Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Applied Mathematics 154 (2006) 1865 – 1876

DISCRETE APPLIED MATHEMATICS

www.elsevier.com/locate/dam

On the combinatorial structure of chromatic scheduling polytopes

Javier Marenco[a, b], Annegret Wagler[c]

[a] Computer Science Dept., FCEN, University of Buenos Aires, Pabellón I, Ciudad Universitaria, (1428) Buenos Aires, Argentina
[b] Sciences Institute, National University of General Sarmiento, J. M. Gutiérrez y J. Verdi, Los Polvorines, (1613) Buenos Aires, Argentina
[c] Institute for Mathematical Optimization, Faculty for Mathematics, Otto-von-Guericke University of Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany

Received 29 December 2004; received in revised form 11 May 2005; accepted 18 January 2006
Available online 19 May 2006

Abstract

Chromatic scheduling polytopes arise as solution sets of the bandwidth allocation problem in certain radio access networks, supplying wireless access to voice/data communication networks for customers with individual communication demands. To maintain the links, only frequencies from a certain spectrum can be used, which typically causes capacity problems. Hence it is necessary to reuse frequencies but no interference must be caused by this reuse. This leads to the bandwidth allocation problem, a special case of so-called chromatic scheduling problems. Both problems are NP-hard, and there do not even exist polynomial time algorithms with a fixed quality guarantee.

As algorithms based on cutting planes have shown to be successful for many other combinatorial optimization problems, the goal is to apply such methods to the bandwidth allocation problem. For that, knowledge on the associated polytopes is required. The present paper contributes to this issue, exploring the combinatorial structure of chromatic scheduling polytopes for increasing frequency spans. We observe that the polytopes pass through various stages—emptyness, non-emptyness but low-dimensionality, full-dimensionality but combinatorial instability, and combinatorial stability—as the frequency span increases. We discuss the thresholds for this increasing "quantity" giving rise to a new combinatorial "quality" of the polytopes, and we prove bounds on these thresholds. In particular, we prove combinatorial equivalence of chromatic scheduling polytopes for large frequency spans and we establish relations to the linear ordering polytope.

Keywords: Bandwidth allocation; Polyhedral combinatorics

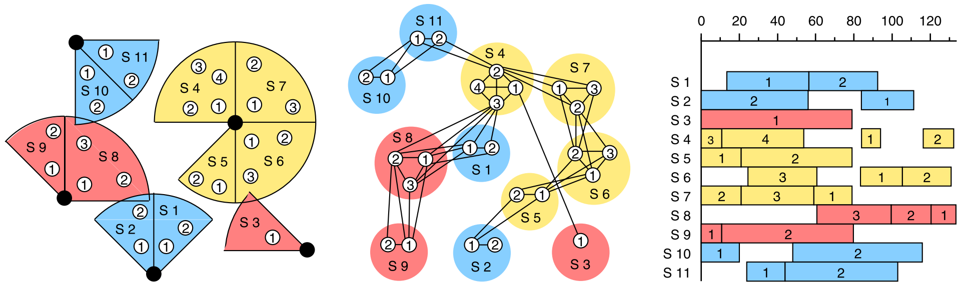
1. Introduction

The purpose of a point-to-multipoint radio access system (PMP-system) is to supply wireless access to voice/data communication networks. Base stations form the access points to the backbone network and customer terminals are linked to base stations by means of radio signals.

There are two main differences between PMP-systems and cellular phone networks. Firstly, each customer is provided a fixed antenna and is assigned to a certain sector of a base station (see Fig. 1a). Secondly, the customers do not have a uniform communication demand but individual ones, hence the task is to assign frequency intervals instead of single channels (see Fig. 1c). A central issue is that a link connecting a customer terminal and a base station may be subject to

E-mail addresses: jmarenco@dc.uba.ar (J. Marenco), wagler@imo.math.uni-magdeburg.de (A. Wagler).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2006.03.032



(a) Sectorization (b) Graph (c) Frequency plan

Fig. 1. Bandwidth allocation in Point-to-Multipoint radio access systems.

interference from another link using the same frequency: links to customers of the same sector must not use the same frequency (since they are served by the same antenna) and, in addition, some links of customers in different sectors may also cause interferences (due to power and direction of the transmitted signals), see the links in Fig. 1b.

To maintain the links in PMP-systems, some specific part of the radio frequency spectrum has to be used. This typically causes capacity problems and, therefore, it is necessary to reuse frequencies. The *bandwidth allocation problem* has to be solved in order to guarantee an interference-free communication. The goal is to assign a frequency interval within the available radio frequency spectrum to each customer (see Fig. 1c), taking into account the individual communication demands and possible interference.

The input of this problem is given as follows. Let $\mathscr{T} = \{t_1, \dots, t_n\}$ be the set of all customer terminals, and $\mathbf{S} = \{S_1, \dots, S_k\}$ be a partition of $\mathscr{T}$ into sectors, providing the information which sector S_j serves the terminal $t_i \in \mathscr{T}$. Let $d = (d_1, \dots, d_n)$ be the vector of communication demands associated with the customer terminals, indicating that customer $t_i \in \mathscr{T}$ has demand $d_i \in \mathbf{Z}$. Additionally, we have a set $\mathscr{E}_X$ of unordered pairs (t_i, t_j) of terminals in *different* sectors that must not use the same frequency due to possible interference. We can represent this setting by a *weighted interference graph* $(G, d) = (V, E, d)$, where the node set V stands for the customer terminals, the edge set E for pairs of interfering customers, and the node weights d for the communication demands. Throughout this paper we denote by $n = |V|$ resp. $m = |E|$ the number of nodes resp. edges of G.

In base stations, oscillators provide the different frequencies—with a possible difference Δ to the required frequency. Thus, between the frequency intervals of possibly interfering links $(t_i, t_j) \in \mathscr{E}_X$ in different sectors, a guard distance $g = 2\Delta$ has to be obeyed. Finally, we have the available radio frequency spectrum $[0, s]$, with $s \in \mathbf{Z}$, where all the frequency intervals have to be placed in. Thus, the problem input consists in the quadruple (G, d, s, g).

The desired output is an assignment of an interval $I(i) = [l_i, r_i] \subseteq [0, s]$, with $l_i, r_i \in \mathbf{Z}$, to each customer $t_i \in \mathscr{T}$ such that $r_i - l_i \geqslant d_i$ for every $t_i \in \mathscr{T}$ and

$$\max\{l_i, l_j\} - \min\{r_i, r_j\} \geqslant \begin{cases} 0 & \text{if } t_i \text{ and } t_j \text{ belong to the same sector,} \\ g & \text{if } (t_i, t_j) \in \mathscr{E}_X \end{cases}$$

for every pair of interfering customers $t_i, t_j \in \mathscr{T}$. For $g = 0$, the problem can be seen as a chromatic scheduling problem [3] or a consecutive coloring problem [4] on the weighted graph (G, d); the problem corresponds to the ordinary graph coloring problem if $d = \mathbf{1}$ holds in addition.

