contain a vertex in V_Q, then we have a contradiction with $|C|\geqslant 5$ (because $|C\cap P|\leqslant 2$ and $|C\cap Q|\leqslant 2$). So, we may assume that $|C\cap V_P|=0$ and $|C\cap V_Q|\geqslant 1$.

Let v_q be a vertex in V_Q contained in C. Its two neighbors in C must be vertices from Q. As the length of C is greater than 4, then $|C\cap V_Q|=1$ (because V_Q is complete to Q).

Hence an induced cycle without chords of length greater than or equal to 5 in H must be formed by two vertices from P, two vertices from Q and one vertex from V_Q.

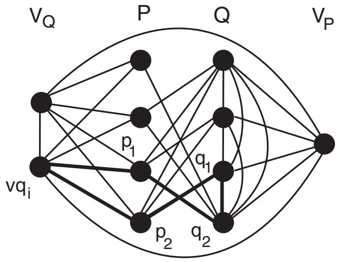

Fig. 30. Induced cycle without chords of length five that may appear in $\overline{H}$.

Both vertices from P must be consecutive in C, otherwise the edge between them would be a chord of C. So, since the vertices from Q are not adjacent between them, the only way of forming C is $\{p_1, p_2, q_2, v_q, q_1\}$, where $p_1, p_2 \in P$, $q_1, q_2 \in Q$ and $v_q \in V_Q$. But then, the edges (p_1, q_1) and (p_2, q_2) are independent in the chain graph $G' = (P, Q, E \backslash F)$, a contradiction.

$\Leftarrow$) Suppose that F is a weakly chordal k-deletion set for $C(G)$. We shall prove that $F \cap E$ is a chain k-deletion set for G. Let $G' = (P, Q, E \backslash F)$. If G' is not a chain graph, then it contains a pair of independent edges (p_1, q_1) and (p_2, q_2), where $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. In $C(G)$, p_1, p_2 are connected by an edge and at least k edge-disjoint paths of length two. Also, q_1, q_2 are connected by at least $k+1$ edge-disjoint paths of length two. Then, p_1, p_2 are yet connected by an induced path of length at most two in $H = (V', E' \backslash F)$ and q_1, q_2 are yet connected by an induced path of length two in H. Then, p_1, q_1, p_2 and q_2 are contained in an induced cycle of length at least 5 in H, a contradiction, since H was weakly chordal. □

Corollary 22. *Weakly chordal completion is NP-complete.*

Proof. By definition, weakly chordal graphs are closed under complement. Then, this result holds from Theorem 21 and the fact that Π-deletion and $\overline{\Pi}$-completion are polynomially equivalent. □

4.5. Bridged deletion

In this subsection, we prove that bridged deletion is NP-complete.

Theorem 23. *Bridged deletion is NP-complete.*

Proof. The problem belongs to NP since bridged graphs can be recognized in polynomial time [13].

We will show a reduction from chain deletion, which is NP-complete [26]. Let $\langle G = (P, Q, E), k\rangle$ be an instance of chain deletion, and construct the following instance $\langle C(G) = (V', E'), k\rangle$ of bridged deletion:

- Define $V' = P \cup Q \cup V_P \cup V_Q$, where $V_P = \{v_1, \ldots, v_k\}$ and $V_Q = \{v_{k+1}, \ldots, v_{2k}\}$.
- Define $E' = E \cup (P \times P) \cup (Q \times Q) \cup (P \times V_P) \cup (Q \times V_Q)$ (see Fig. 31 for an example).

We will show that the chain deletion instance has a solution if and only if the bridged deletion instance has a solution.

$\Rightarrow$) Suppose that F is a chain k-deletion set for G. We claim that F is also a bridged k-deletion set for $C(G)$. Let $H = (V', E' \backslash F)$ the resultant graph. Assume the opposite is true, so H is not a bridged graph. The definition of bridged graphs implies that there is in H an induced cycle C of length greater than 3 and without bridges. If C contains a vertex $v \in V_P$, its two neighbors x and y in C are vertices from P, forming a triangle, a contradiction. The same holds if C contains a vertex $v \in V_Q$. Then, $C \cap V_P = C \cap V_Q = \emptyset$. Since P and Q are complete subgraphs, then C must be of the form $\{p_1, p_2, q_1, q_2\}$, where $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. But then (p_1, q_2) and (p_2, q_1) are independent edges in the chain graph $(P, Q, E \backslash F)$, a contradiction.

