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MULTIPLE SOLUTIONS FOR THE p-LAPLACE EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this note, we show the existence of at least three nontrivial solutions to the quasilinear elliptic equation

$$-\Delta_p u + |u|^{p-2}u = f(x,u)$$

in a smooth bounded domain Ω of $\mathbb{R}^N$ with nonlinear boundary conditions $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u)$ on $\partial\Omega$. The proof is based on variational arguments.

1. Introduction

Let us consider the nonlinear elliptic problem

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = f(x,u) \quad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian and $\partial/\partial\nu$ is the outer unit normal derivative.

Problem (1.1) appears naturally in several branches of pure and applied mathematics, such as the study of optimal constants for the Sobolev trace embedding (see [5, 10, 12, 11]); the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [7, 16]); non-Newtonian fluids, reaction diffusion problems, flow through porus media, nonlinear elasticity, glaciology, etc. (see [1, 2, 3, 6]).

The purpose of this note, is to prove the existence of at least three nontrivial solutions for (1.1) under adequate assumptions on the sources terms f and g. This result extends previous work by the author [8, 9].

Here, no oddness condition is imposed in f or g and a positive, a negative and a sign-changing solution are found. The proof relies on the Lusternik–Schnirelman method for non-compact manifolds (see [14]).

For a related result with Dirichlet boundary conditions, see [15] and more recently [4, 17]. The approach in this note follows the one in [15].

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Throughout this work, by (weak) solutions of (1.1) we understand critical points of the associated energy functional acting on the Sobolev space $W^{1,p}(\Omega)$:

$$\Phi(v) = \frac{1}{p}\int_\Omega |\nabla v|^p + |v|^p \, dx - \int_\Omega F(x,v)\, dx - \int_{\partial\Omega} G(x,v)\, dS, \tag{1.2}$$

where $F(x,u) = \int_0^u f(x,z)\,dz$, $G(x,u) = \int_0^u g(x,z)\,dz$ and dS is the surface measure.

We will denote

$$\mathcal{F}(v) = \int_\Omega F(x,v)\,dx \quad \text{and} \quad \mathcal{G}(v) = \int_{\partial\Omega} G(x,v)\,dS, \tag{1.3}$$

so the functional Φ can be rewritten as

$$\Phi(v) = \frac{1}{p}\|v\|^p_{W^{1,p}(\Omega)} - \mathcal{F}(v) - \mathcal{G}(v).$$

2. Assumptions and statement of the results

The precise assumptions on the source terms f and g are as follows:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x,0) = 0$ for every $x \in \Omega$.

(F2) There exist constants $p < q < p^* = Np/(N-p)$, $s > p^*/(p^*-q)$, $t = sq/(2+(q-2)s) > p^*/(p^*-2)$ and functions $a \in L^s(\Omega)$, $b \in L^t(\Omega)$, such that for $x \in \Omega$, $u, v \in \mathbb{R}$,

$$|f_u(x,u)| \le a(x)|u|^{q-2} + b(x),$$
$$|(f_u(x,u) - f_u(x,v))u| \le (a(x)(|u|^{q-2} + |v|^{q-2}) + b(x))|u-v|.$$

(F3) There exist constants $c_1 \in (0, 1/(p-1))$, $c_2 > p$, $0 < c_3 < c_4$, such that for any $u \in L^q(\Omega)$

$$\begin{aligned} c_3\|u\|^q_{L^q(\Omega)} &\le c_2 \int_\Omega F(x,u)\,dx \le \int_\Omega f(x,u)u\,dx \\ &\le c_1 \int_\Omega f_u(x,u)u^2\,dx \le c_4\|u\|^q_{L^q(\Omega)}. \end{aligned}$$

(G1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $y \in \partial\Omega$. Moreover, $g(y,0) = 0$ for every $y \in \partial\Omega$.