Small instances of the bandwidth allocation problem could be handled by greedy-like heuristics [1], but in order to tackle problem sizes of real world applications, algorithms have to be designed that rely on a deeper insight of the problem structure. The polyhedral approach, consisting of an in-depth investigation of polytopes associated with a combinatorial structure and the application of linear programming based cutting plane techniques, has been very successful in the recent years. To apply such methods to the bandwidth allocation problem, the convex hull of the incidence vectors of all feasible solutions has to be studied. In order to represent a solution, besides the interval bounds l_i and r_i for all $i \in V$, we introduce the ordering variables $x_{ij} \in \{0, 1\}$ for all $ij \in E$, $i < j$, such that $x_{ij} = 1$ if and only if $r_i \leqslant l_j$.

The ordering variables are necessary as the convex hull of the solutions represented only by the interval bounds may contain infeasible integer points. For example, consider the instance $(K_2, d, 5, 0)$, with $d = (1, 2)$. The vectors

$z = (0, 1, 1, 3, 1)$ and $z' = (3, 1, 4, 3, 0)$ represent feasible solutions, but dropping the information given by x_{12}, the convex hull of even these two points contains two infeasible but integral points, namely (1, 1, 2, 3) and (2, 1, 3, 3).

For $ij \in E$, define δ_{ij} to be $\delta_{ij} = 0$ if t_i and t_j belong to the same sector, and $\delta_{ij} = g$ otherwise. A feasible solution is, therefore, an assignment of values to $l_i, r_i \ \forall i \in V$ and $x_{ij} \ \forall ij \in E$ such that the following constraints are satisfied:

$$d_i \leqslant r_i - l_i \quad \forall i \in V, \tag{1}$$

$$0 \leqslant l_i \leqslant r_i \leqslant s \quad \forall i \in V, \tag{2}$$

$$r_i + \delta_{ij} \leqslant l_j + s(1 - x_{ij}) \quad \forall ij \in E,\ i < j, \tag{3}$$

$$r_j + \delta_{ij} \leqslant l_i + s x_{ij} \quad \forall ij \in E,\ i < j, \tag{4}$$

$$x_{ij} \in \{0, 1\} \quad \forall ij \in E,\ i < j, \tag{5}$$

$$l_i,\ r_i \in \mathbf{Z} \quad \forall i \in V. \tag{6}$$

The *demand constraints* (1) and the *bound constraints* (2) assert that the interval $I(i) = [l_i, r_i]$ must satisfy the demand d_i and fit within the available frequency spectrum $[0, s]$. Inequalities (3) and (4) realize the *antiparallelity constraints*, which prevent interfering pairs of intervals from overlapping. Note that the intervals corresponding to the pairs of customers located in the same sector must not overlap, and there must be a distance of at least g between the intervals corresponding to pairs of interfering customers in different sectors. Finally, the *integrality constraints* (5) resp. (6) force the x-variables to be binary resp. the interval bounds to be integral.

Definition 1 (*Chromatic scheduling polytope*). Let (G, d, s, g) be an instance of the bandwidth allocation problem in PMP-systems. We define the *chromatic scheduling polytope* $P(G, d, s, g) \subseteq \mathbf{R}^{2n+m}$ to be the convex hull of all feasible solutions $(l, r, x) \in \mathbf{Z}^{2n+m}$ satisfying constraints (1)–(6).

Chromatic scheduling polytopes admit interesting properties from a combinatorial point of view. As experimentally observed in [5,9] for small instances with co-bipartite and general interference graphs, respectively, the polytopes are empty if the frequency span s is too small and pass through several stages as s increases: from a nonempty but low-dimensional stage to full-dimensionality and, finally, to a combinatorially steady state. In this paper, we shall discuss these combinatorial stages and prove the existence of the corresponding thresholds $s_{\min}(G, d, g)$, $s_{\text{full}}(G, d, g)$, and $s_{\max}(G, d, g)$ as the minimum frequency span s such that $P(G, d, s, g)$ is nonempty, full-dimensional, and combinatorially stable, respectively.

We address in Section 2 the threshold $s_{\min}(G, d, g)$ for nonemptyness. Proving nonemptyness for $P(G, d, s, g)$ is an important task as knowing one feasible solution enables us to run a PMP-system properly. The $\mathcal{NP}$-completeness of the bandwidth allocation problem implies, however, that the exact calculation of $s_{\min}(G, d, g)$ is $\mathcal{NP}$-hard. We, therefore, provide lower and upper bounds for $s_{\min}(G, d, g)$ as certificates for the existence/nonexistence of feasible solutions respectively the nonemptyness/emptyness of the associated polyhedra.

Section 3 presents results related to full-dimensionality. We show that determining the dimension of $P(G, d, s, g)$ is an $\mathcal{NP}$-hard problem. As knowledge on the dimension is crucial for proving which valid inequalities are facets (and, therefore, the best possible cutting planes), we again provide an upper bound on $s_{\text{full}}(G, d, g)$ guaranteeing full-dimensionality.

Section 4 gives a characterization of the extreme points of chromatic scheduling polytopes, which is employed in Section 5 to establish the combinatorial equivalence of all polytopes $\{P(G, d, s, g)\}_{s \geqslant s_{\max}(G,d,g)}$. This implies that frequency spans larger than $s_{\max}(G, d, g)$ do not further simplify the bandwidth allocation problem as all polytopes $\{P(G, d, s, g)\}_{s \geqslant s_{\max}(G,d,g)}$ have the same combinatorial structure of facets and extreme points. We give an upper bound on $s_{\max}(G, d, g)$ guaranteeing combinatorial equivalence.

Finally, we explore some relations between chromatic scheduling polytopes and the linear ordering polytope P^n_{LO}. In particular, we prove that $P(K_n, d, s, 0)$ is affinely isomorphic to P^n_{LO} if $s = \sum d_i$. This result implies that even simple chromatic scheduling polytopes are hard to characterize by means of linear inequalities, since a complete description of $P(K_n, d, s, 0)$ includes all the linear ordering facets.

We close with some concluding remarks and open problems.

2. On emptyness/nonemptyness

This section treats the problem of proving nonemptyness for chromatic scheduling polytopes $P(G, d, s, g)$. Clearly, if the frequency spectrum $[0, s]$ is too small, there exists no feasible schedule for the frequency intervals at all, and so the polytope $P(G, d, s, g)$ is empty. Since the bandwidth allocation problem in PMP-systems is $\mathcal{NP}$-complete [9], the exact calculation of the minimum frequency span $s_{\min}(G, d, g)$ ensuring nonemptyness is an $\mathcal{NP}$-hard problem. We, therefore, provide straightforward bounds on $s_{\min}(G, d, g)$ that guarantee emptyness resp. nonemptyness. It is worth noting that lower bounds for $s_{\min}(G, d, g)$ arise from maximum weighted clique arguments, whereas upper bounds come from coloring assertions.