$\Leftarrow$) Suppose that F is a bridged k-deletion set for $C(G)$, and let us prove that $F \cap E$ is a chain k-deletion set for G. Let $G' = (P, Q, E \backslash F)$. If G' is not a chain graph, then by Theorem 19 it contains a pair of independent edges (p_1, q_1) and (p_2, q_2), where $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. In $C(G)$, the vertices p_1, p_2 (and also q_1, q_2) are connected by an

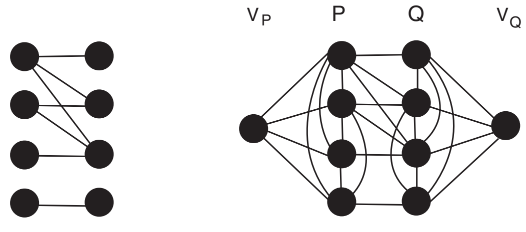

Fig. 31. (a) G is an example of an instance of chain deletion, setting $k=1$. (b) $C(G)$ is its corresponding instance of bridged deletion, keeping $k=1$.

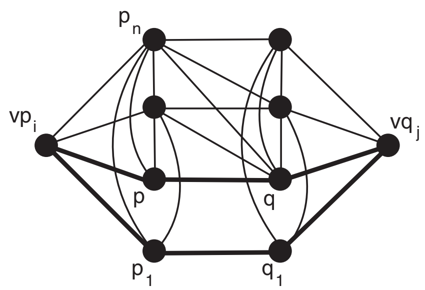

Fig. 32. Cycle of length 6 that can exist in H, composed by vertices $\{vp_i, p_1, q_1, vq_j, q_2, p_2\}$.

edge and at least k edge-disjoint paths of length two. Then, each pair is yet connected by an induced path of length 1 or 2 in $H=(V', E'\backslash F)$, since at most k edges were deleted. So, (p_1, q_1) and (p_2, q_2) must be included in a chordless cycle C of length 4, 5 or 6 in H. If C has length 4 or 5, then it cannot contain any bridge, since the only bridge possible is a chord, a contradiction. If C has length 6, it must be of the form $\{vp_i, p_1, q_1, vq_j, q_2, p_2\}$ as we can see in Fig. 32, and can have a bridge of length 2 joining two vertices separated by three edges in C.

By the construction of $C(G)$, the bridge cannot join a vp_i with vq_j, it can only exist between p_1 and q_2 or between p_2 and q_1. Suppose that the bridge B exists between p_1 and q_2 (the other case is analogous). The vertex in the middle of the bridge cannot belong either to V_P (there are no edges connecting this vertex with q_2) or to V_Q (there are no edges connecting this vertex with p_1). Hence it can only belong to P or Q. Suppose that this vertex belongs to P (the other case is analogous), then B is composed by the edges (p_1, p_n) and (p_n, q_2), with $n \notin \{1, 2\}$. The vertices $\{p_1, p_n, q_2, vq_j, q_1\}$ form a cycle of length 5 in H. The only chord that may appear in this cycle is (p_n, q_1). If (p_n, q_1) is not in H, then there exists a chordless cycle of length 5 in H, a contradiction. If this chord exists, then the vertices $\{p_n, q_2, vq_j, q_1\}$ form a cycle of length 4 in H. This cycle is chordless, again a contradiction. □

4.6. Clique-Helly graphs

Clique-Helly graphs and some of their subclasses have been studied in many previous works. For example, clique-Helly chordal graphs are studied in [34]; clique-Helly circular-arc graphs in [10]; and clique-Helly perfect graphs in [2]. Clique-Helly graphs can be recognized in polynomial time [33]. Thus, if the graph property Π can be recognized in polynomial time, the same holds for the graph property clique-Helly Π.

Theorem 24. *If Π is a graph property such that Π is hereditary and the property remains valid if we add a universal vertex to the graph, then Π-deletion (Π-completion, Π-editing) is polynomially reducible to clique-Helly Π-deletion (clique-Helly Π-completion, clique-Helly Π-editing).*

Proof. We shall prove the following: "*Given a graph G, G satisfies $\Pi \Leftrightarrow G+K_1$ satisfies clique-Helly Π*".

$\Leftarrow$) Trivial, because Π is hereditary.

$\Rightarrow$) $G + K_1$ satisfies Π since only a universal vertex has been added to the graph, and $G + K_1$ is clique-Helly because the new vertex belongs to every clique of $G + K_1$.

