AN OPTIMIZATION PROBLEM FOR THE FIRST EIGENVALUE OF THE $p-$LAPLACIAN PLUS A POTENTIAL

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Abstract. In this paper we study the optimization problem for the first eigenvalue of the $p-$Laplacian plus a potential V with respect to V, when the potential is restricted to a bounded, closed and convex set of $L^q(\Omega)$.

1. Introduction

Eigenvalue problems for second order elliptic differential equations are one of the fundamental problems in mathematical physics and, probably, one of the most studied ones in the past years. See [7].

When studying eigenvalue problems for nonlinear homogeneous operators, the classical linear theory does not work, but some of its ideas can still be applied and partial results are obtained. See, for instance, García Azorero–Peral Alonso [8, 9], Cuesta [6], Anane [2], etc. Some of these results are described in Section 3.

In the theory for eigenvalues of elliptic operators, a relevant problem is the optimization of these eigenvalues with respect to the different parameters under consideration.

We consider Schröedinger operators, that is elliptic operators L under perturbations given by a potential V, in bounded regions. These operators appear in different fields of applications such as quantum mechanics, stability of bulk matter, scattering theory, etc.

In Ashbaugh–Harrell [4] the following problem is studied: Let L be a uniformly elliptic linear operator and assume that $\|V\|_{L^q(\Omega)}$ is constrained but otherwise the potential V is arbitrary. Can the maximal value of the first (fundamental) eigenvalue for the operator $L + V$ be estimated? And the minimal value? There exists *optimal* potentials? (i.e. potentials V^* and V_* such that the first eigenvalue for $L + V^*$ is maximal and the first eigenvalue for $L + V_*$ is minimal).

In [4] these questions are answered in a positive way and, moreover, a characterization of these optimal potentials is given.

We arrive then at the purpose of this work that is the extension of the results of Ashbaugh–Harrell [4] to the nonlinear case. We are also interested in extending

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these results to degenerate/singular operators. As a model of these operators, we take the $p-$Laplacian that is defined as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

This operator has been intensively studied in recent years and is a model for the study of degenerated operators (if $p > 2$) and singular operators (if $1 < p < 2$). In the case $p = 2$ it agrees with the usual Laplacian. This operator also serves as a model in the study of non-Newtonian fluids. See Arcoya–Diaz–Tello [3] and Atkinson–Kalli [5].

Here we prove that, if one consider perturbations of the $p-$Laplacian by a potential V with $\|V\|_{L^q(\Omega)}$ constrained, then there exists optimal potentials in the sense described above and a characterizations of these potentials are given.

We want to remark that the proofs are not straightforward extensions of those in [4] since the proof there are not, in general, variational. Moreover, some new technical difficulties arise since solutions to a $p-$Laplace type equation are not regular and, mostly, since the eigenvalue problem for the $p-$Laplacian is far from being completely understood.

The rest of the paper is divided into two sections. Section 2 consists in an overview of some results for the operator $H_V := -\Delta_p + V(x)$ with $V \in L^q(\Omega)$. Some of these results are well known to experts, but we decided to include them in order to make the paper self contained. Finally, in Section 3, we analyze the existence and characterization problem for optimal potentials.

2. Preliminaries

In this section we review some results regarding solutions of some $p-$Laplace type equations. Most of these results are well known, but we include it here for the sake of completeness.

Given $\Omega \subset \mathbb{R}^N$ a smooth bounded domain and $V \in L^q(\Omega)$ $(1 \le q < \infty)$, consider the operator H_V, which has the form

$$H_V u := -\Delta_p u + V(x)|u|^{p-2}u. \tag{2.1}$$

Suppose that $u \in W^{1,p}(\Omega)$ and $q > N/p$, we say u is a weak solution of $H_V u = 0$ $(\geq 0, \leq 0)$ in Ω if

$$\begin{aligned} \mathfrak{D}(u,v) &:= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w \, dx + \int_\Omega V(x)|u|^{p-2}uw \, dx \\ &= 0 \ (\le 0, \ \ge 0), \end{aligned} \tag{2.2}$$

for each $w \in C_0^1(\Omega)$. Let $f \in L^{p'}(\Omega)$, $u \in W^{1,p}(\Omega)$ is a weak solution of the equation

$$H_V u = f \tag{2.3}$$

in Ω if

$$\mathfrak{D}(u,w) = G(w) := \int_\Omega fw \, dx \ \ \forall w \in C_0^1(\Omega). \tag{2.4}$$

We study the Dirichlet problem for the equation (2.3).

Definition 2.1. *We say $u \in W^{1,p}(\Omega)$ is a weak solution of the Dirichlet problem*

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

if u is a weak solution of (2.3) *and $u \in W_0^{1,p}(\Omega)$.*

Note that

$$\begin{aligned} |\mathfrak{D}(u,w)| &\leq \|\nabla u\|_p^{p-1}\|\nabla w\|_p + \int_\Omega (|V(x)|^{\frac{1}{p'}}|u|^{p-1})(|V(x)|^{\frac{1}{p}}|w|)\,\mathrm{d}x \\ &\leq \|\nabla u\|_p^{p-1}\|\nabla w\|_p + \left(\int_\Omega |V(x)||u|^p\,\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_\Omega |V(x)||w|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \|\nabla u\|_p^{p-1}\|\nabla w\|_p + C\|V\|_q\|u\|_{1,p}^{p-1}\|w\|_{1,p} \\ &\leq (1 + C\|V\|_q)\|u\|_{1,p}^{p-1}\|w\|_{1,p}. \end{aligned}$$

Here and throughout the paper we use the notations

$$\|u\|_p := \Big(\int_\Omega |u|^p\,\mathrm{d}x\Big)^{1/p}, \qquad \|u\|_{1,p} := \|u\|_p + \|\,|\nabla u|\,\|_p,$$

where $|x|$ denotes the euclidean norm of a point $x \in \mathbb{R}^N$.