Let $\omega(G, d)$ denote the weighted clique number of (G, d), i.e., the maximal weight of a clique in G. Then $s_{\min}(G, d, g) \geqslant \omega(G, d)$ clearly holds but a better bound can be achieved taking into account the guard distance g. If $A \subseteq V$, we define $d(A) = \sum_{i \in A} d_i$.

Definition 2 (*Clique bound*). Define $K(G)$ to be the set of all cliques of G and, for $K \in K(G)$, let $p_K = |\{i : S_i \cap K \neq \emptyset\}|$ denote the number of sectors intersecting the clique K. We define the *clique bound* $\omega(G, d, g)$ by

$$\omega(G, d, g) = \max_{K \in K(G)} (d(K) + g(p_K - 1)).$$

Proposition 1. *If $s < \omega(G, d, g)$, then $P(G, d, s, g)$ is empty.*

Proof. Let $K \subseteq V$ be a clique such that $d(K) + g\ (p_K - 1) = \omega(G, d, g)$. Since K is a clique, then the intervals $\{I(i) : i \in K\}$ must be disjoint. Moreover, in every feasible solution there are at least $p_K - 1$ adjacent intervals belonging to different sectors, and since K is a clique they must obey the guard distance, hence at least $p_K - 1$ guard distances must occur among the intervals assigned to the nodes of K. Therefore, we need a frequency span of at least $d(K) + g\ (p_K - 1)$ to assign all these intervals. □

However, $s \geqslant \omega(G, d, g)$ does not provide a certificate for feasibility, as there exist instances where $\omega(G, d, g)$ is strictly smaller than the span of any feasible solution. Note that such instances exist not only for the special case $d = \mathbf{1}$, $g = 0$ of usual graph coloring, but also if $d \neq \mathbf{1}$ by [6]; even such real-world instances are reported in [1], see [9] for more details.

In order to derive an upper bound on $s_{\min}(G, d, g)$, we use some coloring arguments. Let $\chi(G)$ denote the chromatic number of G, i.e., the least s such that the nodes of G can be partitioned into s stable sets. Then obviously $\chi(G) = s_{\min}(G, \mathbf{1}, 0)$ holds and we have to consider the general case $d \neq \mathbf{1}$ and $g \neq 0$.

Definition 3 (*Chromatic bound*). Let $d_{\max} = \max\{d_i : i \in V\}$ denote the maximum node weight of (G, d). We define the *chromatic bound* $\chi(G, d, g)$ as

$$\chi(G, d, g) = (d_{\max} + g)\ \chi(G) - g.$$

Proposition 2. *If $s \geqslant \chi(G, d, g)$, then $P(G, d, s, g)$ is nonempty.*

Proof. Let $k = \chi(G)$ and let $c : V \to \{1, \dots, k\}$ be a coloring of G (i.e., a partition of V into disjoint stable subsets). Construct a feasible solution $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ by setting $z_{l_i} = (c(i) - 1)(d_{\max} + g)$ and $z_{r_i} = z_{l_i} + d_i$, where $c(i)$ is the color assigned to i by c. Note that this assignment is feasible and fits in the frequency spectrum $[0, s]$. Therefore, $P(G, d, s, g)$ is nonempty. □

Thus, Propositions 1 and 2 imply that $s_{\min}(G, d, g)$ can be bounded by the clique bound and the chromatic bound as follows:

$$\omega(G, d, g) \leqslant s_{\min}(G, d, g) \leqslant \chi(G, d, g).$$

Note that the weighted chromatic number $\chi(G, d)$ (i.e., the minimum number of stable sets covering every node i at least d_i times) cannot be used to obtain a better bound than $\chi(G, d, g)$ since the colors assigned to each node cannot be expected to be consecutive.

However, $s_{\min}(G, d, 0)$ equals the interval chromatic number $\chi_{\text{int}}(G, d)$ introduced in [4]. Integrating the guard distance g into the communication demand d by $d'_i = d_i + g$ for all $i \in V$ yields, therefore, $\chi_{\text{int}}(G, d')$ as a further upper bound on $s_{\min}(G, d, g)$ which is even better than the chromatic bound. Unfortunately, determining all the bounds on $s_{\min}(G, d, g)$ is as hard as calculating $s_{\min}(G, d, g)$ itself.

3. Full-dimensional chromatic scheduling polytopes

This section deals with nonempty chromatic scheduling polytopes and addresses the problem of calculating their dimension. We show that this problem is $\mathcal{NP}$-complete in general. However, knowledge on the dimension is important as the best cutting planes are facets, i.e., inequalities defining a face with dimension one less than the polytope itself. We, therefore, establish a lower bound on $s_{\text{full}}(G, d, g)$ guaranteeing full-dimensionality.

The polytope $P(G, d, s, g)$ is nonempty if and only if $s \geqslant s_{\min}(G, d, g)$, but it may not have full dimension even if $s > s_{\min}(G, d, g)$. For example, we have $s_{\min}(C_4, \mathbf{1}, 0) = 2$ but it is a routine to check that the polytopes $P(C_4, \mathbf{1}, 2, 0)$ and $P(C_4, \mathbf{1}, 3, 0)$ are not full-dimensional.

However, every feasible solution of $P(G, d, s, g)$ is also a feasible solution of $P(G, d, s+1, g)$, thus $P(G, d, s, g) \subseteq P(G, d, s+1, g)$ holds and the dimension is a nondecreasing function of the frequency span s. We now prove that chromatic scheduling polytopes are full-dimensional if $s \gg \omega(G, d)$. An ordering of the nodes of a directed graph $D = (V, A)$ is called a *topological ordering* if the node i is located before the node j for every $ij \in A$.

Lemma 1. *Let $\lambda \in \mathbf{R}^{2n+m}$ and $\lambda_0 \in \mathbf{R}$ such that $\lambda^{\mathrm{T}} z = \lambda_0$ for every $z \in P(G, d, s, g)$. If $s > s_{\min}(G, d, g)$, then $\lambda_{l_j} = \lambda_{r_j} = 0$ for every $j \in V$.*

Proof. Let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be an integer feasible solution such that $z_{r_i} - z_{l_i} = d_i$ for every $i \in v$ and such that all the intervals are contained in $[0, s_{\min}(G, d, g)]$. Construct a digraph $D = (V, E_D)$ such that $ij \in E_D$ if and only if $ij \in E$ and $I(j)$ is located before $I(i)$. Note that D is acyclic. Now, let $i_1, \ldots, i_n$ be a topological ordering of the nodes of D and construct $2n$ feasible solutions $\{z^k, w^k\}_{k=1}^n$ as follows. The point z^k is obtained from z by shifting the intervals $I(i_j)$ for $j = 1, \ldots, k$ one unit to the right. The point w^k is obtained from z by shifting the intervals $I(i_j)$ for $j = 1, \ldots, k-1$ one unit to the right, and enlarging $I(i_k)$ one unit to the right. These new points are feasible solutions. Indeed, if the interval $I(i_j)$ has been shifted resp. enlarged to the right in z^k resp. w^k, then all the possible interfering intervals to the right of $I(I_j)$ have already been shifted, since the corresponding nodes are located before i_j in any topological ordering of D. The pair of solutions z^k and w^k for $k = 1, \ldots, n$ only differ in their l_{i_k}-coordinate, hence the l_{i_k}-coordinate of λ must be zero. Moreover, the pair of solutions z^k and z^{k-1} for $k = 1, \ldots, n$ (where we consider $z^0 = z$) only differ in their l_{i_k}- and r_{i_k}-coordinates, hence the r_{i_k}-coordinate of λ must be zero. Therefore, $\lambda_{l_j} = \lambda_{r_j} = 0$ for every $j \in V$. □

This enables us to provide a lower bound on s ensuring full-dimensionality in the general case.

