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LIMITS FOR MONGE-KANTOROVICH MASS TRANSPORT PROBLEMS.

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Abstract. In this paper we study the limit of Monge-Kantorovich mass transfer problems when the involved measures are supported in a small strip near the boundary of a bounded smooth domain, Ω. Given two absolutely continuos measures (with respect to the surface measure) supported on the boundary $\partial\Omega$, by performing a suitable extension of the measures to a strip of width ε near the boundary of the domain Ω we consider the mass transfer problem for the extensions. Then we study the limit as ε goes to zero of the Kantorovich potentials for the extensions and obtain that it coincides with a solution of the original mass transfer problem. Moreover we look for the possible approximations of these problems by solutions to equations involving the $p-$Laplacian for large values of p.

1. Introduction.

The main goal of this article is to obtain a solution to the Monge-Kantorovich mass transport problem for some measures supported on surfaces, as a limit when $\varepsilon \to 0$ of solutions to usual solid mass transport, the masses being supported on small strips of width ε. We will also analyze approximations involving the $p-$Laplacian of these transport problems and its viscosity limits.

First, let us briefly present what are the main features of the problem under consideration. Assume that we have a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$ and a continuous funtion $g : \partial\Omega \mapsto \mathbb{R}$ with

$$\int_{\partial\Omega} g\, d\sigma = \int_{\partial\Omega\cap\{g>0\}} g_+\, d\sigma - \int_{\partial\Omega\cap\{g<0\}} g_-\, d\sigma = 0,$$

where $d\sigma$ denotes the area measure on $\partial\Omega$. Hence, we have two subsets $\Gamma_+ = \partial\Omega \cap \{g > 0\}$, $\Gamma_- = \partial\Omega \cap \{g < 0\}$ and two positive functions

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(densities) g_+ and g_- such that

$$\int_{\Gamma_+} g_+ \, d\sigma = \int_{\Gamma_-} g_- \, d\sigma.$$

Our aim is to solve the following Monge-Kantorovich mass transfer problem: among all mappings $\tilde{T} : \Gamma_+ \to \Gamma_-$ that preserve the measures given by the two densities choose one that minimizes the transport cost

$$C(\tilde{T}) = \int_{\Gamma_+} |x - \tilde{T}(x)| g_+(x) \, d\sigma.$$

Applying the Kantorovich optimality principle to the mass transfer problem for the measures $g^+\mathcal{H}^{N-1} \llcorner \partial\Omega$ and $g^-\mathcal{H}^{N-1} \llcorner \partial\Omega$ that are concentrated on $\partial\Omega$ we obtain the maximization problem

$$\max \left\{ \int_{\partial\Omega} wg \, d\sigma \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \, \|Dw\|_\infty \leq 1 \right\}. \tag{1.1}$$

The maximizers of (1.1) are maximal Kantorovich potentials, see [1] and [15]. Note that one usually defines Kantorovich potentials as maximizers of (1.1) without imposing that $\int_\Omega w = 0$. Here we use this normalization to gain compactness.

As first noticed in [8] (see also [6]), a natural approach to the maximization problem (1.1) is to consider limits of optimization problems involving the p-Laplacian. That is, we consider $u_{p,0}$ the solution of the maximization problem

$$\max \left\{ \int_{\partial\Omega} wg \, d\sigma \colon w \in W^{1,p}(\Omega), \int_\Omega w = 0, \, \|Dw\|_{L^p(\Omega)} \leq 1 \right\}. \tag{1.2}$$

In [9] the limit as $p \to \infty$ of the family $u_{p,0}$ is studied. It is proved there that a uniform limit of a subsequence $\{u_{p_i,0}\}$, $p_i \to \infty$, v_∞, is a solution to (1.1). Since we are interested in large values of p we will assume throughout this paper that $p > N$.

These variational problems can be studied as a singular limit of mass transport problems where the measures are supported in small strips near the boundary. In this sense we get a natural Neumann problem for the p-Laplacian while in the paper [8] the relevant problem is of Dirichlet type.

More precisely, consider the subset of Ω,

$$\omega_\delta = \{ x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) < \delta \} .$$

Note that this set has measure $|\omega_\delta| \sim \delta \mathcal{H}^{N-1}(\Omega)$ for small values of δ (here $\mathcal{H}^{N-1}(\Omega)$ stands for the $N-1$ dimensional measure of $\partial\Omega$). Then for sufficiently small $s \geq 0$ we can define the *parallel* interior boundary

$\Gamma_s = \{z - s\nu(z),\ z \in \partial\Omega\}$ where $\nu(z)$ denotes the outwards normal unit at $z \in \partial\Omega$. Note that $\Gamma_0 = \partial\Omega$. Then we can also look at the set ω_δ as the neighborhood of Γ_0 defined by

$$\omega_\delta = \{y = z - s\nu(z),\ z \in \partial\Omega,\ s \in (0,\delta)\} = \bigcup_{0<s<\delta} \Gamma_s$$

for sufficiently small δ, say $0 < \delta < \delta_0$. We also denote $\Omega_s = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > s\}$ and for s small we have that $\partial\Omega_s = \Gamma_s$.

