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Nielsen–Olesen vortices in noncommutative space

G.S. Lozano [a,1], E.F. Moreno [b,1], F.A. Schaposnik [b,2]

[a] *Departamento de Física, FCEyN, Universidad de Buenos Aires, Pab. 1, Ciudad Universitaria, Buenos Aires, Argentina*
[b] *Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina*

Abstract

We construct an exact regular vortex solution to the self-dual equations of the Abelian Higgs model in noncommutative space for arbitrary values of θ. To this end, we propose an ansatz which is the analogous, in Fock space, to the one leading to exact solutions for the Nielsen–Olesen vortex in commutative space. We compute the flux and energy of the solution and discuss its relevant properties. © 2001 Published by Elsevier Science B.V.

1. Introduction

The recent interest aroused by quantum field theories in noncommutative space [1–3] prompted the search of localized classical solutions in noncommutative geometry. Instantons, solitons carrying various kinds of fluxes, BPS and non-BPS solutions to different noncommutative theories have been presented in Refs. [4–16]. Among these models, the Abelian Higgs model in noncommutative space has received particular attention in connection with vortex like solutions [7–16].

Several vortex solutions that have been discussed up to now are regular at finite noncommutative parameter θ, but they become singular in the limit $\theta \to 0$ [7, 14]. More precisely, the magnetic field B associated to the flux tube behaves as $B \to \delta^{(2)}(x)$ as $\theta \to 0$. This fact is not surprising since these solutions are

obtained through a procedure which is the analogous to performing singular gauge transformations leading to topologically non-trivial solutions from trivial ones in commutative space.

A different class of vortices in noncommutative space has been considered by D.P. Jatkar, G. Mandal and S. Wadia [8], which are closer in spirit to the regular Nielsen–Olesen vortices of the theory in ordinary space. More specifically, in [8], *self-dual* Bogomol'nyi equations were derived and the solutions in the limiting cases $\theta \to 0$ (where they are regular) and $\theta \to \infty$ were considered.

Bogomol'nyi equations for Abelian Higgs model were also discussed in [13] and [16]. The case considered there corresponds, in our terminology, to the *anti-self-dual case*. In fact, as in the model in commutative space, there are two sets of Bogomol'nyi equations, one admitting solutions with positive flux (which we call self-dual), another admitting solutions for negative flux (which we call anti-self-dual). Nevertheless, while these two sets of equations (and their solutions) are trivially related via a parity transformation in commutative space, the presence of the parity breaking pa-

E-mail address: emoreno@venus.fisica.unlp.edu.ar
(E.F. Moreno).

[1] Associated with CONICET.
[2] Associated with CICBA.

rameter θ prevents such a trivial connection in the noncommutative case. Hence the existence and properties of solutions should be checked separately. While this has been done in detail for the anti-self-dual case in [13] and [16], less is known for the self-dual case with the exception of the limiting cases $\theta \to 0, \theta \to \infty$.

We present in this note an ansatz leading to regular vortex solutions in noncommutative space for arbitrary value of θ. Starting from the noncommutative Abelian Higgs Lagrangian, we solve the associated self-dual equations finding an exact solution that is the noncommutative version of the exact one presented long ago for the ordinary Nielsen–Olesen vortices [17] and, remarkably, it shares, qualitatively, all its basic properties. In fact the solution converges to it as θ go to zero while, for large values of θ, its profile differs appreciably from the Nielsen–Olesen solution.