Definition 4. For any instance (G, d, s, g), let

$$\gamma(G, d, g) = s_{\min}(G, d, g) + \max_{jk \in E} (d_j + d_k) + 2g.$$

Theorem 1. *If $s \geqslant \gamma(G, d, g)$ then $P(G, d, s, g)$ is full-dimensional.*

Proof. Let $\lambda^{\mathrm{T}} z = \lambda_0$ for every $z \in P(G, d, s, g)$. By Lemma 1, we have $\lambda_{l_i} = \lambda_{r_i} = 0$ for every $i \in V$. Now, let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be a feasible solution such that $\max_{i \in V} z_{r_i} = s_{\min}(G, d, g)$ (such a solution exists by the definition of the nonemptyness threshold $s_{\min}(G, d, g)$). Consider an arbitrary edge $ij \in E$ and construct the feasible solution z^1 as follows:

$$z^1_{l_k} = \begin{cases} s_{\min}(G, d, g) + g & \text{if } k = i, \\ s_{\min}(G, d, g) + d_i + 2g & \text{if } k = j, \\ z_{l_k} & \text{otherwise.} \end{cases}$$

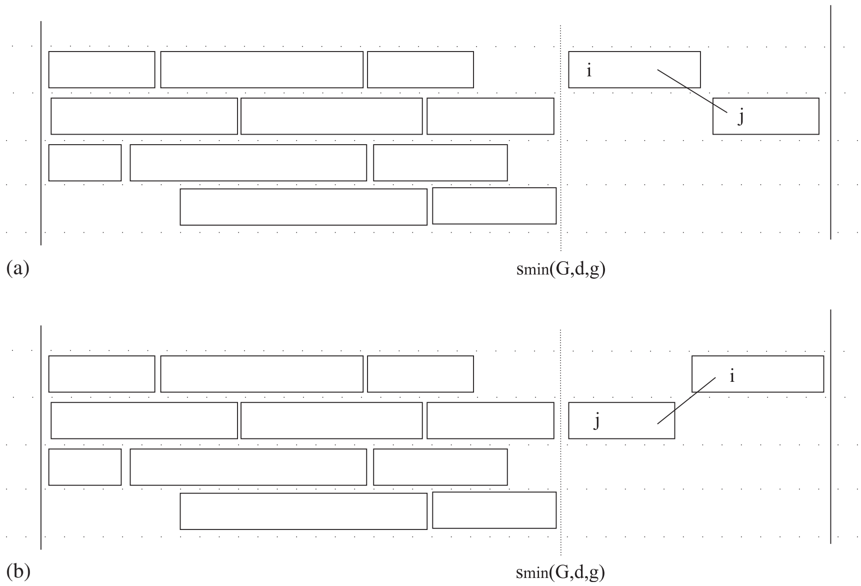


Fig. 2. Constructions for the proof of Theorem 1.

Define further $z^1_{r_k} = z^1_{l_k} + d_k$ for every $k \in V$. Now construct a new feasible solution z^2 from z^1 by swapping the intervals $I(i)$ and $I(j)$ (see Figure 2). These solutions only differ in their l_i-, r_i-, l_j-, r_j- and x_{ij}-coordinates and, therefore, $\lambda_{x_{ij}} = 0$. Since ij is an arbitrarily chosen edge, we have $\lambda = \mathbf{0}$, and so we conclude that $P(G, d, s, g)$ is full-dimensional. □

Theorem 1 implies that the polytope $P(G, d, s, g)$ is full-dimensional if s is large enough. Hence there exists indeed a threshold $s_{\text{full}}(G, d, g)$ for full-dimensionality, and Theorem 1 can be restated as

$$s_{\text{full}}(G, d, g) \leqslant \gamma(G, d, g).$$

This bound is sharp, in the sense that there exist infinitely many graphs G such that $P(G, d, s, g)$ does not have full dimension for $s < \gamma(G, d, g)$, see [9]. However, calculating $s_{\text{full}}(G, d, g)$ turns out to be an $\mathcal{NP}$-hard problem even for uniform instances. Consider the associated decision problem for $g = 0$:

	Full-Dimensionality
Instance:	A weighted graph (G, d) and an integer $s \in \mathbf{Z}_+$.
Question:	Has $P(G, d, s, 0)$ full dimension?

Theorem 2. *Full-Dimensionality is $\mathcal{NP}$-complete.*

Proof. It is not hard to verify that this problem belongs to $\mathcal{NP}$, since we can nondeterministically generate a set of integer feasible solutions and verify in polynomial time whether this set is a set of affinely independent points with the required number of elements or not. To complete the proof, we shall reduce Graph coloring to Full-dimensionality. Let $G = (V, E)$ be an arbitrary graph and construct a graph $H = (V_H, E_H)$ from G by taking

$$V_H = V \cup \{v_1, v_2, v_3, v_4\},$$
$$E_H = E \cup \{v_i\ w : w \in V,\ i = 1, \ldots, 4\} \cup \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}.$$

We claim that $\chi(G) \leqslant s$ if and only if $P(H, \mathbf{1}, s + 4, 0)$ has full dimension. For the forward direction, if $\chi(G) \leqslant s$ then $\gamma(H, \mathbf{1}, 0) = \chi(H) + 2 \leqslant \chi(G) + 4 \leqslant s + 4$, and $P(H, \mathbf{1}, s + 4, 0)$ is full-dimensional by Theorem 1. For the converse direction, suppose $\chi(G) \geqslant s + 1$ and consider any feasible solution $z \in P(H, \mathbf{1}, s + 4, 0) \cap \mathbf{Z}^{2|V_H|+|E_H|}$. This solution must have at least $s + 1$ colors occupied by intervals corresponding to nodes in V, and this leaves at most three colors left for the nodes $\{v_1, \ldots, v_4\}$. Thus, either v_1 and v_3 or v_2 and v_4 have the same color, and only the four configurations