Let us consider the transport problem for a suitable extension of g. To define this extension, as we have mentioned, let us denote by $d\sigma$ and $d\sigma_s$ the surface measures on the sets $\partial\Omega$ and Γ_s respectively. Given a function ϕ defined on $\bar{\Omega}$, and given $y \in \Gamma_s$ (with s small) , there exists $z \in \partial\Omega$ such that $y = z - s\nu(z)$. Hence, we can change variables:

$$\int_{\Gamma_s} \phi(y) d\sigma_s = \int_{\partial\Omega} \phi(z - s\nu(z)) G(s,z)\, d\sigma$$

where $G(s,z)$ depends on Ω (more precisely, it depends on the surface measures $d\sigma$ and $d\sigma_s$), and by the regularity of $\partial\Omega$, $G(s,z) \to 1$ as $s \to 0$ uniformly for $z \in \partial\Omega$.

Using these ideas, we define the following extension of g in Ω. Consider $\eta : [0,\infty) \to [0,1]$ a $\mathcal{C}^\infty$ function such that $\eta(s) = 1$ if $0 \le s \le \frac{1}{2}$, $0 < \eta(s) < 1$ when $\frac{1}{2} < s < 1$, $\eta(s) = 0$ if $s \ge 1$, and $\int_0^\infty \eta(s)\, ds = A$. Defining $\eta_\varepsilon(s) = \frac{1}{A\varepsilon}\eta\left(\frac{s}{\varepsilon}\right)$, we get $\int_0^\infty \eta_\varepsilon(s)\, ds = 1$. For $\varepsilon < \delta_0$ consider Γ_s and let

$$g_\varepsilon(y) = \eta_\varepsilon(s) \frac{g(z)}{G(s,z)}, \qquad y = z - s\nu(z), \qquad \text{for } 0 \le s \le \varepsilon,$$

extended as $g_\varepsilon(y) = 0$ in the rest of Ω; that is, in $\Omega \setminus \omega_\varepsilon$.

We have $g_\varepsilon \in \mathcal{C}(\Omega)$. Moreover,

$$\begin{aligned} \int_\Omega g_\varepsilon(x)\, dx &= \int_0^\varepsilon \int_{\Gamma_s} g_\varepsilon(y)\, d\sigma_s\, ds \\ &= \int_0^\varepsilon \int_{\partial\Omega} g_\varepsilon(z - s\nu(z)) G(s,z)\, d\sigma\, ds \\ &= \int_0^\varepsilon \eta_\varepsilon(s) \int_{\partial\Omega} g(z)\, d\sigma\, ds = 0. \end{aligned}$$

Associated to this extension we could consider the following two variational problems. First, the maximization problem in $W^{1,p}(\Omega)$,

$$\max \left\{ \int_{\omega_\varepsilon} w g_\varepsilon \colon w \in W^{1,p}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^p(\Omega)} \le 1 \right\}, \tag{1.3}$$

and the maximization problem in $W^{1,\infty}(\Omega)$,

$$\max\left\{\int_{\omega_\varepsilon} wg_\varepsilon \colon w\in W^{1,\infty}(\Omega), \int_\Omega w=0, \|Dw\|_{L^\infty(\Omega)}\le 1\right\}. \tag{1.4}$$

We call $u_{p,\varepsilon}$ a solution to (1.3) and $u_{\infty,\varepsilon}$ a solution to (1.4).

Remark 1.1. *Notice that the extremal functions* $u_{p,\varepsilon}, u_{p,0}, u_{\infty,\varepsilon}, u_{\infty,0}$ *satisfy*

$$\|Du_{p,\varepsilon}\|_{L^p(\Omega)} = \|Du_{p,0}\|_{L^p(\Omega)} = \|Du_{\infty,\varepsilon}\|_{L^\infty(\Omega)} = \|Du_{\infty,0}\|_{L^\infty(\Omega)} = 1,$$

unless $g\equiv 0$.

Our first result says that we can take the limits as $\varepsilon\to 0$ and $p\to\infty$ in these variational problems. With the above notations we have the following commutative diagram

$$\begin{array}{cccc} & u_{\infty,\varepsilon} & \longrightarrow & u_{\infty,0} \\ p\to\infty & \uparrow & & \uparrow \\ & u_{p,\varepsilon} & \longrightarrow & u_{p,0} \\ & & \varepsilon\to 0 & \end{array} \tag{1.5}$$

This diagram can be understood in two ways, either taking into account the variational properties satisfied by the functions, or considering the corresponding PDEs that the functions satisfy.

