Selective and Efficient Estimation of Parameters for Quantum Process Tomography

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We present a new method for quantum process tomography enabling the efficient estimation, with fixed precision, of any of the parameters characterizing a quantum channel. The estimation strategy depends upon the set of matrix elements one selects to estimate. Furthermore, we describe a way to efficiently gather all the information required to efficiently estimate any average survival probability of the channel (i.e., to measure any diagonal element of its χ matrix).

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The efficient characterization of the temporal evolution of a quantum system is one of the main tasks one needs to accomplish to achieve quantum information processing. The set of methods used to determine the evolution of a system is generically denoted as quantum process tomography. In general, quantum process tomography is a hard task whose completion requires resources scaling exponentially with the number of qubits in the system (n). In fact, under general assumptions the evolution of the quantum state can be represented by a linear, completely positive, trace preserving map \( \rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}}) \). Choosing a basis of \( D^2 \) operators \( \{ E_m \} \), the map can be written as

\[
\mathcal{E}(\rho) = \sum_{nm} \chi_{nm} E_n \rho E_m^\dagger \quad \text{with} \quad \sum_{nm} \chi_{nm} E_n^\dagger E_m = I.
\]

(1)

Thus, the map is completely characterized by the positive Hermitian matrix \( \chi_{nm} \) that must also satisfy the above trace preserving condition. Therefore, fully characterizing the map requires \( D^4 - D^2 \) real parameters (where \( D = 2^n \) is the dimension of the Hilbert space of \( n \) qubits). However, the exponentially large number of coefficients required is not the main problem: Existing methods require resources scaling exponentially with \( n \) in order to estimate a single coefficient \( \chi_{nm} \).

In this Letter we present a method that enables us to evaluate any coefficient of the \( \chi \) matrix with resources that scale as a polynomial of the number of qubits. The method is “selective” because for each coefficient there is a different estimation strategy, which is always efficient (with fixed precision). For this reason we denote this as selective and efficient estimation of parameters for quantum process tomography. In a further development, we propose a related strategy that provides all the necessary information to estimate any diagonal \( \chi_{nm} \) coefficient using resources that scale as a polynomial of \( n \). This second strategy is non-selective as all the coefficients can be estimated by using results from the same experiments.

First, it is worth reviewing the properties of the main tomographic schemes. Standard quantum process tomography was the first method proposed [1]. It involves preparing a set of input states \( \rho_k \) and performing full quantum state tomography on the output states. By doing this, we measure coefficients \( \chi_{jk} = \text{Tr}(\rho_k \mathcal{E}(\rho_j)) \). However, finding matrix elements \( \chi_{nm} \) requires inverting an exponentially large system of equations relating \( \lambda \) with \( \chi \) [2]. Thus, this method is not only indirect but also inefficient: To estimate any coefficient \( \chi_{nm} \) one needs an exponentially large number of experiments and classical postprocessing.

Direct characterization of quantum dynamics [3,4] requires an ancillary system of \( n \) clean qubits with a clean quantum channel. If such resource is available, the method enables the direct estimation of all diagonal \( \chi_{nm} \) by performing joint Bell measurements on the system plus ancilla ensemble. The method requires a relatively small number of elementary quantum gates \([O(n)]\). However, using direct characterization of quantum dynamics to estimate off-diagonal coefficients \( \chi_{nm'} \) requires the inversion of a system of equations which in the most general case is exponentially large. Thus, for this purpose, the method is inefficient.

Symmetrized characterization of noisy quantum processes [5] (SCNQP) transforms the channel \( \mathcal{E} \) into a symmetrized \( \mathcal{E}' \) via twirling operations, requiring \( O(n) \) single qubit gates with constant depth. After symmetrization, only diagonal \( \chi_{nm} \) coefficients remain, being the averages over the original coefficients of the same Hamming weight. The averaged coefficients are linearly related to output probabilities through an upper diagonal square matrix of size \( n + 1 \). The method is ideally tailored for evaluating the applicability of quantum error correcting codes [6]. However, it is not possible to estimate any of the off-diagonal \( \chi_{nm'} \) coefficients, which are wiped out by the symmetrization protocol, nor to distinguish among specific Pauli errors of the same Hamming weight.