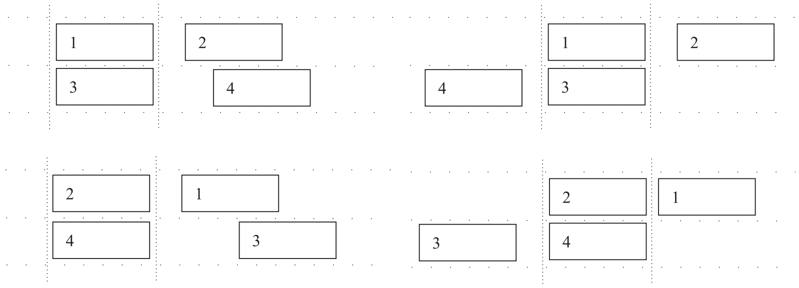


Fig. 3. Illustration for the proof of Theorem 2.

depicted in Fig. 3 (along with their symmetrical solutions) are possible. All of them satisfy $x_{v_1v_2} - x_{v_1v_4} = x_{v_3v_2} - x_{v_3v_4}$, hence $P(H, \mathbf{1}, s+4, 0)$ is not full-dimensional. $\square$

4. A characterization of the extreme points

In this and the following section we explore the combinatorially steady state of chromatic scheduling polytopes. We start by providing a characterization of the extreme points for this kind of polyhedra. For every $ij \in E$, recall that δ_{ij} is the minimum required gap between the intervals $I(i)$ and $I(j)$, i.e., $\delta_{ij} = g$ if i and j belong to different sectors, and $\delta_{ij} = 0$ otherwise.

Definition 5 (*Adjacency graph*). Let $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be a feasible schedule. The *adjacency graph* associated with this schedule is $H(z) = (V', E')$, with

$$V' = \{l_i : i \in V\} \cup \{r_i : i \in V\},$$
$$E' = \{l_i r_i : i \in V \text{ and } z_{r_i} - z_{l_i} = d_i\} \cup \{r_i l_j : ij \in E \text{ and } z_{r_i} + \delta_{ij} = z_{l_j}\}.$$

For example, if H is the interference graph depicted in Fig. 4(a) and $d = \mathbf{1}$, then Fig. 4(b) shows a feasible schedule in $P(H, \mathbf{1}, 6, 0)$, and Fig. 4(c) presents its associated adjacency graph.

Definition 6. A connected component C of $H(z)$ is called a *border component* if there exists some $l_i \in C$ with $z_{l_i} = 0$ or some $r_i \in C$ with $z_{r_i} = s$.

Theorem 3. *The point $z \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ is an extreme point of the polytope $P(G, d, s, g)$ if and only if every connected component of $H(z)$ is a border component.*

Proof. *Only if.* Consider a feasible solution z and its adjacency graph $H(z)$. Suppose that $H(z)$ has a nonborder component C, and construct two feasible schedules $z^1, z^2 \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ from z by shifting the bounds in C one unit to the left resp. to the right, i.e.,

$$z^1_{l_j} = \begin{cases} z_{l_j} - 1 & \text{if } l_j \in C, \\ z_{l_j} & \text{if } l_j \notin C, \end{cases} \qquad z^2_{l_j} = \begin{cases} z_{l_j} + 1 & \text{if } l_j \in C, \\ z_{l_j} & \text{if } l_j \notin C, \end{cases}$$
$$z^1_{r_j} = \begin{cases} z_{r_j} - 1 & \text{if } r_j \in C, \\ z_{r_j} & \text{if } r_j \notin C, \end{cases} \qquad z^2_{r_j} = \begin{cases} z_{r_j} + 1 & \text{if } r_j \in C, \\ z_{r_j} & \text{if } r_j \notin C. \end{cases}$$

Claim: $z^1, z^2 \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$. We first verify that $z^1_{r_j} - z^1_{l_j} \geqslant d_j$ for every $j \in V$. Suppose that $r_j \in C$ but $l_j \notin C$. The construction of $H(z)$ implies $z_{r_j} - z_{l_j} > d_j$, since otherwise l_j would belong to C. Hence z^1 satisfies the demand constraints. It is not difficult to verify that $0 \leqslant z^1_{l_j}$ for every $j \in V$, since the left interval bound l_j is shifted to the left only when l_j belongs to a nonborder component, implying $z_{l_j} > 0$. The opposite constraints $z^1_{l_j} \leqslant s - d_j$ are clearly satisfied.

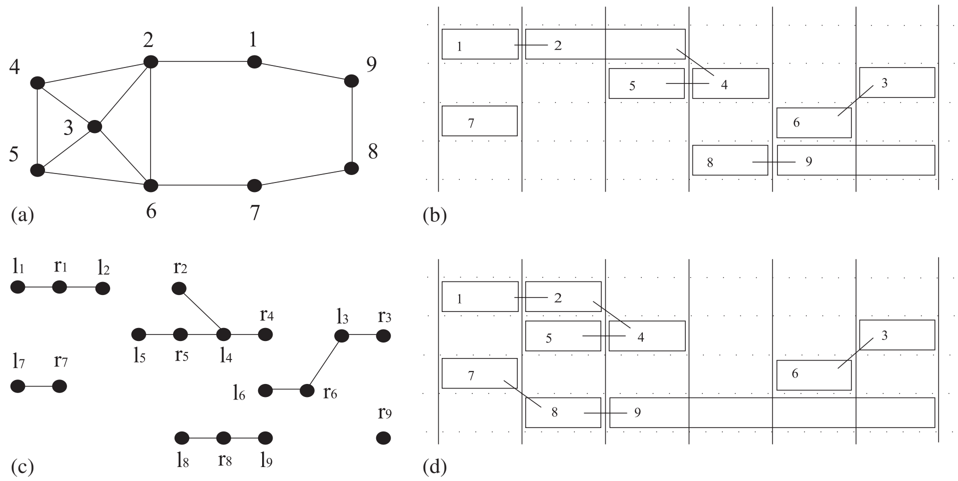


Fig. 4. Examples for Section 4.

To complete the proof of the claim we show that z^1 satisfies the antiparallelity constraints, by verifying that no overlappings are produced by the shifting. In this setting, an overlapping can occur only when $z_{x_{jk}} = 1$ (for $jk \in E$) and z_{l_k} is shifted but z_{r_j} remains unchanged. By construction, this implies $l_k \in C$ and $r_j \notin C$, hence $z_{r_j} + \delta_{jk} < z_{l_k}$ and so $z^1_{r_j} + \delta_{jk} \leqslant z^1_{l_k}$. The schedule z^2 is defined similarly, and the same arguments show that it is feasible.

But now we have $z = \frac{1}{2}(z^1 + z^2)$ and, therefore, z is not an extreme point.

If: Let z be a feasible solution such that every connected component of $H(z)$ is a border component. Further, suppose that $z^1, \ldots, z^p \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ are p extreme points of $P(G, d, s, g)$ such that $z = \sum_{i=1}^{p} \alpha_i z^i$, with $\sum_{i=1}^{p} \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1, \ldots, p$. Since $z_{x_e}, z^i_{x_e} \in \{0, 1\}$ for every edge $e \in E$, then $z_{x_e} = z^i_{x_e}$.