Hence for fixed $u \in W^{1,p}(\Omega)$, the mapping $w \mapsto \mathfrak{D}(u,w)$ is a bounded linear functional on $W_0^{1,p}(\Omega)$. Consequently the validity of the relations (2.2) for $w \in C_0^1(\Omega)$ imply their validity for $w \in W_0^{1,p}(\Omega)$. We remark that for fixed $u \in W^{1,p}(\Omega)$, $H_V u$ may be defined as an element of the dual space of $W_0^{1,p}(\Omega)$, $W^{-1,p'}(\Omega)$, $H_V u(w) = \mathfrak{D}(u,w)$, $w \in W_0^{1,p}(\Omega)$, and hence the Dirichlet problem (2.5) can be studied for $f \in W^{-1,p'}(\Omega)$.

2.1. **Solvability of the Dirichlet problem.**

We need the following notation:

$$S_q := \inf_{v \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p\,\mathrm{d}x}{\Big(\displaystyle\int_\Omega |v|^q\,\mathrm{d}x\Big)^{\frac{p}{q}}}. \tag{2.6}$$

This constant S_q is positive and is the best (largest) constant in the Sobolev–Poincaré inequality

$$S\|\,|\nabla v|\,\|_p^p \leq \|v\|_q^p, \qquad \forall\, v \in W_0^{1,p}(\Omega).$$

We have the following,

Theorem 2.2. *Let $V \in L^q(\Omega)$ with $q > N/p$. If $\|V\|_q < S_{pq'}^{-1}$, or $V \geq -S_p + \delta$ for some $\delta > 0$, then the Dirichlet problem* (2.5) *has a unique weak solution for any $f \in L^{p'}(\Omega)$.*

Proof. The proof of this theorem is standard. First observe that weak solutions of (2.5) are critical points of the functional $\phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{p}\int_\Omega V(x)|u|^p\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x.$$

Now, it is easy to see that ϕ is bounded below, coercive, strictly convex and sequentially weakly lower semi continuous. Therefore it has a unique critical point which is a global minimum. $\square$

It is proved in [11] that solutions to (2.5) are bounded. We state the Theorem for future reference.

Theorem 2.3 ([11], Proposition 1.3). *Let $u \in W_0^{1,p}(\Omega)$ be a solution to* (2.5)*, with $f \in L^q(\Omega)$, $q > N/p$, $p < N$. Then u is bounded. Moreover, there exists a constant $C = C(N, p, |\Omega|)$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}^{1/(p-1)}.$$

2.2. **The Strong Maximum Principle.** Here we recall the classical maximum principles for H_V.

Theorem 2.4 (The Weak Maximum Principle). *Let $V \in L^q(\Omega)$, with $q > N/p$, $f \in L^{p'}(\Omega)$ and let $u \in W_0^{1,p}(\Omega)$ be the weak solution of* (2.5)*. If $\|V\|_q < S_{pq'}^{-1}$ or $V \geq -S_p + \delta$ for some $\delta > 0$, then $f \geq 0$ implies $u \geq 0$ in Ω.*

Proof. The proof follows using u^- as a test function in the weak formulation of (2.5). See [10] for the case $p = 2$. Here is analogous. $\square$

For the strong maximum principle, we need the following

Theorem 2.5 (Harnack's Inequality). *Let u be a weak solution of problem* (2.5) *in a cube $K = K(3\rho) \subset \Omega$, with $0 \leq u < M$ in K. Then*

$$\max_{K(\rho)} u \leq C \min_{K(\rho)} u,$$

where $C = C(N, M, \rho)$.

Proof. See Trudinger [13]. $\square$

Now we can prove the strong maximum principle for weak solutions of (2.5).

Theorem 2.6 (The Strong Maximum Principle). *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of problem* (2.5)*. Then, if $f \geq 0$, $f \not\equiv 0$,*

$$u > 0 \text{ in } \Omega.$$

Proof. It follows from Theorems 2.3, 2.4 and 2.5. $\square$

2.3. **The Eigenvalue Problem.** In this subsection we analyze the (nonlinear) eigenvalue problem,

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

The first (lowest) eigenvalue of this problem is

$$E(V) := \inf\left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V(x)|u|^p \,\mathrm{d}x : u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1 \right\}.$$

By standard compactness arguments, we now prove that there exists u_0 weak solution of (2.7) when $\lambda = E(V)$. Hence, we will say that u_0 is eigenfunction of H_V in $W_0^{1,p}(\Omega)$ with eigenvalue $E(V)$. Since $|u_0|$ is also an eigenfunction, we can construct a nonnegative eigenfunction for (2.7) associated to $E(V)$. By the Strong Maximum Principle it follows that $|u_0| > 0$ in Ω and hence eigenfunctions associated to $E(V)$ has constant sign.

We now recall the arguments of the results just mentioned.

Theorem 2.7. *If $V \in L^q(\Omega)$ with $q > N/p$ then there exists $u_0 \in W_0^{1,p}(\Omega)$ such that*

$$\begin{cases} E(V) = \displaystyle\int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V(x)|u_0|^p \,\mathrm{d}x \\ \|u_0\|_p = 1. \end{cases}$$

Moreover, u_0 is a weak solution of (2.7) *with $\lambda = E(V)$. Finally, $E(V)$ is the lowest eigenvalue of* (2.7)*.*

For the proof we need the following Lemma

Lemma 2.8. *Let $V \in L^q(\Omega)$ with $q > N/p$. Then, given $\varepsilon > 0$, there exists a constant $D_\varepsilon > 0$ such that*

$$\Big| \int_\Omega V(x)|v|^p \,\mathrm{d}x \Big| \le \varepsilon \int_\Omega |\nabla v|^p \,\mathrm{d}x + D_\varepsilon \|V\|_q \int_\Omega |v|^p \,\mathrm{d}x,$$

for any $v \in W_0^{1,p}(\Omega)$.

Proof. Let us observe that $q > N/p$ implies that $pq' < p^*$. Now the Lemma follows from Hölder's inequality and the Sobolev embedding. In fact, let us see that if $1 < r < p^*$, there exists a constant M_ε such that

$$\|v\|_r \le \varepsilon \| \, |\nabla v| \, \|_p + M_\varepsilon \|v\|_p, \qquad \text{for every } v \in W_0^{1,p}(\Omega). \tag{2.8}$$

Assume (2.8) does not hold, then there exists $\varepsilon_0 > 0$ and a sequence $(v_n)_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ such that $\|v_n\|_r = 1$ and

$$\varepsilon_0 \| \, |\nabla v_n| \, \|_p + n\|v_n\|_p < 1.$$

But then $(v_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and $\|v_n\|_p \to 0$. Now, by the Rellich–Kondrashov compactness Theorem, up to a subsequence, $u_n \to u$ in $L^r(\Omega)$, and so $\|u\|_r = 1$. A contradiction.