From the variational viewpoint, we can state our first result:

Theorem 1. *Diagram* (1.5) *is commutative in the following sense:*

(1) *Maximizers of* (1.3)*,* $u_{p,\varepsilon}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{p,0}$ *a maximizer of* (1.2) *as* $\varepsilon\to 0$*.*
(2) *Maximizers of* (1.3)*,* $u_{p,\varepsilon}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,\varepsilon}$ *a maximizer of* (1.4) *as* $p\to\infty$*.*
(3) *Maximizers of* (1.4)*,* $u_{\infty,\varepsilon}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,0}$ *a maximizer of* (1.1) *as* $\varepsilon\to 0$*.*
(4) *Maximizers of* (1.2)*,* $u_{p,0}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,0}$ *a maximizer of* (1.1) *as* $p\to\infty$*.*

We turn now our attention to the PDE verified by the limits in the viscosity sense (see Section 3 for the precise definition) or in the weak sense.

When $p \to \infty$ we find the ∞-Laplacian, a well known nonlinear operator, given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i},$$

see [5], [13]. The ∞-Laplacian appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function f, see [2], [3], and [12].

Up to a Lagrange multiplier λ_p the functions $u_{p,0}$ are viscosity (and weak) solutions to the problem,

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = \lambda_p \, g & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Let us to point out that it is easily seen that $\lambda_p^{1/p} \to 1$ as $p \to \infty$ (see the remark at the end of Section 2.)

In [9] (see also [10]) the limit as $p \to \infty$ of the family $u_{p,0}$ is studied in the viscosity setting. It is proved that the problem that is satisfied by a uniform limit $u_{\infty,0}$ in the viscosity sense is as follows,

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ B(x,u,Du) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

Here

$$B(x,u,Du) \equiv \begin{cases} \min\left\{|Du| - 1 \ , \ \frac{\partial u}{\partial \nu}\right\} & \text{if } g > 0, \\ \max\{1 - |Du| \ , \ \frac{\partial u}{\partial \nu}\} & \text{if } g < 0, \\ H(|Du|)\frac{\partial u}{\partial \nu} & \text{if } g = 0, \end{cases}$$

and $H(a)$ is given by

$$H(a) = \begin{cases} 1 & \text{if } a \geq 1, \\ 0 & \text{if } 0 \leq a < 1. \end{cases}$$

Moreover, the function $u_{\infty,0}$ satisfies the inequalities

$$|Du| \leq 1 \quad \text{and} \quad -|Du| \geq -1$$

in the viscosity sense.

On the other hand, when we deal with the problems in the strips, the functions $u_{p,\varepsilon}$ are weak (and hence viscosity) solutions to the problem,

$$\begin{cases} -\Delta_p u = g_\varepsilon & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Passing to the limit as $p \to \infty$ in these problems we get that the function $u_{\infty,\varepsilon}$ satisfy the following properties in the viscosity sense:

$$\begin{cases} |Du| \le 1 & \text{in } \Omega, \\ -|Du| \ge -1 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

and, in the different regions determined by g_ε:

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega \setminus \omega_\epsilon, \\ |Du| = 1 & \text{in } \{g_\epsilon > 0\}, \\ -|Du| = -1 & \text{in } \{g_\epsilon < 0\}, \\ -\Delta_\infty u \ge 0 & \text{in } \Omega \cap \partial\{g_\epsilon > 0\} \setminus \partial\{g_\epsilon < 0\}, \\ -\Delta_\infty u \le 0 & \text{in } \Omega \cap \partial\{g_\epsilon < 0\} \setminus \partial\{g_\epsilon > 0\}. \end{cases} \tag{1.10}$$

Theorem 2.

(1) *The limit $u_{p,0}$ of a uniformly converging sequence $u_{p,\varepsilon}$ of weak solutions to* (1.8) *as $\varepsilon \to 0$ is a weak solution to* (1.6) *(and hence a viscosity solution).*
(2) *The limit $u_{\infty,0}$ of a uniformly converging sequence $u_{p,0}$ of viscosity solutions to* (1.6) *as $p \to \infty$ is a viscosity solution to* (1.7).

Let us to point out that when $\varepsilon \to 0$, g_ε concentrates on the boundary, and therefore the sequence $\{g_\varepsilon\}$ is not uniformly bounded. This makes it difficult t to pass to the limit in the viscosity sense when $\varepsilon \to 0$. Hence in this case, we consider the variational characterization of the sequence $\{u_{p,\varepsilon}\}$ (that is equivalent to the fact of being a weak solution). To the best of our knowledge, it is not known that the notions of viscosity and weak solutions coincide for solutions to (1.8), cf. [14] where such equivalence is only proved for Dirichlet boundary conditions.

Now, we deal with the rest of the commutative diagram. To pass to the limit in the sequence $u_{\infty,\varepsilon}$ we need the variational characterization and a uniqueness result for the limit problem. The latter has been proved in [9] and it says that:

If Ω is convex and $\{g = 0\}^o = \emptyset$, then there is a unique function which satisfies the extremal property (1.1).

Here $\{g = 0\}^o$ denotes the interior of the set $\{g = 0\}$ in the topology of $\partial\Omega$.

Let us to point out that the hypothesis $\{g = 0\}^o = \emptyset$ implies also the uniqueness of the extremals to (1.4), see [11]. Therefore, under this hypothesis there exists a unique $u_{\infty,\varepsilon}$ reached as a limit of the solutions $u_{p,\varepsilon}$ as $p \to \infty$.

