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COMPRESSIBILITY EFFECTS ON THE GRAVITATIONAL INSTABILITY OF A PLASMA-VACUUM INTERFACE

ALEJANDRO G. GONZALEZ* and JULIO GRATTON†
Laboratorio de Fisica del Plasma, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina
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Abstract—The stability of gravitational compressible surface modes of a plasma-vacuum interface is investigated. Stratified equilibrium profiles of density and magnetic field in the plasma are considered. The corresponding boundary conditions for magnetohydrodynamic linear perturbations are deduced. Three types of surface normal modes (slow, intermediate and fast) may appear. The slow mode is unstable below a critical wavenumber. The stability criterion is affected in general by the compressibility. The growth rate of the unstable modes is increased by compressibility. It is shown that for "flute" or pure interchange modes with respect to the vacuum magnetic field (but not with respect to the equilibrium magnetic field in the plasma) the compressibility enlarges the instability domain of surface modes. The interval of existence of unstable surface perturbations, as well as their growth rates, are larger than those of the internal unstable modes. Intervals of non-existence of surface modes appear for the flute perturbations with respect to the plasma magnetic field. The critical values of the wavenumber below which the modes are unstable, are discussed in general.

1. INTRODUCTION

In this paper we continue the study of the gravity modes of a compressible plasma; the first part, dealing with internal modes (i.e. modes that can be excited even if the plasma is bounded by rigid conducting walls, and that can propagate within the plasma) was published in a previous paper (GRATTON et al., 1988, hereafter referred to as Paper I). There we derived stability criteria and properties of the internal modes and found that compressibility has a destabilizing effect. Here we study surface modes, that is, perturbations which involve motion of a plasma-vacuum interface and are localized in its neighbourhood.

The stability of gravity modes in accelerated plasmas is relevant in many situations which include laboratory experiments as well as astrophysical plasmas. We can mention the cases of the θ-pinch (AMINI, 1985), inertial fusion (TAKABE et al., 1985; BODNER et al., 1987), compression of thin foils (PARKS, 1983), and the plasma focus (BILBAO and BRUZZONE, 1984) among the experiments; a review oriented to the astrophysical problems can be found in PRIEST (1983); additional references can be found in Paper I, and in GONZALEZ (1988). The emphasis of the research on the role of compressibility has been mainly limited to the consideration of internal modes, see for instance the classical paper of NEWCOMB (1961), and more recently PRITCHETT et al., (1978), PRIEST (1983), and Paper I (in which a summary of the extensive literature on the theory of the Rayleigh–Taylor instability, in particular with reference to the role of compressibility, as well as additional references, can be found).

When interfaces (plasma–vacuum, or plasma–plasma) are present, as is the case

* Fellow of the Argentine Consejo Nacional de Investigaciones Científicas y Técnicas.
† Member of the Argentine Consejo Nacional de Investigaciones Científicas y Técnicas.
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for many experiments like those mentioned above, surface modes (i.e. perturbations that are localized near the interface and require its motion) can be excited. Magnetohydrodynamic surface waves in compressible plasmas without gravity have been discussed by Wentzel (1979), Roberts (1981), Rae and Roberts (1983) and Uberoi and Narayanan (1986). The instability of surface gravity modes in stratified, incompressible configurations has been studied by Mikaelian (1982), see also Gupta and Lawande (1986), in the absence of magnetic field. The effect of compressibility has been investigated by Plesset and Hsieh (1964), Baker (1983) and Bernstein and Book (1983) for ordinary fluids, and by Parks (1983) for a plasma slab with no internal magnetic field. Tasi et al. (1981) and Powell (1986) have considered isothermal perturbations of accelerated plasma slabs with an internal magnetic field, for special orientations of the wavenumber; their configurations are different from that considered here, and their equations cannot easily be solved and require numerical analysis for the particular cases considered.

In this paper we derive the dispersion relation of the gravity surface modes of a plasma-vacuum interface. In the unperturbed state the plasma is isothermally stratified and is supported against gravity by a vacuum magnetic field. The magnetic field inside the plasma, and that in the vacuum, are parallel to the interface but are otherwise arbitrarily oriented. We examine in detail the limiting case of very small compressibility, and show that the domain of unstable wavenumbers, and the growth rate of the instability, are increased with respect to the incompressible case, as happens with internal modes. For the general case of finite compressibility we derive the stability criteria, and present a graphical method of analysis that allows us to find the solutions for any case of interest. Finally we analyze the cases of flute perturbations, both with respect to the vacuum magnetic field, or to the plasma magnetic field. Other properties of surface modes are also discussed.