Now, it is easy to check that (2.8) implies the Lemma since $q > N/p$. □

Proof of Theorem 2.7. Let $V \in L^q(\Omega)$ and let $(u_n)_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be a minimizing sequence for $E(V)$, i.e.

$$\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega V(x)|u_n|^p \,\mathrm{d}x \to E(V), \qquad \|u_n\|_p = 1 \;\forall\, n \in \mathbb{N}.$$

Then there exists $C > 0$ such that

$$\int_\Omega |\nabla u_n|^p \,\mathrm{d}x + \int_\Omega V(x)|u_n|^p \,\mathrm{d}x \le C \quad \forall\, n \in \mathbb{N}.$$

Since $q > N/p$, by Lemma 2.8, given $\varepsilon > 0$ there exists D_ε such that

$$\Big|\int_\Omega V(x)|u_n|^p\,\mathrm{d}x\Big| \le \varepsilon\|\,|\nabla u_n|\,\|_p^p + D_\varepsilon\|V\|_q\|u_n\|_p^p,$$

for any $n\in\mathbb{N}$. Then

$$(1-\varepsilon)\int_\Omega |\nabla u_n|^p\,\mathrm{d}x - D_\varepsilon\|V\|_q \le \int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega V(x)|u_n|^p\,\mathrm{d}x \le C, \quad \forall\, n\in\mathbb{N}.$$

Fixing $\varepsilon < 1$, we get

$$\int_\Omega |\nabla u_n|^p\,\mathrm{d}x \le \frac{C + \|V\|_q D_\varepsilon}{1-\varepsilon}, \quad \forall\, n\in\mathbb{N}.$$

Therefore $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.

Now, as $pq' < p^*$, there exists a function $u_0\in W_0^{1,p}(\Omega)$ such that, for a subsequence that we still call $(u_n)_{n\in\mathbb{N}}$,

$$u_n \rightharpoonup u_0, \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{2.9}$$

$$u_n \to u_0, \quad \text{strongly in } L^p(\Omega), \tag{2.10}$$

$$u_n \to u_0, \quad \text{strongly in } L^{pq'}(\Omega). \tag{2.11}$$

By (2.10), $\|u_0\|_p = 1$ so $u_0\neq 0$ and by (2.9) and (2.11)

$$E(V) = \lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x + \int_\Omega V(x)|u_n|^p\,\mathrm{d}x \ge \int_\Omega |\nabla u_0|^p\,\mathrm{d}x + \int_\Omega V(x)|u_0|^p\,\mathrm{d}x$$

It is clear that u_0 is an eigenfunction of H_V with eigenvalue $E(V)$.

Finally, let λ be an eigenvalue of problem (2.7) with associated eigenfunction $w\in W_0^{1,p}(\Omega)$. Then

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla w|^p\,\mathrm{d}x + \int_\Omega V(x)|w|^p\,\mathrm{d}x}{\displaystyle\int_\Omega |w|^p\,\mathrm{d}x} \ge E(V).$$

This finishes the proof. □

Now, we prove that u_0 has constant sign in Ω.

Theorem 2.9. *Let $u_0\in W_0^{1,p}(\Omega)$ be an eigenfunction of* (2.7) *associated to $E(V)$. Then $|u_0| > 0$ in Ω.*

Proof. Since u_0 is a weak solution of (2.7) with eigenvalue $E(V)$, by the variational characterization of $E(V)$, $|u_0|$ is also an eigenfunction associated to $E(V)$.

Since $V\in L^q(\Omega)$ with $q > N/p$, by Theorem 2.3, $|u_0|\in L^\infty(\Omega)$ and then by the Theorem 2.6 $|u_0| > 0$ in Ω. □

Now, we recall a couple of results regarding the eigenvalue problem (2.7). We do not use these results in the rest of the paper, but we include them here for completeness.

Proposition 2.10. *If $V \in L^q(\Omega)$, with $q > N/p$, then there exists a increasing, unbounded sequence of eigenvalues for the problem* (2.7).

Proof. It is similar to García Azorero–Peral Alonso [8, 9]. □

Proposition 2.11. *If $V \in L^q(\Omega)$ with $q > N/p$, then $E(V)$ is isolated in the spectrum.*

Proof. It is similar to Cuesta [6]. □

Now we prove the simplicity of $E(V)$. This is, the only eigenfunctions of H_V associated to $E(V)$ are multiples of a single one, u_0. For this we need the following lemma.

Lemma 2.12 (Picone's Identity)**.** *Let $v > 0$, $u \geq 0$ be differentiable and let $p \geq 1$. Denote*

$$\begin{aligned} L(u,v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u|\nabla v|^{p-2}\nabla v \\ R(u,v) &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v. \end{aligned}$$

Then $L(u,v) = R(u,v)$. Moreover

(1) $L(u,v) \geq 0$.
(2) *$L(u,v) = 0$ a.e. Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. Ω, i.e. $u = kv$ for some constant k in each component of Ω.*

Proof. See Allegretto–Huang [1]. □

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^N$ be a connected smooth bounded domain, $V \in L^q(\Omega)$ with $q > N/p$ and let $u_0 \in W_0^{1,p}(\Omega)$ be a nonnegative eigenfunction of H_V associated with $E(V)$ normalized $\|u_0\|_p = 1$.*

Then any eigenfunction $w \in W_0^{1,p}(\Omega)$ of H_V associated to $E(V)$ is a scalar multiple of u_0, i.e. there exists $k \in \mathbb{R}_+$ such that $w = ku_0$ a.e. in Ω.

Proof. We can assume that w is nonnegative. By Theorem 2.3, it follows that w is bounded.