Next, we state our second theorem.

Theorem 3.

(1) *The limit $u_{\infty,\varepsilon}$ of a uniformly converging sequence $u_{p,\varepsilon}$ of viscosity solutions to* (1.8) *as $p \to \infty$ is a viscosity solution to* (1.9)-(1.10).

(2) *Assume that Ω is convex and $\{g = 0\}^o = \emptyset$. Consider the viscosity solutions $u_{\infty,\varepsilon}$ to* (1.9)-(1.10), *obtained as a uniform limit as $p \to \infty$ of the solutions $u_{p,\varepsilon}$. Then, the sequence $\{u_{\infty,\varepsilon}\}$ converges uniformly to a viscosity solution to* (1.7), *$u_{\infty,0}$.*

Note that in (2) we are considering solutions $u_{\infty,\varepsilon}$ that are limits of $u_{p,\varepsilon}$ as $p \to \infty$. We note that whether a similar statement holds for arbitrary solutions is an open question.

The rest of the paper is organized as follows: in Section 2 we pass to the limit in the variational sense and prove Theorem 1 and in Section 3 we deal with viscosity solutions and prove Theorems 2 and 3.

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1 that shows that diagram (1.5) commutes in the variational sense.

Proof of Theorem 1. The proof of the uniform convergence (along subsequences) of $u_{p,0}$ to $u_{\infty,0}$ is contained in [9].

Let us prove that $u_{p,\varepsilon}$ converges to $u_{p,0}$ as $\varepsilon \to 0$. We have

$$\|Du_{p,\varepsilon}\|_{L^p(\Omega)} \leq 1.$$

Note that we can assume $\|Du_{p,\varepsilon}\|_{L^p(\Omega)} = 1$ unless $g = 0$.

Therefore we can extract a subsequence (that we still call $u_{p,\varepsilon}$) such that

$$u_{p,\varepsilon} \rightharpoonup v, \qquad \text{as } \varepsilon \to 0,$$

weakly in $W^{1,p}(\Omega)$ and, since $p > N$,

$$u_{p,\varepsilon} \to v, \qquad \text{as } \varepsilon \to 0,$$

uniformly in Ω (in fact, convergence holds in a Hölder space $C^{0,\beta}$ for some suitable $\beta > 0$). This limit v verifies the normalization constraint

$$\int_\Omega v = 0$$

and moreover

$$\|Dv\|_{L^p(\Omega)} \le 1.$$

On the other hand, thanks to the uniform convergence and to the definition of the extension g_ε we obtain,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\omega_\varepsilon} g_\varepsilon u_{p,\varepsilon} &= \lim_{\varepsilon\to 0}\int_0^\varepsilon \int_{\Gamma_s} g_\varepsilon(y) u_{p,\varepsilon}(y)\, d\sigma_s\, ds \\
&= \lim_{\varepsilon\to 0}\int_0^\varepsilon \int_{\partial\Omega} g_\varepsilon(z - s\nu(z)) u_{p,\varepsilon}(z - s\nu(z)) G(s,z)\, d\sigma\, ds \\
&= \lim_{\varepsilon\to 0}\int_0^\varepsilon \eta_\varepsilon(s) \int_{\partial\Omega} g(z) u_{p,\varepsilon}(z - s\nu(z))\, d\sigma\, ds \\
&= \int_{\partial\Omega} gv\, d\sigma
\end{aligned}$$

and hence

$$\int_\Omega |Dv|^p - \int_{\partial\Omega} gv\, d\sigma \le \liminf_{\varepsilon\to 0}\left(\int_\Omega |Du_{p,\varepsilon}|^p - \int_{\omega_\varepsilon} g_\varepsilon u_{p,\varepsilon}\right). \tag{2.1}$$

On the other hand for every $w \in C^1(\overline{\Omega})$ we have

$$\int_\Omega |Dw|^p - \int_{\partial\Omega} gw\, d\sigma = \lim_{\varepsilon\to 0}\int_\Omega |Dw|^p - \int_{\omega_\varepsilon} g_\varepsilon w.$$

Taking $w \in C^1(\overline{\Omega})$ with $\|Dw\|_{L^p(\Omega)} = 1$,and $\int_\Omega w = 0$, by the extremal characterization of $u_{p,\varepsilon}$, we have

$$\begin{aligned}
&\int_\Omega |Dw|^p - \int_{\partial\Omega} gw d\sigma \\
&\qquad \ge \liminf_{\varepsilon\to 0}\int_\Omega |Du_{p,\varepsilon}|^p - \int_{\omega_\varepsilon} g_\varepsilon u_{p,\varepsilon}.
\end{aligned}$$

Therefore by (2.1) we obtain

$$\begin{aligned}
&\inf_{w\in W^{1,p}(\Omega), \int_\Omega w=0, \|Dw\|_{L^p(\Omega)}=1}\left\{\int_\Omega |Dw|^p - \int_{\partial\Omega} gw d\sigma\right\} \\
&\qquad \ge \int_\Omega |Dv|^p - \int_{\partial\Omega} gv\, d\sigma,
\end{aligned}$$

and hence it follows that all possible limits $v = u_{p,0}$ satisfy the extremal property (1.2).