Our method has a similar flavor to SCNQP, adding the possibility to determine any coefficients \( \chi_{nm'} \) with polynomial resources. It is based on two observations: The first is the fact that any matrix element \( \chi_{nm'} \) can be related to an average survival probability of input states under the action
of the channel (or a related quantity as described below). The average involved here is over the entire Hilbert space using the Haar measure. The second observation is that such averages can be efficiently estimated by sampling over a finite set of states (a 2-design, as described below).

Before describing these two crucial observations, we point out that a choice of basis to define the $X_{mm'}$ matrix is required. We use an operator basis $\{E_m\}$ satisfying $\text{Tr}(E_mE_{m'}^\dagger) = \delta_{m,m'}$ (orthonormality) and $E_mE_m^\dagger = I$ (unitarity). We also assume $E_0 = I$. A convenient choice is the basis of Pauli operators $\{P_m\}$, which are decomposable as tensor products and have a convenient group structure: $P_mP_{m'} = i^{\delta_{m,m'}}P_{mm'}$, where $\ast$ is the commutative group operation.

Here we describe a tomographic method to determine the $\chi$ matrix on a unitary basis $E_m$. This type of tomography is desirable from the point of view of gathering dynamical information about the quantum process and designing error correction strategies. However, it must be pointed out that selective tomographic schemes to determine matrix elements of $\mathcal{E}$ in nonunitary basis exist. For the Choi matrix (where the operator basis is of the form $|k\rangle\langle j|$ for computational states $|k\rangle$, $|j\rangle$) they are described in [1].

We start by noticing that the average fidelity of the map $\mathcal{E}$ is defined as [7]

$$F(\mathcal{E}) = \int \langle \psi \Omega | \mathcal{E}(\psi) \Omega | \psi \rangle d\psi$$

(2)

$F(\mathcal{E})$ is nothing but the survival probability averaged over all pure states $|\psi\rangle$, which turns out to be directly related with the coefficient $\chi_{00}$. To see this, we should notice that for any operators $O_1$ and $O_2$ [8]

$$\int \langle \psi | O_1P_\psi O_2 | \psi \rangle d\psi = \frac{\text{Tr}(O_1)\text{Tr}(O_2) + \text{Tr}(O_1O_2)}{D(D+1)},$$

(3)

where $P_\psi = |\psi\rangle\langle \psi|$ is the projector on state $|\psi\rangle$. Using this, the $\chi$-matrix representation for the map $\mathcal{E}$, and the definition of the average fidelity (2), we find $F(\mathcal{E}) = \chi_{00} + \frac{1}{D+1}$. All other diagonal coefficients $\chi_{mm}$ are related to average fidelities of slightly modified channels. In fact, for the channel $\mathcal{E}_m(\rho) = E_m^\dagger \rho E_m$ we get

$$F(\mathcal{E}_m) = \int \langle \psi | E_m^\dagger\mathcal{E}(P_\psi)E_m | \psi \rangle d\psi = \frac{D\chi_{mm} + 1}{D+1}.$$  

(4)

Note that all the above fidelities have a lower bound of $1/(D+1)$. To measure the diagonal coefficients $\chi_{mm}$ we must perform the experiment described in Fig. 1 and average over all states $|\psi\rangle$.

The off-diagonal elements $\chi_{mm'}$ can also be related to average quantities by using the following identity:

$$\int \langle \psi | E_m^\dagger P_\psi E_{m'} | \psi \rangle d\psi = \frac{D\chi_{mm'} + \delta_{mm'}}{D+1}.$$  

(5)

This equation can be obtained by using the $\chi$ representation for the map $\mathcal{E}$, together with Eq. (3) and the trace preserving condition for $\mathcal{E}$:

$$\sum_{m,m'} \chi_{mm'} \text{Tr}(E_m^\dagger E_{m'} E_l) = \text{Tr}(E_l E_l) = D \delta_{m,l}.$$  

(6)

To measure the complex off-diagonal coefficients $\chi_{mm'}$ we can proceed as follows: We add an extra clean qubit and consider the map $\mathcal{E}_{mm'}$ to be the one described by the circuit in the dashed box of Fig. 2.