Now, given $n \in \mathbb{N}$, we consider $u_0 + \frac{1}{n} > 0$. Thus

$$\begin{aligned} \int_\Omega L(w, u_0 + \frac{1}{n})\,\mathrm{d}x &= \int_\Omega R(w, u_0 + \frac{1}{n})\,\mathrm{d}x \\ &= \int_\Omega \left[|\nabla w|^p - |\nabla u_0|^{p-2}\nabla u_0 \nabla\left(\frac{w^p}{(u_0+\frac{1}{n})^{p-1}}\right)\right]\mathrm{d}x. \end{aligned}$$

As $\frac{w^p}{(u_0+\frac{1}{n})^{p-1}} \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega L(w,u_0+\frac{1}{n})\,\mathrm{d}x &= \int_\Omega \Big[|\nabla w|^p - |\nabla u_0|^{p-2}\nabla u_0 \nabla\Big(\frac{w^p}{(u_0+\frac{1}{n})^{p-1}}\Big)\Big]\,\mathrm{d}x \\ &= \int_\Omega |\nabla w|^p\,\mathrm{d}x - E(V)\int_\Omega w^p \frac{u_0^{p-1}}{(u_0+\frac{1}{n})^{p-1}}\,\mathrm{d}x \\ &\quad + \int_\Omega V(x) w^p \frac{u_0^{p-1}}{(u_0+\frac{1}{n})^{p-1}}\,\mathrm{d}x.\end{aligned}$$

By the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla w|^p\,\mathrm{d}x - E(V)\int_\Omega w^p\frac{u_0^{p-1}}{(u_0+\frac{1}{n})^{p-1}}\,\mathrm{d}x + \int_\Omega V(x)w^p \frac{u_0^{p-1}}{(u_0+\frac{1}{n})^{p-1}}\,\mathrm{d}x\right)$$

$$= \int_\Omega |\nabla w|^p\,\mathrm{d}x - E(V)\int_\Omega w^p\,\mathrm{d}x + \int_\Omega V(x)w^p\,\mathrm{d}x.$$

Hence,

$$\lim_{n\to\infty}\int_\Omega L(w,u_0+\frac{1}{n})\,\mathrm{d}x = \int_\Omega |\nabla w|^p\,\mathrm{d}x - E(V)\int_\Omega w^p\,\mathrm{d}x + \int_\Omega V(x)w^p\,\mathrm{d}x.$$

By Fatou's lemma

$$\int_\Omega L(w,u_0)\,\mathrm{d}x \le \int_\Omega |\nabla w|^p\,\mathrm{d}x - E(V)\int_\Omega w^p\,\mathrm{d}x + \int_\Omega V(x)w^p\,\mathrm{d}x.$$

Then, since w is an eigenfunction,

$$\int_\Omega L(w,u_0)\,\mathrm{d}x \le 0.$$

As $L(w,u_0) \ge 0$ in Ω, it follows that $L(w,u_0) = 0$ a.e. in Ω, thus there exists $k \in \mathbb{R}_+$ such that $w = ku_0$ a.e. in Ω. □

3. Maximal and Minimal Potentials

Let $\Omega \subset \mathbb{R}^N$ be a connected smooth bounded domain. We consider the differential operator

$$H_V u := -\Delta_p u + V(x)|u|^{p-2}u$$

where $V \in L^q(\Omega)$ and $1 < p < \infty$ and let $E(V)$ be the lowest eigenvalue of H_V in $W_0^{1,p}(\Omega)$.

In this section we analyze the following problems: If $B \subset L^q(\Omega)$ is a convex, bounded and closed set,

(1) find $\sup_B E(V)$ and $V \in B$, if any, where this value is attained.
(2) find $\inf_B E(V)$ and $V \in B$, if any, where this value is attained.

Here we answer these questions positively, following the approach of Ashbaugh–Harrell's work for the case $p = 2$ and $1 \le N \le 3$, [4, 12].

3.1. **Properties of $E(\cdot)$.** We begin by proving some important properties of $E(\cdot)$.

Lemma 3.1. *$E : B \to \mathbb{R}$ is concave.*

Proof. Let V_1, $V_2 \in B$ and $0 \le t \le 1$. Then

$$\begin{aligned}
E(tV_1 + (1-t)V_2) &= \inf\left\{ J_{tV_1+(1-t)V_2}(u) : u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1 \right\} \\
&= \inf\left\{ tJ_{V_1}(u) + (1-t)J_{V_2}(u) : u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1 \right\} \\
&\ge \inf\left\{ tJ_{V_1}(u) : u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1 \right\} \\
&\quad + \inf\left\{ (1-t)J_{V_2}(u) : u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1 \right\} \\
&= tE(V_1) + (1-t)E(V_2),
\end{aligned}$$

as we wanted to prove. □

Next we set M for which $\|V\|_q \le M$ for all $V \in B$.

Proposition 3.2. *There exists a constant $C > 0$, depending only on p, q, M and Ω such that*

$$E(V) \le C \quad \textit{for every } V \in B.$$

Proof. Let $u_0 \in C_0^1(\Omega)$ be such that $\|u_0\|_p = 1$.

$$\begin{aligned}
E(V) &\le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V(x)|u_0|^p \,\mathrm{d}x \le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \|u_0\|_\infty^p \int_\Omega V(x) \,\mathrm{d}x \\
&\le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \|u_0\|_\infty^p |\Omega|^{1/q'} \|V\|_q \le \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \|u_0\|_\infty^p |\Omega|^{1/q'} M \\
&= C(p, q, M, \Omega).
\end{aligned}$$

□

3.2. **Maximizing Potentials.** In this subsection we prove that there exists an unique $V^* \in B$ such that

$$E(V^*) = \sup\{E(V) : V \in B\}$$

and we characterize it.

Theorem 3.3. *Let $q > N/p$. Then there exists $V^* \in B$ that maximizes $E(V)$. Moreover if $V_i \in B$, $i = 1, 2$, are two maximizing potentials and $u_i \in W_0^{1,p}(\Omega)$, $i = 1, 2$, are the eigenfunctions of H_{V_i} associated to $E(V_i)$ respectively, then $u_1 = u_2$ a.e. in Ω and $V_1 = V_2$ a.e. in Ω.*

Proof. Let $E^* = \sup\{E(V) : V \in B\}$ and let $(V_n)_{n\in\mathbb{N}} \subset B$ be a maximizing sequence, i.e.

$$\lim_{n\to\infty} E(V_n) = E^*.$$

Note that, by Proposition 3.2, E^* is finite. As $(V_n)_{n\in\mathbb{N}} \subset B$ and B is bounded, there exists $V^* \in L^q(\Omega)$ and a subsequence of $(V_n)_{n\in\mathbb{N}}$, which we denote again by $(V_n)_{n\in\mathbb{N}}$, such that

$$V_n \rightharpoonup V^* \qquad \text{weakly in } L^q(\Omega).$$

By Mazur's Theorem (see [14]), $V^* \in B$.