Next we prove that $u_{\infty,\varepsilon}$ converges to $u_{\infty,0}$, a maximizer of (1.1). Recall that $u_{\infty,\varepsilon}$ is a solution to the problem

$$M_\varepsilon = \max\left\{\int_{\omega_\varepsilon} wg_\varepsilon \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_\infty \le 1\right\}.$$

That is, we have

$$M_\varepsilon = \int_{\omega_\varepsilon} u_{\infty,\varepsilon} g_\varepsilon.$$

Therefore $u_{\infty,\varepsilon}$ is bounded in $W^{1,\infty}(\Omega)$ and then there exists a subsequence (that we still denote by $u_{\infty,\varepsilon}$) such that,

$$\begin{aligned} &u_{\infty,\varepsilon} \overset{*}{\rightharpoonup} v \text{ weakly-* in } W^{1,\infty}(\Omega) \text{ and} \\ &u_{\infty,\varepsilon} \to v \text{ uniformly in } \overline{\Omega}, \end{aligned} \tag{2.2}$$

as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon\to 0} \int_{\omega_\varepsilon} u_{\infty,\varepsilon} g_\varepsilon = \int_{\partial\Omega} vg\, d\sigma.$$

On the other hand, for every $z \in C^1(\overline{\Omega})$ it holds that

$$\lim_{\varepsilon\to 0} \int_{\omega_\varepsilon} g_\varepsilon z = \int_{\partial\Omega} gz\, d\sigma.$$

Hence, if we call

$$M = \max\left\{ \int_{\partial\Omega} wg\, d\sigma \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_\infty \le 1 \right\}, \tag{2.3}$$

we obtain, from (2.2),

$$M \le \liminf_{\varepsilon\to 0} M_\varepsilon = \int_{\partial\Omega} vg\, d\sigma.$$

We can conclude that $v = u_{\infty,0}$ is a maximizer of (2.3), as we wanted to prove.

Finally, let us prove that $u_{p,\varepsilon} \to u_{\infty,\varepsilon}$. Recall that

$$\int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon = \max\left\{ \int_{\omega_\varepsilon} wg_\varepsilon \colon w \in W^{1,p}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^p(\Omega)} \le 1 \right\}.$$

In particular, for any $q < p$

$$\|Du_{p,\varepsilon}\|_{L^q(\Omega)} \le \|Du_{p,\varepsilon}\|_{L^p(\Omega)} \left(|\Omega|^{\frac{p-q}{p}} \right)^{1/q} \le (|\Omega| + 1)^{\frac{p-q}{pq}}.$$

Hence, we can extract a subsequence (still denoted by $u_{p,\varepsilon}$) such that,

$$u_{p,\varepsilon} \to u, \qquad \text{uniformly in } \overline{\Omega},$$

as $p \to \infty$ with

$$\|Du\|_{L^q(\Omega)} \le (|\Omega| + 1)^{\frac{1}{q}}.$$

Letting $q \to \infty$ and using that $\|Du\|_{L^q(\Omega)} \to \|Du\|_{L^\infty(\Omega)}$ we get

$$\|Du\|_{L^\infty(\Omega)} \le 1.$$

Then we have

$$\int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon \to \int_{\omega_\varepsilon} u g_\varepsilon, \qquad \text{as } p \to \infty.$$

This limit u verifies that

$$\int_{\omega_\varepsilon} u g_\varepsilon \le \max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}.$$

Let us prove that we have an equality here. If not, there exists a function v such that $v \in W^{1,\infty}(\Omega)$, $\int_\Omega v = 0$, $\|Dv\|_{L^\infty(\Omega)} \le 1$ with

$$\int_{\omega_\varepsilon} u g_\varepsilon < \int_{\omega_\varepsilon} v g_\varepsilon.$$

If we normalize, taking $\varphi_p = v/|\Omega|^{1/p}$, we obtain a function in $W^{1,p}(\Omega)$ with $\int_\Omega \varphi_p = 0$, $\|D\varphi_p\|_{L^p(\Omega)} \le 1$ and such that

$$\lim_{p\to\infty} \int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon = \int_{\omega_\varepsilon} u g_\varepsilon < \int_{\omega_\varepsilon} v g_\varepsilon = \lim_{p\to\infty} |\Omega|^{1/p} \int_{\omega_\varepsilon} \varphi_p g_\varepsilon.$$

Note that

$$\int_{\omega_\varepsilon} \varphi_p g_\varepsilon \le \int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon,$$

for any p, and hence we arrive to a contradiction.

This contradiction proves that

$$\int_{\omega_\varepsilon} u g_\varepsilon = \max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}.$$

This ends the proof. □

Let us close this section with the following remark.

Remark. The limits of the solutions to the maximization problems (1.2) and (1.3) coincide with the limits of the solutions to the corresponding PDEs (1.6) and (1.8) when $p \to \infty$.