2. THE MODEL AND BASIC EQUATIONS

We shall study a model based on ideal MHD in the presence of a uniform constant gravity field, so that the principal equations are given by:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]

\[ \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \rho \mathbf{g} - \frac{1}{4\pi} \mathbf{B} \times (\nabla \times \mathbf{B}), \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \]

(1)

Here, \( \mathbf{g} = -\rho \mathbf{e} \), denotes gravity, \( \mathbf{B} \) the magnetic field, \( \rho \) the density, \( \mathbf{v} \) the velocity and \( p \) the pressure. The unperturbed configuration is as follows; the plasma fills the region \( y > 0 \), while the half space \( y < 0 \) corresponds to the vacuum and has a uniform horizontal magnetic field \( \mathbf{B} \), which supports the plasma; the plasma is assumed to be stratified in \( y = \text{const.} \) planes, there is no mass flow, and the magnetic field is everywhere horizontal. The equilibrium condition is then given by:
We shall assume in addition that the equilibrium is isothermal and that $p, \rho, B$ decrease exponentially with $y$ in the plasma region:

$$p = p_0 e^{-2\gamma y}, \quad \rho = \rho_0 e^{-2\gamma y}, \quad B = B_0 e^{-\gamma y}$$

so that the sound velocity $C_s = (\gamma p/\rho)^{1/2}$, where $\gamma$ is the adiabatic exponent, and the Alfvén velocity $C_A = B/(4\pi \rho)^{1/2}$ are constant in space. Here $1/2\gamma$ is the effective height of the profile. The equilibrium condition (2) can be expressed as:

$$\frac{1}{\gamma} + \frac{1}{2} \frac{M_A^2}{C_s^2} = \frac{g}{2qC_s^2} = \hat{g} \quad M_A = C_A/C_s.$$  

We assume that $B$, the unperturbed magnetic field in the plasma, is unidirectional but not necessarily parallel to $B_v$, the vacuum magnetic field. The fact that the adiabatic exponent $\gamma$ appears in expression (4) for the equilibrium condition is a consequence of having introduced the sound velocity to obtain dimensionless parameters. This, as well as the other definitions of the parameters that will be given later on, are useful for our purposes, but one must be careful when considering the incompressible and very low compressibility limits.

We shall consider an adiabatic behaviour of the motions with the exponent $\gamma$. It is shown in Paper I that the linear normal modes of this equilibrium which involve vertical displacements of the plasma consist of perturbations of the form

$$\zeta = \zeta_0 \exp \left[ q(1 - \Gamma)y + i(k_x x + k_z z - \omega t) \right]$$

where $\zeta$ represents the vertical displacement of the plasma, $\zeta_0$ is a constant amplitude, $k = (k_x, 0, k_z)$ is the vector wavenumber that describes the modulation of the plasma–vacuum boundary, $\omega$ is the frequency of the perturbation, and $\Gamma$ is given by:

$$\Gamma^2 = 1 - \frac{-4\hat{g}(V_A^2 - (1 + M_A^2 - V_b^2)\nu^2) - 4\hat{g}^2(V^2 - V_A^2)}{(V^2 - V_A^2)(1 + M_A^2)(V^2 - V_b^2)}.$$  

In (6) we have introduced the following dimensionless quantities:

$$V = \omega/kC_s, \quad V_A^2 = \frac{1}{2} \left\{ 1 + M_A^2 \pm \sqrt{(1 + M_A^2)^2 - 4M_A^2 \cos^2 \psi} \right\},$$

$$V_b^2 = M_A^2 \cos^2 \psi, \quad V_A^2 = M_A^2 \cos^2 \psi/(1 + M_A^2),$$

$$\chi = q/k, \quad \cos \psi = k \cdot B/kB.$$  

According to Paper I the perturbations can be classified in two types: (a) internal modes, corresponding to $\Gamma$ pure imaginary, and (b) surface modes, that have $\Gamma$ real.
Internal modes were studied in the reference mentioned above, where their stability is discussed; they may exist even if the plasma is bounded by rigid conducting walls. Surface modes, on the other hand, always require a motion of the interface, so that their existence is precluded if the plasma boundaries are rigid, then in this case $\Gamma$ is related to the penetration depth of the perturbation in the plasma. As we shall show later on, these modes should be considered when one is concerned with the stability of interfaces, since their growth rates are larger than those of the unstable internal modes with the same wavenumber. For completeness it should be said that modes of a third type are also possible, they consist of purely horizontal motions and are always stable (Gratton and Gonzalez, 1989). They are the equivalent for a plasma of the Lamb mode, well known for an ordinary gas atmosphere (e.g. see Eckart, 1960). They will not be studied here, since they are not relevant to the stability analysis.