Let us see that $E^* = E(V^*)$. Given $\varepsilon > 0$, there exists $u_0 \in C_0^1(\Omega)$ such that

$$E(V^*) \geq \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V^*(x)|u_0|^p \,\mathrm{d}x - \varepsilon.$$

Since Ω is bounded,

$$\lim_{n\to\infty} \int_\Omega V_n(x)|u_0|^p \,\mathrm{d}x = \int_\Omega V^*(x)|u_0|^p \,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
E(V^*) + \varepsilon &\geq \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V^*(x)|u_0|^p \,\mathrm{d}x \\
&= \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \lim_{n\to\infty} \int_\Omega V_n(x)|u_0|^p \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V_n(x)|u_0|^p \,\mathrm{d}x \\
&\geq \lim_{n\to\infty} E(V_n) = E^*.
\end{aligned}$$

Then, as $V^* \in B$, $E(V^*) = E^*$.

We just proved existence. Let us now show uniqueness.

Suppose we have V_1 and V_2 two maximizing potentials and let $V_3 = \frac{V_1+V_2}{2}$. Since B is convex and $E(\cdot)$ is concave, we have $V_3 \in B$ and

$$E(V_3) \geq \frac{E(V_1) + E(V_2)}{2} = E^*,$$

therefore V_3 is also a maximizing potential.

We denote the associated normalized, positive eigenfunctions by u_1, u_2 and u_3 respectively. If $u_3 \neq u_1$ or $u_3 \neq u_2$, since, by Theorem 2.13, there exists only one normalized nonnegative eigenfunction,

$$\begin{aligned}
E^* &= E(V_3) = \int_\Omega |\nabla u_3|^p \,\mathrm{d}x + \int_\Omega V_3(x)|u_3|^p \,\mathrm{d}x \\
&= \frac{1}{2}\left(\int_\Omega |\nabla u_3|^p \,\mathrm{d}x + \int_\Omega V_1(x)|u_3|^p \,\mathrm{d}x + \int_\Omega |\nabla u_3|^p \,\mathrm{d}x + \int_\Omega V_2(x)|u_3|^p \,\mathrm{d}x \right) \\
&> \frac{E(V_1) + E(V_2)}{2} \\
&= E^*,
\end{aligned}$$

a contradiction. Thus $u_1 = u_2 = u_3$. Now we write,

$$-\Delta_p u_1 + V_1(x)|u_1|^{p-2}u_1 = E^*|u_1|^{p-2}u_1 \tag{3.1}$$

$$-\Delta_p u_1 + V_2(x)|u_1|^{p-2}u_1 = E^*|u_1|^{p-2}u_1. \tag{3.2}$$

Subtracting (3.2) from (3.1), we get

$$(V_1(x) - V_2(x))|u_1|^{p-2}u_1 = 0 \qquad \text{a.e. in } \Omega,$$

then $V_1 = V_2$ a.e. in Ω. $\square$

Remark 3.4. *In the proof of Theorem 3.3 we only used $q > N/p$ to show the existence of an eigenfunction for the lowest eigenvalue.*

Assume now that the convex set B is the ball in $L^q(\Omega)$. Then we can prove that $E^*(M) := \max\{E(V) : V \in L^q(\Omega),\ \|V\|_q \leq M\}$ is increasing in M. We will need this in the sequel.

Theorem 3.5. *Let $E^* : \mathbb{R}_{\geq 0} \to \mathbb{R}$*

$$E^*(M) = \max\{E(V) : V \in L^q(\Omega),\ \|V\|_q \leq M\}.$$

Then $E^*(\cdot)$ increases monotonically.

Proof. Let $0 \leq M_1 < M_2$. Then, by Theorem 3.3, there exists $V_1 \in \overline{B(0, M_1)}$ such that $E^*(M_1) = E(V_1)$. Since $\|V_1\|_q \leq M_1 < M_2$, there exists $t \in \mathbb{R}_{>0}$ such that $\|V_1 + t\|_q \leq M_2$.

Now, given $u \in W_0^{1,p}(\Omega)$, with $\|u\|_p = 1$, we have

$$\begin{aligned}\int_\Omega |\nabla u|^p \, dx + \int_\Omega (V_1(x) + t)|u|^p \, dx &= \int_\Omega |\nabla u|^p \, dx + \int_\Omega V_1(x)|u|^p \, dx + t \\ &\geq E(V_1) + t.\end{aligned}$$

Thus

$$E(V_1 + t) \geq E(V_1) + t > E(V_1).$$

As $(V_1 + t) \in \overline{B(0, M_2)}$,

$$E^*(M_2) \geq E(V_1 + t) > E(V_1) = E^*(M_1).$$

Then $E^*(\cdot)$ increases monotonically. $\square$

Remark 3.6. *In the proof that $E^*(\cdot)$ increases monotonically, what is actually proved is that $E^*(M) \nearrow \infty$ as $M \nearrow \infty$.*

Let $q > N/p$ and consider the case $B = \overline{B(0, M)} \subset L^q(\Omega)$, for simplicity we take $M = 1$. Observe that B is a convex, closed and bounded set.

Let $V^* \in B$ be such that $E(V^*) = \max\{E(V) : V \in B\}$ and $V_0 = \frac{|V^*|}{\|V^*\|_q} \in S := \partial B$.

Let $u_0 \in W_0^{1,p}(\Omega)$ be a normalized eigenfunction of H_{V_0} associated to $E(V_0)$, i.e. $\|u_0\|_p = 1$ and

$$E(V_0) = \int_\Omega |\nabla u_0|^p \, dx + \int_\Omega \frac{|V^*(x)|}{\|V^*\|_q} |u_0|^p \, dx.$$

Then

$$\begin{aligned}E(V_0) &\geq \int_\Omega |\nabla u_0|^p \, dx + \int_\Omega V^*(x)|u_0|^p \, dx \\ &\geq E(V^*) = E^*.\end{aligned}$$

Thus, from uniqueness, $V_0 = V^*$, from where $\|V^*\|_q = 1$ and $V^* \geq 0$.