We consider space–time with coordinates X^μ ($\mu = 0, 1, 2, 3$) obeying the following noncommutative relations

$$\left[X^\mu, X^\nu\right] = i\theta^{\mu\nu}. \tag{1}$$

We take $\theta^{0i} = 0$ ($i = 1, 2, 3$). Concerning θ^{ij}, it can be can be brought into its canonical (Darboux) form by an appropriate orthogonal rotation

$$\left[X^1, X^2\right] = i\theta, \qquad \left[X^1, X^3\right] = \left[X^2, X^3\right] = 0. \tag{2}$$

One way to describe field theories in noncommutative space is by introducing a Moyal product $*$ between ordinary functions. To this end, one can establish a one to one correspondence between operators $\hat{f}$ and ordinary functions f through a Weyl ordering

$$\hat{f}\left(X^1, X^2\right) = \frac{1}{2\pi} \int d^2k \, \tilde{f}(k_1, k_2) \\ \times \exp\left(i\left(k_1 X^1 + k_2 X^2\right)\right). \tag{3}$$

Then, the product of two Weyl ordered operators $\hat{f}\hat{g}$ corresponds to a function $f * g(x)$ defined as

$$f * g(x) = \exp\left(\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})\right) \\ \times f(x_1, x_2)g(y_1, y_2)\Big|_{x_1 = x_2, y_1 = y_2}. \tag{4}$$

Given a $U(1)$ gauge field $A_\mu(x)$, the field strength $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu * A_\nu - A_\nu * A_\mu). \tag{5}$$

We shall couple the gauge field to a complex scalar field ϕ with covariant derivative

$$D_\mu \phi = \partial_\mu \phi - i A_\mu * \phi. \tag{6}$$

Dynamics for the model will be governed by the Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu} * F^{\mu\nu} + \overline{D_\mu \phi} * D^\mu \phi \\ - \frac{1}{2}(\phi * \bar{\phi} - \eta^2)^2. \tag{7}$$

Here we have chosen coefficient of the symmetry breaking potential at the Bogomol'ny point [17,18]. We are looking for static axially symmetric Nielsen–Olesen vortices with $A_0 = A_3 = 0$. Then, the only relevant coordinates in the problem will be $i = 1, 2$.

The alternative approach to noncommutative field theories is to directly work with operators in the phase space (X^1, X^2), with commutator (2). In this case the $*$ product is just the product of operators and integration over the (X^1, X^2) plane is a trace,

$$\int dx^1 dx^2 f\left(x^1, x^2\right) = 2\pi\theta \operatorname{Tr} \hat{f}\left(X^1, X^2\right). \tag{8}$$

In this framework, we introduce complex variables z and $\bar{z}$

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2), \qquad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2), \tag{9}$$

and annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ in the form

$$\hat{a} = \frac{1}{\sqrt{2\theta}}\left(X^1 + iX^2\right),$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2\theta}}\left(X^1 - iX^2\right) \tag{10}$$

so that (2) becomes

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{11}$$

With this conventions, derivatives are given by

$$\partial_z = -\frac{1}{\sqrt{\theta}}\left[\hat{a}^\dagger, \right], \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{\theta}}[\hat{a},]. \tag{12}$$

The field strength takes then the form

$$\hat{F}_{z\bar{z}} = \partial_z \hat{A}_{\bar{z}} - \partial_{\bar{z}} \hat{A}_z - i\left[\hat{A}_z, \hat{A}_{\bar{z}}\right] \\ = -\frac{1}{\sqrt{\theta}}\left(\left[\hat{a}^\dagger, \hat{A}_z\right] + \left[\hat{a}, \hat{A}_{\bar{z}}\right] + i\sqrt{\theta}\left[\hat{A}_z, \hat{A}_{\bar{z}}\right]\right) \\ \equiv i\hat{B}, \tag{13}$$

with $\hat{B}$ the magnetic field. Concerning covariant derivatives

$$D_{\bar{z}}\hat{\phi} = \partial_{\bar{z}}\hat{\phi} - i A_{\bar{z}}\hat{\phi} = \frac{1}{\sqrt{\theta}}[\hat{a}, \hat{\phi}] - i \hat{A}_{\bar{z}}\hat{\phi},$$

$$D_z\hat{\bar{\phi}} = \partial_z\hat{\bar{\phi}} + i \hat{A}_z\hat{\bar{\phi}} = -\frac{1}{\sqrt{\theta}}[\hat{a}^\dagger, \hat{\bar{\phi}}] + i \hat{A}_z\hat{\bar{\phi}}, \qquad (14)$$