Denoting the controlled $E_m$ operation as $C - E_m$, we have $\mathcal{E}_{mm'}(\sigma) = \mathcal{E}(C - E_m^\dagger C - E_m^\dagger \sigma H \sigma C - E_m C - E_m)$, where $\sigma$ is the joint state of the ancillary qubit and the original system. Thus, using (5) we realize that $\text{Re}(\chi_{mm'})$ can be obtained from the polarization of the ancillary qubit (i.e., the expectation value of $\sigma_2$ conditioned on the survival of the state $|\psi\rangle$, averaged over all states $|\psi\rangle$). This is because

$$\frac{D\text{Re}(\chi_{mm'}) + \delta_{mm'}}{D+1}. $$

(7)

If instead of measuring the expectation of $\sigma_2 \otimes P_\psi$ we measure the expectation value of $\sigma_2 \otimes P_\psi$, the average over $|\psi\rangle$ yields $\chi_{0m}$. In summary, so far we showed that all matrix elements $\chi_{mm'}$ can be directly related to average survival probabilities. In fact, the diagonal coefficients $\chi_{mm}$ are related to average fidelities of the channel $\mathcal{E}_m$. The off-diagonal coefficients are related to the average polarization of the ancillary qubit conditioned to the survival of the state after the evolution with the channel $\mathcal{E}_{mm'}$.

Experimentally measuring these averages over the Hilbert space seems completely unrealistic. However, the exceptional recent work on the theory of 2-designs [9–12] provides the means for doing so. Delsarte, Goethals, and Seidel [13] showed how integrating polynomials on the sphere could be reduced to averaging the integrand on a finite set of points coined spherical designs. The same idea can be extended to integrals over the entire Hilbert space. Here, we need only a state 2-design $X$ that satisfies

$$\int |\psi\rangle O_1 P_\psi O_2 |\psi\rangle d\psi = \frac{1}{|X|} \sum_{\psi \in X} \langle \psi | O_1 P_\psi O_2 |\psi\rangle.$$  

(8)

FIG. 1. Circuit for measuring $\chi_{mm}$ for a channel $\mathcal{E}$.

FIG. 2. Circuit for measuring $\text{Re}(\chi_{mm'})$ for a channel $\mathcal{E}$.
for all operators $O_1$, $O_2$. Thus, averaging over Hilbert space is equivalent to averaging over the finite set $X$. A state 2-design with a finite (but exponentially large) number of states exists. Although the computation of the exact average over the 2-design is still exponentially hard, it is now possible to efficiently obtain an estimate for the average: This average can be estimated by randomly sampling over initial states $|\psi\rangle$ chosen from the set $X$. This is the final piece of our method.

Luckily, it is simple to find a state 2-design for $n$ qubits. One possibility is using the $D + 1$ mutually unbiased bases (MUBs), which automatically form a state 2-design [2]. Each basis will be labeled with $J = 0, \ldots, D$ and the states within each basis will be labeled with $m = 1, \ldots, D$. In order for the orthonormal bases to be unbiased, the $D(D + 1)$ states of the MUBs must satisfy $|\langle \psi_m^J, \psi_n^K \rangle|^2 = \frac{1}{D}$ for all $J \neq K$. Since generalized Pauli operators may be partitioned into $D + 1$ maximally sets of $D$ commuting operators so that each pair of sets hold only the identity as a common element [14], there are $D + 1$ MUBs, each one diagonalizing one of the commuting subsets of Paulis [15]. The states of these MUBs cannot only be efficiently described in this way but they can also be efficiently generated with circuits containing $O(n^2)$ one and two qubit gates [16]. But one could also use other types of 2-designs. For example, Dankert et al. [9,10] propose using approximate unitary 2-designs (i.e., designs on the space of unitary operators) showing that they can be efficiently approximated. Approximate unitary 2-design with accuracy $\epsilon = 1/D^2$ can be obtained by employing only $O(n \log^2 \frac{1}{\epsilon})$ gates. Unitary 2-designs acting on any fixed state induce state 2-designs fitting into the previous scheme. Dually the action of the random unitaries may be interpreted as symmetrizing the channel $E$ through twirling. Following this line, we may also use weaker symmetrization protocols as in SCNQP [5] for estimating fidelities of modified channels (1, 2).