(G2) There exist constants $p < r < p_* = (N-1)p/(N-p)$, $\sigma > p_*/(p_*-r)$, $\tau = \sigma r/(2+(r-2)\sigma) > p_*/(p_*-2)$ and functions $\alpha \in L^\sigma(\partial\Omega)$, $\beta \in L^\tau(\partial\Omega)$, such that for $y \in \partial\Omega$, $u, v \in \mathbb{R}$,

$$|g_u(y,u)| \le \alpha(y)|u|^{r-2} + \beta(y),$$
$$|(g_u(y,u) - g_u(y,v))u| \le (\alpha(y)(|u|^{r-2} + |v|^{r-2}) + \beta(y))|u-v|.$$

(G3) There exist constants $k_1 \in (0, 1/(p-1))$, $k_2 > p$, $0 < k_3 < k_4$, such that for any $u \in L^r(\partial\Omega)$

$$\begin{aligned} k_3\|u\|^r_{L^r(\partial\Omega)} &\le k_2 \int_{\partial\Omega} G(x,u)\,dS \le \int_{\partial\Omega} g(x,u)u\,dS \\ &\le k_1 \int_{\partial\Omega} g_u(x,u)u^2\,dx \le k_4\|u\|^r_{L^r(\partial\Omega)}. \end{aligned}$$

Remark 2.1. Assumptions (F1)–(F3) imply, since the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $1 < q < p^*$ is compact, that $\mathcal{F}$ is C^1 with compact derivative. Analogously, (G1)–(G3) implies the same facts for $\mathcal{G}$ by the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ for $1 < r < p_*$.

The main result of the paper reads as follows.

Theorem 2.2. *Under assumptions* (F1)–(F3), (G1)–(G3)*, there exist three different, nontrivial, (weak) solutions of* (1.1)*. Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

3. Proof of the Theorem

The proof uses the same approach as in [15]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that Φ has a critical point in K_i. These sets will be subsets of smooth manifolds $M_i \subset W^{1,p}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$M_1 = \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} u_+ \, dS > 0,\ \|u_+\|^p_{W^{1,p}(\Omega)} = \langle \mathcal{F}'(u), u_+\rangle + \langle \mathcal{G}'(u), u_+\rangle\},$$
$$M_2 = \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} u_- \, dS > 0,\ \|u_-\|^p_{W^{1,p}(\Omega)} = \langle \mathcal{F}'(u), u_-\rangle + \langle \mathcal{G}'(u), u_-\rangle\},$$
$$M_3 = M_1 \cap M_2,$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of u, and $\langle \cdot, \cdot \rangle$ is the duality pairing of $W^{1,p}(\Omega)$.

Finally we define

$$K_1 = \{u \in M_1 \mid u \geq 0\}, \quad K_2 = \{u \in M_2 \mid u \leq 0\}, \quad K_3 = M_3.$$

For the proof of the main theorem, we need the following Lemmas.

Lemma 3.1. *There exist $c_j > 0$ such that, for every $u \in K_i$, $i = 1, 2, 3$,*

$$\|u\|^p_{W^{1,p}(\Omega)} \leq c_1 \Big(\int_\Omega f(x,u)u \, dx + \int_{\partial\Omega} g(x,u)u \, dS \Big) \leq c_2 \Phi(u) \leq c_3 \|u\|^p_{W^{1,p}(\Omega)}.$$

Proof. Since $u \in K_i$, we have

$$\|u\|^p_{W^{1,p}(\Omega)} = \int_\Omega f(x,u)u \, dx + \int_{\partial\Omega} g(x,u)u \, dS.$$

This proves the first inequality. Now, by (F3) and (G3)

$$\int_\Omega F(x,u) \, dx \leq \frac{1}{k_2} \int_\Omega f(x,u)u \, dx,$$
$$\int_{\partial\Omega} G(x,u) \, dS \leq \frac{1}{c_2} \int_{\partial\Omega} g(x,u)u \, dS.$$

So, for $C = \max\{\frac{1}{k_2}; \frac{1}{c_2}\} < \frac{1}{p}$, we have

$$\Phi(u) \leq \big(\frac{1}{p} - C\big) \|u\|^p_{W^{1,p}(\Omega)}.$$

This proves the third inequality.

To prove the middle inequality we proceed as follows:

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \int_\Omega F(x,u)\,dx - \int_{\partial\Omega} G(x,u)\,dS\\ &= \frac{1}{p}\Big(\int_\Omega f(x,u)u\,dx + \int_{\partial\Omega} g(x,u)u\,dS\Big) - \Big(\int_\Omega F(x,u)\,dx + \int_{\partial\Omega} G(x,u)\,dS\Big)\\ &\ge (\frac{1}{p} - C)\Big(\int_\Omega f(x,u)u\,dx + \int_{\partial\Omega} g(x,u)u\,dS\Big).\end{aligned}$$