Let C be a connected component of $H(z)$. Since C is a border component, then either (a) $l_t \in C$ and $z_{l_t} = 0$ or (b) $r_t \in C$ and $z_{r_t} = s$, for some $t \in V$. Assume w.l.o.g. that the former holds. For $k \in C$, define γ_k to be the distance from node k to l_t in $H(z)$ (note that $\gamma_{l_t} = 0$). We now verify by induction on γ that $z_{l_j} = z^i_{l_j}$ for every $l_j \in C$ and $z_{r_j} = z^i_{r_j}$ for every $r_j \in C$. Let $k \in C$. If $\gamma_k = 0$ then $k = l_t$, so $z_{l_t} = 0$. But $z^i_{l_t} \geqslant 0$ for $i = 1, \ldots, p$, implying $z^i_{l_t} = 0$. On the other hand, if $\gamma_k > 0$, then either $k = l_j$ or $k = r_j$ for some $j \in V$. Suppose w.l.o.g. the former and consider the following cases:

- If there exists some $r_l \in C$ such that $z_{l_j} + \delta_{jl} = z_{r_l}$ and $\gamma_{r_l} = \gamma_{l_j} - 1$, by inductive hypothesis we have $z_{r_l} = z^i_{r_l}$ for $i = 1, \ldots, p$. Since z and z^i have the same ordering among the intervals, then $z^i_{l_j} \geqslant z^i_{r_l} - \delta_{jl} = z_{r_l} - \delta_{jl} = z_{l_j}$, implying $z^i_{l_j} = z_{l_j}$ for $i = 1, \ldots, p$.
- On the other hand, if $z_{r_j} - z_{l_j} = d_j$ and $\gamma_{r_j} = \gamma_{l_j} - 1$, the inductive hypothesis implies $z^i_{r_j} = z_{r_j}$ for $i = 1, \ldots, p$. Since $z^i_{l_j} \leqslant z^i_{r_j} - d_j = z_{r_j} - d_j = z_{l_j}$, then $z^i_{l_j} = z_{l_j}$ for $i = 1, \ldots, p$.

The same arguments apply to the case $k = r_j$. This way we show that $z = z^i$ for $i = 1, \ldots, p$ and, therefore, z is an extreme point of $P(G, d, s, g)$. □

In the example above, the feasible schedule depicted in Fig. 4(b) is not an extreme point of $P(H, \mathbf{1}, 6, 0)$, whereas Fig. 4(d) presents a solution whose incidence vector is an extreme point of $P(H, \mathbf{1}, 6, 0)$.

5. Combinatorial equivalence for large frequency spans

The main result of this section asserts the existence of a value $s_{\max}(G, d, g) \in \mathbf{Z}_+$ such that the polytopes $\{P(G, d, s, g)\}_{s \geqslant s_{\max}(G,d,g)}$ are pairwise affinely isomorphic and, hence, combinatorially equivalent. Moreover, we establish an upper bound on $s_{\max}(G, d, g)$.

Definition 7. The polytopes $P \subseteq \mathbf{R}^n$ and $Q \subseteq \mathbf{R}^m$ are *affinely isomorphic*, denoted by $P \cong Q$, if there is an affine map $f : \mathbf{R}^n \to \mathbf{R}^m$ that is a bijection between the two polytopes.

Note that the definition asks for an affine bijection between all the points of the polytopes, and this is equivalent to finding an affine bijection between the extreme points of P and Q, since affine bijections preserve convex combinations of points. Moreover, if f is a bijection in the ambient spaces, then P and Q are basically "the same polytope" with respect to an affine change of coordinates. From the combinatorial point of view, if P and Q are affinely isomorphic, then they share the same facial structure. In particular, the affine map gives an isomorphism between their extreme points, and between their facets [11].

Definition 8. Let $\tau(G, d, g)$ denote the minimum frequency spectrum length s such that there exists a solution for every possible ordering among the intervals.

In order to prove the equivalence of $P(G, d, s, g)$ and $P(G, d, s + 1, g)$, we define a different representation for feasible schedules in terms of binary variables. For every node $i \in V$ and every $k \in \{0, \ldots, s - 1\}$, define the binary *position variables* q_{ik} and u_{ik} as

$$q_{ik} = \begin{cases} 1 & \text{if } l_i \geqslant k, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$u_{ik} = \begin{cases} 1 & \text{if } r_i \geqslant k, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

We also consider the ordering variables x_{ij}, for $ij \in E$, with the usual meaning. To every extreme point $z = (l, r, x)$ of $P(G, d, s, g)$ we can associate a point $w^z = (q, u, x) \in \mathbf{Z}^{2ns+m}$ with q resp. u defined by (7) resp. (8). We define $\mathscr{P}(G, d, s, g) \subseteq \mathbf{R}^{2ns+m}$ to be the convex hull of all the points constructed this way, i.e.,

$$\mathscr{P}(G, d, s, g) = \text{conv}\{w^z : z \text{ is an extreme point of } P(G, d, s, g)\}.$$

Since the extreme points $z_1, \ldots, z_t$ of $P(G, d, s, g)$ are pairwise distinct, then $w^{z_1}, \ldots, w^{z_t}$ are pairwise distinct as well. Moreover, $w^{z_1}, \ldots, w^{z_t}$ are 0/1-vectors and, therefore, none of them can be written as a convex combination of the remaining ones. Hence $\mathscr{P}(G, d, s, g)$ has the same number of extreme points as $P(G, d, s, g)$.

Lemma 2. $P(G, d, s, g) \cong \mathscr{P}(G, d, s, g)$.

Proof. Consider the affine map $f : \mathscr{P}(G, d, s, g) \cap \mathbf{Z}^{2ns+m} \to P(G, d, s, g)$ defined by

$$f(w)_{l_i} = \sum_{k=0}^{s-1} w_{q_{ik}} \quad \forall i \in V,$$

$$f(w)_{r_i} = \sum_{k=0}^{s-1} w_{u_{ik}} \quad \forall i \in V,$$

$$f(w)_{x_{ij}} = w_{x_{ij}} \quad \forall ij \in E.$$

This function maps the point $w = (q, u, x)$ onto the point $f(w) = (l, r, x)$. Therefore, f maps extreme points of $\mathscr{P}(G, d, s, g)$ onto extreme points of $P(G, d, s, g)$. This mapping is clearly injective and, since the sets of the extreme points of both polytopes have the same cardinality, it follows that f is a bijection between these sets. Since f is an affine bijection between the set of extreme points of $\mathscr{P}(G, d, s, g)$ and the set of extreme points of $P(G, d, s, g)$, then f is a bijection between $\mathscr{P}(G, d, s, g)$ and $P(G, d, s, g)$ and, therefore, these polytopes are affinely isomorphic. □

Lemma 3. *If* $s > 2\tau(G, d, g)$, *then* $\mathscr{P}(G, d, s, g) \cong \mathscr{P}(G, d, s + 1, g)$.