3. BOUNDARY CONDITIONS AT A PLASMA–VACUUM INTERFACE AND DISPERSION RELATION

The continuity of the total (kinetic plus magnetic) pressure across the perturbed plasma–vacuum interface leads to the linearized boundary condition:

$$
\left[ g\rho\zeta - \delta p - \frac{\delta \mathbf{B} \cdot \mathbf{B}}{4\pi} \right]_{y = 0^+} = \left[ -\frac{\delta \mathbf{B}_y \cdot \mathbf{B}_x}{4\pi} \right]_{y = 0^-}
$$

(8)

where the prefix $\delta$ indicates a perturbation of the corresponding quantity. The expressions for $\delta p$ and $\delta \mathbf{B}$ in terms of $\zeta$, which we omit for brevity, can be derived (see Paper I) from the linearized version of equations (1). The expression for $\delta \mathbf{B}$, as a function of $\zeta$ can be obtained considering that: (a) the magnetic field must be irrotational inside the vacuum region, which extends from the interface to $y \to -\infty$, (b) its normal component (with respect to the perturbed interface) must be continuous, and (c) the magnetic field lines are frozen in the plasma (for a detailed discussion see Gonzalez, 1988, also Gratton and Gonzalez, 1986). One then obtains in a linear approximation:

$$
\left[ -\frac{\delta \mathbf{B}_y \cdot \mathbf{B}_x}{4\pi} \right]_{y = 0^-} = \left[ \frac{(k \cdot \mathbf{B}_x)^2}{4\pi k} \right]_{y = 0^-}.
$$

(9)

Introducing the expressions for $\delta p$, $\delta \mathbf{B}$, and (9) in (8) yields:

$$
h \left( \frac{1}{M} - 1 \right) \frac{d\zeta}{dy} + \frac{k^2 g}{\omega^2} \left( 1 - \frac{h}{M} \right) \zeta = \frac{k(k \cdot \mathbf{B}_x)^2}{4\pi \rho \omega^2} \zeta.
$$

(10)

In equation (10) all quantities must be evaluated at $y = 0$, and the following definitions have been used:

$$
M = 1 - \frac{k^2 (C_A^2 + C_S^2)}{\omega^2} + \frac{k^4 C_S^2 C_A^2}{\omega^4} \cos^2 \psi = (V^2 - V^2_+) (V^2 - V^2) V^{-4}
$$
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\[ h = \left( 1 - \frac{k^2 C_s^2 \cos^2 \psi}{\omega^2} \right) = (V^2 - V_s^2) V^{-2}. \]  

(11)

We notice that the boundary condition (10) is neither restricted to an isothermal profile nor to a unidirectional magnetic field in the unperturbed state, but is quite general.

For internal modes (\( \Gamma \) imaginary) the appropriate solution for \( y > 0 \) is of the form:

\[ \zeta \mid \begin{align*} & = (A_+ e^{i k y} + A_- e^{-i k y}) \ e^{qy + i(k y + k_x - \omega t)} \end{align*} \]  

(12)

with \( k_y = -iq\Gamma \), and \( A_+ , A_- \) arbitrary constants. Each one of the terms of the r.h.s. of (12) corresponds to a non-homogeneous [due to the factor \( e^{qy} \)] hydromagnetic wave, which is analogous to a plane wave in a uniform plasma. The non-homogeneity is a consequence of the exponential stratification of the basic state. It can be verified that the energy density of each individual wave in (12) is constant in space. The solution (12) represents an incident wave which is reflected at the interface. When the boundary condition (10) for the plasma–vacuum interface is applied to this solution, one obtains a relationship between the amplitudes of the incoming and reflected waves. It can be shown that the reflection coefficient in this case is unity, as it obviously should be for physical reasons. It should also be evident that the presence of an interface cannot change the stability properties of the internal modes, which are determined by conditions inside the plasma (see for example, Newcomb, 1961, also Paper I), so that all the results of Paper I can be used for the present configuration.

For surface modes (which are also possible due to the presence of the interface), the situation is quite different since now \( \zeta \) must be of the form (5) with \( \Gamma \) real and positive. Only the positive root of (6) can be accepted since the energy density of the mode, \( \rho (\partial \zeta / \partial t)^2 \), must tend to zero as \( y \to \infty \). Using (5) in (10) one obtains

\[ \frac{V^2 h}{2 \dot{\gamma}} \left( \frac{1}{M} - 1 \right) (1 - \Gamma) + \left( 1 - \frac{h}{M} \right) - \frac{S}{2 \dot{\gamma}} = 0 \]  

(13)

where \( S \) is defined by

\[ S = \frac{B_s^2 \cos^2 \psi_s}{4 \pi \rho_s C_s^2} = 2 \dot{\gamma} \cos^2 \psi_s, \quad \cos \psi_s = k \cdot B_s / k B_s . \]  

(14)

Equation (13), with \( \Gamma \) defined as the positive definite root of (6), is the dispersion relation for the surface modes of a plasma–vacuum interface.