Let us summarize the complete method. The estimation of diagonal coefficients $\chi_{mm'}$ of the quantum map $E$ can be efficiently done by evaluating the fidelity of the channel $E_m$ averaged over a random sample of a state 2-design. The real and imaginary parts of the off-diagonal elements $\chi_{mm'}$ are obtained from the $x$ and $y$ polarizations of the ancillary qubit conditioned on the survival of the states randomly chosen from the same 2-design, when the evolution of the combined system + ancilla is described by the map $E_{mm'}$. The strongest requirements for this method are the production of states of the 2-design and the implementation of the controlled Pauli operators used to create the $E_{mm'}$ channels.

It is worth emphasizing the main virtue of this method: Estimating any $\chi_{mm'}$ requires resources that depend polynomially on the number of qubits. In fact, to estimate fidelities $F(E_m)$ we average results of experiments with binary outcomes (i.e., survival or no survival). So Chernoff bounds apply to this problem giving a lower bound to the number of experiments $M$ required to obtain an estimation error lower than $\epsilon$ with probability $p$. Thus, Chernoff bounds implies that $M$ must satisfy the condition $M \geq \ln[2(1 - p)^{-1}]/(2\epsilon^2)$. Therefore the required number of experiments does not increase with the number of qubits.

Only the gate complexity of each experiment has a polynomial dependence on $n$ as $O(n^2)$ (logarithmic in $D$). The same argument applies to the estimation of the off-diagonal coefficients in a simple way: When measuring quantities smaller than $D^{m+1}$, the experiment has an extra outcome arising from measurement of the state of the auxiliary qubit described above. Therefore in this case the same accuracy requires multiplying the number of experiments by 4.

The above method can be extended in an interesting way: We will show below that the estimation of any diagonal coefficient $\chi_{mm'}$ can be efficiently done by performing an experiment which is independent of the coefficient one wants to estimate. In turn, the experiment does depend on the operator basis used to define the $\chi$ matrix of $E$. Thus, the crucial step is to prepare initial states in the 2-design formed by the MUBs associated to the operator basis $E_m$ and to measure transition probabilities between such states (and not only survival probabilities as before).

To fix these ideas, we assume from now on that $E_m$ are Pauli operators. Each one of the MUBs is determined by $n$ commuting Pauli operators $P_1^n, \ldots, P_n^n$, which are the generators of the Abelian group of operators that are diagonal in that particular basis. We can label the basis elements $|\psi_1^J\rangle$ according to the eigenvalues of the $n$ generators. Thus, the label $k$ is an $n$-component binary vector: the $i$th component $k_i$ determines the $\pm 1$ eigenvalue of $P_i^k$ as $P_i^k |\psi_1^J\rangle = (-1)^{k_i} |\psi_1^J\rangle$. The action of any Pauli $P$ on the state $|\psi_1^J\rangle$ transforms it into another state of the same basis $J$. Those transition rules are fully determined by the binary vector $p$ that encodes the commutation pattern between $P$ and the generators of the $J$th basis [i.e., the vector $p$ is such that $(-1)^p PP_i^k = P_i^k P^p$]. Thus, $P^p |\psi_1^J\rangle \approx (\phi_1^{J+p})$ (up to a phase), where $k$ and $p$ are added bitwise modulo 2. Hence, the probability to detect the state $|\psi_1^{J+p}\rangle$ in the final measurement after evolving with any unitary operator $U$ is identical to the probability of detecting the state $|\psi_2^J\rangle$ after evolving with the operator obtained as the product $P \times U$. Applying this observation we conclude that in order to measure any diagonal coefficient $\chi_{mm'}$ we could modify the strategy we described above by removing the final $E_m$ gate proceeding as follows: We compute $\chi_{mm'}$ by averaging the transition probability to the state $|\psi_2^{J+p}_{mm'}\rangle$ whenever the
state $|\psi_E^k\rangle$ is prepared ($p_m$ is the commutation vector of $E_m$ and the generators of the basis $J$).