Therefore if we take $S = \partial B(0,1)$, there exists $V_0 \geq 0$ in S such that

$$E(V_0) = \max\{E(V) : V \in S\} = \max\{E(V) : V \in B\}.$$

We now try to characterize V_0. For this, we need the following notation: For any $V \in S$, we denote by $T_V(S)$ the tangent space of S at V. It is well known that

$$T_V(S) = \Big\{W \in L^q(\Omega) \mid \int_\Omega |V|^{q-2} V W \,\mathrm{d}x = 0\Big\}.$$

Now, let $W \in T_{V_0}(S)$ and $\alpha : (-1,1) \to L^q(\Omega)$ be a differentiable curve such that

$$\alpha(t) \in S \quad \forall t \in (-1,1), \quad \alpha(0) = V_0 \quad \text{and} \quad \dot\alpha(0) = W.$$

We denote by $V_t = \alpha(t)$ and $\lambda(t) = E(\alpha(t))$.

Let $u_t \in W_0^{1,p}(\Omega)$ be the nonnegative normalized eigenfunction of H_{V_t} with eigenvalue $\lambda(t)$, i.e. $\|u_t\|_p = 1$ and

$$\lambda(t) = \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x.$$

We have the following,

Lemma 3.7. *$\lambda(t)$ is continuos at $t = 0$, i.e.*

$$\lim_{t\to 0} \lambda(t) = \lambda(0) = E(V_0) = E^*.$$

Proof. By Proposition 3.2, there exists $C = C(\Omega, q, p) > 0$ such that

$$C > \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x$$

and as $q > N/p$, by Lemma 2.8, given $\varepsilon > 0$ there exists D_ε such that

$$\left| \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x \right| \leq \varepsilon \|\nabla u_t\|_p^p + D_\varepsilon \|u_t\|_p^p$$

for any t. Thus if $\varepsilon < 1$

$$\|\nabla u_t\|_p^p \leq \frac{C + D_\varepsilon}{1-\varepsilon}.$$

Then $(u_t)_{t\in(-1,1)}$ is bounded in $W_0^{1,p}(\Omega)$ and therefore it is bounded in $L^{pq'}(\Omega)$. Since

$$\lim_{t\to 0} V_t = V_0 \qquad \text{in } L^q(\Omega),$$

then

$$\lim_{t\to 0} \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x = 0.$$

Thus

$$\begin{aligned}
\lambda(t) &= \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_t|^p \,\mathrm{d}x \\
&= \int_\Omega |\nabla u_t|^p \,\mathrm{d}x + \int_\Omega V_0(x)|u_t|^p \,\mathrm{d}x + \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x \\
&\geq \lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}\lambda(0) &= \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V_0(x)|u_0|^p \,\mathrm{d}x \\ &= \int_\Omega |\nabla u_0|^p \,\mathrm{d}x + \int_\Omega V_t(x)|u_0|^p \,\mathrm{d}x + \int_\Omega (V_0(x) - V_t(x))|u_0|^p \,\mathrm{d}x \\ &\geq \lambda(t) + \int_\Omega (V_0(x) - V_t(x))|u_0|^p \,\mathrm{d}x.\end{aligned}$$

Therefore

$$\lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_0|^p \,\mathrm{d}x \geq \lambda(t) \geq \lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_t|^p \,\mathrm{d}x.$$

Hence,

$$\lim_{t\to 0} \lambda(t) = \lambda(0),$$

as we wanted to show. □

Lemma 3.8. *$\lambda(t)$ is differentiable at $t = 0$ and*

$$\dot{\lambda}(0) = \int_\Omega W(x)|u_0|^p \,\mathrm{d}x.$$

Proof. Let $(t_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} t_n = 0$. As $(u_{t_n})_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}} \subset (t_n)_{n\in\mathbb{N}}$ and $u \in W_0^{1,p}(\Omega)$ such that

$$u_{t_{n_k}} \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.3}$$

$$u_{t_{n_k}} \to u \quad \text{strongly in } L^r(\Omega), \tag{3.4}$$

for any $1 < r < p^*$. Let us see that $u = u_0$.

In fact, by (3.4) we have $\|u\|_p = 1$ and by (3.3), we have

$$\liminf_{k\to\infty} \int_\Omega |\nabla u_{t_{n_k}}|^p \,\mathrm{d}x \geq \int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

Again by (3.4) and as, by Lemma 3.7, $V_{t_{n_k}} \to V_0$ in $L^q(\Omega)$, we get

$$\lim_{k\to\infty} \int_\Omega V_{t_{n_k}}(x)|u_{t_{n_k}}|^p \,\mathrm{d}x = \int_\Omega V_0(x)|u|^p \,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}\lambda(0) &= \lim_{k\to\infty} \int_\Omega |\nabla u_{t_{n_k}}|^p \,\mathrm{d}x + \int_\Omega V_{t_{n_k}}(x)|u_{t_{n_k}}|^p \,\mathrm{d}x \\ &\geq \int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega V_0(x)|u|^p \,\mathrm{d}x \geq \lambda(0).\end{aligned}$$

Hence u is a nonnegative, normalized eigenfunction associated to $\lambda(0)$. By Theorem 2.13, we have that $u = u_0$. Since the limit u_0 is independent of the sequence $(t_{n_k})_{k\in\mathbb{N}}$, it follows that (3.3)–(3.4) hold for the limit $t \to 0$.