The standard procedure for solving a dispersion relation of this type is to transform it into a polynomial in the dimensionless phase velocity \( V \) (or in the dimensionless wavenumber \( 1/\alpha \)). This involves taking the square of \( \Gamma \), so that some of the roots of the polynomial may correspond to negative values of \( \Gamma \), which are not solutions of the dispersion relation (13), these spurious roots should be discarded. In addition, other solutions of the polynomial are related to negative values of \( \Gamma^2 \) so that they do not correspond to surface modes, but to internal ones.
After some algebra one obtains from (13) and (6) the polynomial:

\[ a^{-2}S'(V'-V_i)(V'-V_l) + (V^2 - V_A^2)(V^2 - V_B^2) - 2S_\pi^{-1}\{ (V^2 - V_A^2) \times (1 + M_i)(V^2 - V_0^2) + 2\theta [V_A^2 - (1 + M_A^2 - V_0^2)V^2] \} - 4\theta^2(1 + M_A^2)(V^2 - V_0^2) = 0, \]  

(15)

with

\[ V_0^2 \equiv V_A^2(1 + V_A^2)/(1 + M_A^2). \]  

(16)

The l.h.s. of (15) is a cubic polynomial in \( V' \). For a fixed \( a \) there are three roots, but in certain intervals of the parameters some of them may be spurious, or correspond to internal modes. The remaining solutions define the three branches of the spectrum of surface modes: slow, intermediate and fast, as will be shown below. For practical purposes, however, we observe that as (15) is quadratic in \( a \), it is actually easier to solve it analytically for \( a(V') \) and then find \( V^2 \) by a graphical (or numerical) inversion.

Although this problem admits an analytic solution in closed form the expressions are exceedingly awkward and not very illuminating, so that it is preferable to concentrate our discussion on some important special situations and limiting cases to elucidate the effect of compressibility and stratification on the properties of surface modes. Other solutions of the dispersion relation can be obtained in a straightforward way whenever necessary.

4. EFFECTS OF VERY SMALL COMPRESSIBILITY

Let us first consider the limiting case of negligible compressibility. For the solenoidal perturbations of a stratified plasma, it is easy to obtain from (6) and (13) by taking the limit \( \gamma \rightarrow \infty \):

\[ v_f^2 = \frac{\omega^2}{k^2} = C_A^2 \cos^2 \psi + \frac{g}{k} \cos^2 \psi, \quad \pm \sqrt{\left( \frac{g}{k} \cos^2 \psi \right)^2 + \left( \frac{(B_c)^2}{4\pi \rho_0} \cos^2 \psi - \frac{g}{k} \right) \frac{\mu}{2}}, \]  

(17)

where \( v_f \) is the phase velocity of the perturbation and it should be remembered that the sign of \( \Gamma \) must be checked in order to discard spurious roots, as said before.

We now discuss some special cases of equation (17), as follows.

(a) Flute modes with respect to the vacuum magnetic field (\( \cos \psi_v = 0 \)) : the stability criterion is given by

\[ C_A^2 \cos^2 \psi > \frac{g}{k}. \]  

(18)

This result is the same as for an incompressible uniform plasma (Chandrasekhar, 1961). This is a consequence of the particular stratification we are considering, which has uniform magnetic tensions throughout the plasma.

(b) Flute modes with respect to the plasma magnetic field (\( \cos \psi = 0 \)) : it can be seen from equation (17) that \( v_f \) is zero when \( k = k_{cs} \) with \( k_{cs} \) given by
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Using (6) and (13) it can be verified that if \( k > k_{cs} \), the root corresponding to the minus sign in (17) is spurious, so that the unstable modes are obtained only for \( k < k_{cs} \). The stability condition is then

\[
\frac{B_z^2}{4\pi \rho_0} \cos^2 \psi \nu > \frac{g}{k}
\]

as in the analogous case of a uniform plasma.

For arbitrary \( \cos \psi \), it can be shown that modes with \( k > k_{cs} \), where

\[
k_{cs} = \frac{L(g+zkC_s^2 \cos^2 \psi) - \sqrt{L^2(g+zkC_s^2 \cos^2 \psi)^2 - g^2(L^2 - C_s^4 \cos^4 \psi)}}{L^2 - C_s^4 \cos^4 \psi}
\]

are stable.

Although the stability criteria (18) and (20) are not affected by the stratification, the growth rate of the unstable modes, and the phase velocity of the stable ones are not equal to those of a uniform plasma.

If the compressibility is small, but not negligible, we can expand the phase velocity as a series of inverse powers of \( C_s \):

\[
v^2 = v_0^2 + \frac{v_1^2}{C_s^2} + \ldots
\]

Using this expansion in the polynomial (15) and keeping the lowest order corrections to the incompressible case one obtains

\[
v_1^2 = -\frac{1}{2} \frac{g}{k} \left( \frac{gL}{g} v_0^2 - C_s^2 \cos^2 \psi \right) \left( \frac{kL}{g} v_0^2 - C_s^2 \cos^2 \psi \right) - \cos^2 \psi \nu \left( v_0^2 - C_s^2 \cos^2 \psi \right)
\]

In equation (23), \( v_0 \) is given by (17). It can be observed that when \( v_0 \) is pure imaginary (unstable modes), correction (23) increases the growth rate, so that to first approximation the introduction of compressibility tends to destabilize the plasma.