The procedure to estimate any diagonal $\chi_{mm}$ coefficient must be clear now. The associated fidelity $F(E_m)$ is estimated as the relative frequency of the events $(J, k, k')$ that satisfy the condition $k + k' = p_m$ (where $p_m$ is the commutation vector of the operator $E_m$ and the generators of the basis $J$). These are the $m$-type events. Performing this check requires knowing the vector $p_m$ [which requires $O(n^2)$ classical operations]. The classical complexity for this process is $O(n^2M)$. However, the most demanding aspect of our proposal is the generation and measurement of MUB states requiring $O(n^2)$ single and 2-qubit elemental quantum gates. Errors corresponding to the estimations of the different $\chi_{mm}$ are correlated. However, the variance of any such estimator behaves in the same way as the one corresponding to the selective method to evaluate a single $\chi_{mm}$ at a time.

We can also use this method to devise a test to efficiently detect those coefficients $\chi_{mm}$ with values that are above a certain $D$-independent threshold (i.e., the threshold cannot be exponentially small). The idea is that if the coefficient $\chi_{mm}$ is large, then the condition $k + k' = p_m$ must be satisfied frequently enough in the experimental data. Given the triplets corresponding to two experiments performed on a different basis, we may efficiently determine the unique operator $E_m$ satisfying the condition for both of them [using $O(n^3)$ operations]. The number of such pairs of triplets is bounded by $M(M - 1)/2$. Therefore we can find out the operators $E_m$ satisfying the condition $k + k' = p_m$ for at least two bases $J$ with an overhead that depends at most quadratically on $M$. All the operators satisfying such criterion will be obtained with $O(M^2n^3)$ classical operations. After sieving the operators in this way we can focus on those that passed the test to study them further. The conclusion is that in this way, it is not only possible to efficiently identify but also to efficiently estimate all large $\chi_{mm}$ coefficients. This method enables us to determine if the coefficients are larger than a fixed $D$-independent threshold. The number of experiments required to estimate all $\chi_{mm}$ larger than $\epsilon$ with probability $p$ is still independent of the number of qubits $n$ [it is a factor $\log(1/\epsilon)$ larger than the one required for estimating a single coefficient]. We may also simultaneously estimate groups of off-diagonal $\chi_{m'n'}$ coefficients for which $E_{m'}E_m$ is the same. The required bookkeeping is similar to the one used for diagonal coefficients but requires additional care regarding the numerical phases that arise.

We have shown how any particular $\chi_{m'n'}$ coefficient for a channel $E$ may be estimated from the average survival probabilities of at most two modified channels requiring at most one extra clean qubit. This approach, together with the use of randomized fidelity measurement schemes, is the first to allow the efficient estimation of any desired $\chi_{m'n'}$ coefficient investing resources that depend as a polynomial on the number of qubits. The method shows its strength when only partial tomography is desired, since the resources required are polynomial in the number of subsystems (see [17] for an analysis of resources required by other methods). We further presented an extension requiring preparation and detection of random initial states belonging to the MUBs associated with the operator basis used in the channel representation. This allows us to profit from the information contained in transition probabilities to estimate any diagonal coefficient $\chi_{mm}$. Finally, we proved the possibility of efficiently characterizing Pauli channels whose $\chi$ matrices have few coefficients above a $D$-independent threshold.

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