By the differentiability of V_t and by (3.4) we obtain

$$\lim_{t\to 0} \int_\Omega \left(\frac{V_t(x) - V_0(x)}{t}\right)|u_t|^p \,\mathrm{d}x = \int_\Omega W(x)|u_0|^p \,\mathrm{d}x.$$

In the proof of Lemma 3.7 we have showed that

$$\lambda(0) + \int_\Omega (V_t(x) - V_0(x))|u_0|^p \,\mathrm{d}x \geq \lambda(t) \geq \lambda(0) + \int_\Omega (V_t(x) - V_0(x))(x)|u_t|^p \,\mathrm{d}x.$$

Thus, for $t > 0$,

$$\int_\Omega \left(\frac{V_t(x) - V_0(x)}{t} \right) |u_0|^p \,\mathrm{d}x \geq \frac{\lambda(t) - \lambda(0)}{t} \geq \int_\Omega \left(\frac{V_t(x) - V_0(x)}{t} \right) |u_t|^p \,\mathrm{d}x,$$

and an analogous inequality for $t < 0$. Then $\lambda(t)$ is differentiable at $t = 0$ and

$$\dot{\lambda}(0) = \int_\Omega W(x)|u_0|^p \,\mathrm{d}x.$$

The proof is now complete. $\square$

Remark 3.9. *Since λ has maximum at $t = 0$, we have*

$$\int_\Omega W(x)|u_0|^p \,\mathrm{d}x = 0 \qquad \textit{for every } W \in T_{V_0}S. \tag{3.5}$$

The following proposition characterize the support of the maximal potential.

Proposition 3.10. $\Omega \subseteq$ *supp*(V_0).

Proof. Suppose not. Then, let $x \in \Omega$ such that $x \notin \mathrm{supp}(V_0)$. As $\mathrm{supp}(V_0)$ is closed there exists $r > 0$ such that

$$B(x, r) \subset \Omega \qquad \text{and} \qquad B(x, r) \cap \mathrm{supp}(V_0) = \emptyset.$$

Then $W = \chi_{B(x,r)} \in T_{V_0}S$ and by (3.5),

$$\int_{B(x,r)} |u_0|^p \,\mathrm{d}x = 0.$$

Hence $u_0 = 0$ a.e. in $B(x, r)$, a contradiction. $\square$

Finally we arrive at the following characterization of the maximal potential.

Theorem 3.11. *Let V_0 be a maximal potential and let u_0 be the eigenfunction associated to $E(V_0)$. Then there exists a constant k such that*

$$|u_0|^p = k|V_0|^{q-1} \textit{ in } \Omega. \tag{3.6}$$

Proof. Let T_1 and T_2 be subsets of $\mathrm{supp}(V_0)$. We denote

$$W(x) = \frac{\chi_{T_1}(x)}{\displaystyle\int_{T_1} |V_0|^{q-1} \,\mathrm{d}x} - \frac{\chi_{T_2}(x)}{\displaystyle\int_{T_2} |V_0|^{q-1} \,\mathrm{d}x}.$$

Let us see that $W \in T_{V_0}S$. In fact, as V_0 is a maximal potential, $V_0 \geq 0$. Then

$$\begin{aligned} \int_\Omega |V_0|^{q-2} V_0 W \,\mathrm{d}x &= \int_\Omega V_0^{q-1} W \,\mathrm{d}x \\ &= \frac{\displaystyle\int_{T_1} V_0^{q-1} \,\mathrm{d}x}{\displaystyle\int_{T_1} V_0^{q-1} \,\mathrm{d}x} - \frac{\displaystyle\int_{T_2} V_0^{q-1} \,\mathrm{d}x}{\displaystyle\int_{T_2} V_0^{q-1} \,\mathrm{d}x} \\ &= 0 \end{aligned}$$

thus $W \in T_{V_0} S$, as we wanted to see.

By (3.5), we have

$$0 = \int_\Omega W |u_0|^p \, \mathrm{d}x = \frac{\displaystyle\int_{T_1} |u_0|^p \, \mathrm{d}x}{\displaystyle\int_{T_1} |V_0|^{q-1} \, \mathrm{d}x} - \frac{\displaystyle\int_{T_2} |u_0|^p \, \mathrm{d}x}{\displaystyle\int_{T_2} |V_0|^{q-1} \, \mathrm{d}x}.$$

Then

$$\frac{\displaystyle\int_{T_1} |u_0|^p \, \mathrm{d}x}{\displaystyle\int_{T_1} |V_0|^{q-1} \, \mathrm{d}x} = \frac{\displaystyle\int_{T_2} |u_0|^p \, \mathrm{d}x}{\displaystyle\int_{T_2} |V_0|^{q-1} \, \mathrm{d}x}.$$

Thus, there exists a constant k such that

$$\frac{\displaystyle\int_{T} |u_0|^p \, \mathrm{d}x}{\displaystyle\int_{T} |V_0|^{q-1} \, \mathrm{d}x} = k$$

for each $T \subset \operatorname{supp}(V_0)$. In particular, if we take

$$T = \{x \in \operatorname{supp}(V_0) : k|V_0(x)|^{q-1} > |u_0(x)|^p\}$$

we get

$$\int_T |u_0|^p \, \mathrm{d}x = k \int_T |V_0|^{q-1} \, \mathrm{d}x$$

thus

$$k \int_T |V_0|^{q-1} \, \mathrm{d}x - \int_T |u_0|^p \, \mathrm{d}x = 0.$$

Since $k|V_0(x)|^{q-1} > |u_0(x)|^p$ for any $x \in T$, the measure of T is zero. In the same way, we obtain that

$$\{x \in \operatorname{supp}(V_0) : k|V_0(x)|^{q-1} < |u_0(x)|^p\}$$

has measure zero. Thus

$$|u_0|^p = k|V_0|^{q-1} \qquad \text{a.e. in } \operatorname{supp}(V_0).$$

By Proposition 3.10,

$$|u_0|^p = k|V_0|^{q-1} \text{ in } \Omega.$$

This ends the proof. □

Equation (3.6) gives us purely algebraic relationship between the optimizing potentials and their associated eigenfunction. Since the eigenvalue equation is homogeneous of degree p in the eigenfunction, we can choose the constant in (3.6) to be equal to one, this can be obtained by taking $\frac{u_0}{k^p}$ as the eigenfunction instead of u_0. Replacing in equation (2.7), we see that the eigenfunction associated to the maximal eigenvalue satisfies

$$-\Delta_p u + u^\alpha = E u^{p-1} \tag{3.7}$$

where E is the maximal potential eigenvalue and the equation can be written in terms of the associated eigenfunction. An interesting consequence of Theorem 3.3

is, in this context, a proof of existence and certain properties of solution of equation (3.7). More precisely, we have

Corollary 3.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 < p < \infty$ and $\alpha \in \mathbb{R}$. For any $\lambda > E(0)$, where $E(0)$ is the principal eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$, the nonlinear eigenvalue problem*

$$\begin{cases} -\Delta_p u + u^\alpha = \lambda u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

has a solution in the following cases:

(1) *If $1 < p < 2$, we take $\alpha < \max\{\frac{2p-2}{2-p}, \frac{(p-1)N}{N-p}\}$.*
(2) *If $p \geq 2$, we take $\alpha > 1$.*

Proof. The existence of a potential V_0 maximizing of $-\Delta_p + V$ subject to $\|V\|_q = M$, for any $M > 0$ is known from Theorem 3.3, with $\alpha = \frac{pq-q+1}{q-1}$. If the maximized eigenvalue is $E^* = E(V_0)$, then the necessary condition (3.7) becomes (3.8) with $u = u_0$ and $\lambda = E^*$.