Then, the result holds by Proposition 3. □

Corollary 25. *Clique-Helly circular-arc (interval) (chordal) (perfect) (comparability) (permutation) edge modification problems (completion, deletion and editing) are NP-complete.*

Proof. Circular-arc, interval, chordal, perfect, comparability and permutation graphs verify the hypotheses of Theorem 24. The results of Table 1 and Theorem 24 imply this corollary. □

Bipartite edge modification problems can be defined in analogous way to edge modification problems. Let Π be a bipartite graph property. In the *Π-bipartite editing* problem the input is given by a graph $G = (V_1, V_2, E)$, and our goal is to find a minimum set $F \subseteq V_1 \times V_2$ such that $G' = (V_1, V_2, E \triangle F)$ satisfies Π. In the *Π-bipartite deletion* problem we can only delete edges (that is, $F \subseteq E$). This problem is equivalent to *Π-deletion*. In the *Π-bipartite completion* problem we can only add edges (that is, $F \cap E = \emptyset$).

A *bicomplete* in a graph $G = (V, E)$ is a pair of stable sets (B_1, B_2) of G, so that every vertex in B_1 is adjacent to every vertex in B_2. A *biclique* is a maximal bicomplete of a graph. Bicliques in graphs were studied in [28,27]. A graph is biclique-Helly if its bicliques verify the Helly property. A *universal bicomplete* of a bipartite graph $G = (V_1, V_2, E)$ is a bicomplete $B = (B_1, B_2)$ of G such that $B_1 \subseteq V_1$, $B_2 \subseteq V_2$, $V_1 \times B_2 \subseteq E$ and $V_2 \times B_1 \subseteq E$.

The following results are analogous to Propositions 2 and 3 for bipartite graphs.

Proposition 26. *Let $G = (V_1, V_2, E)$ and $G' = (V_1 \cup S_1, V_2 \cup S_2, E)$ be two bipartite graphs, where S_1 and S_2 are disjoint, non-empty stable sets and disjoint from V_1 and V_2. If Π and Π' are (bipartite) graph properties such that G satisfies Π if and only if G' satisfies Π', then Π-deletion is polynomially reducible to Π'-deletion. If in addition Π is hereditary, then Π-bipartite completion (Π-bipartite editing) is polynomially reducible to Π'-bipartite completion (Π'-bipartite editing).*

Proposition 27. *Let $G = (V_1, V_2, E)$ and $G' = (V_1 \cup S_1, V_2 \cup S_2, E \cup E')$ be two bipartite graphs, where $E' = S_1 \times S_2 \cup V_1 \times S_2 \cup S_1 \times V_2$ (S_1 and S_2 are disjoint, disjoint with V_1 and V_2 and non-empty), that is, we add a universal bicomplete. If Π and Π' are (bipartite) graph properties such that G satisfies Π if and only if G' satisfies Π', then Π-bipartite completion is polynomially reducible to Π'-bipartite completion. If in addition Π is hereditary, then Π-deletion (Π-bipartite editing) is polynomially reducible to Π'-deletion (Π'-bipartite editing).*

Biclique-Helly graphs can be recognized in polynomial time [19]. If the graph property Π can be recognized in polynomial time, then the same holds for the property biclique-Helly Π.

Theorem 28. *If Π is a bipartite graph property such that Π is hereditary and the property remains valid if we add a universal bicomplete to the graph, then Π-deletion (Π-bipartite completion, Π-bipartite editing) is polynomially reducible to biclique-Helly Π-deletion (biclique-Helly Π-bipartite completion, biclique-Helly Π-bipartite editing).*

Proof. We will prove the following: "*Given a bipartite graph $G = (V_1, V_2, E)$, G satisfies $\Pi \Leftrightarrow G'$ is biclique-Helly and satisfies Π, where $G' = (V_1 \cup S_1, V_2 \cup S_2, E \cup E')$, and $E' = S_1 \times S_2 \cup V_1 \times S_2 \cup S_1 \times V_2$ (i.e., we add a universal bicomplete)*".

$\Leftarrow$) Trivial, because Π is hereditary and G is an induced subgraph of G'.

$\Rightarrow$) The graph G' satisfies Π since only a universal bicomplete has been added to the graph, and G' is biclique-Helly since $S_1 \cup S_2$ belongs to every biclique of G'.

Then, the result holds by Proposition 27. □

Corollary 29. *Biclique-Helly chain bipartite completion, and biclique-Helly chain deletion are NP-complete.*

Proof. Chain graphs verify the hypotheses of Theorem 28, hence the results of Table 1 and Theorem 28 imply this corollary. □

Note that the complexity of chain editing is still unknown (see Table 1). We only know that this problem is reducible in polynomial time to biclique-Helly chain bipartite editing.

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