It may be interesting to ascertain how a small compressibility alters the domain of instability. It can be found (we omit details for brevity) that to the lowest order correction, \( k_{cs} \) is increased by an amount.
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\[ \Delta k = \frac{1}{2} \left( g \frac{C_A}{C_S} \cos \psi \right)^2 \left[ L^2 (g + xkC_A^2 \cos^2 \psi)^2 - g^2 (L^2 - C_A^4 \cos^4 \psi) \right]^{-1/2} \]  

(24)

so that the instability domain is enlarged.

Summarizing, compressibility destabilizes the modes, through an increase of the growth rates and an enlargement of the unstable \( k \) interval. This result is similar to that obtained in Paper I for the internal modes of the same plasma configuration, and is in agreement with the comparison theorem of NEWCOMB (1983).

5. FINITE COMPRESSIBILITY EFFECTS

The analysis of the dispersion relation in the general case when compressibility is finite and \( B_y, B_z \) and \( k \) have arbitrary directions does not present any special difficulty, but requires rather cumbersome algebra. It can be carried out whenever required following a procedure analogous to that used in Paper I. We shall briefly describe the main steps involved, which will later be developed in detail when discussing specific configurations.

It can be recognized that the dispersion relation involves \( V^2, z \), and four independent dimensionless parameters that can be taken to be \( M_A, g, \psi \) and \( \psi' \). Of these, three \((M_A, g \) and \( \psi - \psi')\) are only dependent on the plasma and magnetic field configurations in the equilibrium, being independent of the perturbation. Then, for a given unperturbed state, the dispersion relation depends, in general, on three dimensionless variable parameters: \( V^2 \) (related to the phase velocity), \( z \) (related to the wavenumber), and an angle (e.g. \( \psi \)) that specifies the orientation of \( k \). To avoid the complications of interpreting a three-dimensional graph, in our previous work we represented the dispersion relation for internal modes in the plane \((V^2, z^{-2})\), for fixed values of \( \psi \) (notice that only the \( z^{-2} > 0 \) half plane is physically meaningful). In the present case, we shall use the same type of graph, in order to make the comparison between the surface and the internal modes easier.

As the dispersion relation for surface modes contains terms in \( z^{-1} \), the polynomial (15) yields two branches of \( z^{-1}(V^2) \) that can be represented as curves in the \((V^2, z^{-2})\) plane. However, only certain portions of these curves actually represent the solutions of the dispersion relation [equation (13)], for two reasons, as follows.

(a) By definition \( z \) is a positive quantity, so that any negative \( z^{-1} \) solution of (15) should be discarded.

(b) The solutions of (15) include those of equation (13) for both positive and negative values of \( \Gamma \); they can then be classified in two branches: the "true" branch (TB), which comprises the roots of (13) with \( \Gamma \) positive, and a spurious branch (SB) encompassing the roots of (13) with negative \( \Gamma \) values. The latter should of course be discarded.

The true and spurious branches are disconnected, except for those isolated points where \( \Gamma \) [equation (6)] is zero, or passes through infinity, where they may touch each other. The loci \( L \) defined by the zeros and poles of \( \Gamma \) are precisely the boundaries of the regions of the \((V^2, z^{-2})\) plane corresponding to the spectrum of the internal modes (see also Paper I) for \( z^{-2} > 0 \). Then the contact between branches can occur only at those points where they touch the boundaries of the spectrum of the internal modes.
It is easy to ascertain in which regions of the \((V^2, x^{-2})\) plane the true branch lies. They are given by the condition \(\Gamma(V^2, x^{-2}) > 0\), i.e.

\[
P(V, x) = 1 + \frac{2\delta [V^2_0 - (1 - V^2_0) V^2] - S x^{-1} (1 + M^2_1)^{-1} (V^2 - V^2_0) (V^2 - V^2_0)}{(V^2 - V^2_0)(V^2 - V^2_0)} > 0
\]

as can be verified from (13). The curves \(P\) defined by zeros and poles of \(P(V, x)\), with \(x > 0\), separate regions of the plane in which equation (13) admits solutions for positive \(\Gamma\) from regions where this is not possible. The true branch lies entirely within the former region. Therefore the points of contact between the true and the spurious branch are given by the intersections of the \(L\) and \(P\) curves. In this way, it is possible to obtain the spectrum for arbitrary equilibrium configurations and any value of \(k\). Some examples of this procedure will be given later on when discussing specific instances.

It can be seen that the spectrum of surface modes usually consists of three disjoint parts which can be called the slow (SS), intermediate (IS) and fast (FS) surface modes. This is analogous to what happens with the internal modes, in which one also finds three spectral bands. The difference is that for each value of \(x^{-2}\) one finds here discrete values of \(V^2\), while for internal modes one has continuous ranges of \(V^2\), i.e. bands. Then the spectrum of surface modes is represented by curves in our diagram while that of the internal modes consists of areas.