This completes the proof. □

Lemma 3.2. *There exists $c > 0$ such that*

$$\begin{gathered}\|u_+\|_{W^{1,p}(\Omega)} \ge c \quad \text{for } u \in K_1,\\ \|u_-\|_{W^{1,p}(\Omega)} \ge c \quad \text{for } u \in K_2,\\ \|u_+\|_{W^{1,p}(\Omega)},\ \|u_-\|_{W^{1,p}(\Omega)} \ge c \quad \text{for } u \in K_3.\end{gathered}$$

Proof. By the definition of K_i, by (F3) and (G3), we have

$$\begin{aligned}\|u_\pm\|^p_{W^{1,p}(\Omega)} &= \int_\Omega f(x,u)u_\pm\,dx + \int_{\partial\Omega} g(x,u)u_\pm\,dS\\ &\le c(\|u_\pm\|^q_{L^q(\Omega)} + \|u_\pm\|^r_{L^r(\partial\Omega)}).\end{aligned}$$

Now the proof follows by the Sobolev immersion Theorem and by the Sobolev trace Theorem, as $p < q, r$. □

Lemma 3.3. *There exists $c > 0$ such that $\Phi(u) \ge c\|u\|^p_{W^{1,p}(\Omega)}$ for every $u \in W^{1,p}(\Omega)$ such that $\|u\|_{W^{1,p}(\Omega)} \le c$.*

Proof. By (F3), (G3) and the Sobolev immersions we have

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \mathcal{F}(u) - \mathcal{G}(u)\\ &\ge \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c(\|u\|^q_{L^q(\Omega)} + \|u\|^r_{L^r(\partial\Omega)})\\ &\ge \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c(\|u\|^q_{W^{1,p}(\Omega)} + \|u\|^r_{W^{1,p}(\Omega)})\\ &\ge c\|u\|^p_{W^{1,p}(\Omega)},\end{aligned}$$

if $\|u\|_{W^{1,p}(\Omega)}$ is small enough, as $p < q, r$. □

The following lemma describes the properties of the manifolds M_i.

Lemma 3.4. *M_i is a $C^{1,1}$ sub-manifold of $W^{1,p}(\Omega)$ of co-dimension 1 ($i = 1, 2$), 2 ($i = 3$) respectively. The sets K_i are complete. Moreover, for every $u \in M_i$ we have the direct decomposition*

$$T_u W^{1,p}(\Omega) = T_u M_i \oplus \operatorname{span}\{u_+, u_-\},$$

where $T_u M$ is the tangent space at u of the Banach manifold M. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of M_i.

Proof. Let us denote

$$\begin{gathered}\bar{M}_1 = \Big\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} u_+ \, dS > 0\Big\},\\ \bar{M}_2 = \Big\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} u_- \, dS > 0\Big\},\\ \bar{M}_3 = \bar{M}_1 \cap \bar{M}_2.\end{gathered}$$

Observe that $M_i \subset \bar{M}_i$.

By the Sobolev trace Theorem, the set $\bar{M}_i$ is open in $W^{1,p}(\Omega)$, therefore it is enough to prove that M_i is a smooth sub-manifold of $\bar{M}_i$. In order to do this, we will construct a $C^{1,1}$ function $\varphi_i : \bar{M}_i \to \mathbb{R}^d$ with $d = 1$ $(i = 1, 2)$, $d = 2$ $(i = 3)$ respectively and M_i will be the inverse image of a regular value of φ_i.

In fact, we define: For $u \in \bar{M}_1$,

$$\varphi_1(u) = \|u_+\|^p_{W^{1,p}(\Omega)} - \langle \mathcal{F}'(u), u_+\rangle - \langle \mathcal{G}'(u), u_+\rangle.$$

For $u \in \bar{M}_2$,

$$\varphi_2(u) = \|u_-\|^p_{W^{1,p}(\Omega)} - \langle \mathcal{F}'(u), u_-\rangle - \langle \mathcal{G}'(u), u_-\rangle.$$

For $u \in \bar{M}_3$,

$$\varphi_3(u) = (k_1(u), k_2(u)).$$

Obviously, we have $M_i = \varphi_i^{-1}(0)$. We need to show that 0 is a regular value for φ_i. To this end we compute, for $u \in M_1$,

$$\begin{aligned}\langle \nabla\varphi_1(u), u_+\rangle =& p\|u_+\|^p_{W^{1,p}(\Omega)} - \int_\Omega f_u(x,u)u_+^2 + f(x,u)u_+ \, dx\\ &- \int_{\partial\Omega} g_u(x,u)u_+^2 + g(x,u)u_+ \, dS\\ =& (p-1)\int_\Omega f(x,u)u_+ \, dx - \int_\Omega f_u(x,u)u_+^2 \, dx\\ &+ (p-1)\int_{\partial\Omega} g(x,u)u_+ \, dS - \int_{\partial\Omega} g_u(x,u)u_+^2 \, dS.\end{aligned}$$