In fact, the unique maximizer of (1.2), u_p, is a weak solution to

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ |Du_p|^{p-2} \frac{\partial u_p}{\partial \nu} = \lambda_p \, g & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Here λ_p is a Lagrange multiplier. If we take

$$\tilde{u}_p = (\lambda_p)^{1/(p-1)} u_p$$

we get a solution to (1.6), that is,

$$\begin{cases} -\Delta_p \tilde{u}_p = 0 & \text{in } \Omega, \\ |D\tilde{u}_p|^{p-2} \frac{\partial \tilde{u}_p}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

From the weak form of (2.4) and our previous results we get

$$\lim_{p\to\infty} \lambda_p = \lim_{p\to\infty} \left(\int_{\partial\Omega} g u_p \, d\sigma \right)^{-1} = \left(\int_{\partial\Omega} g u_\infty \, d\sigma \right)^{-1} \neq 0.$$

Therefore,

$$\lim_{p\to\infty} \tilde{u}_p = \lim_{p\to\infty} (\lambda_p)^{1/(p-1)} u_p = \lim_{p\to\infty} u_p.$$

In a completely analogous way it can be proved that the limits as $p \to \infty$ of the solutions to the maximization problems (1.3) and solutions to the PDEs (1.8) coincide.

3. Proofs of Theorems 2 and 3

In this section we deal with the PDE version of the commutative diagram (1.5). To this end it is natural to consider solutions in the viscosity sense.

Following [4] let us recall the definition of viscosity solution for elliptic problems with general boundary conditions. Assume

$$F : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N\times N} \to \mathbb{R}$$

a continuous function. The associated equation

$$F(x, Du, D^2u) = 0$$

is called (degenerate) elliptic if

$$F(x, \xi, X) \leq F(x, \xi, Y) \qquad \text{if } X \geq Y.$$

Definition 3.1. *Consider the boundary value problem*

$$\begin{cases} F(x, Du, D^2u) = 0 & \text{in } \Omega, \\ B(x, u, Du) = 0 & \text{on } \partial\Omega. \end{cases}$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$, we have the inequality*

$$\max\{B(x_0, \phi(x_0), D\phi(x_0)),\ F(x_0, D\phi(x_0), D^2\phi(x_0))\} \geq 0$$

and if $x_0 \in \Omega$ then we require

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\psi \in C^2(\overline{\Omega})$ such that $u-\psi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\min\{B(x_0, \psi(x_0), D\psi(x_0)),\ F(x_0, D\psi(x_0), D^2\psi(x_0))\} \le 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, D\psi(x_0), D^2\psi(x_0)) \le 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

In the sequel, we will use the same notation as in the definition: ϕ stands for the test functions touching from below the graph of u, and ψ stands for the test functions touching from above the graph of u.

First, we point out a lemma that is implicit in the arguments in [5] (see Propositions 5.1 and 5.2 in [5]) and in [8].

Lemma 3.2. *The extremal functions in Theorem 1 satisfy in the viscosity sense:*

$$|Du_{\infty,\varepsilon}| \le 1\,; \quad |Du_{\infty,0}| \le 1\,; \quad -|Du_{\infty,\varepsilon}| \ge -1\,; \quad -|Du_{\infty,0}| \ge -1\,.$$

On the other hand, at level p we can pass from weak solutions to solutions in the sense of viscosity:

Lemma 3.3. *Let $u_{p,0}$ be a continuous weak solution of* (1.6) *for $p > N$. Then $u_{p,0}$ is a viscosity solution to*

$$\begin{cases} -\Delta_p u_{p,0} = 0 & \text{in } \Omega, \\ |Du_{p,0}|^{p-2} \frac{\partial u_{p,0}}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

Proof. See [9], Lemma 2.3. □

Proof of Theorem 2. We decompose the proof in several steps.

Step 1. First, assume that a sequence of viscosity solutions to (1.6), $u_{p,0}$ converge, as $p \to \infty$, uniformly in Ω to a limit $u_{\infty,0}$, then it is proved in [9] that $u_{\infty,0}$ is a viscosity solution to (1.7).

Step 2. As we have mentioned in the introduction at this part of the proof we have to deal with weak solutions since the right hand side of the problem (1.8) is not uniformly bounded in ε.

Assume that we have a sequence of weak solutions to (1.8), $u_{p,\varepsilon}$ that converge, as $\varepsilon \to 0$, uniformly in Ω to a limit $u_{p,0}$ then let us prove that $u_{p,0}$ is a weak (and a viscosity) solution of (1.6).

As it was proved in the previous section, we can pass to the limit in the variational formulation (that is equivalent to the weak formulation) of (1.8) to obtain that every limit of $u_{p,\varepsilon}$ is a variational solution (and hence a weak solution) to (1.6).

To finish we just have to observe that a continuous weak solution to (1.6) is in fact a viscosity solution, thanks to Lemma 3.3. □

Next, we pass to the proof of Theorem 3.

Proof of Theorem 3. We recall that, thanks to lemma 3.2, we have the estimates $|Du_{\infty,\varepsilon}| \leq 1$; $|Du_{\infty,0}| \leq 1$, $-|Du_{\infty,\varepsilon}| \geq -1$; $-|Du_{\infty,0}| \geq -1$, in the sense of viscosity, in all the domain Ω.