Proof. Let z be an extreme point of $P(G, d, s, g)$, and let C be a connected component of $H(z)$. Since C is a border component, there exists some $i \in C$ such that either $z_{l_i} = 0$ or $z_{r_i} = s$ holds. If $z_{l_i} = 0$, $s > 2\tau(G, d, g)$ implies $\max_{j \in C} z_{l_j} < \lfloor s/2 \rfloor$ and $\max_{j \in C} z_{r_j} < \lfloor s/2 \rfloor$. Similarly, if $z_{r_i} = s$, $s > 2\tau(G, d, g)$ implies $\min_{j \in C} z_{l_j} > \lfloor s/2 \rfloor$ and

$\min_{j\in C} z_{r_j} > \lfloor s/2 \rfloor$. Hence $z_{l_i} \neq \lfloor s/2 \rfloor$ and $z_{r_i} \neq \lfloor s/2 \rfloor$ for every $i \in V$, and thus the interval bounds can be partitioned into two subsets, namely the bounds located in $[0, s/2-1]$ and the bounds located in $[s/2+1, s]$.

Now, if w^z is the corresponding feasible solution of $\mathscr{P}(G, d, s, g)$, we denote by $\mathrm{shift}(w^z)$ the corresponding extreme point of $\mathscr{P}(G, d, s+1, g)$, which has the same configuration, but the interval bounds located in $[s/2+1, s]$ are now shifted one unit to the right. Therefore, the point $\mathrm{shift}(w^z)$ can be written as:

$$\mathrm{shift}(w^z)_{q_{ik}} = \begin{cases} w_{q_{ik}} & \text{if } k < \lfloor s/2 \rfloor, \\ w_{q_{i,k-1}} & \text{if } k \geqslant \lfloor s/2 \rfloor, \end{cases}$$
$$\mathrm{shift}(w^z)_{u_{ik}} = \begin{cases} w_{u_{ik}} & \text{if } k < \lfloor s/2 \rfloor, \\ w_{u_{i,k-1}} & \text{if } k \geqslant \lfloor s/2 \rfloor, \end{cases}$$
$$\mathrm{shift}(w^z)_{x_{ij}} = w_{x_{ij}}.$$

This mapping shifts the left resp. right interval bounds located in $[s/2+1, s]$ (and therefore with $q_{i,s/2} = 1$ resp. $u_{i,s/2} = 1$) one unit to the right, and lets the remaining bounds unchanged. Moreover, it is an affine bijection between the extreme points of $\mathscr{P}(G, d, s, g)$ and $\mathscr{P}(G, d, s+1, g)$ implying that they are affinely isomorphic. □

Theorem 4. *If* $s > 2\tau(G, d, g)$, *then* $P(G, d, s, g) \cong P(G, d, s+1, g)$.

Proof. From Lemma 2 and Lemma 3 follows $P(G, d, s, g) \cong \mathscr{P}(G, d, s, g) \cong \mathscr{P}(G, d, s+1, g) \cong P(G, d, s+1, g)$. □

Remark. The definition of $\mathscr{P}(G, d, s, g)$ presented in this section was inspired by the construction given in [8] for characterizing the integer hull of a general polytope. It is also worth noting that an alternative proof of a weaker version of Theorem 4 can be obtained by proving that the Fourier–Motzkin elimination method [11] performs the same operations on $P(G, d, s, g)$ and $P(G, d, s+1, g)$ when $s \gg \omega(G, d)$.

The previous results ensure the existence of a threshold $s_{\max}(G, d, g)$ on the frequency span such that $P(G, d, s, g) \cong P(G, d, s+1, g)$ for all $s \geqslant s_{\max}(G, d, g)$. Theorem 4 implies

$$s_{\max}(G, d, g) \leqslant 2\tau(G, d, g) + 1$$

but computational evidence suggests $s_{\max}(G, d, g) = \tau(G, d, g) + 1$, see [9]. The latter has been verified for disjoint unions of cliques as interference graphs in [9] but remains open for arbitrary interference graphs.

6. Relations to the linear ordering polytope

A *linear ordering* of a finite set $V = \{1, \ldots, n\}$ is a bijective mapping $\sigma : V \to \{1, \ldots, n\}$. For $i \in V$ and $j \in V$, $i \neq j$, we say that i is *before* j in σ if $\sigma(i) < \sigma(j)$. With each linear ordering σ we associate a characteristic vector $x^\sigma \in \mathbf{R}^{n(n-1)}$, defined as follows:

$$x^\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i, j \in V,\ i \neq j.$$

The *linear ordering polytope* P^n_{LO} on n nodes is the convex hull of the characteristic vectors of all linear orderings of $\{1, \ldots, n\}$ [7]. Complete descriptions of P^n_{LO} are known for $n \leqslant 7$, with 87.472 facets for $n = 7$. A conjectured complete description for $n = 8$ contains over 480 million facets [2].

Chromatic scheduling polytopes share many structural properties with the linear ordering polytope, since the ordering variables have the same meaning in both settings. Not surprisingly, some of the simplest cases of chromatic scheduling polytopes, namely the instances defined over complete graphs, with minimum frequency spectrum length are equivalent to P^n_{LO}.

Theorem 5. *If* $s = \sum_{i=1}^n d_i$, *then* $P(K_n, d, s, 0) \cong P^n_{\mathrm{LO}}$.

Proof. Since $s = \omega(K_n, d, 0)$ then $P(K_n, d, s, 0)$ is nonempty. Moreover, in every feasible solution all intervals $I(i)$ have exactly length d_i and there is no gap between two intervals left; thus the feasible solutions distinguish only in the order of the intervals. Therefore, the following linear equations are satisfied by every feasible solution of $P(K_n, d, s, 0)$:

$$l_i = \sum_{j \neq i} d_j \ x_{ji}, \quad i = 1, \ldots, n,$$
$$r_i = \sum_{j \neq i} d_j \ x_{ji} + d_i, \quad i = 1, \ldots, n.$$

Hence the interval bound variables can be written as affine combinations of the ordering variables, which are precisely the linear ordering variables. Moreover, this affine mapping is a bijection, since every linear ordering generates a feasible schedule in $P(K_n, d, s, 0)$ and conversely. Thus, $P(K_n, d, s, 0) \cong P^n_{\mathrm{LO}}$. □

This result implies that even simple chromatic scheduling polytopes are hard to characterize. A complete description of $P(K_n, d, s, 0)$ in terms of its facets should include all the linear ordering facets, which amount to several millions of valid inequalities even for small instances. A similar relationship holds for chromatic scheduling polytopes over arbitrary interference graphs, as shown in the remaining of this section.

Definition 9. If $\pi^{\mathrm{T}} x \leqslant \pi_0$ is a valid inequality of P^n_{LO}, let S_π denote the set of edges having nonzero coefficients in the inequality (i.e., $S_\pi = \{e \in E : \pi_e \neq 0\}$).