The corollary will thus be proved if it is shown that E^* increases continuously from $E(0)$ to ∞ as M goes from 0 to ∞. By Remark 3.5, $E^*(\cdot)$ is increases monotonically from $E(0)$ to ∞ as $M \nearrow \infty$. It remains to prove the continuity.

We denote with V_0^M the maximal potential associated to $E^*(M)$. If $t > 0$, then

$$E(V_0^{M+t}) = E^*(M+t) \geq E^*(M).$$

Take $V = \frac{M}{M+t} V_0^{M+t}$, note that $\|V\|_q = M$, then $E(V) \leq E^*(M)$. Given $u \in W_0^{1,p}(\Omega)$, $\|u\|_p = 1$, we have

$$\begin{aligned} \int_\Omega |\nabla u|^p \, dx + \int_\Omega V(x)|u|^p \, dx &= \int_\Omega |\nabla u|^p \, dx + \int_\Omega \frac{M}{M+t} V_0^{M+t}(x)|u|^p \, dx \\ &= \frac{M}{M+t}\left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega V_0^{M+t}(x)|u|^p \, dx \right) \\ &\quad + (1 - \frac{M}{M+t}) \int_\Omega |\nabla u|^p \, dx \\ &\geq \frac{M}{M+t}\left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega V_0^{M+t}(x)|u|^p \, dx \right). \end{aligned}$$

Thus

$$E(V) = E(\frac{M}{M+t} V_0^{M+t}) \geq \frac{M}{M+t} E(V_0^{M+t}) = \frac{M}{M+t} E^*(M+t)$$

then, as $E(V) \leq E^*(M)$,

$$\frac{M}{M+t} E^*(M+t) \leq E^*(M) \leq E^*(M+t). \tag{3.9}$$

Similarly,

$$E^*(M-t) \leq E^*(M) \leq \frac{M-t}{M} E^*(M-t). \tag{3.10}$$

Then, taking limits in (3.9) and (3.10),

$$\lim_{t\to 0} E^*(M+t) = E^*(M).$$

This completes the proof. □

3.3. **Minimizing Potentials.** In this subsection we present the results for minimizing potentials. Since the results and the proof are completely analogous to those of the previous subsection we only state the main results and point out only the significant differences.

Theorem 3.13. *If $q > N/p$, there exists $V_* \in B$ that minimizes $E(V)$.*

Proof. Is analogous to that of Theorem 3.3. □

As in the previous subsection, we consider the case $B = \overline{B(0,M)} \subset L^q(\Omega)$, and to simplify the computations, we take $M = 1$.

As a concave function defined over a convex set achieves its minimum at the extreme points of the convex, there exists $V_0 \in \partial B$ such that $E(V_0) = \min\{E(V) : V \in \partial B\} = \min\{E(V) : V \in B\}$. Moreover, since $-|V_0| \le V_0$ and $E(\cdot)$ is nondecreasing we may assume that $V_0 \le 0$.

Let us now try to characterize V_0. As before, let $\alpha : (-1,1) \to L^q(\Omega)$ be a differentiable curve such that

$$\alpha(t) \in S := \partial B, \qquad \alpha(0) = V_0 \qquad \text{and} \qquad \dot\alpha(0) = W \in T_{V_0}S.$$

We denote by $V_t = \alpha(t)$ and $\lambda(t) = E(\alpha(t))$. Let u_t the normalized, nonnegative eigenfunction of H_{V_t} associated to $\lambda(t)$. Observe that Lemmas 3.7 and 3.8 apply. Hence, as λ has a minimum at $t = 0$ we have

$$\int_\Omega W(x)|u_0|^p \,\mathrm{d}x = 0 \, \forall W \in T_{V_0}S \tag{3.11}$$

As for maximizing potential, we have,

Proposition 3.14. $\Omega \subseteq$ *supp*(V_0).

Proof. Analogous to that of Lemma 3.10. □

Proposition 3.15. *Let V_0 be a minimal potential and let u_0 be the normalized, nonnegative eigenfunction of H_{V_0} associated to $E(V_0)$. Then, there exists a constant $k \in \mathbb{R}_+$ such that*

$$|u_0|^p = k|V_0|^{q-1} \tag{3.12}$$

in Ω.

Proof. Analogous to that of Lema 3.11. □

As before, from (3.12) we obtain a purely algebraic relationship between minimal potential and their associated eigenfunctions. Using the homogeneity of the equation, we can choose the constant in (3.12) to be 1. Replacing in (2.7) we obtain that the eigenfunction associated to the minimal potential satisfies

$$-\Delta_p u - u^\alpha = Eu^{p-1} \tag{3.13}$$

where E is the minimal eigenvalue and $\alpha = \frac{pq-q+1}{q-1}$.

Therefore, we obtain the following corollary

Corollary 3.16. *Let $\Omega \subset \mathbb{R}^N$ be a smooth open and bounded set, $1 < p < \infty$ and $\alpha \in \mathbb{R}$. For every $\lambda < E(0)$, where $E(0)$ is the principal eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, the nonlinear eigenvalue problem*

$$\begin{cases} -\Delta_p u - u^\alpha = \lambda u^{p-1} & \text{in } \Omega \\ u > 0 & \text{en } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

has a solution in the cases

(1) *If $1 < p < 2$, taking $\alpha < \frac{(p-1)N}{N-p}$.*
(2) *If $p \geq 2$, taking $\alpha > 1$.*

Proof. Analogous to that of Corollary 3.12 □

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