To find the equation that $u_{\infty,\varepsilon}$ satisfies in the viscosity sense, assume that $u_{\infty,\varepsilon} - \phi$ has a strict minimum at $x_0 \in \Omega$. Depending on the location of the point x_0 we have different cases. First, suppose that $x_0 \in \Omega \setminus \omega_\varepsilon$. By the uniform convergence of $u_{p_i,\varepsilon}$ to $u_{\infty,\varepsilon}$ there exists points x_{p_i} such that $u_{p_i,\varepsilon} - \phi$ has a minimum at x_{p_i} with $x_{p_i} \in \Omega \setminus \omega_\varepsilon$ for p_i large. Using that u_{p_i} is a viscosity solution to (1.8) we obtain

$$-\Delta_p \phi(x_{p_i}) = -\mathrm{div}(|D\phi|^{p_i-2}D\phi)(x_{p_i}) \geq 0.$$

Therefore

$$-(p_i-2)|D\phi|^{p_i-4}\Delta_\infty \phi(x_{p_i}) - |D\phi|^{p_i-2}\Delta\phi(x_{p_i}) \geq 0.$$

If $D\phi(x_0) = 0$ we get $-\Delta_\infty \phi(x_0) = 0$. If this is not the case, we have that $D\phi(x_{p_i}) \neq 0$ for large i and then

$$-\Delta_\infty \phi(x_{p_i}) \geq \frac{1}{p_i-2}|D\phi|^2\Delta\phi(x_{p_i}) \to 0, \quad \text{as } p_i \to \infty.$$

We conclude that

$$-\Delta_\infty \phi(x_0) \geq 0. \tag{3.1}$$

That is $u_{\infty,\varepsilon}$ is a viscosity supersolution of $-\Delta_\infty u_{\infty,\varepsilon} = 0$ in $\Omega \setminus \omega$.

The fact that it is a viscosity subsolution of $-\Delta_\infty u_{\infty,\varepsilon} = 0$ in $\Omega \setminus \omega$ is completely analogous.

Next suppose that x_0 lies on the region where $g_\varepsilon > 0$.

Assume that we have a test function ϕ touching from below the graph of $u_{\infty,\varepsilon}$, that is, $u_{\infty,\varepsilon} - \phi$ has a strict minimum at the point x_0 . Then, for p_i large, there exist points x_{p_i} such that $u_{p_i,\varepsilon} - \phi$ has a minimum at x_{p_i} with $g_\varepsilon(x_{p_i}) > 0$. Using that u_{p_i} is a viscosity solution to (1.8) we obtain

$$-(p_i-2)|D\phi|^{p_i-4}\Delta_\infty \phi(x_{p_i}) - |D\phi|^{p_i-2}\Delta\phi(x_{p_i}) \geq g_\varepsilon(x_{p_i}) > 0.$$

In particular $|D\phi(x_{p_i})| \neq 0$ and therefore

$$-\Delta_\infty \phi(x_{p_i}) \geq \frac{1}{p_i - 2}|D\phi|^2 \Delta\phi(x_{p_i}) + \frac{g_\varepsilon(x_{p_i})}{(p-2)|D\phi(x_{p_i})|^{p-4}}.$$

Since $g_\varepsilon(x_{p_i}) \to g(x_0) > 0$, this means that $|D\phi(x_0)|$ cannot be smaller than 1 (in this case the right hand side of the inequality tends to infinity). Therefore, we conclude that $|D\phi(x_0)| \geq 1$.

Assume now that we have a test function ψ that touches from above the graph of $u_{\infty,\varepsilon}$. Then, for p_i large, there exists points x_{p_i} such that $u_{p_i,\varepsilon} - \psi$ has a minimum at x_{p_i} with $g_\varepsilon(x_{p_i}) > 0$. Using that u_{p_i} is a viscosity solution to (1.8) we obtain

$$-(p_i - 2)|D\psi|^{p_i-4}\Delta_\infty \psi(x_{p_i}) - |D\psi|^{p_i-2}\Delta\psi(x_{p_i}) \leq g_\varepsilon(x_{p_i})(> 0).$$

Then, if $|D\psi(x_0)| > 1$, it follows that $-\Delta_\infty \psi(x_0) \leq 0$. Therefore, the condition on ψ reads

$$\min\{|D\psi(x_0)| - 1, -\Delta_\infty \psi(x_0)\} \leq 0. \tag{3.2}$$

But notice that this condition is always satisfied, since we know that $|Du_{\infty,\varepsilon}| \leq 1$ in the sense of viscosity. Therefore, (3.1) and (3.2) imply that, if $g_\varepsilon(x_0) > 0$, then $|Du_{\infty,\varepsilon}(x_0)| = 1$.

Similar computations give us that if we look at a point x_0 such that $g_\varepsilon(x_0) < 0$ then $-|Du_{\infty,\varepsilon}(x_0)| = -1$.