As for internal modes, unstable surface modes can arise from the slow surface portion of the spectrum. In the present case, however, one always finds instability for sufficiently long wavelengths \((x^{-2} \to 0)\), while for the internal modes the instability may not exist for certain ranges of \(\beta\) and \(\psi\) (see Paper I). On the other hand, as \(x^{-2} \sim k^2\) increases, the stabilizing effect of the magnetic tensions increases, and for sufficiently large values of \(k\) \((k > k_c)\) there are no unstable surface oscillations. In this connection, two situations may arise, according to the location of the intersection of \(P\) and \(L\), as illustrated in the diagram (Fig. 1).

(a) \(L\) and \(P\) intersect for positive \(V^2\). In this case, as \(k\) increases and reaches \(k_c\), the growth rate becomes zero. For \(k > k_c\) there is a stable slow surface mode, which disappears at point 1 of the graph, where \(P\) and \(L\) intersect. (b) \(L\) and \(P\) cross each other for negative \(V^2\) (point 1' in the graph). In this case the unstable mode disappears at this point, where the growth rate has a finite value. There is no stable slow surface mode for \(k > k_c\).

It can be noticed that when an unstable surface mode exists its growth rate is larger than that of any internal mode having the same \(k\). In case (a), there is an interval of \(k\) where the only possible unstable modes are surface perturbations, as the internal modes are stable. However, in case (b), there is a quite different situation, since now for a certain \(k\) interval there are no slow surface modes, but some unstable internal modes are still present. This means that for a plasma configuration with an interface, as we are considering, the surface modes are generally more critical for stability than the internal ones.

Let us call \(x_0^{-2}\) the ordinate of the intersection of \(P\) with the \(V^2 = 0\) axis, given by

\[
x_0^{-2} = \left(\frac{V^2_0 + 2\delta}{S}\right).
\]
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Fig. 1.—Sketch of the spectrum of unstable surface modes showing the location of the critical \( \alpha \): TB and SB represent the true and spurious branch, respectively (see text for explanation). (a) \( L \) and \( P \) intersect for \( V^2 > 0 \); the growth rate vanishes at \( \alpha_{ci} \); there is a stable slow surface mode. (b) \( L \) and \( P \) intersect for \( V^2 < 0 \); the growth rate is finite for \( \alpha = \alpha_{ci} \); in the interval \( \alpha_{ci} < \alpha < \alpha_{ci}^* \) there is no slow surface mode, but there are unstable internal modes.

The critical \( k \) where the internal modes become stable (intersection of \( L \) with the \( V^2 = 0 \) axis) is given by

\[
\alpha_{ci}^{-2} = \left[ 4 \frac{\hat{g}(\hat{g} - 1)}{V^2} - 1 \right]
\]

(27)

as can easily be shown. Clearly when \( \alpha_{ci} > \alpha_0 \) one has case (a), while the opposite
situation leads to case (b). One finds in the first case that the slow surface modes become unstable for \( \alpha_{cs} < \alpha \), where

\[
\alpha_{cs} = \frac{S(V_A^2 + 2\delta) - V_A^2 \sqrt{4\delta^2(1 + V_A^2) - S^2\alpha^{-2}}}{S^2 - V_A^4}.
\]  

(28)

This expression is obtained by solving (15) for \( V^2 = 0 \); only the root given by (28) has to be considered since the other one (corresponding to the plus sign in front of the radical) is either negative, or leads to a negative \( \Gamma \).

In case (b) it is not possible to find a simple formula for \( \alpha_{cs} \), as this requires one to find the intersections of \( P \) and \( L \), which are given by the roots of a third-degree polynomial in \( V^2 \) or \( \alpha^{-2} \). Nevertheless, \( \alpha_0^{-2} \) may be used as an upper bound of \( \alpha_{cs}^{-2} \) in this case.

It can be verified that if

\[
4\delta^2(1 + V_A^2) < \alpha_0^{-2},
\]  

(29)

expression (28) has no real values, and therefore the situation corresponds to case (b). If on the contrary:

\[
4\delta^2(1 + V_A^2) > \alpha_0^{-2}
\]  

(30)

and then (28) has a real value, the situation may correspond to case (a) or (b) according to the relation between \( \alpha_0 \) and \( \alpha_{cs} \), as mentioned above.

It can be observed that the stability of the surface modes depends on the magnetic tensions in the plasma, as well as on the vacuum magnetic tensions (to which \( S \) is related); on the other hand, the stability of internal modes depends only on the plasma magnetic tensions. When the magnetic tensions within the plasma are weak, as compared to the vacuum ones, it is possible to have unstable internal modes, but no surface instability [i.e. case (b)].

Let us now consider some special cases in more detail.