where

$$\hat{A}_z = \frac{1}{\sqrt{2}}(\hat{A}_1 - i \hat{A}_2),$$

$$\hat{A}_{\bar{z}} = \frac{1}{\sqrt{2}}(\hat{A}_1 + i \hat{A}_2). \qquad (15)$$

The energy functional associated to action (7) can be then written as [8]

$$E = 2\pi\theta \operatorname{Tr}\left(\frac{1}{2}\hat{B}^2 + D_{\bar{z}}\hat{\phi}D_z\hat{\bar{\phi}} + D_z\hat{\phi}D_{\bar{z}}\hat{\bar{\phi}} \right.$$
$$\left. + \frac{1}{2}(\hat{\phi}\hat{\bar{\phi}} - \eta^2)^2\right). \qquad (16)$$

We want to find static solutions minimizing the energy. To this end, we shall proceed à la Bogomol'nyi writing the energy E in the two following forms [8]

$$E = 2\pi\theta \operatorname{Tr}\left(\frac{1}{2}(\hat{B} + (\hat{\phi}\hat{\bar{\phi}} - \eta^2))^2 + 2D_{\bar{z}}\hat{\phi}D_z\hat{\bar{\phi}} \right.$$
$$\left. + (\hat{T}^s + \eta^2\hat{B})\right) \qquad (17)$$

with $\hat{T}^s$ defined as

$$\hat{T}^s = \partial_z((D_{\bar{z}}\hat{\phi})\hat{\bar{\phi}}) - \partial_{\bar{z}}((D_z\hat{\phi})\hat{\bar{\phi}}) \qquad (18)$$

or

$$E = 2\pi\theta \operatorname{Tr}\left(\frac{1}{2}(\hat{B} - (\hat{\phi}\hat{\bar{\phi}} - \eta^2))^2 + 2D_{\bar{z}}\hat{\bar{\phi}}D_z\hat{\phi} \right.$$
$$\left. - (\hat{T}^a + \eta^2\hat{B})\right) \qquad (19)$$

with

$$\hat{T}^a = -\hat{T}^s. \qquad (20)$$

Now, one can easily see that $\operatorname{Tr}\hat{T}^a = 0$ [8] and hence the energy is bounded by the magnetic flux, as in the case of vortices in ordinary space. The bound is attained when the following first order Bogomol'nyi equations hold

$$\hat{B} = \eta^2 - \hat{\phi}\hat{\bar{\phi}}, \qquad D_{\bar{z}}\hat{\phi} = 0$$

(self-dual equations), $\qquad (21)$

or

$$-\hat{B} = \eta^2 - \hat{\phi}\hat{\bar{\phi}}, \qquad D_z\hat{\phi} = 0$$

(anti self-dual equations) $\qquad (22)$

We have fixed in Eqs. (21), (22) our terminology. Equations (21) are called *self-dual equations* while equations (22) are the corresponding *anti self-dual equations*. Solutions to Eq. (21) correspond to positive magnetic flux, while those to (22) give negative magnetic flux. Note that our convention coincide with that in ([8]) and is the opposite to that in [16], where, in our terminology, anti-self-dual solutions are discussed in detail and a critical value of the noncommutative parameter is found, $\theta_c = 1/\eta^2$, such that solutions cease to exist when $\theta > \theta_c$. Now, as stressed above, in the noncommutative case, the presence of the parity breaking θ parameter renders the connection between the anti-self-dual and the self-dual case non-trivial, in contrast to what happens in the commutative case where it is straightforward.