The next case to consider, is when $g_\varepsilon(x_0) = 0$ and the point x_0 can be reached as a limit of points x_{p_i} that could be contained in the region $\{g_\varepsilon > 0\}$ or in the region $\{g_\varepsilon = 0\}$. In other words, $x_0 \in \Omega \cap \partial\{g_\varepsilon > 0\} \cap (\partial\{g_\varepsilon < 0\})^C$.

In this case, if we consider a test function ϕ touching from below the graph of $u_{\infty,\varepsilon}$ at x_0, then we get a sequence $\{x_{p_i}\}$ converging to x_0, such that $u_{p_i,\varepsilon} - \phi$ has a strict minimum at x_{p_i}. Passing to a subsequence if necessary, we have two possibilities: either $g_\varepsilon(x_{p_i}) = 0$, or $g_\varepsilon(x_{p_i}) > 0$. If we assume $g_\varepsilon(x_{p_i}) = 0$, then

$$-(p_i - 2)|D\phi|^{p_i-4}\Delta_\infty \phi(x_{p_i}) - |D\phi|^{p_i-2}\Delta\phi(x_{p_i}) \geq 0.$$

Then, if $|D\phi(x_{p_i})| \neq 0$ it follows that $-\Delta_\infty \phi(x_0) \geq 0$. On the other hand, if $|D\phi(x_{p_i})| = 0$ for infinitely many indexes, then $-\Delta_\infty \phi(x_0) = 0$.

If we assume $g_\varepsilon(x_{p_i}) > 0$, then $|D\phi(x_{p_i})| \neq 0$ and therefore passing to the limit we get $-\Delta_\infty \phi(x_0) \geq 0$.

Concerning the test functions ψ touching from above the graph of $u_{\infty,\varepsilon}$, when $g_\varepsilon(x_{p_i}) = 0$, then we have

$$-(p_i - 2)|D\psi|^{p_i-4}\Delta_\infty \psi(x_{p_i}) - |D\psi|^{p_i-2}\Delta\psi(x_{p_i}) \leq 0.$$

This implies that $-\Delta_\infty \psi(x_0) \le 0$. But if $g_\varepsilon(x_{p_i}) > 0$, then, as in a previous case, we get that $\min\{|D\psi(x_0)| - 1, -\Delta_\infty \psi(x_0)\} \le 0$, and this condition is always satisfied because $|Du_{\infty,\varepsilon}| \le 1$.

As a conclusion, if $x_0 \in \Omega \cap \partial\{g_\varepsilon > 0\} \cap (\partial\{g_\varepsilon < 0\})^C$, we have in the sense of viscosity that $-\Delta_\infty u_{\infty,\varepsilon} \ge 0$ (jointly with the general viscosity estimates on the gradient, valid in all Ω).

In an analogous way, if $x_0 \in \Omega \cap (\partial\{g_\varepsilon > 0\})^C \cap \partial\{g_\varepsilon < 0\}$, we have in the sense of viscosity that $-\Delta_\infty u_{\infty,\varepsilon} \le 0$ (jointly with the general viscosity estimates on the gradient, valid in the whole domain Ω).

The next region consists on the points $x_0 \in \Omega$ that can be reached as limits of sequences contained either in $\{g_\varepsilon > 0\}$, either in $\{g_\varepsilon = 0\}$, either in $\{g_\varepsilon < 0\}$. That is, $x_0 \in \Omega \cap \partial\{g_\varepsilon > 0\} \cap \partial\{g_\varepsilon < 0\}$. The same arguments as before give us that in this set the equation satisfied in the sense of viscosity is simply $|Du_{\infty,\varepsilon}| \le 1$ and $-|Du_{\infty,\varepsilon}| \ge -1$.

Finally, the boundary condition satisfied by $u_{\infty,\varepsilon}$ in the sense of viscosity is

$$\frac{\partial u_{\infty,\varepsilon}}{\partial \nu} = 0.$$

To see this fact, we use that the p-Laplacian satisfies hypothesis of the "strict monotonicity in the direction of the normal" as it is stated in [4]. Then, for instance, the boundary condition at level p reads simply

$$|D\phi(x_p)|^{p-2} \frac{\partial \phi}{\partial \nu}(x_p) \ge 0,$$

for any test function ϕ touching the graph of $u_{p,\varepsilon}$ from below at a point $x_p \in \partial\Omega$. Test functions touching the graph from above can be handled in a similar way.

The last step of the proof consists of taking limits on the sequence $\{u_{\infty,\varepsilon}\}$. Notice that these functions, as limits of the sequence $\{u_{p,\varepsilon}\}$ as $p \to \infty$, satisfy the extremal property (1.4). Therefore, by Theorem 1, the limit $u_{\infty,0}$ is an extremal of (1.1). From hypothesis $\{g = 0\}^o = \emptyset$, we obtain a uniqueness result for $u_{\infty,0}$ (see [9]), and then it must be the same function that we reach as the limit of the sequence $\{u_{p,0}\}$ when $p \to \infty$, and, as it was proved in [9], it satisfies (1.7). □

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