Proposition 3. *Let $\pi^{\mathrm{T}} x \leqslant \pi_0$ be a valid inequality of P^n_{LO} with $S_\pi \subseteq E$. Then the inequality $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leqslant \pi_0$ is valid for $P(G, d, s, g)$.*

Proof. Let $(l, r, x) \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$ be an integer feasible solution. The vector x specifies a partial ordering among the intervals, and can be extended into a linear ordering $x' \in P^n_{\mathrm{LO}}$ satisfying $\pi^{\mathrm{T}} x' \leqslant \pi_0$. Since $S_\pi \subseteq E$, then $\pi^{\mathrm{T}} x' = \sum_{ij \in S_\pi} \pi_{ij} x'_{ij} = \sum_{ij \in S_\pi} \pi_{ij} x_{ij}$, implying that $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leqslant \pi_0$ is valid for $P(G, d, s, g)$. □

Theorem 6. *Let $\pi^{\mathrm{T}} x \leqslant \pi_0$ be a facet-defining inequality of P^n_{LO} with $S_\pi \subseteq E$. If $s \gg \omega(G, d)$, then $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leqslant \pi_0$ defines a facet of $P(G, d, s, g)$.*

Proof. Since the equations $x_{ij} + x_{ji} = 1 \ \forall i \neq j$ are a maximal equation system for P^n_{LO}, there exist $k = n(n-1)/2$ affinely independent integer points $x^1, \ldots, x^k \in P^n_{\mathrm{LO}}$ such that $\pi^{\mathrm{T}} x^i = \pi_0$ for $i = 1, \ldots, k$. These points have $n(n-1)/2$ coordinates, one for each edge of K_n. Delete the coordinates corresponding to the edges that are not present in G. That way we obtain the new points $\mathrm{proj}_x(x^1), \ldots, \mathrm{proj}_x(x^k) \in \mathbf{R}^m$, and we can find m affinely independent points among them. Since $s \gg \omega(G, d)$, we can extend $\bar{x}^i = \mathrm{proj}_x(x^i)$ to a feasible schedule $z^i \in P(G, d, s, g) \cap \mathbf{Z}^{2n+m}$, by assigning the intervals in such a way that the precedence relation indicated by $\bar{x}^i$ is satisfied.

We now construct $2n$ more affinely independent points from z^1 as follows. Let $D = (V, E_D)$ be a digraph such that $ij \in E_D$ if and only if $ij \in E$ and $I(j)$ is located before $I(i)$ in z^1. Let $i_1, \ldots, i_n$ be a topological ordering of D, and construct n feasible solutions $u^1, \ldots, u^n \in P(G, d, s, g)$ by setting

$$u^i_{l_j} = \begin{cases} z^1_{l_j} + 1 & \text{if } \ j = i_t \quad \text{for } t \leqslant i, \\ z^1_{l_j} & \text{if } \ j = i_t \quad \text{for } t > i, \end{cases}$$
$$u^i_{r_j} = u^i_{l_j} + d_j.$$

Now, for $j = 1, \ldots, n$, construct a point $w^j \in P(G, d, s, g)$ from u^j by enlarging the interval $I(i_j)$ one unit to the left. These new schedules are affinely independent with respect to $z^1, \ldots, z^n$. This way we complete a set of $2n + m$ affinely points and, therefore, $\sum_{ij \in S_\pi} \pi_{ij} x_{ij} \leqslant \pi_0$ defines a facet of the (full-dimensional) polytope $P(G, d, s, g)$. □

7. Concluding remarks and open problems

The present paper explores the different combinatorial stages of chromatic scheduling polytopes for increasing frequency spans, from nonemptyness to full-dimensionality, and finally to a combinatorial steady state. As the associated decision problems are $\mathcal{NP}$-hard, we provided bounds on the corresponding thresholds giving rise to a new combinatorial "quality" of the polytopes.

Deciding emptyness/nonemptyness for $P(G, d, s, g)$ is a crucial issue with strong practical implications; we provided the clique bound and the chromatic bound as certificates for emptyness and nonemptyness, respectively. Clearly, further strengthening or refining these bounds is of interest in order to obtain better conditions ensuring feasibility/infeasibility of the original bandwidth allocation problem.

It turned out that determining the dimension of chromatic scheduling polytopes is a difficult task, both from the computational and the theoretical point of view. Clearly, the dimension is a nondecreasing function of the frequency span; we could prove that $P(G, d, s, g)$ is full-dimensional if $s \geqslant \gamma(G, d, g)$, but nothing is known about the dimension in the case $s < \gamma(G, d, g)$. In particular, we do not even have a complete characterization of the dimension for usual graph coloring instances $(G, \mathbf{1}, s, 0)$. It would also be interesting to develop lower bounds of $s_{\text{full}}(G, d, g)$.

Moreover, we proved combinatorial equivalence of all polytopes $P(G, d, s, g)$ for $s > 2\tau(G, d, g)$, but empirical evidence suggests $s_{\max}(G, d, g) = \tau(G, d, g) + 1$. As the proof technique applied in Theorem 4 cannot be trivially adapted to this case, further ideas related to the combinatorial stability are required in order to get a better bound.

Finally, we related chromatic scheduling polytopes with the linear ordering polytopes and proved that chromatic scheduling polytopes associated with complete interference graphs are affinely isomorphic to linear ordering polytopes. This implies that even chromatic scheduling polytopes associated with the simplest instances are hard to characterize.

References

[1] A. Bley, A. Eisenblätter, M. Grötschel, A. Wagler, R. Wessäly, Frequenzplanung für Punkt- zu Mehrpunkt-Funksysteme, Abschlußbericht eines Projekts der BOSCH Telecom GmbH und des Konrad-Zuse-Zentrums für Informationstechnik, Berlin, 1999.

[2] T. Christof, G. Reinelt, Decomposition and parallelization techniques for enumerating the facets of 0/1 polytopes. Preprint, Heidelberg University (1998).

[3] D. de Werra, Y. Gay, Chromatic scheduling and frequency assignment, Discrete Appl. Math. 49 (1994) 165–174.

[4] D. de Werra, A. Hertz, Consecutive colorings of graphs, ZOR 32 (1988) 1–8.

[5] A. Gerhardt, Polyedrische Untersuchungen von Zwei-Maschinen-Scheduling-Problemen mit Antiparallelitätsbedingungen, Master thesis, Technische Universität Berlin, 1999.

[6] M.C. Golumbic, Algorithmic graph theory and perfect graphs, Academic Press, New York, 1980.

[7] M. Grötschel, M. Jünger, G. Reinelt, Facets of the linear ordering polytope, Math. Programming 33 (1985) 43–60.

[8] J. Lasserre, The integer hull of a convex rational polytope. Technical Report # 03018, LAAS, Toulouse, 2003.

[9] J. Marenco, Chromatic scheduling polytopes coming from the bandwidth allocation problem in point-to-multipoint radio access systems, PhD Thesis, Universidad de Buenos Aires, Argentina, 2005.

[11] G. Ziegler, Lectures on Polytopes, Springer, New York, 1995.