5.1. **Flute modes with respect to the vacuum magnetic field**

In this case \( \cos \psi_v = 0 \) (\( S = 0 \)), so that the vacuum magnetic field has no effect on the perturbations, and the stability of the surface modes is determined only by plasma properties. The roots of the polynomial (15) are represented in the \( (V^2, \alpha^{-2}) \) plane by the curve

\[
\alpha^{-2} = 4\delta^2 \frac{V^2 - V_0^2}{(V^2 - V_A^2)(V^2 - V_0^2)}.
\]  

(31)

Since we are now in the case (a) discussed above, the stability criterion is given by

\( k > k_{cs} \), where
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\[ k_{cs} = \frac{g \left( 1 + \frac{C_A}{C_S} \cos^2 \psi \right)^{1/2}}{C_A \cos^2 \psi} . \]  

(32)

As for the case of small compressibility (Section 4), we find that the presence of a finite \( C_S \) leads to an enlargement of the interval of unstable modes with respect to the incompressible case. Notice that formula (32) is valid for any value of \( C_S \), unlike (24). It can be observed that for \( C_S \to 0 \), \( k_{cs} \to \infty \) for any finite value of \( C_A \). One concludes that the stabilizing effect of the magnetic tensions is opposed by the destabilizing effect of compressibility, so that as compressibility is increased larger magnetic fields are required to achieve stability.

The growth rate is given by a cubic polynomial in \( \omega^2 \) which reduces to a quadratic form in two special cases, as follows.

(i) \( \cos \psi = 0 \), when \( \omega^2 = \pm k g \). Then \( \omega \) does not depend on \( C_S \) (but the penetration depth still depends on the compressibility). This case is similar to that of an ordinary gas.

(ii) \( \cos \psi = 1 \), one then obtains from equation (15)

\[ \omega^2 = \frac{C_A^2 k^2 (C_A^2 + 2C_S^2)}{2(C_A^2 + C_S^2)} \pm \frac{g^2 k^2 + \frac{C_A^8 k^4}{4(C_A^2 + C_S^2)^2}}{ \pm \sqrt{g^2 k^2 + \frac{C_A^8 k^4}{4(C_A^2 + C_S^2)^2} }} \]  

(33)

and

\[ \omega^2 = C_A^2 k^2 . \]  

(34)

The instability corresponds to the minus sign in (33). The slow surface mode [minus sign in (33)] exists for all values of \( k \). Its critical value for stability is given by (28). The maximum growth rate of the instability is given by:

\[ n_m^2 = -\omega_m^2 = \frac{g^2 (C_A^2 + C_S^2)}{C_A^2 [C_A^2 + 2C_S^2 + 2C_S (C_A^2 + C_S^2)^{1/2}]} \]  

(35)

which occurs for the value

\[ k_m^2 = \frac{g^2 (C_A^2 + C_S^2)^{3/2}}{C_A^2 C_S [C_A^2 + 2C_S^2 + 2C_S (C_A^2 + C_S^2)^{1/2}]} . \]  

(36)

For \( C_S \) decreasing from \( \infty \) to 0 (i.e. compressibility varying in its whole range), \( n_m \) increases by a factor of 2 from its value for the incompressible case \( (g/2C_A) \).

For \( \cos \psi = 1 \), the curve \( P \) degenerates into two vertical lines: (i) \( V^2 = V_2^2 \) and (ii) \( V^2 = V_2^2 \) with

\[ V_2^2 = \frac{2 \left( M_A + \frac{1}{\gamma} \right)}{M_A^2 + 1} . \]  

(37)
It can be verified that the true branch lies in the regions $V^2 > V_2^2$ and $V^2 < V_\alpha^2$. Then, two cases are possible, as follows. (a) $M_\alpha^2 > V_2^2$, i.e.

$$M_\alpha^2(M_\alpha^2 - 1) > \frac{2}{\gamma} \quad (38)$$

in which case the root (34) corresponds to the intermediate surface mode; the fast surface mode, given by the $(+)$ sign in (33), also exists for all values of $k$. (b) $M_\alpha^2 < V_2^2$; then (34) gives the intermediate internal mode, and there is no intermediate surface mode. The fast surface mode exists for small $k$ values, as shown in Fig. 2. The lower bound of the phase velocity of this mode is given by $V_2^2$, which marks in the diagram the disappearance of the modes.

For other values of $\cos\psi$ a graphical analysis can be carried on as indicated at the beginning of this section. The typical aspect of the spectrum is represented in Fig. 3. It can be observed that the slow surface mode exists for all values of $k$, and corresponds to $V^2 < V_\alpha^2$. The same is true for the intermediate surface mode, which lies in the interval $V_\alpha^2 < V^2 < V_2^2$, and is always present except in the degenerate cases $\cos^2\psi = 0, 1$. The spectrum of this mode has a positive slope. This is associated with anomalous dispersion, as is characteristic of ordinary gravity capillary waves for short wavelengths. In the present case, the magnetic tensions are playing a role analogous to that of ordinary surface tension. The fast surface mode does not exist for large $k$ values. Notice that, for $S = 0$, $P(V, \alpha) = 0$ on the two vertical lines $V_\alpha^2$, and $V_2^2$ given by:

$$V_{\alpha,2}^2 = \frac{1}{4} \left\{ V_\alpha^2 + V_\alpha^2 + 2\delta(1 - V_\alpha^2) \pm \sqrt{[V_\alpha^2 + V_\alpha^2 + 2\delta(1 - V_\alpha^2)]^2 - 4V_\alpha^2(V_\alpha^2 + 2\delta)} \right\}. \quad (39)$$