In what follows, we construct *exact solutions to the self-dual equations* (21) for arbitrary values of θ and in this sense, our calculation complements those in [8] and [16]. To this end, we propose the following ansatz

$$\hat{A}_z = \frac{i}{\sqrt{\theta}}\sum_n (\sqrt{n+1} - \sqrt{n+2} + e_n)|n+1\rangle\langle n|, \qquad (23)$$

$$\hat{\phi} = \eta \sum_n f_n|n\rangle\langle n+1|. \qquad (24)$$

Notice that the Higgs field can be rewritten as

$$\hat{\phi} = \eta \frac{f(\hat{N})}{\sqrt{\hat{N}+1}} \frac{X^1 + iX^2}{\sqrt{2\theta}}, \qquad (25)$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$ and $\langle|f(\hat{N})|n\rangle = f_n$. This should be compared with the ansatz in the commutative case,

$$\phi = \eta\, g(|z|)\, z \qquad (26)$$

with $g(0)$ to be determined by solving the Bogomol'nyi equations and requiring that at infinity $g(|z|) \sim 1/|z|$. This has been done in [17] with the result

$$g(0)^2 = 0.72791. \qquad (27)$$

In the same way, introducing the ansatz (23), (24) we expect to derive a recurrence relation for f_n whose solution is uniquely determined by requiring that $f(\infty) \to 1$.

Notice also that the flux-tube solutions presented in [6,14] correspond to the choice of coefficients $e_n = 0$ and $f_n = 1$, leading to "quasi pure gauge" solutions (which, in the $\theta \to 0$ limit give singular vortex solutions with magnetic field $B = \delta^{(2)}(x)$). What we are looking for here is to determine, through recurrence relations deriving from (21)–(24), the non-trivial values for e_n, f_n that correspond to exact solutions, which should lead to the regular ones found in [17] in the commutative $\theta \to 0$ case. In fact, this ansatz can be seen as the analogous to performing, in the commutative case, a $U(1)$ singular gauge transformation $\exp(in\varphi)$ on $|\phi(r)|$; the condition $|\phi(0)| = 0$ ensures the regularity of the solution. In noncommutative space, the equivalent of such a procedure is to apply an operator S^n with $\hat{S}$ the shift operator defined as [6]

$$\hat{S} = \sum_k |k\rangle\langle k+1|. \tag{28}$$

Ansatz (24) just corresponds to a combination of bra and kets like in S but with arbitrary coefficients f_n. It is easy to also see that the compatible ansatz for the gauge field is just (23).

Now, in order to determine the up to now arbitrary coefficients f_n, e_n, we plug ansatz (23), (24) in Eq. (21) getting the following recurrence relations

$$\sqrt{(n+2)}(f_{n+1} - f_n) - e_n f_{n+1} = 0,$$
$$2\sqrt{(n+1)}\, e_{n-1} - e_{n-1}^2 - 2\sqrt{(n+2)}\, e_n + e_n^2$$
$$= -\theta\eta^2\big(f_n^2 - 1\big). \tag{29}$$

This coupled system can be combined to give for f_n

$$f_1^2 = \frac{2f_0^2}{1 + \theta\eta^2 - \theta\eta^2(f_0^2)},$$
$$f_{n+1}^2 = \frac{(n+2)f_n^4}{f_n^2 - \theta\eta^2 f_n^2(f_n^2 - 1) + (n+1)f_{n-1}^2},$$
$$n > 0. \tag{30}$$

Given a value for f_0 one can then determine all f_n's from (30). The correct value for f_0 should make $f_n^2 \to 1$ asymptotically so that boundary conditions are satisfied. The values of these coefficients will depend on the choice of the dimensionless parameter $\theta\eta^2$.

For small θ we have checked that we reobtain the values for the commutative solution. Indeed,

$$\frac{f_0^2}{2\eta^2\theta} = 0.72792, \quad \theta \ll 1 \tag{31}$$

(compare with Eq. (27)), while for large θ we reobtain the result of Ref. [8]

$$f_0^2 = 1 - \frac{1}{\eta^2\theta}, \quad \theta \gg 1. \tag{32}$$

Exploring the whole range of $\theta\eta^2$, one finds that the vortex solution with $+1$ units of magnetic flux exists in all the intermediate range. As an example, we list three representative values,

$$\theta\eta^2 = 0.5, \quad f_0^2 = 0.40069\ldots,$$
$$\theta\eta^2 = 1.0, \quad f_0^2 = 0.56029\ldots,$$
$$\theta\eta^2 = 2.0, \quad f_0^2 = 0.70670\ldots. \tag{33}$$