By (F3) and (G3) the last term is bounded by

$$(p-1-c_1^{-1})\int_\Omega f(x,u)u_+ \, dx + (p-1-k_1^{-1})\int_{\partial\Omega} g(x,u)u_+ \, dS.$$

Recall that $c_1, k_1 < 1/(p-1)$. Now, by Lemma 3.1, this is bounded by

$$-c\|u_+\|^p_{W^{1,p}(\Omega)}$$

which is strictly negative by Lemma 3.2. Therefore, M_1 is a smooth sub-manifold of $W^{1,p}(\Omega)$. The exact same argument applies to M_2.

Since trivially

$$\langle \nabla\varphi_1(u), u_-\rangle = \langle \nabla\varphi_2(u), u_+\rangle = 0$$

for $u \in M_3$, the same conclusion holds for M_3.

To see that K_i is complete, let u_k be a Cauchy sequence in K_i, then $u_k \to u$ in $W^{1,p}(\Omega)$. Moreover, $(u_k)_\pm \to u_\pm$ in $W^{1,p}(\Omega)$. Now it is easy to see, by Lemma 3.2 and by continuity that $u \in K_i$.

Finally, by the first part of the proof we have the decomposition

$$T_u W^{1,p}(\Omega) = T_u M_i \oplus \operatorname{span}\{u_+, u_-\}.$$

Now let $v \in T_u W^{1,p}(\Omega)$ be a unit tangential vector, then $v = v_1 + v_2$ where v_i are given by

$$v_2 = (\nabla\varphi_i(u)|_{\text{span}\{u_+,u_-\}})^{-1}\langle\nabla\varphi_i(u), v\rangle \in \text{span}\{u_+, u_-\},$$
$$v_1 = v - v_2 \in T_u M_i.$$

From these formulas and from the estimates given in the first part of the proof, the uniform continuity follows. □

Now, we need to check the Palais-Smale condition for the functional Φ restricted to the manifold M_i.

Lemma 3.5. *The functional $\Phi|_{K_i}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_k\} \subset K_i$ be a Palais-Smale sequence, that is $\Phi(u_k)$ is uniformly bounded and $\nabla\Phi|_{K_i}(u_k) \to 0$ strongly. We need to show that there exists a subsequence u_{k_j} that converges strongly in K_i.

Let $v_j \in T_{u_j} W^{1,p}(\Omega)$ be a unit tangential vector such that

$$\langle\nabla\Phi(u_j), v_j\rangle = \|\nabla\Phi(u_j)\|_{(W^{1,p}(\Omega))'}.$$

Now, by Lemma 3.4, $v_j = w_j + z_j$ with $w_j \in T_{u_j} M_i$ and $z_j \in \text{span}\{(u_j)_+, (u_j)_-\}$.

Since $\Phi(u_j)$ is uniformly bounded, by Lemma 3.1, u_j is uniformly bounded in $W^{1,p}(\Omega)$ and hence w_j is uniformly bounded in $W^{1,p}(\Omega)$. Therefore

$$\|\Phi(u_j)\|_{(W^{1,p}(\Omega))'} = \langle\nabla\Phi(u_j), v_j\rangle = \langle\nabla\Phi|_{K_i}(u_j), v_j\rangle \to 0.$$

As u_j is bounded in $W^{1,p}(\Omega)$, there exists $u \in W^{1,p}(\Omega)$ such that $u_j \rightharpoonup u$, weakly in $W^{1,p}(\Omega)$. As it is well known that the unrestricted functional Φ satisfies the Palais-Smale condition (cf. [9] and [13]), the lemma follows. See [15] for the details. □

We obtain immediately the following result.

Lemma 3.6. *Let $u \in K_i$ be a critical point of the restricted functional $\Phi|_{K_i}$. Then u is also a critical point of the unrestricted functional Φ and hence a weak solution to* (1.1).

With all this preparatives, the proof of the Theorem follows easily.

Proof of the Theorem. The proof now is a standard application of the Lusternik–Schnirelman method for non-compact manifolds. See [14]. □

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