![Fig. 2.—Spectrum of the surface (full lines) and of the internal modes (hatched areas) for flute perturbations with respect to the vacuum magnetic field ($V_\alpha^2 = M_\alpha^2 = 1.1, \gamma = 5/3$). The dotted lines correspond to the spurious branch.](image-url)
The $V^2$ intervals corresponding to the surface modes are represented as functions of $M^2_{\lambda}$ for a fixed value of $\cos \psi$ in Fig. 4. It can be observed that the fast surface mode has a lower boundary given by the contact of $P$ with $L$ at $V^2\lambda$, which is always larger than $V^2\pi$. Then this point is located on the boundary of the fast internal mode spectrum. As $V^2_B < V^2_\perp < V^2_\parallel$, the vertical line $V^2_\parallel$ does not intersect $L$ in the upper half plane, so that it is of no consequence in limiting the spectra of the intermediate and slow surface modes. For any fixed $k$, the phase velocity of any stable surface mode is always smaller than that of the corresponding internal modes.

5.2. **Flute modes with respect to the plasma magnetic field**

Now, the magnetic tensions within the plasma have no effect on the perturbations. For $\cos \psi = 0$, but an arbitrary $\psi_v$, one has $V^2 = V^2_{\lambda} = V^2_{\parallel} = 0$ so that the analysis is simplified. The slow surface mode is unstable for $k < k_{cs}$, with $k_{cs}$ given by (19), as for the incompressible case; notice, however, that the growth rate is different, increasing with compressibility. A typical spectrum is shown in Fig. 5 where the three surface modes are shown as well as the internal ones. The slow mode is always unstable, and exists only for $k < k_{cs}$. The intermediate surface mode exists for large $k$ values, and disappears at a certain $k^*$ (see Fig. 5), larger than $k_{cs}$. It can be observed that the upper portion (in the diagram) of the spectrum of this mode has a positive slope, and a small lower portion of it has a negative slope. In the long wavelength range of the spectrum, just before this mode disappears, the effect of the vacuum magnetic tensions is small and the dispersion becomes normal. The fast surface mode exists only for long wavelengths; the upper bound for $k$ is lower than $k_{cs}$. Therefore, there is a small interval $k_{cs} < k < k^*$ for which surface modes are impossible.
Fig. 4.—Intervals of existence of surface modes for flute perturbations with respect to the vacuum magnetic field, as functions of $M_A^2$ (hatched areas). The value of $\psi$ is fixed ($\cos^2 \psi = 0.8$).

Fig. 5.—Spectrum of the surface (full lines), and of the internal modes (hatched areas) for flute perturbations with respect to the plasma magnetic field ($M_A^2 = 1$, $\cos^2 \psi = 0.5$, $\gamma = 5/3$). The dotted lines correspond to the spurious branch.
If \( M_A = 0 \), and \( \gamma = 1 \), the present case coincides with a situation studied by Parks (1983).

6. CONCLUSIONS

This paper complements previous research on internal gravity modes (Paper I). We analyze here the surface gravity modes that may appear at an interface between a stratified, compressible plasma and a vacuum magnetic field supporting it. It is shown that if the compressibility is small, there is an increase of the growth rate and of the instability domain of the Rayleigh–Taylor modes, with respect to the incompressible limit. A general graphical procedure to derive the properties of the modes for finite compressibility is presented. Our main results may be summarized as follows.

(1) There are in general three surface gravity modes, called “fast”, “intermediate” and “slow”, according to their diminishing phase velocities. Only the slow mode may lead to instability.

(2) For a given plasma and magnetic field equilibrium configuration, and for a fixed orientation of the wavenumber of the perturbation, the modes have an unstable behaviour if \( k \) is smaller than a critical value \( k_{cs} \). Unlike the classical incompressible case, for certain configurations the unstable mode has a finite growth rate in the limit \( k \to k_{cs} \), where it disappears (at this point the penetration depth becomes infinity, but \( V^2 \) is finite). This happens when the magnetic tensions within the plasma are weak, as compared with the vacuum ones.

(3) The growth rate of the instability, when it exists, is larger than that of the unstable internal modes with the same \( k \). Nevertheless, notice that in the above-mentioned situation, there is an interval of \( k \), for \( k > k_{cs} \), in which there are only unstable internal modes but no surface mode.

(4) Generally speaking, an increase in the compressibility results in an increase in the growth rate and in \( k_{cs} \).

(5) Usually the dispersion of the intermediate gravity surface mode is anomalous; however, for sufficiently long wavelengths, and when the vacuum magnetic tensions prevail with respect to the internal ones, the dispersion may be normal.

(6) The fast gravity surface mode, when it exists, has a phase velocity smaller than that of the fast internal modes for the same \( k \). For very short wavelengths this surface mode does not exist, except for some special degenerate cases.

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