Once all f_n's and e_n's are calculated, one can compute the magnetic field, using for example the formula

$$\hat{B} = \eta^2 \sum_{n=0}^{\infty}\big(1 - f_n^2\big)|n\rangle\langle n| \tag{34}$$

or, using the explicit formula for $|n\rangle\langle n|$ in configuration space [6]

$$B(r) = 2\eta^2 \sum_{n=0}^{\infty}(-1)^n\big(1 - f_n^2\big)$$
$$\times \exp\big(-r^2/\theta\big)L_n\big(2r^2/\theta\big), \tag{35}$$

where L_n are the Laguerre polynomials.

We show in Fig. 1 the resulting magnetic field B as a function of $r\eta$. For $\theta = 0$ one recovers the result for self dual Nielsen–Olesen vortices in ordinary space [17]. As θ grows, the maximum for B decreases and the vortex is less localized with total area such that the magnetic flux remains equal to 1. It is important to stress that we have found noncommutative self-dual vortex solutions in the whole range of θ, in agreement with the analysis for large and small θ presented in [8].

As θ becomes larger, one needs more and more precision in order to match the value of f_0 so that the vortex has the adequate behavior at infinity, but a solution can be always found (this should be contrasted with the anti self-dual case discussed in [16]). One can easily integrate $B(r)$ in (35) and check that the magnetic
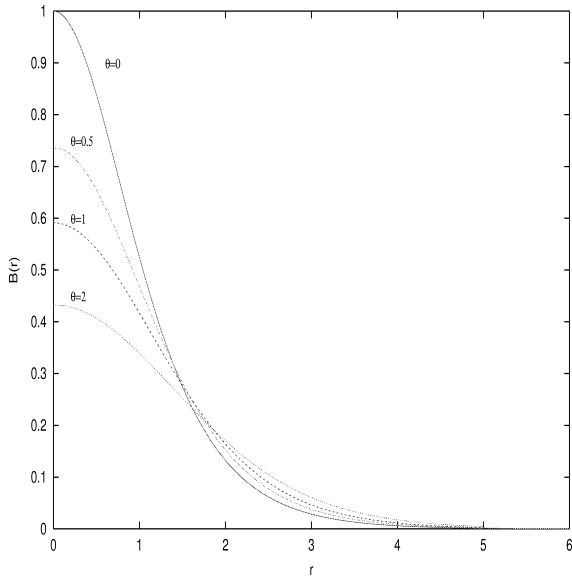
Fig. 1. Magnetic field of the vortex as a function of the radial coordinate (in units of η) for different values of the anticommuting parameter θ (in units of η^2). The curve for $\theta = 0$ coincides with that of the ordinary Nielsen–Olesen vortex.

flux Φ, which can also be written as

$$\Phi = 2\pi\theta \operatorname{Tr} \hat{B} \tag{36}$$

gives, for the exact solution,

$$\frac{\Phi}{2\pi} = 1. \tag{37}$$

We have also computed the energy by inserting our vortex solution directly in Eq. (17). As expected, the solution saturates the bound giving

$$E = 2\pi\eta^2. \tag{38}$$

In summary, we have constructed exact regular vortex solutions to the self-dual equations of Abelian Higgs model in noncommutative space for arbitrary values of θ. The solution corresponds to a magnetic flux $\Phi = 2\pi$ (solutions with $\Phi = 2\pi n$ can be constructed by straightforward generalization of our procedure). For $\theta \to 0$ it converges to the commutative Nielsen–Olesen solution while, for growing θ the flux tube becomes more and more delocalized. The connection between self-dual and anti-self-dual solutions deserves a thorough investigation which we hope to present elsewhere.

Acknowledgements

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